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Improved Bounds for Incidences Between Points and Circles

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We establish an improved upper bound for the number of incidences between m points and n circles in three dimensions. The previous best known bound, originally established for the planar case and later extended to any dimension $\geqslant 2$, is $O^*(m^{2/3}n^{2/3} + m^{6/11}n^{9/11} + m + n)$, where the $O^*(\cdot)$ notation hides polylogarithmic factors. Since all the points and circles may lie on a common plane (or sphere), it is impossible to improve the bound in $\mathbb{R}^3$ without first improving it in the plane.

Nevertheless, we show that if the set of circles is required to be 'truly three-dimensional' in the sense that no sphere or plane contains more than q of the circles, for some $q \ll n$, then for any $\varepsilon > 0$ the bound can be improved to

$$O\big(m^{3/7+\varepsilon}n^{6/7} + m^{2/3+\varepsilon}n^{1/2}q^{1/6} + m^{6/11+\varepsilon}n^{15/22}q^{3/22} + m + n\big).$$

For various ranges of parameters (*e.g.*, when $m = \Theta(n)$ and $q = o(n^{7/9})$), this bound is smaller than the lower bound $\Omega^*(m^{2/3}n^{2/3} + m + n)$, which holds in two dimensions.

We present several extensions and applications of the new bound.

(i) For the special case where all the circles have the same radius, we obtain the improved bound $O(m^{5/11+\varepsilon}n^{9/11} + m^{2/3+\varepsilon}n^{1/2}q^{1/6} + m + n)$.
(ii) We present an improved analysis that removes the subpolynomial factors from the bound when $m = O(n^{3/2-\varepsilon})$ for any fixed $\varepsilon > 0$.
(iii) We use our results to obtain the improved bound $O(m^{15/7})$ for the number of mutually similar triangles determined by any set of m points in $\mathbb{R}^3$.

Our result is obtained by applying the polynomial partitioning technique of Guth and Katz using a constant-degree partitioning polynomial (as was also recently used by Solymosi and

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Tao). We also rely on various additional tools from analytic, algebraic, and combinatorial geometry.

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1. Introduction

Recently, Guth and Katz [24] presented the *polynomial partitioning technique* as a major technical tool in their solution of the famous planar distinct distances problem of Erdős [18]. This problem can be reduced to an incidence problem involving points and lines in $\mathbb{R}^3$ (following the reduction that was proposed in [17]), which can be solved by applying the aforementioned polynomial partitioning technique. The Guth–Katz result prompted various other incidence-related studies that rely on polynomial partitioning (*e.g.*, see [9, 19, 22, 29, 30, 44, 45, 50]). One consequence of these studies is that they have led to further developments and enhancements of the technique itself (evidenced for example by the use of induction in [45], and the use of two partitioning polynomials in [29, 50]). Also, the technique was recently applied to some problems that are not incidence related. It was used to provide an alternative proof of the existence of spanning trees with small crossing number in any dimension [30], and to obtain improved algorithms for range searching with semi-algebraic sets [2]. Thus, it seems fair to say that applications and enhancements of the polynomial partitioning technique form an active contemporary area of research in combinatorial and computational geometry.

In this paper we study incidences between points and circles in three dimensions. Let $\mathcal{P}$ be a set of m points and let $\mathcal{C}$ be a set of n circles in $\mathbb{R}^3$. We denote the number of point–circle incidences in $\mathcal{P} \times \mathcal{C}$ as $I(\mathcal{P},\mathcal{C})$. When the circles have arbitrary radii, the current best bound for any dimension $d \geqslant 2$ (originally established for the planar case in [3, 8, 34], and later extended to higher dimensions by Aronov, Koltun and Sharir [7]) is

$$I(\mathcal{P},\mathcal{C}) = O^*\big(m^{2/3}n^{2/3} + m^{6/11}n^{9/11} + m + n\big). \tag{1.1}$$

The precise best known upper bound (see [34]) is

$$O\big(m^{2/3}n^{2/3} + m^{6/11}n^{9/11}\log^{2/11}(m^3/n) + m + n\big).$$

Since the three-dimensional case also allows $\mathcal{P}$ and $\mathcal{C}$ to lie on a single common plane or sphere,[1] the point–circle incidence bound in $\mathbb{R}^3$ cannot be improved without first improving the planar bound (1.1) (which has been an open problem for about ten years). Nevertheless, as we show in this paper, an improved bound can be obtained if the configuration of points and circles is 'truly three-dimensional' in the sense that no sphere or plane contains too many circles from $\mathcal{C}$. (Guth and Katz [24] impose a similar assumption on the maximum number of lines that can lie in a common plane or regulus; see also Sharir and Solomon [44].) Our main result is given in the following theorem.

[1] There is no real difference between the cases of coplanarity and cosphericality of the points and circles, since the latter case can be reduced to the former (and *vice versa*) by means of the stereographic projection.

Theorem 1.1. *Let $\mathcal{P}$ be a set of m points and let $\mathcal{C}$ be a set of n circles in $\mathbb{R}^3$, let ε be an arbitrarily small positive constant, and let $q \leqslant n$ be an integer. If no sphere or plane contains more than q circles of $\mathcal{C}$, then*

$$I(\mathcal{P},\mathcal{C}) = O\big(m^{3/7+\varepsilon}n^{6/7} + m^{2/3+\varepsilon}n^{1/2}q^{1/6} + m^{6/11+\varepsilon}n^{15/22}q^{3/22} + m + n\big),$$

where the constant of proportionality depends on ε.

Remarks. **(1)** In the planar case, the best known lower bound for the number of point–circle incidences is $\Omega^*(m^{2/3}n^{2/3} + m + n)$ (*e.g.*, see [39]). Theorem 1.1 implies that for certain ranges of m, n, and q, a smaller upper bound holds in $\mathbb{R}^3$. This is the case, for example, when $m = \Theta(n)$ and $q = o(n^{7/9})$.

(2) When $m > n^{3/2}$, we have $m^{3/7}n^{6/7} < m$ and $m^{6/11}n^{15/22}q^{3/22} < m^{2/3}n^{1/2}q^{1/6}$. Hence, we have

$$I(\mathcal{P},\mathcal{C}) = O\big(m^{2/3+\varepsilon}n^{1/2}q^{1/6} + m^{1+\varepsilon}\big).$$

In fact, as the analysis in this paper will show, the first term in this bound only arises from bounding incidences on certain potentially 'rich' planes or spheres. For $q = O(m^2/n^3)$ we have $I(\mathcal{P},\mathcal{C}) = O(m^{1+\varepsilon})$. Informally, for such small values of q, when the number of points is $\Omega(n^{3/2})$, a typical point can lie in only a small (nearly constant) number of circles.

(3) When $m \leqslant n^{3/2}$, any of the terms except for m can dominate the bound. However, if in addition $q = O((n^3/m^2)^{3/7})$, then the bound becomes

$$I(\mathcal{P},\mathcal{C}) = O\big(m^{3/7+\varepsilon}n^{6/7} + n^{1+\varepsilon}\big),$$

for any $\varepsilon > 0$. Note also that the interesting range of parameters is $m = \tilde{\Omega}(n^{1/3})$ and $m = \tilde{O}(n^2)$, where the $\tilde{\Omega}(\cdot)$ and $\tilde{O}(\cdot)$ notations hide terms of the form m^ε and n^ε; in the complementary ranges both the old and new bounds become (almost) linear in $m + n$. In the interesting range, the new bound is asymptotically smaller than the planar bound given in (1.1) for q sufficiently small (*e.g.*, when $q = O((n^3/m^2)^{3/7})$ as above), and as noted, it is also smaller than the best known worst-case lower bound in the planar case for certain ranges of m and n.

(4) Interestingly, the 'threshold' value $m = \Theta(n^{3/2})$ where a quantitative change in the bound takes place (as noted in remarks (2) and (3) above) also arises in the study of incidences between points and lines in $\mathbb{R}^3$ [16, 23, 24]. See Section 6 for a discussion of this threshold phenomenon.

The proof of Theorem 1.1 is based on the polynomial partitioning technique of Guth and Katz [24], where we use a constant-degree partitioning polynomial in a manner similar to that used by Solymosi and Tao [45]. (The use of constant-degree polynomials and the inductive arguments it leads to are essentially the only similarities with the technique of [45], which does not apply to circles in any dimension since it cannot handle situations where arbitrarily many curves can pass between any specific pair of points. Constant degree partitioning polynomials were also recently used in [22, 44].) The application of this technique to incidences involving circles leads to new problems, involving the handling of points that are incident to many circles that are entirely contained in the zero set of the partitioning polynomial. To handle this situation we turn these circles into lines using an inversion transformation. We then analyse the geometric and algebraic structure of the transformed zero set using a variety of tools such as *flecnode polynomials* (as used in [24]), additional classical nineteenth-century results in analytic geometry from [42] (mostly related to *ruled surfaces*), a very recent technique for analysing surfaces that are ‘ruled’ by lines and circles [38], and some traditional tools from combinatorial geometry. We note that while our results refer to the real affine case, part of our analysis considers the setup in complex projective spaces, in which more classical results from algebraic geometry can be brought to bear. By transporting the results back to the real affine case, we obtain the properties that we wish to establish. See Section 4 for details.

Removing the epsilons. One disadvantage of the current use of constant-degree partitioning polynomials is that it leads to an upper bound involving ε in some of the exponents (as stated in Theorem 1.1). In Section 3.1 we present an alternative approach, which uses partitioning polynomials of higher degree but requires a more involved analysis, for partially removing ε. It yields the following theorem.

Theorem 1.2. *Let $\mathcal{P}$ be a set of m points and let $\mathcal{C}$ be a set of n circles in $\mathbb{R}^3$, let $q \leqslant n$ be an integer, and let $m = O(n^{3/2-\delta})$, for some fixed arbitrarily small constant $\delta > 0$. If no sphere or plane contains more than q circles of $\mathcal{C}$, then*

$$I(\mathcal{P},\mathcal{C}) \leqslant A_{m,n}\big(m^{3/7}n^{6/7} + m^{2/3}n^{1/2}q^{1/6} + m^{6/11}n^{15/22}q^{3/22}\log^{2/11} m + m + n\big), \tag{1.2}$$

where

$$A_{m,n} = A^{\left\lceil \frac{3}{2}\cdot\frac{\log(m/n^{1/3})}{\log(n^{3/2}/m)}\right\rceil + 1},$$

for some absolute constant $A > 1$.

Note that $A_{m,n}$ grows slowly with the quantity $\log(m/n)$. For example, it is A for $m \leqslant n^{1/3}$, A^2 for $n^{1/3} < m \leqslant n^{4/5}$, and A^3 for $n^{4/5} < m \leqslant n$.

Recently, several other situations in which ε can be removed were described in [51]. Our alternative technique seems to be sufficiently general, and we hope that appropriate variants of it could yield similar improvements of other bounds that were obtained with constant-degree partitioning polynomials, such as the ones in [45].

Unit circles. In the special case where all the circles of $\mathcal{C}$ have the same radius, we derive the following improved bound.

Theorem 1.3. *Let $\mathcal{P}$ be a set of m points and let $\mathcal{C}$ be a set of n unit circles in $\mathbb{R}^3$, let ε be an arbitrarily small positive constant, and let $q \leqslant n$ be an integer. If no plane or sphere contains more than q circles of $\mathcal{C}$, then*

$$I(\mathcal{P},\mathcal{C}) = O\big(m^{5/11+\varepsilon}n^{9/11} + m^{2/3+\varepsilon}n^{1/2}q^{1/6} + m + n\big),$$

where the constant of proportionality depends on ε.

This improvement is obtained through the following steps.

(i) We use the improved planar (or spherical) bound $O(m^{2/3}n^{2/3}+m+n)$ for incidences with coplanar (or co-spherical) unit circles (*e.g.*, see [46]).
(ii) We show that the number of unit circles incident to at least three points in a given set of m points in $\mathbb{R}^3$ is only $O(m^{5/2})$.
(iii) We use this bound as a bootstrapping tool for deriving the bound asserted in the theorem. The full details are presented in Section 5.

Here too we can refine the bound of Theorem 1.3 and get rid of the ε in the exponents, for $m = O(n^{3/2-\varepsilon})$ for any $\varepsilon > 0$. The resulting refinement, analogous to that of Theorem 1.1, is given in Section 5.

An application: similar triangles. Given a finite point set $\mathcal{P}$ in $\mathbb{R}^3$ and a triangle Δ, we denote by $F(\mathcal{P},\Delta)$ the number of triangles that are spanned by points of $\mathcal{P}$ and are similar to Δ. Let

$$F(m) = \max_{|\mathcal{P}|=m,\Delta} F(\mathcal{P},\Delta).$$

The problem of obtaining good bounds for $F(m)$ is motivated by questions in exact pattern matching, and has been studied in several previous works (see [1, 4, 6, 12]). Theorem 1.2 implies the bound $F(m) = O(m^{15/7})$, which slightly improves upon the previous bound of $O^*(m^{58/27})$ from [6]; see also [1]. The new bound is an almost immediate corollary of Theorem 1.2, while the previous bound requires a more complicated analysis. This application is presented in Section 6.

