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# AN ABSTRACT ALGEBRAIC LOGIC STUDY OF DA COSTA'S LOGIC $\mathcal{C}_1$ AND SOME OF ITS PARACONSISTENT EXTENSIONS

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**Abstract.** Two famous negative results about da Costa's paraconsistent logic $\mathcal{C}_1$ (the failure of the Lindenbaum–Tarski process [44] and its non-algebraizability [39]) have placed $\mathcal{C}_1$ seemingly as an exception to the scope of Abstract Algebraic Logic (AAL). In this paper we undertake a thorough AAL study of da Costa's logic $\mathcal{C}_1$. On the one hand, we strengthen the negative results about $\mathcal{C}_1$ by proving that it does not admit any algebraic semantics whatsoever in the sense of Blok and Pigozzi (a weaker notion than algebraizability also introduced in the monograph [6]). On the other hand, $\mathcal{C}_1$ is a protoalgebraic logic satisfying a Deduction-Detachment Theorem (DDT). We then extend our AAL study to some paraconsistent axiomatic extensions of $\mathcal{C}_1$ covered in the literature. We prove that for extensions $\mathcal{S}$ such as $\mathcal{C}ilo$ [26], every algebra in $\mathsf{Alg}^*(\mathcal{S})$ contains a Boolean subalgebra, and for extensions $\mathcal{S}$ such as $\mathbf{P^1}$, $\mathbf{P^2}$, or $\mathbf{P^3}$ [16, 53], every subdirectly irreducible algebra in $\mathsf{Alg}^*(\mathcal{S})$ has cardinality at most 3. We also characterize the quasivariety $\mathsf{Alg}^*(\mathcal{S})$ and the intrinsic variety $\mathbb{V}(\mathcal{S})$, with $\mathcal{S} = \mathbf{P^1}, \mathbf{P^2}$, and $\mathbf{P^3}$.

**§1. Introduction.** Paraconsistent logics were developed in the last century as a direct challenge to the principle of explosion (valid in both classical and intuitionistic logics) that from a contradiction anything follows. Concretely, *ex contradictione sequitur quodlibet* is dropped, thus allowing for situations where $\varphi, \neg\varphi \nvdash \psi$. The study of paraconsistency has emerged in order to cope with inconsistency in a logical way, as motivated for instance by paradoxes of self-reference, or more simply in order to reason soundly in the presence of contradictory information. There are several competing approaches to paraconsistency, most notably, *discussive logics*, *adaptive logics*, *relevance logics*, and *logics of formal inconsistency*. For a historical account of the development of paraconsistent logics and an overview of some of its applications, ranging from philosophy to artificial intelligence, we refer the reader to [2, 17, 34, 42, 46–49, 55].

Da Costa's paraconsistent logic $\mathcal{C}_1$, and more generally da Costa's $\mathcal{C}$-systems $\mathcal{C}_n$, originally introduced in [18], constitute the seminal examples of *logics of formal inconsistency*. The logics are defined axiomatically, by weakening classical logic ($\mathcal{CL}$) in such a way that the principle of explosion

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does not hold in general. However, distinctively, these logics are able to express consistency internally, and enjoy a controlled form of explosion. In $\mathcal{C}_1$, for instance, the consistency (or classicality) of a formula $\varphi$ can be expressed as $\varphi^\circ := \neg(\varphi \wedge \neg\varphi)$ and $\varphi^\circ, \varphi, \neg\varphi \nvdash \psi$.