2. Algebraic preliminaries

We briefly review in this section the machinery needed for our analysis, including the polynomial partitioning technique of Guth and Katz and several basic tools from algebraic geometry.

Polynomial partitioning. In what follows, we regard the dimension d of the ambient space as a (small) constant, and we ignore the dependence on d of the various constants of proportionality in the bounds. Consider a set $\mathcal{P}$ of m points in $\mathbb{R}^d$. Given a polynomial $f \in \mathbb{R}[x_1,\ldots,x_d]$, we define the *zero set* of f to be $Z(f) = \{p \in \mathbb{R}^d \mid f(p) = 0\}$. For $1 < r \leqslant m$, we say that $f \in \mathbb{R}[x_1,\ldots,x_d]$ is an *r-partitioning polynomial* for $\mathcal{P}$ if no connected component of $\mathbb{R}^d \setminus Z(f)$ contains more than m/r points of $\mathcal{P}$. Notice that there is no restriction on the number of points of $\mathcal{P}$ that lie in $Z(f)$.

The following result is due to Guth and Katz [24]. A detailed proof can also be found in [30].

Theorem 2.1 (polynomial partitioning [24]). *Let $\mathcal{P}$ be a set of m points in $\mathbb{R}^d$. Then, for every $1 < r \leqslant m$, there exists an r-partitioning polynomial $f \in \mathbb{R}[x_1,\ldots,x_d]$ of degree $O(r^{1/d})$.* □

To use such a partitioning effectively, we also need a bound on the maximum possible number of cells of the partition. Such a bound is provided by the following theorem.

Theorem 2.2 (Warren's theorem [48]; see also [5]). *For a polynomial $f \in \mathbb{R}[x_1,\dots,x_d]$ of degree k, the number of connected components of $\mathbb{R}^d \setminus Z(f)$ is $O((2k)^d)$.* □

Consider an r-partitioning polynomial f for a point-set $\mathcal{P}$, as provided in Theorem 2.1. The number of cells in the partition is equal to the number of connected components of $\mathbb{R}^d \setminus Z(f)$, which, by Theorem 2.2, is $O((r^{1/d})^d) = O(r)$ (recall that f is of degree $O(r^{1/d})$ and that d is treated as a fixed constant, in our case 3). It follows that the bound on the number of points in each cell, namely m/r, is asymptotically best possible.

We will also rely on the following classical result, somewhat similar to Warren's theorem (the following formulation is taken from [5]).

Theorem 2.3 (Milnor–Thom theorem [36, 47]). *Let V be a real variety in $\mathbb{R}^d$, which is the solution set of the real polynomial equations*

$$f_i(x_1,\dots,x_d) = 0 \quad (i = 1,\dots,m),$$

and suppose that the degree of each polynomial f_i is at most k. Then the number of connected components of V is at most $k(2k-1)^{d-1}$. □

Since this paper studies incidences in a three-dimensional space, we will only apply the above theorems for $d = 3$.

Real and complex varieties. Let $F = \mathbb{R}$ or $\mathbb{C}$ be the underlying field, and let $I \subset F[x_1,x_2,x_3]$ be an ideal. We define $Z(I) \subset F^3$ to be the variety corresponding to I; this is the common vanishing locus of all the polynomials $f \in I$. When it is not clear from the context whether $F = \mathbb{R}$ or $\mathbb{C}$, we resolve the ambiguity by writing $Z_{\mathbb{R}}(I)$ or $Z_{\mathbb{C}}(I)$, respectively. If $f \in F[x_1,x_2,x_3]$, we will abuse notation and write $Z(f)$ instead of $Z((f))$ (where (f) is the ideal generated by f). If $Z \subset F^3$ is a variety, we define $\mathbf{I}(Z) \subset F[x_1,x_2,x_3]$ to be the ideal of polynomials vanishing on Z (we will use bold typeface $\mathbf{I}$ to distinguish it from the inversion transform $I : \mathbb{R}^3 \to \mathbb{R}^3$ that will appear in Section 4).

Let $R = \mathbb{R}[x_1,\dots,x_d]$, and let $I \subset R$ be an affine ideal. We say that I is a *real ideal* if for every set $g_1,\dots,g_t \in R$, we have

$$g_1^2 + \ldots + g_t^2 \in I \implies g_i \in I \quad \text{for } i = 1,\dots,t.$$

Intuitively, algebraic geometry over the reals tends to be more pathological than algebraic geometry over $\mathbb{C}$. Many of the problematic cases do not occur if we only work with real ideals. For example, the following theorem presents two properties of real ideals.

Theorem 2.4.

(i) *Let $J \subset \mathbb{R}[x_1,\dots,x_d]$ be an ideal. Then $J = \mathbf{I}(Z_{\mathbb{R}}(J))$ if and only if J is a real ideal.*
(ii) *Let $f \in \mathbb{R}[x_1,\dots,x_d]$ be an irreducible polynomial. Then (f) is a real ideal if and only if* $\dim Z(f) = d-1$.

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A nice introduction to real ideals can be found in [11]. The two parts of the above theorem are Theorems 4.1.4 and 4.5.1 of [11], respectively.

Given a variety $Z \subset \mathbb{R}^3$, the *complexification* $Z^* \subset \mathbb{C}^3$ of Z is the smallest complex variety that contains Z (in the sense that any other complex variety that contains Z also contains Z^*, *e.g.*, see [41, 49]). As shown in [49, Lemma 6], such a complexification always exists, and Z is precisely the locus of real points of Z^*. More specifically, $Z^* = Z_{\mathbb{C}}(\mathbf{I}(Z))$.

According to [49, Lemma 7], there is a bijection between the set of irreducible components of Z and the set of irreducible components of Z^*, such that each real component is the real part of its corresponding complex component. Specifically, the complexification of an irreducible variety is irreducible.

Bézout's theorem. We also need the following basic property of zero sets of polynomials in the plane (for further discussion see [14, 15]).

Theorem 2.5 (Bézout's theorem). *Let f, g be two polynomials in $\mathbb{R}[x_1, x_2]$ or $\mathbb{C}[x_1, x_2]$ of degrees D_f and D_g, respectively.*

(i) *If $Z(f)$ and $Z(g)$ have a finite number of common points, then this number is at most $D_f D_g$.*
(ii) *If $Z(f)$ and $Z(g)$ have an infinite number of (or just more than $D_f D_g$) common points, then f and g have a common (non-trivial) factor.* □

The following result, also used below, is somewhat related to Bézout's theorem, and holds in complex projective spaces of any dimension (*e.g.*, see [21]; for a formal definition of three-dimensional complex projective space $\mathbb{C}\mathbf{P}^3$, in which we will apply the following theorem and other tools, see Section 4, and recall the comments made earlier concerning the passage between the real and complex setups).

Theorem 2.6. *Let Z_1 and Z_2 be pure-dimensional varieties (every irreducible component of a pure-dimensional variety has the same dimension) in d-dimensional complex projective space, with $\operatorname{codim} Z_1 + \operatorname{codim} Z_2 = d$. Then, if $Z_1 \cap Z_2$ is a zero-dimensional set of points, this set is finite.* □

The following lemma is a consequence of Theorem 2.5. Its proof is given in Guth and Katz [23, Corollary 2.5] and in Elekes, Kaplan and Sharir [16, Proposition 1].

Lemma 2.7 (Guth and Katz [23]). *Let f and g be two polynomials in $\mathbb{R}[x_1, x_2, x_3]$ (or in $\mathbb{C}[x_1, x_2, x_3]$) of respective degrees D_f and D_g, such that f and g have no common factor. Then there are at most $D_f D_g$ lines on which both f and g vanish identically.* □

Flecnode polynomial. A *flecnode* of a surface Z in $\mathbb{C}^3$ is a point $p \in Z$ for which there exists a line that passes through p and agrees with Z at p to order three. That is, if $Z = Z(f)$ and the direction of the line is $v = (v_1, v_2, v_3)$, then

$$f(p) = 0, \quad \nabla_v f(p) = 0, \quad \nabla_v^2 f(p) = 0, \quad \nabla_v^3 f(p) = 0, \tag{2.1}$$

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where $\nabla_v f, \nabla_v^2 f, \nabla_v^3 f$ are, respectively, the first-, second-, and third-order derivatives of f in the direction v.

If $Z = Z(f)$ is a surface in $\mathbb{R}^3$, we say that $p \in Z$ is a flecnode of Z if p is a flecnode of the corresponding complex surface $(Z(f))^*$.

The *flecnode polynomial* of f, denoted by FL_f, is the polynomial obtained by eliminating v from the last three equations in (2.1). Note that the corresponding polynomials of the system are homogeneous in v (of respective degrees 1, 2, and 3). We thus have a system of three equations in six variables. Eliminating the variables v_1, v_2, v_3 results in a single polynomial equation in $p = (x_1, x_2, x_3)$, which is the desired flecnode polynomial. By construction, the flecnode polynomial of f vanishes on all the flecnodes of $Z(f)$. The following results, also mentioned in [24, Section 3], are taken from Salmon [42, Chapter XVII, Section III].

Lemma 2.8. *Let $Z \subset \mathbb{R}^3$ be a surface, with $Z = Z(f)$ for a polynomial $f \in \mathbb{R}[x_1, x_2, x_3]$ of degree $d \geqslant 3$. Then FL_f is a real polynomial* (*i.e., an element of* $\mathbb{R}[x_1, x_2, x_3]$), *and it has degree at most* $11d - 24$. □

Definition. An algebraic surface S in three-dimensional space (we restrict our attention to $\mathbb{R}^3$, $\mathbb{C}^3$, and $\mathbb{C}\mathbf{P}^3$) is said to be *ruled* if every point of S is incident to a straight line that is fully contained in S. Equivalently, S is a (two-dimensional) union of lines.[2] We say that an irreducible surface S is *triply ruled* if for every point on S there are (at least) three straight lines contained in S and passing through that point. As is well known, the only triply ruled surfaces are planes (*e.g.*, see [20, Lecture 16]; while this reference only provides proofs for the case of $\mathbb{R}^3$, proofs for the cases of $\mathbb{C}^3$ and $\mathbb{C}\mathbf{P}^3$ are also known). We say that an irreducible surface S is *doubly ruled* if it is not triply ruled and for every point on S there are (at least) two straight lines contained in S and passing through that point. It is well known that the only doubly ruled surfaces are the hyperbolic paraboloid and the hyperboloid of one sheet (again, see [20, Lecture 16]). Finally, we say that an irreducible ruled surface is *singly ruled* if it is neither doubly nor triply ruled.

Lemma 2.9. *Let $Z \subset \mathbb{R}^3$ be a surface with $Z = Z(f)$ for a polynomial $f \in \mathbb{R}[x_1, x_2, x_3]$ of degree $d \geqslant 3$. Then every line that is fully contained in Z is also fully contained in $Z(\mathsf{FL}_f)$.*

Proof. Every point on any such line is a flecnode of Z, so FL_f vanishes identically on the line. □

Theorem 2.10 (Cayley–Salmon [42]). *Let $Z \subset \mathbb{R}^3$ be a surface with $Z = Z(f)$ for a polynomial $f \in \mathbb{R}[x_1, x_2, x_3]$ of degree $d \geqslant 3$. Then Z is ruled if and only if $Z \subseteq Z(\mathsf{FL}_f)$.* □

[2] We do not insist on the more restrictive definition used in differential (or in algebraic) geometry, which requires the ruling lines to form a smooth one-parameter family; see [10, Chapter III] and [26, Section V.2].

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Corollary 2.11. *Let $Z \subset \mathbb{R}^3$ be a surface with $Z = Z(f)$ for an irreducible polynomial $f \in \mathbb{R}[x_1, x_2, x_3]$ of degree $d \geqslant 3$. If Z contains more than $d(11d-24)$ lines then Z is a ruled surface.*

Proof. Lemmas 2.7 and 2.9 imply that in this case f and FL_f have a common factor. Since f is irreducible, f divides FL_f, and Theorem 2.10 completes the proof. □

A modern treatment of the Cayley–Salmon theorem can be found in a more recent work by Landsberg [32]. (The results in [32] are considerably more general, but we state here only the special case related to the Cayley–Salmon theorem.)

Theorem 2.12 (Landsberg [32]). *Let Z be a surface in $\mathbb{C}^3$, and let $Z = Z(f)$ for a polynomial f of degree $d \geqslant 3$. Then Z is ruled if and only if $Z \subseteq Z(\mathsf{FL}_f)$.* □

3. The main theorem

In this section we prove Theorem 1.1, which we restate for the convenience of the the reader.

Theorem 1.1. *Let $\mathcal{P}$ be a set of m points and let $\mathcal{C}$ be a set of n circles in $\mathbb{R}^3$, let ε be an arbitrarily small positive constant, and let $q \leqslant n$ be an integer. If no sphere or plane contains more than q circles of $\mathcal{C}$, then*

$$I(\mathcal{P},\mathcal{C}) = O\big(m^{3/7+\varepsilon}n^{6/7} + m^{2/3+\varepsilon}n^{1/2}q^{1/6} + m^{6/11+\varepsilon}n^{15/22}q^{3/22} + m + n\big),$$

where the constant of proportionality depends on ε.

Proof. The proof proceeds by induction on $m+n$. Specifically, we prove by induction that, for any fixed $\varepsilon > 0$, there exist constants α_1, α_2 such that

$$I(\mathcal{P},\mathcal{C}) \leqslant \alpha_1\big(m^{3/7+\varepsilon}n^{6/7} + m^{2/3+\varepsilon}n^{1/2}q^{1/6} + m^{6/11+\varepsilon}n^{15/22}q^{3/22}\big) + \alpha_2(m+n).$$

Let n_0 be a constant (whose concrete choice will be made later). The base case where $m + n < n_0$ can be dealt with by choosing α_1 and α_2 sufficiently large.

We start by recalling a well-known simple, albeit weaker bound. The incidence graph $G \subseteq \mathcal{P} \times \mathcal{C}$, whose edges are the incident pairs in $\mathcal{P} \times \mathcal{C}$, cannot contain $K_{3,2}$ as a subgraph, because two circles have at most two intersection points. By the Kővári–Sós–Turán theorem (*e.g.*, see [35, Section 4.5]), $I(\mathcal{P},\mathcal{C}) = |G| = O(n^{2/3}m + n)$. This immediately implies the theorem if $m = O(n^{1/3})$ (the resulting bound is $O(n)$ in this case). Thus we may assume that $n = O(m^3)$.

We next apply the polynomial partitioning technique. Specifically, we set r as a sufficiently large constant (whose value depends on ε and will be determined later), and apply the polynomial partitioning theorem (Theorem 2.1) to obtain an r-partitioning polynomial f. According to the theorem, f is of degree $D = O(r^{1/3})$ and $Z(f)$ partitions $\mathbb{R}^3$ into maximal connected cells, each containing at most m/r points of $\mathcal{P}$. As already noted, Warren's theorem (Theorem 2.2) implies that the number of cells is $O(r)$.

Let $\mathcal{C}_0$ denote the subset of circles of $\mathcal{C}$ that are fully contained in $Z(f)$, and let $\mathcal{C}' = \mathcal{C} \setminus \mathcal{C}_0$. Similarly, set $\mathcal{P}_0 = \mathcal{P} \cap Z(f)$ and $\mathcal{P}' = \mathcal{P} \setminus \mathcal{P}_0$. Notice that

$$I(\mathcal{P},\mathcal{C}) = I(\mathcal{P}_0,\mathcal{C}_0) + I(\mathcal{P}_0,\mathcal{C}') + I(\mathcal{P}',\mathcal{C}'). \tag{3.1}$$

The terms $I(\mathcal{P}_0,\mathcal{C}')$ and $I(\mathcal{P}',\mathcal{C}')$ can be bounded using techniques (detailed below) that are by now fairly standard. On the other hand, bounding $I(\mathcal{P}_0,\mathcal{C}_0)$ is the main technical challenge in this proof. Other works that have applied the polynomial partitioning technique, such as [29, 30, 45, 50, 51], also spend most of their efforts on incidences with curves that are fully contained in the zero set of the partitioning polynomial (where these curves are either original input curves specified in the statement of the problem, or the intersections of input surfaces with the zero set of a partitioning polynomial).

Bounding $I(\mathcal{P}_0,\mathcal{C}')$ and $I(\mathcal{P}',\mathcal{C}')$. For a circle $C \in \mathcal{C}'$, let Π_C be the plane that contains C, and let f_C denote the restriction of f to Π_C. Since C is not contained in $Z(f_C)$, f_C and the irreducible quadratic equation of C within Π_C do not have any common factor. Thus by Bézout's theorem (Theorem 2.5), C and $Z(f_C)$ have at most $2 \cdot \deg(f_C) = O(r^{1/3})$ common points. This immediately implies

$$I(\mathcal{P}_0,\mathcal{C}') = O(nr^{1/3}). \tag{3.2}$$

Next, let us denote the cells of the partition as $K_1,\dots,K_s$ (recall that $s = O(r)$ and that the cells are open). For $i = 1,\dots,s$, put $\mathcal{P}_i = \mathcal{P} \cap K_i$ and let $\mathcal{C}_i$ denote the set of circles in $\mathcal{C}'$ that intersect K_i. Put $m_i = |\mathcal{P}_i|$ and $n_i = |\mathcal{C}_i|$, for $i = 1,\dots,s$. Note that $|\mathcal{P}'| = \sum_{i=1}^{s} m_i$, and recall that $m_i \leqslant m/r$ for every i. The above bound of $O(r^{1/3})$ on the number of intersection points of a circle $C \in \mathcal{C}'$ and $Z(f)$ implies that each circle enters $O(r^{1/3})$ cells (a circle has to intersect $Z(f)$ when moving from one cell to another). This implies $\sum_{i=1}^{s} n_i = O(nr^{1/3})$.

Notice that

$$I(\mathcal{P}',\mathcal{C}') = \sum_{i=1}^{s} I(\mathcal{P}_i,\mathcal{C}_i),$$

so we proceed to bound the number of incidences within a cell K_i. From the induction hypothesis, we get

$$\begin{aligned} I(\mathcal{P}',\mathcal{C}') &\leqslant \sum_{i=1}^{s} \left(\alpha_1 \left(m_i^{3/7+\varepsilon} n_i^{6/7} + m_i^{2/3+\varepsilon} n_i^{1/2} q^{1/6} + m_i^{6/11+\varepsilon} n_i^{15/22} q^{3/22}\right) + \alpha_2(m_i + n_i)\right) \\ &\leqslant \sum_{i=1}^{s} \left(\alpha_1 \left(\left(\frac{m}{r}\right)^{3/7+\varepsilon} n_i^{6/7} + \left(\frac{m}{r}\right)^{2/3+\varepsilon} n_i^{1/2} q^{1/6} + \left(\frac{m}{r}\right)^{6/11+\varepsilon} n_i^{15/22} q^{3/22}\right)\right) \\ &\quad + \alpha_2\left(|\mathcal{P}'| + \sum_{i=1}^{s} n_i\right). \end{aligned} \tag{3.3}$$

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Since $\sum_{i=1}^{s} n_i = O(nr^{1/3})$, Hölder's inequality implies

$$\sum_{i=1}^{s} n_i^{6/7} = O\big((nr^{1/3})^{6/7} \cdot r^{1/7}\big) = O(n^{6/7}r^{3/7}),$$
$$\sum_{i=1}^{s} n_i^{1/2} = O\big((nr^{1/3})^{1/2} \cdot r^{1/2}\big) = O(n^{1/2}r^{2/3}), \tag{3.4}$$
$$\sum_{i=1}^{s} n_i^{15/22} = O\big((nr^{1/3})^{15/22} \cdot r^{7/22}\big) = O(n^{15/22}r^{6/11}).$$

By combining (3.3) and (3.4), we obtain

$$I(\mathcal{P}',\mathcal{C}') \leqslant \alpha_1 \cdot \frac{c\big(m^{3/7+\varepsilon}n^{6/7} + m^{2/3+\varepsilon}n^{1/2}q^{1/6} + m^{6/11+\varepsilon}n^{15/22}q^{3/22}\big)}{r^{\varepsilon}} + \alpha_2(|\mathcal{P}'| + cnr^{1/3}),$$

for a suitable constant $c > 0$. Notice that the bound in (3.2) is proportional to the last term in this bound, and that this term is dominated by $O(m^{3/7}n^{6/7})$ since we assume that $n = O(m^3)$ and that r is constant. Choosing r to be sufficiently large, so that $r^{\varepsilon} > 4c$, and choosing $\alpha_1 \gg \alpha_2 r^{1/3}$, we can ensure that

$$I(\mathcal{P}_0 \cup \mathcal{P}',\mathcal{C}') \leqslant \frac{\alpha_1}{3}\big(m^{3/7+\varepsilon}n^{6/7} + m^{2/3+\varepsilon}n^{1/2}q^{1/6} + m^{6/11+\varepsilon}n^{15/22}q^{3/22}\big) + \alpha_2|\mathcal{P}'|. \tag{3.5}$$

Bounding $I(\mathcal{P}_0,\mathcal{C}_0)$: handling shared points. We are left with the task of bounding the number of incidences between the set $\mathcal{P}_0$ of points of $\mathcal{P}$ that are contained in $Z(f)$ and the set $\mathcal{C}_0$ of circles of $\mathcal{C}$ that are fully contained in $Z(f)$. We call a point of $\mathcal{P}_0$ *shared* if it is contained in the zero sets of at least two distinct irreducible factors of f, and otherwise we call it *private*. We first consider the case of shared points; in this argument it is simplest to work over $\mathbb{C}$.

Let $\mathcal{P}_s$ denote the subset of points in $\mathcal{P}_0$ that are shared, and put $m_s = |\mathcal{P}_s|$. Let f_1 be the square-free (over $\mathbb{C}$) part of f, so in particular $Z_{\mathbb{C}}(f) = Z_{\mathbb{C}}(f_1)$. If $p \in \mathcal{P}_s$ is a shared point, then p lies in at least two distinct irreducible (over $\mathbb{R}$) components of $Z_{\mathbb{R}}(f)$, and thus p lies in at least two irreducible (over $\mathbb{C}$) components of $Z_{\mathbb{C}}(f_1)$. Thus p lies in the singular set of $Z_{\mathbb{C}}(f_1)$, and in particular $p \in Z_{\mathbb{C}}(f_1) \cap Z_{\mathbb{C}}(f_1')$, where $f_1' = e \cdot \nabla f_1$ and e is a generically chosen unit vector (*i.e.*, f_1' is a partial derivative of f_1 in a generic direction). Note that $\deg f_1, \deg f_1' \leqslant D$, so $\gamma = Z_{\mathbb{C}}(f_1) \cap Z_{\mathbb{C}}(f_1')$ is an algebraic space curve of degree at most D^2 (*e.g.*, see [25, Exercise 11.6]). The curve γ contains at most D^2 irreducible components, and thus γ contains at most D^2 (complex) circles. We conclude that there are at most $D^2 m_s$ incidences between points from $\mathcal{P}_s$ and circles whose complexification is contained in $Z_{\mathbb{C}}(f_1) \cap Z_{\mathbb{C}}(f_1')$.

It remains to bound the number of incidences between points in $\mathcal{P}_s$ and circles of $\mathcal{C}_0$ whose complexification is not contained in $Z_{\mathbb{C}}(f_1) \cap Z_{\mathbb{C}}(f_1')$ (that is, circles that are contained in $Z_{\mathbb{R}}(f)$ but whose complexification is not fully contained in $Z_{\mathbb{C}}(f_1')$). Let C be a circle whose complexification C^* is not contained in $Z_{\mathbb{C}}(f_1')$, and let $\Pi \subset \mathbb{C}^3$ be the 2-plane containing C^*. If we identify Π with $\mathbb{C}^2$, then the restriction of f_1' to Π is a polynomial $\tilde{f}_1' \in \mathbb{C}[x_1, x_2]$. By Bézout's theorem (Theorem 2.5), C^* and $Z_{\mathbb{C}}(\tilde{f}_1')$ intersect in

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at most $2D$ points. Thus C^* and $Z_{\mathbb{C}}(f_1')$ intersect in at most $2D$ points. This in turn implies that $|C \cap \mathcal{P}_s| \leqslant 2D$. Therefore, by taking α_2 (and consequently also α_1) to be sufficiently large, we have

$$I(\mathcal{P}_s, \mathcal{C}_0) \leqslant \frac{1}{2}D^2 m_s + 2Dn \leqslant \alpha_2(m_s + n/3). \tag{3.6}$$

Bounding $I(\mathcal{P}_0, \mathcal{C}_0)$: handling private points. Let $\mathcal{P}_p = \mathcal{P}_0 \setminus \mathcal{P}_s$ denote the set of private points in $\mathcal{P}_0$. Recall that each private point is contained in the zero set of a single irreducible factor of f. Let $f_1, f_2, \ldots, f_t$ be the factors of f whose zero sets are planes or spheres. For $i = 1, \ldots, t$, set $\mathcal{P}_{p,i}^{(1)} = \mathcal{P}_p \cap Z(f_i)$ and $m_{p,i} = |\mathcal{P}_{p,i}^{(1)}|$. Put

$$\mathcal{P}_p^{(1)} = \bigcup_{i=1}^{t} \mathcal{P}_{p,i}^{(1)} \quad \text{and} \quad m_p^{(1)} = |\mathcal{P}_p^{(1)}| = \sum_{i=1}^{t} m_{p,i}.$$