A couple of papers soon followed the seminal work of da Costa in 1963, namely [3, 25], which further developed these recent logical systems. Among the first results about $\mathcal{C}_1$, one can emphasize the decidability of $\mathcal{C}_1$, the fact that da Costa's $\mathcal{C}$-systems form a hierarchy of proper extensions, and the fact that they are not complete w.r.t. finite matrices.[1]

Semantical investigations of da Costa's $\mathcal{C}$-systems were also present since the beginning. Da Costa himself proposed the notion of $\mathcal{C}_n$-algebras as early as 1966 in [19], defined the notions of filters and ideals for $\mathcal{C}_n$-algebras in [20], and later in a collaboration with Sette also defined the notion of $\mathcal{C}_\omega$-algebras in [27]. But perhaps the most successful semantics was the non-truth-functional two-valued semantics developed by Alves and da Costa in [22, 23] and establishing a completeness theorem for $\mathcal{C}_1$. The quest for an algebraic counterpart for the logic $\mathcal{C}_1$ was yet to have one further development with the so-called *da Costa algebras* investigated in [14, 52] and generalizing the $\mathcal{C}_1$-algebras introduced in [19].

The definite reference on da Costa's logic would finally arrive in [21], where the theory of inconsistent formal systems is presented fully developed and where several results from the previous papers mentioned are compiled.

The next big advance on da Costa's logic would appear in 1980 and tore apart any hope of algebraizing $\mathcal{C}_1$ according to the classical Lindenbaum–Tarski process. Indeed, Mortensen proved in [44] that the only congruence on the formula algebra compatible with the theorems of $\mathcal{C}_1$ is the identity congruence, and as a consequence the Lindenbaum–Tarski algebra $\boldsymbol{Fm}/\boldsymbol{\Omega}^{\boldsymbol{Fm}}(\mathrm{Thm}_{\mathcal{C}_1})$ is isomorphic to the algebra of formulas $\boldsymbol{Fm}$. Still in the 80s, two further contributions to the study of da Costa's $\mathcal{C}$-systems were published, namely [45, 54]. It is worth mentioning that Mortensen's paper [45] already contained material about the logics $\mathbf{P}^1$ and $\mathbf{P}^2$, under the names $\mathrm{C}_{0.1}$ and $\mathrm{C}_{0.2}$, respectively. As to Urbas' paper [54], it provided several counterexamples for very simple properties which fail in $\mathcal{C}_1$. Perhaps the most surprising one is that the consistency formula itself, $\varphi^\circ := \neg(\varphi \wedge \neg\varphi)$, turns out *not* to be interderivable with the formula $\neg(\neg\varphi \wedge \varphi)$. These somehow odd properties made $\mathcal{C}_1$ a paradigmatic exception among non-classical logics, and in particular among paraconsistent logics.

One last negative result was put forward in 1991, when Lewin, Mikenberg, and Schwarze proved in [39] that $\mathcal{C}_1$ is not algebraizable according to

[1] The proof of the first fact is credited to Fidel in [21, Theorem 11], but was only published in [30], and the proof of the latter fact is credited to Arruda in footnote 3 of [18] and [21, Theorem 10].

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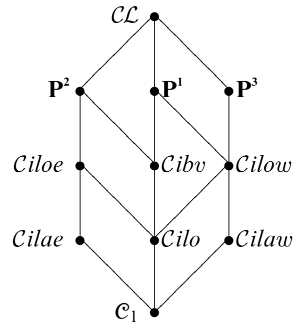


FIGURE 1. Extensions of $\mathcal{C}_1$ under study.

Blok and Pigozzi's theory. This seemed to leave da Costa's logic outside of any standard attempt of algebraization. As a result, two alternative algebraizations were proposed, namely behavioral algebraization [12, 13] and possible-translations algebraization [9].

The motivation behind the present work is to study da Costa's logic from an Abstract Algebraic Logic (AAL) perspective. As we have seen, $\mathcal{C}_1$ had already been subject to extensive research in the literature, but apart from its famous non-algebraizability [39] and the side-remark on its non-equivalentiality [36, pp. 425–426], a thorough AAL investigation of $\mathcal{C}_1$ is still missing.