Let $n_{p,i}$ denote the number of circles of $\mathcal{C}_0$ that are fully contained in $Z(f_i)$. Notice that (i) $t \leqslant D = O(r^{1/3})$, (ii) $n_{p,i} \leqslant q$ for every i, and (iii) $\sum_{i=1}^{t} n_{p,i} \leqslant n$ (we may ignore circles that are fully contained in more than one component, since these will not have incidences with private points). Applying (1.1) and using the fact that there are no hidden polylogarithmic terms in the linear part of (1.1), we obtain[3]

$$\begin{aligned} I(\mathcal{P}_p^{(1)}, \mathcal{C}_0) &= \sum_{i=1}^{t} \Big(O^*\big(m_{p,i}^{2/3} n_{p,i}^{2/3} + m_{p,i}^{6/11} n_{p,i}^{9/11}\big) + O(m_{p,i} + n_{p,i}) \Big) \\ &= \sum_{i=1}^{t} \Big(O^*\big(m_{p,i}^{2/3} n_{p,i}^{1/3} q^{1/3} + m_{p,i}^{6/11} n_{p,i}^{5/11} q^{4/11}\big) + O(m_{p,i} + n_{p,i}) \Big) \\ &= O^*\big(m^{2/3} n^{1/3} q^{1/3} + m^{6/11} n^{5/11} q^{4/11}\big) + O(m_p^{(1)} + n), \end{aligned}$$

where the last step uses Hölder's inequality; it bounds (twice) $\sum_i m_{p,i} = m_p^{(1)}$ by m. Since $q \leqslant n$, it follows that when α_1 and α_2 are sufficiently large, we have

$$I(\mathcal{P}_p^{(1)}, \mathcal{C}_0) \leqslant \frac{\alpha_1}{3}\big(m^{2/3+\varepsilon} n^{1/2} q^{1/6} + m^{6/11+\varepsilon} n^{15/22} q^{3/22}\big) + \alpha_2(m_p^{(1)} + n/3). \tag{3.7}$$

Let $\mathcal{P}_p^{(2)} = \mathcal{P}_p \setminus \mathcal{P}_p^{(1)}$ be the set of private points that lie on the zero sets of factors of f that are neither planes nor spheres, and put $m_p^{(2)} = |\mathcal{P}_p^{(2)}|$. To handle incidences with these points we require the following lemma, which constitutes a major component of our analysis and which is proved in Section 4 (somewhat similar results can be found in [28, 33]). First, a definition.

Definition. Let g be an irreducible polynomial in $\mathbb{R}[x_1, x_2, x_3]$ such that $Z(g)$ is a two-dimensional surface. We say that a point $p \in Z(g)$ is *popular* if it is incident to at least $44(\deg g)^2$ circles that are fully contained in $Z(g)$.

[3] Notice that the dependency of this bound on n and q is better than the one in the bound of the theorem. This latter worse bound is the one that is preserved under the partition-based induction.

Lemma 3.1. *An irreducible algebraic surface that is neither a plane nor a sphere cannot contain more than two popular points.*

The lemma implies that the number of incidences between popular points of $\mathcal{P}_p^{(2)}$ (within their respective irreducible components of $Z(f)$, whose number is at most $D/2$) and circles of $\mathcal{C}_0$ is at most $2(D/2)n = Dn \leqslant \alpha_2 n/3$ (the latter inequality holds if α_2 is chosen sufficiently large with respect to D). The number of incidences between non-popular points of $\mathcal{P}_p^{(2)}$ and circles of $\mathcal{C}_0$ is at most $m_p^{(2)} \cdot 44D^2 \leqslant \alpha_2 m_p^{(2)}$ (again for a sufficiently large value of α_2). Combining this with (3.1), (3.5), (3.6), and (3.7), we get

$$I(\mathcal{P},\mathcal{C}) \leqslant \alpha_1\big(m^{3/7+\varepsilon}n^{6/7} + m^{2/3+\varepsilon}n^{1/2}q^{1/6} + m^{6/11+\varepsilon}n^{15/22}q^{3/22}\big) + \alpha_2(m+n).$$

This establishes the induction step, and thus completes the proof of the theorem. □

Remarks. **(1)** Note that we have actually shown that

$$I(\mathcal{P}_0,\mathcal{C}_0) = O(mD^2 + nD),$$

regardless of the degree D of f. It is the term mD^2 that becomes too large when D itself is too large. In this part of the analysis we addressed this issue by taking D to be a constant. In the refined analysis given in the next subsection we use non-constant, albeit still small, values for D, thereby slightly refining the bound.

(2) To see why $m^{3/7+\varepsilon}n^{6/7}$ is the best choice for the leading term, let us denote the leading term as $m^{a+\varepsilon}n^b$ and observe the following restrictions on a and b.

(i) For r to cancel itself in the analysis of the cells of the partition (up to a power of ε), we require $a \geqslant 1 - 2b/3$.

(ii) For $n = O(m^3)$ to imply $n = O(m^a n^b)$, we must have $a + 3b \geqslant 3$.

Combining both constraints, with equalities, results in the term $m^{3/7+\varepsilon}n^{6/7}$.

(3) We believe that the terms in our bounds that depend on q can be significantly improved by using a more careful analysis. The exponents in these terms are chosen so as to make them satisfy the induction step. However, in doing so, in each cell of the partition, we use the same bound q on the maximum number of coplanar or cospherical circles among those that cross the cell. Since the number of circles that cross a cell goes down, the bound q should also decrease. We do not know how to handle this issue rigorously, and leave it as an open problem for further research.

3.1. Removing the epsilons

In this section we will show that, for any $\delta > 0$, when $n = O(m^{3/2-\delta})$, the epsilons from the bound of Theorem 1.1 can be removed. This is what Theorem 1.2 asserts; we repeat its statement for the convenience of the reader.

Theorem 1.2. *Let $\mathcal{P}$ be a set of m points and let $\mathcal{C}$ be a set of n circles in $\mathbb{R}^3$, let $q \leqslant n$ be an integer, and let $m = O(n^{3/2-\delta})$, for some fixed arbitrarily small constant $\delta > 0$. If no sphere or plane contains more than q circles of $\mathcal{C}$, then*

$$I(\mathcal{P},\mathcal{C}) \leqslant A_{m,n}\big(m^{3/7}n^{6/7} + m^{2/3}n^{1/2}q^{1/6} + m^{6/11}n^{15/22}q^{3/22}\log^{2/11} m + m + n\big),$$

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where

$$A_{m,n} = A^{\left\lceil \frac{3}{2} \cdot \frac{\log(m/n^{1/3})}{\log(n^{3/2}/m)} \right\rceil + 1},$$

for some absolute constant $A > 1$.

Proof. We define $\mathcal{P}$, $\mathcal{P}_0$, $\mathcal{C}$, $\mathcal{C}'$, *etc.*, as in the proof of Theorem 1.1. The proof is similar to the one of Theorem 1.1, except that it works in stages, so that in each stage we enlarge the range of m where the bound applies (with an appropriate larger constant $A_{m,n}$). At each stage we construct a partitioning polynomial as before, but of a non-constant degree. We then use the bound obtained in the previous stage to control the number of incidences inside the cells of the polynomial partitioning. Finally, we use a separate argument (essentially the one given in the second part of the proof of Theorem 1.1) to bound the number of incidences with the points that lie on the zero set of the polynomial. Each stage increases the constant of proportionality in the bound by a constant factor, which is why the 'constant' $A_{m,n}$ increases as m approaches $n^{3/2}$. For $j = 1, 2, \ldots,$ the jth stage asserts the bound specified in the theorem when $m \leqslant n^{\alpha_j}$. The sequence of exponents $\{\alpha_j\}$ increases from stage to stage, and approaches $3/2$. Each stage has its own constant of proportionality $A^{(j)}$. The specific values of the exponents α_j (and the constants of proportionality) will be set later. For the 0th, vacuous stage we use $\alpha_0 = 1/3$, and the bound $O(n)$ that was noted above for $m \leqslant n^{\alpha_0}$, with an implied initial constant of proportionality $A^{(0)}$.

In handling the jth stage, we assume that $n^{\alpha_{j-1}} < m \leqslant n^{\alpha_j}$; if $m \leqslant n^{\alpha_{j-1}}$ there is nothing to do as we can use the (better) bound from the previous stage. We construct an r-partitioning polynomial f, just as in the proof of Theorem 1.1, except that its degree is not required to be a constant. Put $\alpha = \alpha_{j-1}$. To apply the bound from the previous stage uniformly within each cell, we want to have a uniform bound on the number of circles entering a cell. The average number of circles entering a cell is proportional to $n/r^{2/3}$ (assuming that the number of cells is $\Theta(r)$, an assumption made only for the sake of intuition). A cell that intersects $tn/r^{2/3}$ circles, for $t > 1$, induces $\lceil t \rceil$ subproblems, each involving all the points in the cell and up to $n/r^{2/3}$ circles. It is easily checked that the number of subproblems remains $O(r)$, with a somewhat larger constant of proportionality, and that each subproblem now involves at most m/r points and at most $n/r^{2/3}$ circles. Moreover, in cells that have strictly fewer than $n/r^{2/3}$ circles, we will assume that there are exactly $n/r^{2/3}$ circles, *e.g.*, by adding dummy circles. This will not decrease the number of incidences.

We assume that the number of cells is at most br, for some absolute constant b. To apply the bound from the previous stage, we need to choose r that will guarantee that

$$\frac{m}{r} \leqslant \left(\frac{n}{r^{2/3}}\right)^{\alpha}, \quad \textit{i.e.} \;\; r^{1-2\alpha/3} \geqslant \frac{m}{n^{\alpha}}, \quad \textit{i.e.} \;\; r \geqslant \frac{m^{3/(3-2\alpha)}}{n^{3\alpha/(3-2\alpha)}}.$$

We choose r to be equal to the last expression. We note that (i) $r \geqslant 1$, because m is assumed to be greater than n^{α} and $\alpha < 3/2$, and (ii) $r \leqslant m$, because $m \leqslant n^{3/2}$. Because of the somewhat weak bound that we will derive below on the number of incidences with points that lie on $Z(f)$ (the same bound as in the proof of Theorem 1.1), this choice of

r will work only when m is not too large. The resulting constraint on m, of the form $m \leqslant n^{\alpha_j}$, will define the new range in which the bound derived in the present stage applies.

In more detail, the number of incidences within the partition cells is

$$\begin{aligned} I(\mathcal{P}', \mathcal{C}') &\leqslant A^{(j-1)} \sum_{i=1}^{br} \big((m/r)^{3/7} (n/r^{2/3})^{6/7} + (m/r)^{2/3} (n/r^{2/3})^{1/2} q^{1/6} \\ &\qquad + (m/r)^{6/11} (n/r^{2/3})^{15/22} q^{3/22} \log^{2/11}(m/r) + m/r + n/r^{2/3} \big) \\ &\leqslant bA^{(j-1)} \big(m^{3/7} n^{6/7} + m^{2/3} n^{1/2} q^{1/6} + m^{6/11} n^{15/22} q^{3/22} \log^{2/11} m + m + nr^{1/3} \big). \end{aligned}$$

We claim that our choice of r ensures that $nr^{1/3} \leqslant m^{3/7} n^{6/7}$. That is,

$$r^{1/3} = \frac{m^{1/(3-2\alpha)}}{n^{\alpha/(3-2\alpha)}} \leqslant \frac{m^{3/7}}{n^{1/7}}.$$

Indeed, this is easily seen to hold because $1/3 \leqslant \alpha < 3/2$ and $m \leqslant n^{3/2}$. Recall that we also have $I(\mathcal{P}_0, \mathcal{C}') \leqslant A' nr^{1/3}$ for some constant A' (see (3.2)). By choosing $A^{(0)} > A'$ (so that $A^{(j-1)} > A'$ for every j), we have

$$\begin{aligned} I(\mathcal{P}, \mathcal{C}') &= I(\mathcal{P}_0, \mathcal{C}') + I(\mathcal{P}', \mathcal{C}') \\ &\leqslant 3bA^{(j-1)} \big(m^{3/7} n^{6/7} + m^{2/3} n^{1/2} q^{1/6} + m^{6/11} n^{15/22} q^{3/22} \log^{2/11} m + m \big). \end{aligned} \tag{3.8}$$

As proved in Theorem 1.1,

$$I(\mathcal{P}_s, \mathcal{C}_0) + I(\mathcal{P}_p^{(2)}, \mathcal{C}_0) = O(mr^{2/3} + nr^{1/3}). \tag{3.9}$$

This follows by substituting $D = O(r^{1/3})$ in the bounds in the proof of Theorem 1.1, which are

$$I(\mathcal{P}_s, \mathcal{C}_0) \leqslant mD^2/2 + 2nD \quad \text{and} \quad I(\mathcal{P}_p^{(2)}, \mathcal{C}_0) \leqslant 44mD^2 + nD.$$

It remains to bound $I(\mathcal{P}_p^{(1)}, \mathcal{C}_0)$. For this, we again use an analysis similar to the one in Theorem 1.1. Let $f_1, f_2, \ldots, f_t$ be the factors of f whose zero sets are planes or spheres. For $i = 1, \ldots, t$, set $\mathcal{P}_{p,i}^{(1)} = \mathcal{P}_p \cap Z(f_i)$ and $m_{p,i} = |\mathcal{P}_{p,i}^{(1)}|$. Let $n_{p,i}$ denote the number of circles of $\mathcal{C}_0$ that are fully contained in $Z(f_i)$ (ignoring, as before, circles that lie in more than one of these surfaces). Put

$$\mathcal{P}_p^{(1)} = \bigcup_{i=1}^{t} \mathcal{P}_{p,i}^{(1)}.$$

Notice that (i) $t = O(r^{1/3})$, (ii) $n_{p,i} \leqslant q$ for every i, and (iii) $\sum_i n_{p,i} \leqslant n$. Applying (1.1), we obtain

$$\begin{aligned} I(\mathcal{P}_p^{(1)}, \mathcal{C}_0) &= \sum_{i=1}^{t} O\big(m_{p,i}^{2/3} n_{p,i}^{2/3} + m_{p,i}^{6/11} n_{p,i}^{9/11} \log^{2/11}(m_{p,i}^3 / n_{p,i}) + m_{p,i} + n_{p,i} \big) \\ &= \sum_{i=1}^{t} O\big(m_{p,i}^{2/3} n_{p,i}^{1/3} q^{1/3} + m_{p,i}^{6/11} n_{p,i}^{5/11} q^{4/11} \log^{2/11}(m_{p,i}^3) + m_{p,i} + n_{p,i} \big) \\ &= O\big(m^{2/3} n^{1/3} q^{1/3} + m^{6/11} n^{5/11} q^{4/11} \log^{2/11} m + m + n \big), \end{aligned} \tag{3.10}$$

where the last step uses Hölder's inequality.

We would like to combine (3.8), (3.9), and (3.10) to obtain the bound asserted in Theorem 1.2. All the elements in these bounds add up to the latter bound, with an appropriate sufficiently large choice of $A^{(j)}$, except for the term $O(mr^{2/3})$, which might exceed the bound of the theorem if m is too large. Thus, we restrict m to satisfy

$$mr^{2/3} \leqslant m^{3/7}n^{6/7}, \quad \text{i.e.} \;\; r \leqslant \frac{n^{9/7}}{m^{6/7}}.$$

Substituting the chosen value of r, we thus require that

$$\frac{m^{3/(3-2\alpha)}}{n^{3\alpha/(3-2\alpha)}} \leqslant \frac{n^{9/7}}{m^{6/7}}.$$

That is, we require

$$m \leqslant n^{\frac{9+\alpha}{13-4\alpha}}.$$

Recalling that we write the (upper bound) constraint on m at the jth stage as $m \leqslant n^{\alpha_j}$, we have the recurrence

$$\alpha_j = \frac{9+\alpha_{j-1}}{13-4\alpha_{j-1}}.$$

To simplify this, we write $\alpha_j = 3/2 - 1/x_j$, and obtain the recurrence

$$x_j = x_{j-1} + \frac{4}{7},$$

with the initial value $x_0 = 6/7$ (this gives the initial constraint $m \leqslant n^{1/3}$). In other words, we have $x_j = (4j+6)/7$, and

$$\alpha_j = \frac{3}{2} - \frac{7}{4j+6}.$$

The first few values are $\alpha_0 = 1/3$, $\alpha_1 = 4/5$, $\alpha_2 = 1$, and $\alpha_3 = 10/9$. Note that every $m < n^{3/2}$ is covered by the range of some stage. Specifically, given such an m, it is covered by stage j, where j is the smallest integer satisfying

$$m \leqslant n^{3/2-7/(4j+6)},$$

and straightforward calculations show that

$$j = \left\lceil \frac{3}{2} \cdot \frac{\log(m/n^{1/3})}{\log(n^{3/2}/m)} \right\rceil.$$

Inspecting the preceding analysis, we see that the bound holds for the jth stage if we choose $A^{(j)} = A \cdot A^{(j-1)}$, where A is a sufficiently large absolute constant. Hence, for m in the jth range, the bound on $I(\mathcal{P}, \mathcal{C})$ has A^{j+1} as the constant of proportionality. This completes the description of the stage, and thus the proof of Theorem 1.2. □

Remarks. **(1)** The analysis holds for any $m < n^{3/2}$. However, when m is very close to $n^{3/2}$, say it is proportional to $n^{3/2}$, then $j = \Theta(\log n)$, and the 'constant' $A^{(j)}$ is no longer a constant. The requirement $m \leqslant n^{3/2-\delta}$ in the theorem is made to ensure that $A_{m,n}$ does not exceed some constant threshold (which depends on δ).

(2) The case $m > n^{3/2}$ is not considered for this improvement, but we believe that it too can be handled by similar techniques. Note that $m^{3/7}n^{6/7} = O(m)$ when $m > n^{3/2}$, so, ignoring the terms that depend on q, the overall bound is $O(m^{1+\varepsilon})$, for any $\varepsilon > 0$. One should be able to remove this dependence on ε, as we did in the case $m < n^{3/2}$.

4. The number of popular points in an irreducible variety

It remains to prove Lemma 3.1. To do so, we will use the three-dimensional *inversion transformation* $I : \mathbb{R}^3 \to \mathbb{R}^3$ about the origin (*e.g.*, see [27, Chapter 37]). The transformation $I(\cdot)$ maps the point $p = (x_1, x_2, x_3) \neq (0,0,0)$ to the point $\bar{p} = I(p) = (\bar{x}_1, \bar{x}_2, \bar{x}_3)$, where

$$\bar{x}_i = \frac{x_i}{x_1^2 + x_2^2 + x_3^2}, \quad i = 1,2,3.$$

A proof for the following lemma can be found in [27, Chapter 37].

Lemma 4.1.

(a) *Let C be a circle incident to the origin. Then $I(C)$ is a line not passing through the origin.*

(b) *Let C be a circle* not *incident to the origin. Then $I(C)$ is a circle not passing through the origin.*

(c) *The converse statements of both* (a) *and* (b) *also hold.* □

Proof of Lemma 3.1. Consider an irreducible surface $Z = Z(g)$ which is neither a plane nor a sphere, and let $E = \deg(g)$. Assume, for contradiction, that there exist three popular points $z_1, z_2, z_3 \in Z$. By translating the axes we may assume that z_1 is the origin. We apply the inversion transformation to Z. Since I is its own inverse, $I(Z)$ can be written as $Z(g \circ I)$. To turn $g \circ I$ into a polynomial, we clear the denominators resulting from this transformation by multiplying $g \circ I$ by a suitable (minimal) power of $x_1^2 + x_2^2 + x_3^2$. This does not change the (real) zero set of $g \circ I$ (except for possibly adding the origin 0 to the set). We refer to the resulting polynomial as $\bar{g}$. Notice that the degree of $\bar{g}$ is strictly smaller than $2E$, since to clear denominators we need to multiply $g \circ I$ by at most $(x_1^2 + x_2^2 + x_3^2)^E$, and the highest-degree terms will be the ones that were previously the linear terms (if they exist; since $Z(g)$ contains the origin, g has no constant term). Since we have multiplied $g \circ I$ by the minimum power of $x_1^2 + x_2^2 + x_3^2$, we may assume that $\bar{g}$ is not divisible by $x_1^2 + x_2^2 + x_3^2$. If some other polynomial divided $\bar{g}$, then after applying the inversion again and clearing denominators we would obtain a non-trivial polynomial different from g that divides g. Thus, since g is irreducible, we conclude that $\bar{g}$ is also irreducible.

By assumption, $Z(g)$ contains $44E^2$ circles incident to the origin z_1. Lemma 4.1(a) thus implies that $Z(\bar{g})$ contains at least $44E^2$ lines. We claim that $Z(\bar{g})$ is a ruled surface. If $\deg(\bar{g}) = 2$, then we can consider all types of quadratic trivariate polynomials and observe that the ones whose zero sets may contain more than $44E^2 = 176$ lines are all ruled (namely, they are pairs of planes, cones, cylinders, one-sheeted hyperboloids, or hyperbolic paraboloids). (If $\deg(\bar{g}) = 1$, $Z(\bar{g})$ is a plane.) If $\deg(\bar{g}) \geqslant 3$, then since $\bar{g}$ is irreducible of degree at most $2E$ and $Z(\bar{g})$ contains at least $44E^2 > 2E(11 \cdot 2E - 24)$ lines, Corollary 2.11 implies that $Z(\bar{g})$ is ruled. Since no regulus contains a point and more than two lines through that point, $Z(\bar{g})$ is not a regulus. Moreover, $Z(\bar{g})$ is not a plane since $Z(g)$ is neither a plane nor a sphere. That is, $Z(\bar{g})$ is singly ruled.

Thus, $Z = Z(g)$ can be written as the union of a set of circles and a (possibly empty) set of lines, all of which are incident to z_1; these are the images under the inverse inversion of the lines spanning $Z(\bar{g})$ (see Lemma 4.1(c), and observe that lines through z_1 are mapped to themselves by the inversion). By a symmetric argument, this property also holds for z_2 and for z_3. This implies that, for $i = 1, 2, 3$, every point u in Z is incident to a circle or a line that is also incident to z_i. These three circles or lines are not necessarily distinct, but they can all coincide only when u lies on the unique circle or line γ that passes through z_1, z_2, z_3, and then all the above three circles or lines coincide with γ.

The original surface Z may or may not be ruled. Recall that the only doubly ruled surfaces are the hyperbolic paraboloid and the hyperboloid of one sheet. Since both of these surfaces do not contain a point that is incident to infinitely many lines or circles contained in the surface, we conclude that Z is not doubly ruled. Since we have assumed that Z is not a plane, it is not triply ruled either. Thus Z is either not ruled or only singly ruled.

We define a point $u \in Z$ to be *exceptional* if there are infinitely many lines contained in Z that pass through u (think, for example, of the case where $Z(g)$ is a cone with this point as an apex). By Corollary 3.6 from [24], if Z is singly ruled, then Z contains at most one exceptional point. According to Corollary 2.11, if Z is not ruled, it contains only finitely many lines, and thus it cannot contain any exceptional points. (Corollary 2.11 does not apply to quadratic surfaces, but it can be verified that the above property also holds in this case, by checking all the possible types of quadratic surfaces.) Therefore, Z contains at most one exceptional point, and in particular we may assume that z_2, z_3 are not exceptional points. Since z_2 (resp. z_3) is popular but not exceptional, there are infinitely many circles passing through z_2 (resp. z_3) and contained in Z. On the other hand, as already observed, at most one circle can pass through the triplet z_1, z_2, z_3. Thus, after possibly interchanging the roles of z_1, z_2 and z_3, we may assume that there exists an infinite collection of circles contained in Z that are incident to z_2 but not to z_1.

Consider the image $Z(\bar{g}) \subset \mathbb{R}^3$ of Z after applying the inversion transform around the point z_1 (which we have translated to become the origin) and let $\bar{z}_2 = I(z_2)$. According to Lemma 4.1(b), the infinite family of circles contained in Z that are incident to z_2 but not to z_1 are transformed into an infinite family of circles that are contained in $Z(\bar{g})$ and incident to $\bar{z}_2$. We denote the latter family as $\bar{\mathcal{C}}$.

Consider a plane Π and notice that $\Pi \cap Z(\bar{g})$ is an algebraic curve of degree at most $2E$. This implies that Π contains at most E circles of $\bar{\mathcal{C}}$. Since this holds for any plane, there exists an infinite subset $\bar{\mathcal{C}}' \subset \bar{\mathcal{C}}$ such that no two circles in $\bar{\mathcal{C}}'$ are coplanar. (Π cannot be contained in $Z(\bar{g})$ since the latter surface is irreducible, and if it is a plane then Z is either a sphere or a plane.)

Since $Z(\bar{g})$ is two-dimensional, by Theorem 2.4 we have $(\bar{g}) = \mathbf{I}(Z(\bar{g}))$. Since $Z(\bar{g})$ is ruled, Theorem 2.10 implies $Z(\bar{g}) \subset Z(\mathsf{FL}_{\bar{g}}) \subset \mathbb{R}^3$. Hence $\mathbf{I}(Z(\mathsf{FL}_{\bar{g}})) \subset \mathbf{I}(Z(\bar{g}))$, and in particular $\mathsf{FL}_{\bar{g}} \in \mathbf{I}(Z(\bar{g})) = (\bar{g})$. That is, $\bar{g}$ divides $\mathsf{FL}_{\bar{g}}$ in $\mathbb{R}[x_1, x_2, x_3]$, and thus also in $\mathbb{C}[x_1, x_2, x_3]$. In particular, this means that $Z_{\mathbb{C}}(\bar{g}) \subset Z_{\mathbb{C}}(\mathsf{FL}_{\bar{g}})$. By Theorem 2.12 this implies that Z^* is also ruled.