The main new result about da Costa's logic established in this work is the fact that $\mathcal{C}_1$ does not admit any algebraic semantics in the sense of Blok and Pigozzi (Theorem 5.6), as originally defined in [6]. Since every algebraizable logic possesses an algebraic semantics (in fact, every truth-equational logic possesses an algebraic semantics), this result generalizes the non-algebraizability of $\mathcal{C}_1$ established by Lewin, Mikenberg, and Schwarze [39]. Although this fact may reinforce the belief that $\mathcal{C}_1$ is an exception to the standard AAL methods, it satisfies nevertheless a wealth of properties within the theory of protoalgebraic logics. Namely, the class $\mathsf{Mod}(\mathcal{C}_1)$ enjoys the Filter Extension Property (FEP), the lattice of $\mathcal{C}_1$-filters of an arbitrary algebra is distributive, and the join-semilattice of finitely generated $\mathcal{C}_1$-theories is dually Brouwerian (Theorem 3.10).

The study of $\mathcal{C}_1$ eventually led us to consider the logic $\mathcal{C}ilo$ [24], a paraconsistent extension of $\mathcal{C}_1$ introduced by da Costa, Béziau, and Bueno in [24], and coined as $\mathcal{C}ilo$ in [16], as well as other paraconsistent axiomatic extensions of $\mathcal{C}_1$ covered in the literature, including the algebraizable logics $\mathbf{P^1}$, $\mathbf{P^2}$, and $\mathbf{P^3}$. The choice of which extensions of $\mathcal{C}_1$ to consider was mainly guided by [16, Section 3.10] and is depicted in Figure 1, along with the inclusion relations among them.

In particular, we will show that every algebra in $\mathsf{Alg}^*(\mathcal{C}ilo)$ contains a Boolean subalgebra (Proposition 9.10). Moreover, for $\mathcal{S} = \mathbf{P^1}, \mathbf{P^2}, \mathbf{P^3}$, we will prove that every non-trivial subdirectly $\mathsf{Alg}^*(\mathcal{S})$-irreducible algebra in $\mathsf{Alg}^*(\mathcal{S})$ has cardinality at most 3 (Theorems 11.9, 11.10, and 12.8 of the Leibniz operator). We further classify these logics within the Leibniz

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hierarchy—the main classification of logics in AAL, which characterizes its member classes in terms of algebraic properties enjoyed by the Leibniz operator—and characterize the quasivariety $\mathsf{Alg}^*(\mathcal{S})$ and the intrinsic variety $\mathbb{V}(\mathcal{S})$, with $\mathcal{S} = \mathbf{P^1}, \mathbf{P^2}, \mathbf{P^3}$ (Propositions 11.11, 11.13, and 12.9 and Corollaries 11.12, 11.14, and 12.10), although the results for Sette's logic $\mathbf{P^1}$ were originally established in [50].

Finally, we correct a couple of claims found in the literature, namely:

- We prove that $\mathcal{C}ilo$ is not equivalential (Proposition 9.6), and hence not algebraizable either, correcting [42, p. 183] and [16, p. 79].
- We provide new proofs for the algebraizability of $\mathbf{P^2}$ and $\mathbf{P^3}$. These logics are among the 8k three-valued logics considered in [16, Fact 3.82] or [17, Theorem 135]. However, the proofs of the cited results make use of relations which may not be *congruence* relations.[2]

We decided to organize the paper in three parts.

In Part I of our study, we undertake a thorough investigation of the logic $\mathcal{C}_1$ from an AAL perspective. We start by formally introducing da Costa's logic $\mathcal{C}_1$ in Section 3, as well as briefly reviewing the semantics put forward in the literature for $\mathcal{C}_1$. In Section 3.1 we fully classify $\mathcal{C}_1$ within the Leibniz and Frege hierarchies (both classifications were known in the literature, but we compile these scattered results) and in Section 3.2 we state some consequences of the Deduction-Detachment Theorem for $\mathcal{C}_1$. In Section 4 we provide two sets of congruence formulas for the logic $\mathcal{C}_1$. In Section 5.1 we study the class $\mathsf{Alg}^*(\mathcal{C}_1)$ and the intrinsic variety $\mathbb{V}(\mathcal{C}_1)$, and in Section 5.2 we prove that $\mathcal{C}_1$ admits no algebraic semantics in the sense of Blok and Pigozzi.