In the remainder of the proof we will work mainly in complex projective 3-space $\mathbb{C}\mathbf{P}^3$ instead of real affine space, which we have considered so far.

Projectivization. The *projectivization* of a point $p = (p_1, p_2, p_3) \in \mathbb{C}^3$ is obtained by passing to *homogeneous coordinates*, and by assigning p to $p^\dagger = (1, p_1, p_2, p_3)$ (where all non-zero scalar multiples of $p^\dagger$ are identified with $p^\dagger$). To distinguish between such homogeneous coordinates and coordinates in the affine spaces $\mathbb{R}^3$ and $\mathbb{C}^3$, we write them as $[x_0 : x_1 : x_2 : x_3]$ (a rather standard notation: see [37]), and we will also use a similar notation for the projective Plücker 5-space of lines in 3-space. The space of all points $[x_0 : x_1 : x_2 : x_3] \neq 0$ is denoted as $\mathbb{C}\mathbf{P}^3$. As noted, two points $[x_0 : x_1 : x_2 : x_3]$, $[x_0' : x_1' : x_2' : x_3'] \in \mathbb{C}\mathbf{P}^3$ are considered to be equivalent if there exists a non-zero constant $\lambda \in \mathbb{C}$ such that $x_0 = \lambda x_0'$, $x_1 = \lambda x_1'$, $x_2 = \lambda x_2'$, and $x_3 = \lambda x_3'$. Given a point $[p_0 : p_1 : p_2 : p_3]$ with $p_0 \neq 0$, its *dehomogenization* with respect to p_0 is the affine point $(p_1/p_0, p_2/p_0, p_3/p_0) \in \mathbb{C}^3$. For more details, see [15, Chapter 8].

If $h \in \mathbb{C}[x_1, x_2, x_3]$ is a polynomial of degree E, we can write

$$h = \sum_I a_I x^I,$$

where each index I is of the form (I_1, I_2, I_3) with $I_1 + I_2 + I_3 \leqslant E$, and $x^I = x_1^{I_1} x_2^{I_2} x_3^{I_3}$. Define

$$h^\dagger = \sum_I a_I x_0^{E - I_1 - I_2 - I_3} x_1^{I_1} x_2^{I_2} x_3^{I_3}.$$

Then $h^\dagger$ is a homogeneous polynomial of degree E, referred to as the *homogenization* of h. We define the *projectivization* of the complex surface $Z(h)$ to be the zero set of $h^\dagger$ in the three-dimensional complex projective space $\mathbb{C}\mathbf{P}^3$. We define the *complex projectivization* of a real surface $S = Z(h)$ to be the projectivization of the complexification S^* of S.

Let $\hat{Z} \subset \mathbb{C}\mathbf{P}^3$ be the complex projectivization of the surface $Z(\bar{g})$. For the next steps of the analysis, we introduce the so-called *absolute conic* in $\mathbb{C}\mathbf{P}^3$ (*e.g.*, see [38])

$$\Gamma = \{[x_0 : x_1 : x_2 : x_3] \mid x_0 = 0,\ x_1^2 + x_2^2 + x_3^2 = 0\}.$$

Notice that Γ is contained in the plane at infinity $x_0 = 0$, and does not contain any real point (a point all of whose coordinates are real).

We will need the following simple lemma.

Lemma 4.2. *Let $f \in \mathbb{C}[x_0, x_1, x_2, x_3]$ be a homogeneous polynomial of degree D, and let $S = Z(f) \subset \mathbb{C}\mathbf{P}^3$. Let p be a point in S, let v be a direction in $\mathbb{C}\mathbf{P}^3$, and let ℓ be the line incident to p with direction v. If all partial derivatives of f of order at most D vanish at p in the direction v, then $\ell \subset S$.*

Proof. Let f_0 be the restriction of f to the line ℓ, and notice that f_0 is a univariate polynomial of degree at most D. Since all derivatives of f_0 of degree at most D vanish at p, f_0 must be identically 0. $\square$

Lemma 4.3. *$\hat{Z} \subset \mathbb{C}\mathbf{P}^3$ is irreducible and singly ruled.*

Proof. According to the above discussion, the complexification Z^* of $Z(\bar{g})$ is irreducible and ruled. It is singly ruled since otherwise Z^* would be a complex plane or regulus, and it cannot be that the real part of a complex plane or regulus is a singly ruled surface. It remains to consider the projectivization Z^* of the complexification.

If the homogenization $f^\dagger$ of a non-constant polynomial f is divisible by a polynomial f_0, then f is divisible by the polynomial obtained by substituting $x_0 = 1$ in f_0. Moreover, $f^\dagger$ is not divisible by any polynomial of the form x_0^a, for an integer $a > 0$ (since f is not a constant). Thus, if f is irreducible, then so is $f^\dagger$. This in turn implies that $\hat{Z}$ is irreducible.

By definition, for every point p in the complexification Z^* of $Z(\bar{g})$, there exists a line ℓ that is incident to p and fully contained in Z^*. Notice that the projective point $p^\dagger \in \hat{Z}$ is incident to the projective line $\ell^\dagger \subset \hat{Z}$ (*i.e.*, the locus of the projectivizations of the points of ℓ). Thus, for any point $p = [p_0 : p_1 : p_2 : p_3] \in \hat{Z}$ for which $p_0 \neq 0$, there is a line incident to p and fully contained in $\hat{Z}$. It remains to consider points $p = [p_0 : p_1 : p_2 : p_3] \in \hat{Z}$ for which $p_0 = 0$. Since $\hat{Z}$ is irreducible and cannot be the plane $Z(x_0)$, then $\hat{Z} \cap Z(x_0)$ is a one-dimensional curve. Since p must have at least one non-zero coordinate, we may assume without loss of generality that $p_3 \neq 0$. Let $Z_0 \subset \mathbb{C}^3$ be the dehomogenization $\hat{Z} \setminus \{x_3 = 0\}$ with respect to x_3, and note that the image of p is contained in Z_0. Notice that every projective line that is fully contained in Z but not contained in $Z(x_3)$ corresponds to a line fully contained in Z_0. Since $\hat{Z}$ contains a line through every point with a non-zero x_0-coordinate (and the set of points with zero x_0-coordinate form a proper sub-variety of $\hat{Z}$), Z_0 contains infinitely many lines. Applying Theorem 2.12 implies that Z_0 is ruled. Thus, Z_0 contains a line ℓ_0 incident to p. The projectivization of ℓ_0 is fully contained in $\hat{Z}$ and incident to p. In conclusion, $\hat{Z}$ is ruled; it remains to show that it is singly ruled.

Notice that there is a bijection between the lines in affine space and the lines in projective space that are not fully contained in the plane $Z(x_0)$. Consider a projective point $p^\dagger \in Z(\bar{g}^\dagger) \setminus Z(x_0)$ and its corresponding dehomogenized affine point p. If $p^\dagger$ is incident to two lines that are fully contained in $Z(\bar{g}^\dagger)$, then the two corresponding affine lines are incident to p and are fully contained in $Z(\bar{g})$. Thus, $Z(\bar{g}^\dagger)$ is singly ruled, since if $Z(\bar{g}^\dagger)$ were doubly or triply ruled then $Z(\bar{g})$ would also have to be doubly or triply ruled, which is not the case. $\square$

Using the Plücker representation of lines. With all these preparations, we reach the following scenario. We have an irreducible singly ruled surface $\hat{Z}$ in $\mathbb{C}\mathbf{P}^3$ (the complex projectivization of $Z(\bar{g})$), which contains an infinite family $\hat{C}$ of circles (the complex projectivization of the circles of $\bar{C}$), no pair of which are coplanar. The following arguments are based on the recent work of Nilov and Skopenkov [38] concerning surfaces that are 'ruled' by lines and circles. Before taking the circles into account, we first probe deeper into the structure of our ruled surface $\hat{Z}$ by considering the Plücker representation of lines in 3-space. Specifically, let

$$\Lambda = \{[x_0 : \ldots : x_5] \mid x_0x_5 + x_1x_4 + x_2x_3 = 0\} \subset \mathbb{C}\mathbf{P}^5 \tag{4.1}$$

be the *Plücker quadric*. Given a point $p = [x_0 : \ldots : x_5] \in \Lambda$, at least two of the four 'canonical' points $[0 : x_0 : x_1 : x_2]$, $[x_0 : 0 : -x_3 : -x_4]$, $[x_1 : x_3 : 0 : -x_5]$, and $[x_2 : x_4 : x_5 : 0]$ cannot be the undefined point $[0 : 0 : 0 : 0]$ because each of the six coordinates $x_0, \ldots, x_5$, not all zero, appears as a coordinate of two of these points. Then there exists a unique line ℓ_p in $\mathbb{C}\mathbf{P}^3$ that passes through all non-zero canonical points of p. We refer to the map $p \to \ell_p$ as the *Plücker map*, and observe that it is a bijection between the points $p \in \Lambda$ and the lines $\ell_p \subset \mathbb{C}\mathbf{P}^3$. Further details about the Plücker map and the Plücker quadric can be found, for example, in [15, Section 8.6].

Let $\Lambda_{\hat{Z}} = \{p \in \Lambda \mid \ell_p \subset \hat{Z}\}$; that is, $\Lambda_{\hat{Z}}$ is the set of all points in Λ that correspond to lines that are fully contained in $\hat{Z}$. We claim that $\Lambda_{\hat{Z}}$ is an algebraic variety in $\mathbb{C}\mathbf{P}^5$ that is composed of a single one-dimensional irreducible component, possibly together with an additional finite set of points. Example 6.19 of [25] establishes that $\Lambda_{\hat{Z}}$ is a projective variety (indeed, it is an example of a Fano variety). We must now show that an irreducible two-dimensional surface in $\mathbb{C}\mathbf{P}^3$ that does not contain any planes cannot contain a two-dimensional family of lines. This implies that $\Lambda_{\hat{Z}}$ is a one-dimensional set.

Lemma 4.4. *Let $S \subset \mathbb{C}\mathbf{P}^3$ be a ruled surface that does not contain any planes and let $\gamma \subset \mathbb{C}\mathbf{P}^5$ be the set of points on the Plücker quadric that correspond to lines contained in S. Then γ is one-dimensional.*

Proof. Assume, for contradiction, that γ contains a two-dimensional irreducible component γ_2, and let $\Pi \subset \mathbb{C}\mathbf{P}^3$ be a generic plane. Since we are in projective space, Π intersects every line of γ_2. Each line contained in S whose corresponding point is in γ_2 intersects the curve $\sigma = S \cap \Pi$. Since the singular points of S are contained in a one-dimensional variety (this follows, for example, from Sard's lemma [43]), and since Π is a generic plane, we may assume that Π contains only finitely many singular points of S. Let $F : \gamma_2 \to \sigma$ be a mapping that sends each point $p \in \gamma_2$ to the intersection point of σ with the line corresponding to p.

We claim that σ contains a non-singular point q of S such that $F^{-1}(q)$ is infinite (*i.e.*, infinitely many lines that correspond to points of γ_2 are incident to q). Indeed, first notice that it is impossible for $F^{-1}(q)$ to be two-dimensional, for any $q \in \sigma$. Indeed, if $F^{-1}(q)$ were two-dimensional, then the intersection of the corresponding lines with any sphere around q would also be two-dimensional, contradicting the fact that S is two-dimensional. Therefore, for every point $q \in \sigma$, $F^{-1}(q)$ is either one-dimensional or zero-dimensional. Moreover, it cannot be that all of these pre-images are zero-dimensional, since a one-dimensional set of zero-dimensional varieties cannot cover the entire two-dimensional family of lines; see, *e.g.*, [26, Exercise 3.22(b)]. Thus, there are infinitely many points $q \in \sigma$, such that $F^{-1}(q)$ is infinite. Since there are finitely many singular points of S in σ, there are non-singular points with such an infinite pre-image.

We can now complete the proof of Lemma 4.4. We have a smooth point q of S such that there are infinitely many lines that pass through q and are contained in S. These lines must also be contained in the tangent plane T_qS. Thus by Lemma 2.7, S must contain the tangent plane T_qS. This contradicts the fact that S does not contain any planes.

The above proof still holds when assuming that the dimension of γ is larger than 2. Notice that γ cannot be zero-dimensional, since then the higher-dimensional extension of Bézout's theorem (Theorem 2.6) would imply that it is finite, and the union of the corresponding lines will not be two-dimensional. □

Lemma 4.4 implies that $\Lambda_{\hat{Z}}$ is one-dimensional. To prove that it consists of a single one-dimensional irreducible component (possibly with additional zero-dimensional components), we shall first require the following two lemmas.

Lemma 4.5. *Let $\gamma \subset \mathbb{C}\mathbf{P}^5$ be a projective subvariety of the Plücker quadric Λ. Then $\bigcup_{p\in\gamma} \ell_p \subset \mathbb{C}\mathbf{P}^3$ is also a projective variety.*

Lemma 4.5 is a special case of Proposition 6.13 from [25].