In Part II we study logics $\mathcal{S}$ such that $\mathcal{C}_1 \leq \mathcal{S} \leq \mathcal{CL}$ and consider two general conditions upon $\mathcal{S}$ under which stronger algebraic results hold, namely:

1. For every $\boldsymbol{A}$, $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and $a, b \in A^\circ$,[3]
$$\langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}(F) \quad \Leftrightarrow \quad a \leftrightarrow^{\boldsymbol{A}} b \in F.$$
2. $\mathcal{S}$ is finitary and finitely equivalential with a set of congruence formulas $\boldsymbol{\rho}(x, y) = \{x \leftrightarrow y, \neg x \leftrightarrow \neg y\}$.

The abstract investigation of $\mathsf{Alg}^*(\mathcal{S})$ under these assumptions paves the way for the individual study of the envisaged axiomatic extensions of $\mathcal{C}_1$.

Part III is devoted to the individual detailed analysis of each of the extensions of $\mathcal{C}_1$. We begin in Section 9 with $\mathcal{C}ilo$. In Section 9.1 we fully classify $\mathcal{C}ilo$ within the Leibniz hierarchy and in Section 9.2 we investigate some algebraic properties of the class $\mathsf{Alg}^*(\mathcal{C}ilo)$. In Section

[2]Using the notation $\circ$ (introduced on page 487), if one defines $\varphi \equiv \psi$ whenever $\vdash (\varphi \leftrightarrow \psi) \wedge \varphi^\circ \wedge \psi^\circ$, as done in [16, Fact 3.81], reflexivity may not hold in general, and if one defines $\varphi \equiv \psi$ whenever $\vdash (\varphi \leftrightarrow \psi) \wedge (\varphi^\circ \to \psi^\circ)$, as done in [16, Fact 3.75] or [17, Theorem 134], compatibility with the connectives does not seem to be trivial.

[3]See page 501.

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10 we consider the extension $\mathcal{C}ilow$ which allows us to prove that $\mathcal{C}ilo$ is not equivalential. Finally, in Sections 11 and 12 we investigate $\mathbf{P^1}$, $\mathbf{P^2}$, and $\mathbf{P^3}$. It is worth mentioning that Sette's logic $\mathbf{P^1}$ had already been subject to an AAL investigation in [50], but we present it here in the unified setting of equivalential extensions of $\mathcal{C}_1$ with set of congruence formulas $\boldsymbol{\rho}(x, y) = \{x \leftrightarrow y, \neg x \leftrightarrow \neg y\}$.

**§2. Preliminaries.** We adopt the standard terminology and notation employed in Abstract Algebraic Logic (AAL), and refer the reader to [32] for a recent and thorough treatment on the subject. We will here only briefly introduce the AAL notions needed in the sequel. These often rely on ground concepts from Universal Algebra and the theory of sentential logics. Classical references on these subjects are [10, 56].

We trust the reader is acquainted with the notions of algebra, homomorphism, congruence, variety, and quasivariety. We ought however to fix some notation. Given two algebras $\boldsymbol{A}, \boldsymbol{B}$, we denote a homomorphism $h$ from $\boldsymbol{A}$ to $\boldsymbol{B}$ simply by $h : \boldsymbol{A} \to \boldsymbol{B}$, and the least congruence on $\boldsymbol{A}$ containing a subset $X \subseteq A \times A$ by $\Theta^{\boldsymbol{A}}(X)$. The set of all congruences on $\boldsymbol{A}$ will be denoted by $\operatorname{Co}\boldsymbol{A}$; and given a class of algebras $\mathsf{K}$, the set of all congruences $\theta \in \operatorname{Co}\boldsymbol{A}$ such that $\boldsymbol{A}/\theta \in \mathsf{K}$ will be denoted by $\operatorname{Co}_{\mathsf{K}}\boldsymbol{A}$, and will be referred to as the set of congruences of $\boldsymbol{A}$ relative to $\mathsf{K}$, or simply the set $\mathsf{K}$-relative congruences of $\boldsymbol{A}$.