Lemma 4.6. *Let $\ell \subset \mathbb{C}\mathbf{P}^3$ be a projective line, and let $V \subset \mathbb{C}\mathbf{P}^5$ be the set of points on the Plücker quadric Λ that correspond to lines that intersect ℓ. Then V is a projective variety.*

Lemma 4.6 is a special case of Example 6.14 from [25].

Let γ be an irreducible one-dimensional component of $\Lambda_{\hat{Z}}$. According to Lemma 4.5, $\bigcup_{p\in\gamma} \ell_p$ is a two-dimensional algebraic variety that is fully contained in $\hat{Z}$. Since $\hat{Z}$ is irreducible,

$$\bigcup_{p\in\gamma} \ell_p = \hat{Z},$$

and γ corresponds to a generating family of $\hat{Z}$. Let $q \in \mathbb{C}\mathbf{P}^5$ be a point of $\Lambda_{\hat{Z}} \setminus \gamma$, and let $\ell_q \subset \mathbb{C}\mathbf{P}^3$ be the line that corresponds to q. According to Lemma 4.6, the set $V_q \subset \mathbb{C}\mathbf{P}^5$, of points that correspond to lines that intersect ℓ_q, is a variety. Note that $\gamma \cap V_q$ is infinite, because every point on ℓ_q lies on some generator line ℓ_p for $p \in \gamma$. Since γ is irreducible, then $\gamma \subset V_q$. That is, every line in the generating family $\{\ell_p\}_{p\in\gamma}$ of $\hat{Z}$ intersects ℓ_q. If there are at least three points in $\Lambda_{\hat{Z}} \setminus \gamma$, then each line in the generating family of $\hat{Z}$ intersects three given lines, which implies that $\hat{Z}$ is either a regulus or a plane.[4] Since reguli and planes are not singly ruled, it follows that $\Lambda_{\hat{Z}}$ is composed of an irreducible one-dimensional curve, and at most two other points (the additional points correspond to non-generating lines that are fully contained in $\hat{Z}$).[5]

Adding Γ to the analysis. Consider a line ℓ whose pre-image under the Plücker map is the point $[p_0 : \ldots : p_5] \in \Lambda$, such that ℓ does not lie in the plane at infinity. Then ℓ intersects Γ if and only if $p_0^2 + p_1^2 + p_2^2 = 0$. Indeed, recall that ℓ contains the point $[0 : p_0 : p_1 : p_2]$, and this is the only point of ℓ on the plane at infinity $Z(x_0)$, for otherwise

[4] A nice proof for this claim, which holds in $\mathbb{R}^3, \mathbb{C}^3$, and $\mathbb{C}\mathbf{P}^3$, can be found in `http://math.mit.edu/~lguth/PolyMethod/lect10.pdf` (version of June 2013).

[5] This also implies that the reguli are the only doubly ruled surfaces in $\mathbb{C}\mathbf{P}^3$.

ℓ would be fully contained in that plane. (It cannot be that $p_0 = p_1 = p_2 = 0$, since then all four points $[0 : p_0 : p_1 : p_2]$, $[p_0 : 0 : -p_3 : -p_4]$, $[p_1 : p_3 : 0 : -p_5]$, and $[p_2 : p_4 : p_5 : 0]$ would have a zero x_0-coordinate, implying that ℓ is contained in $Z(x_0)$.) Thus, the set $\Gamma_\Lambda = \{p \in \Lambda \mid \ell_p \cap \Gamma \neq \emptyset\}$ is an algebraic variety of codimension 1 in Λ. Since the irreducible one-dimensional component of $\Lambda_{\hat{Z}}$ is also a variety, either it is fully contained in Γ_Λ, or the intersection $\Lambda_{\hat{Z}} \cap \Gamma_\Lambda$ is a zero-dimensional variety, and therefore finite according to the higher-dimensional extension of Bézout's theorem (Theorem 2.6). If the former case occurs, then at most two lines in $\hat{Z}$ do not intersect Γ. However, since $\hat{Z}$ is the complex projectivization of a real ruled surface, $\hat{Z}$ contains infinitely many real lines (lines whose defining equations involve only real coefficients) that are not contained in the plane $Z(x_0)$, and if ℓ is such a line then $\ell \cap Z(x_0)$ consists of real points. This is a contradiction since the curve Γ contains no real points. Therefore, the intersection $\Lambda_{\hat{Z}} \cap \Gamma_\Lambda$ is finite.

Lemma 4.7. *Every line intersects* Γ *in at most two points.*

Proof. By Bézout's theorem (Theorem 2.5), applied in the plane at infinity $h = Z(x_0)$, Γ has at most two intersection points with any line that is not fully contained in Γ (clearly, lines not contained in h can meet Γ at most once). Thus, it suffices to prove that no line is fully contained in Γ.

Regard h as the standard complex projective plane $\mathbb{C}\mathbf{P}^2$, with homogeneous coordinates $[x_1 : x_2 : x_3]$. A line $\ell \in h$ has an equation of the form $a_1x_1 + a_2x_2 + a_3x_3 = 0$, and there exists at least one coordinate, say x_3, with $a_3 \neq 0$. This allows us to write the equation of ℓ as $x_3 = \alpha x_1 + \beta x_2$, so its intersection with Γ satisfies the equation

$$x_1^2 + x_2^2 + (\alpha x_1 + \beta x_2)^2 = 0, \quad \text{or} \quad (1+\alpha^2)x_1^2 + 2\alpha\beta x_1x_2 + (1+\beta^2)x_2^2 = 0.$$

This is a quadratic equation, whose coefficients cannot all vanish, as is easily checked. Hence it has at most two solutions, which is what the lemma asserts. □

Lemma 4.7 implies that $\Gamma \cap \hat{Z}$ is a finite set. Indeed, if this were not the case, then there would exist infinitely many points of Γ that lie in $\hat{Z}$ and each of them is therefore incident to a line contained in $\hat{Z}$. Since every line meets Γ in at most two points, Γ would have intersected infinitely many lines contained in $\hat{Z}$. This is a contradiction since, as argued above, $\Lambda_{\hat{Z}} \cap \Gamma_\Lambda$ is a finite intersection.

Adding the circles to the analysis. Let $\bar{C}'$ be the collection of circles described earlier; that is, an infinite set of pairwise non-coplanar circles that are fully contained in $Z(\bar{g})$ and incident to $\bar{z}_2$. Let $\hat{C}'$ be the corresponding collection of the complex projectivizations of these circles. As just argued, all of the intersection points between the circles of $\hat{C}'$ and Γ must lie in the finite intersection $\Gamma \cap \hat{Z}$.

Lemma 4.8. *Each circle* $\hat{C}$ *in* $\hat{C}'$ *intersects* Γ *in precisely two points.*

Proof. Each circle $\hat{C}$ in $\hat{C}'$ is the complex projectivization of a real circle C. Consider C as the intersection of its supporting plane, whose equation is $ax_0 + bx_1 + cx_2 + dx_3 = 0$,

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for appropriate parameters a,b,c,d, with some suitable sphere whose equation is $(x_1 - a'x_0)^2 + (x_2 - b'x_0)^2 + (x_3 - c'x_0)^2 = d'x_0^2$, for appropriate parameters a',b',c',d'. Since C is real, all coefficients of these equations can be assumed to be real, and $d' > 0$.

By combining these equations of C with the equations $x_0 = 0$ and $x_1^2 + x_2^2 + x_3^2 = 0$ of the absolute conic Γ, we obtain the system

$$bx_1 + cx_2 + dx_3 = 0, \quad x_1^2 + x_2^2 + x_3^2 = 0$$

(where the second equation arises twice), which always has two distinct solutions when b,c,d are real. Indeed, using the notation in the proof of Lemma 4.7, the intersection points satisfy a quadratic equation of the form

$$(1+\alpha^2)x_1^2 + 2\alpha\beta x_1x_2 + (1+\beta^2)x_2^2 = 0$$

(or a similar, symmetrically defined equation in another pair of variables), where α and β are real. The discriminant of this equation is

$$4\alpha^2\beta^2 - 4(1+\alpha^2)(1+\beta^2) = -4(\alpha^2 + \beta^2 + 1),$$

which is always non-zero (and strictly negative) when α and β are real. Also, the coefficients of x_1^2, x_2^2 are both non-zero, and thus the equation has exactly two (complex conjugate) solutions. □

The final stretch. Since $\hat{C}'$ contains infinitely many circles and $\Gamma \cap \hat{Z}$ is finite, by the pigeonhole principle there must exist two circles C_1, C_2 in $\hat{C}'$ such that the sets $C_1 \cap \Gamma$ and $C_2 \cap \Gamma$ are identical (each being a set of two points). By construction, C_1 and C_2 are contained in two distinct planes Π_1 and Π_2. Consider the line $\ell = \Pi_1 \cap \Pi_2$ and notice that it contains $C_1 \cap C_2$. Thus, ℓ contains the two intersection points of C_1, C_2 with Γ. Since these two points are contained in the plane $\{x_0 = 0\}$, ℓ is also contained in this plane. However, this is impossible, since ℓ also contains $\bar{z}_2$ (common to all circles of $\hat{C}'$), which is not in the plane $\{x_0 = 0\}$. This contradiction completes, at long last, the proof of Lemma 3.1. □

5. Unit circles

In this section we consider the special case where all circles in $\mathcal{C}$ have the same radius, say 1. The analysis is very similar to the general case, except for two key issues.

(a) In the general case we have used the fact that the incidence graph in $\mathcal{P} \times \mathcal{C}$ does not contain $K_{3,2}$ as a subgraph, to derive the weaker 'bootstrapping' bound $I(\mathcal{P},\mathcal{C}) = O(n^{2/3}m + n)$. Here, in Lemma 5.1 below, we replace this estimate by an improved one, exploiting the fact that all circles are congruent.[6]

[6] If all our circles were coplanar or cospherical, life would have been simpler, since then the incidence graph does not contain $K_{2,3}$ as a subgraph, which is the basis for deriving the improved planar bound $I(\mathcal{P},\mathcal{C}) = O(m^{2/3}n^{2/3} + m + n)$. In three dimensions the incidence graph can contain $K_{2,q}$ for any value of q, making the analysis more involved and subtler.

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(b) When considering the case of circles that lie in a common plane or sphere, we use the improved planar bound for unit circles $I(\mathcal{P},\mathcal{C}) = O(|\mathcal{P}|^{2/3}|\mathcal{C}|^{2/3} + |\mathcal{P}| + |\mathcal{C}|)$ (*e.g.*, see [46]).

These two improvements result in the sharper bound of Theorem 1.3, which we restate here for the convenience of the reader.

Theorem 1.3. *Let $\mathcal{P}$ be a set of m points and let $\mathcal{C}$ be a set of n unit circles in $\mathbb{R}^3$, let ε be an arbitrarily small positive constant, and let $q \leqslant n$ be an integer. If no plane or sphere contains more than q circles of $\mathcal{C}$, then*

$$I(\mathcal{P},\mathcal{C}) = O\big(m^{5/11+\varepsilon}n^{9/11} + m^{2/3+\varepsilon}n^{1/2}q^{1/6} + m + n\big),$$

where the constant of proportionality depends on ε.

Proof. We first establish the following lemma, which improves the weaker bound on $I(\mathcal{P},\mathcal{C})$, as discussed in (a) above.

Lemma 5.1. *Let $\mathcal{P}$ be a set of m points in $\mathbb{R}^3$ and let $\mathcal{C}$ be a collection of unit circles in $\mathbb{R}^3$, so that each circle of $\mathcal{C}$ is incident to at least three points of $\mathcal{P}$. Then $|\mathcal{C}| = O(m^{5/2})$. A stronger statement is that the number of circles of $\mathcal{C}$ that pass through any fixed point $o \in \mathcal{P}$ and through at least two other points is $O(m^{3/2})$.*

Proof. It clearly suffices to establish only the second claim of the lemma. Fix one point o of $\mathcal{P}$ and let $\mathcal{P}' = \mathcal{P}\backslash\{o\}$. For each $a \in \mathcal{P}'$, let σ_a be the locus of all points $w \in \mathbb{R}^3$ such that o, a, and w lie on a common unit circle. The set σ_a is an algebraic surface of revolution, obtained by taking any unit circle passing through o and a and by rotating it around the line oa. If o and a are diametral, that is, if $|oa| = 2$, then σ_a is a sphere. If $|oa| > 2$ then σ_a is empty. Otherwise, σ_a is easily seen to be an irreducible surface of degree 4; the 'outside' portion of σ_a resembles a sphere pinched at o and a, which are the only singular points of σ_a; the 'inner' portion resembles a pointy (American) football.

Let $\mathcal{S} = \{\sigma_a : a \in \mathcal{P}'\}$. In order to prove the second claim of the lemma, it suffices to show that $I(\mathcal{P}',\mathcal{S}) = O(m^{3/2})$. We require the following lemma.

Lemma 5.2. *There exists an absolute constant s such that for all triples $a,b,c \in \mathcal{P}'$, we have $|\sigma_a \cap \sigma_b \cap \sigma_c| \leqslant s$*

Proof. If $|\sigma_a \cap \sigma_b \cap \sigma_c|$ is finite, then Milnor's theorem (Theorem 2.3) implies that this number is at most some constant E. By setting s to be, say, $E+1$, we ensure that the intersection must be infinite.

Consider then the case where $\sigma_a \cap \sigma_b \cap \sigma_c$ is a one-dimensional curve γ (it cannot be two-dimensional because σ_a, σ_b, σ_c are distinct irreducible varieties, no pair of which can overlap in a two-dimensional subset), and let w be a smooth point on γ. Let τ be the tangent to γ at w. Then τ is orthogonal to the three respective normals $\mathbf{n}_a$, $\mathbf{n}_b$, $\mathbf{n}_c$ to σ_a, σ_b, σ_c at w. In other words, these normals must be coplanar. Now, because σ_a is the surface of revolution of a circle, $\mathbf{n}_a$ lies on the ray $\vec{\xi_a w}$, where ξ_a is the centre of the unit circle

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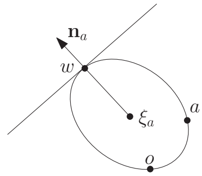

Figure 1. The normal $\mathbf{n}_a$ to σ_a lies on the ray $\vec{\xi_a w}$, where ξ_a is the centre of the unit circle passing through o, a, and w.

passing through o, a, and w; an illustration is provided in Figure 1. Symmetric properties hold for σ_b and σ_c, with respective centres ξ_b, ξ_c.

In other words, the argument implies that w, ξ_a, ξ_b, and ξ_c all lie in a common plane h. However, all three centres ξ_a, ξ_b, and ξ_c must lie on the perpendicular bisector plane π of ow, which does not contain w, so $\pi \neq h$, and these centres then have to lie on the intersection line $\ell = h \cap \pi$. This is impossible if ξ_a, ξ_b, and ξ_c are distinct, because it is impossible for three distinct collinear points to be at the same distance (namely, 1) from o. Assume then that $\xi_a = \xi_b$, say. That is, we have two distinct unit circles passing through o and w with a common centre $\xi = \xi_a = \xi_b$, which is possible only when $|ow| = 2$ (that is, ow is a diameter of both circles). Moreover, ξ lies at distance 1 from o, a, and b, so it is the centre of a unit ball that passes through these points. There can be at most two such balls, so there are only two possible locations for ξ. Since ξ is the midpoint of ow (recall that ow is a diameter of the sphere σ_a), it follows that there are only two possible locations for w. That is, γ has at most two smooth points, which is impossible, as follows, say, from Sard's theorem (*e.g.*, see [43]). □

We can now apply Theorem 2 from [50] to conclude that

$$I(\mathcal{P}', \mathcal{S}) \leqslant |\mathcal{P}'|^{3/4}|\mathcal{S}|^{3/4} + |\mathcal{P}'| + |\mathcal{S}|.$$

Since $|\mathcal{P}'| = O(m)$ and $|\mathcal{S}| = O(m)$, Lemma 5.1 follows. □

Consider pairs of the form (p, c) where $p \in \mathcal{P}$ and the circle $c \in \mathcal{C}$ is incident to p and to at least two other points of $\mathcal{P}$. By Lemma 5.1, every point of $\mathcal{P}$ can participate in at most $O(m^{3/2})$ such pairs, and thus the number of pairs is $O(m^{5/2})$. This implies that $I(\mathcal{P}, \mathcal{C}) = O(m^{5/2} + n)$, so it is $O(n)$ for $m = O(n^{2/5})$ (recall that in the general case this could be claimed only for $m = O(n^{1/3})$).

The proof of Theorem 1.3 now proceeds in complete analogy with the proof of Theorem 1.1, except for the modifications mentioned in (a) and (b) above. Specifically, we construct an r-partitioning polynomial, of degree $O(r^{1/3})$, for a sufficiently large constant parameter r, and consider separately points of $\mathcal{P}$ in the cells of the partition, and points on $Z(f)$. The bound for the former kind of points is handled via induction, in much the same way as before, except that we replace the term $O(n)$, towards the derivation of (a bound analogous to the one in) (3.5), by $O(m^{5/11}n^{9/11})$, which holds for $n = O(m^{5/2})$. We also remove the terms of the form $O(m^{6/11+\varepsilon}n^{9/11})$ (see below for a justification). This

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results in the modified bound

$$I(\mathcal{P}_0 \cup \mathcal{P}', \mathcal{C}') \leqslant \frac{\alpha_1}{3}\big(m^{5/11+\varepsilon} n^{9/11} + m^{2/3+\varepsilon} n^{1/2} q^{1/6}\big) + \alpha_2 |\mathcal{P}'|. \tag{5.1}$$

This ‘explains’ why we can use here the improved exponents $5/11$ and $9/11$ instead of the weaker respective ones $3/7$ and $6/7$.

The second modification is in handling incidences involving private points on $Z(f)$ that lie in planes or spheres that are zero sets of respective irreducible factors of f. Here, in the derivation of a bound analogous to the one in (3.7), we use the sharper planar bound

$$I(\mathcal{P}, \mathcal{C}) = O\big(|\mathcal{P}|^{2/3} |\mathcal{C}|^{2/3} + |\mathcal{P}| + |\mathcal{C}|\big),$$

which also holds when the points and circles are all cospherical. This replaces (3.7) with the sharper bound

$$I(\mathcal{P}_p^{(1)}, \mathcal{C}_0) \leqslant \frac{\alpha_1}{3} m^{2/3+\varepsilon} n^{1/2} q^{1/6} + \alpha_2 (m_p^{(1)} + n/3). \tag{5.2}$$

The rest of the analysis remains unchanged, and leads to the bound asserted in the theorem. □

By applying the techniques presented in Section 3.1, we obtain the following theorem.

Theorem 5.3. *Let $\mathcal{P}$ be a set of m points and let $\mathcal{C}$ be a set of n unit circles in $\mathbb{R}^3$, let $q \leqslant n$ be an integer, and let $m = O(n^{3/2-\delta})$, for some fixed arbitrarily small constant $\delta > 0$. If no sphere or plane contains more than q circles of $\mathcal{C}$, then*

$$I(\mathcal{P}, \mathcal{C}) \leqslant A_{m,n}\big(m^{5/11} n^{9/11} + m^{2/3} n^{1/2} q^{1/6} + m + n\big),$$

where

$$A_{m,n} = A^{\left\lceil \frac{\log(m^5/n^2)}{3\log(n^{3/2}/m)} \right\rceil + 1},$$

for some absolute constant $A > 1$. □

Since the proof of Theorem 5.3 is almost identical to the proof of Theorem 1.2, we omit it.

6. Applications

High-multiplicity points. The following is an easy but interesting consequence of Theorems 1.1 and 1.3.

Corollary 6.1.

(a) *Let $\mathcal{C}$ be a set of n circles in $\mathbb{R}^3$, and let $q \leqslant n$ be an integer so that no sphere or plane contains more than q circles of $\mathcal{C}$. Then there exists a constant k_0 (independent of $\mathcal{C}$) such that for any $k \geqslant k_0$, the number of points incident to at least k circles of $\mathcal{C}$ is*

$$\tilde{O}\left(\frac{n^{3/2}}{k^{7/4}} + \frac{n^{3/2} q^{1/2}}{k^3} + \frac{n^{3/2} q^{3/10}}{k^{11/5}} + \frac{n}{k}\right). \tag{6.1}$$

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In particular, if $q = O(1)$, the number of such points is

$$\tilde{O}\left(\frac{n^{3/2}}{k^{7/4}} + \frac{n}{k}\right).$$

(b) *If the circles of $\mathcal{C}$ are all congruent the bound improves to*

$$\tilde{O}\left(\frac{n^{3/2}}{k^{11/6}} + \frac{n^{3/2}q^{1/2}}{k^3} + \frac{n}{k}\right). \tag{6.2}$$

In particular, if $q = O(1)$, the number of such points is

$$\tilde{O}\left(\frac{n^{3/2}}{k^{11/6}} + \frac{n}{k}\right).$$

Proof. Let m be the number of points incident to at least k circles of $\mathcal{C}$, and observe that these points determine at least mk incidences with the circles of $\mathcal{C}$. Comparing this lower bound with the upper bound in Theorem 1.1 (for (a)), or in Theorem 1.3 (for (b)), the claims follow. □

Remarks. **(1)** It is interesting to compare the bounds in (6.1) and (6.2) with the various recent bounds on incidences between points and lines in three dimensions [16, 23, 24]. In all of them the threshold value $m = \Theta(n^{3/2})$ plays a significant role. Specifically, we have the following.

(i) The number of joints in a set of n lines in $\mathbb{R}^3$ is $O(n^{3/2})$, a tight bound in the worst case [23].

(ii) If no plane contains more than $\sqrt{n}$ lines, the number of points incident to at least $k \geqslant 3$ lines is $O(n^{3/2}/k^2)$ [24].

(iii) A related bound where $m = n^{3/2}$ is a threshold value, under different assumptions, is given in [16].

The bounds in (6.1) and (6.2) are somewhat weaker (because of the extra small factors hidden in the $\tilde{O}(\cdot)$ notation, the rather restrictive constraints on q, and the constraint $k \geqslant k_0$) but they belong to the same class of results. It would be interesting to understand how general this phenomenon is. For example, does it also show up in incidences with other classes of curves in $\mathbb{R}^3$? We tend to conjecture that this is the case, under reasonable assumptions concerning those curves. Similar threshold phenomena should exist in higher dimensions. 'Extrapolating' from the results of [31, 40], these thresholds should be at $m = n^{d/(d-1)}$.

(2) The bounds can be slightly tightened by using Theorem 1.2 or Theorem 5.3 instead of Theorem 1.1 or Theorem 1.3, respectively, but we leave these slight improvements to the interested reader.

Similar triangles. Another application of Theorem 1.1 (or rather of Theorem 1.2) is an improved bound on the number of triangles spanned by a set $\mathcal{P}$ of t points in $\mathbb{R}^3$ and similar to a given triangle Δ. Let $F(\mathcal{P}, \Delta)$ be the number of triangles spanned by $\mathcal{P}$ that are similar to Δ, and let $F(t)$ be the maximum of $F(\mathcal{P},\Delta)$ as $\mathcal{P}$ ranges over all sets of t points and Δ ranges over all triangles. We then have the following result.

Theorem 6.2.

$$F(t) = O(t^{15/7}) = O(t^{2.143}).$$

Proof. Let $\mathcal{P}$ be a set of t points in $\mathbb{R}^3$ and let $\Delta = uvw$ be a given triangle. Suppose that pqr is a similar copy of Δ, where $p,q,r \in \mathcal{P}$. If p corresponds to u and q to v, then r has to lie on a circle c_{pq} that is orthogonal to the segment pq, whose centre lies at a fixed point on this segment, and whose radius is proportional to $|pq|$. Thus, the number of possible candidates for the point r, for p,q fixed, is exactly the number of incidences between $\mathcal{P}$ and c_{pq}. There are $2\binom{t}{2} = t(t-1)$ such circles, and no circle arises more than twice in this manner. It follows that $F(t)$ is bounded by twice the number of incidences between the t points of $\mathcal{P}$ and the $t(t-1)$ circles c_{pq}. We now apply Theorem 1.2 with $m = t$ and $n = t(t-1)$. (The theorem applies for these values, which satisfy $m \approx n^{1/2}$, much smaller than the threshold $n^{3/2}$; in fact, m lies in the second range $[n^{1/3}, n^{4/5}]$.) It remains to show that the expression (1.2) is $O(t^{15/7})$.

The first term of (1.2) is $O(t^{15/7})$. To control the remaining terms, it suffices to show that at most $O((n^3/m^2)^{3/7}) = O(t^{12/7})$ of the circles lie on a common plane or sphere. In fact, we claim that at most $O(t)$ circles can lie on a common plane or sphere. Indeed, let Π be a plane. Then for any circle c_{pq} contained in Π, pq must be orthogonal to Π, pass through the centre of c_{pq}, and each of p and q must lie at a fixed distance from Π (the distances are determined by the triangle Δ and by the radius of c_{pq}). This implies that each point of $\mathcal{P}$ can generate at most two circles on Π. The argument for cosphericality is essentially the same. The only difference is that one point of $\mathcal{P}$ may lie at the centre of the given sphere σ, and then it can determine up to $2(t-1)$ distinct circles on σ. Still, the number of circles on σ is $O(t)$. As noted above, this completes the proof of the theorem. □

As mentioned in the Introduction, this slightly improves a previous bound of $O^*(n^{58/27})$ in [6] (see also [1]).

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