Let $\boldsymbol{A}$ be an algebra. A subset $B \subseteq A$ is a *subuniverse of* $\boldsymbol{A}$, if it is closed under the operations of $\boldsymbol{A}$. That is, if for every $n$-ary operation symbol $f \in \mathcal{L}$ and for every $b_1, \dots, b_n \in B$, it holds that $f^{\boldsymbol{A}}(b_1, \dots, b_n) \in B$. An algebra $\boldsymbol{B}$ of the same similarity type as $\boldsymbol{A}$ is a *subalgebra of* $\boldsymbol{A}$, fact which we shall denote by $\boldsymbol{B} \leq \boldsymbol{A}$, if $B \subseteq A$ and for every operation symbol $f \in \mathcal{L}$, it holds that $f^{\boldsymbol{B}} = f^{\boldsymbol{A}}{\restriction_B}$. Clearly, if $\boldsymbol{B}$ is a subalgebra of $\boldsymbol{A}$, then $B$ is a subuniverse of $\boldsymbol{A}$. Given an algebra $\boldsymbol{A}$ and a subset $X \subseteq A$, the *subuniverse generated by* $X$, which we shall denote by $\operatorname{Sg}^{\boldsymbol{A}}_{\mathcal{L}}(X)$, is the least subuniverse of $\boldsymbol{A}$ containing $X$.

Let $\boldsymbol{A}$ be an algebra. A congruence $\theta \in \operatorname{Co}\boldsymbol{A}$ is *compatible* with a subset $F \subseteq A$, if whenever $\langle a, b\rangle \in \theta$ and $a \in F$, then $b \in F$; in other words, $\theta$ does not identify elements inside $F$ with elements outside $F$. The least congruence on $\boldsymbol{A}$ compatible with a given $F \subseteq A$ is the identity congruence on $\boldsymbol{A}$, hereby denoted by $id_A$, while the largest congruence on $\boldsymbol{A}$ compatible with $F$, which always exists, is known as the *Leibniz congruence of* $F$, and is denoted by $\boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. The Leibniz congruence can be lifted to the powerset. Given $\mathcal{C} \subseteq \mathcal{P}(A)$, the *Tarski congruence* of $\mathcal{C}$ is the largest congruence on $\boldsymbol{A}$ compatible with every $F \in \mathcal{C}$; that is, $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathcal{C}) := \bigcap\{\boldsymbol{\Omega}^{\boldsymbol{A}}(F) : F \in \mathcal{C}\}$.

Given a class of algebras $\mathsf{K}$, we denote by $\mathbb{Q}\mathsf{K}$ and $\mathbb{V}\mathsf{K}$ the least quasivariety containing $\mathsf{K}$ and the least variety containing $\mathsf{K}$, respectively, and we denote by $\mathbb{I}$, $\mathbb{P}$, $\mathbb{P}_{\mathrm{S}}$, $\mathbb{S}$, and $\mathbb{H}$ the isomorphism, direct product, subdirect product, subalgebra, and homomorphic image operators, respectively. Recall that a *trivial* algebra is one with a single element universe, and observe that every quasivariety (and hence every variety as well) contains all trivial algebras.

We assume the reader is familiar with the notion of *subdirectly irreducible algebra*, as well as Birkhoff's famous result characterizing varieties in terms of their subdirectly irreducible elements [5, Theorem 2]. Namely, if $\mathsf{K}$ is a variety, then $\mathsf{K} = \mathbb{IP}_\mathbb{S}(\mathsf{K}_{\text{s.i.}})$, where $\mathsf{K}_{\text{s.i.}}$ denotes the set of subdirectly irreducible elements of $\mathsf{K}$—see for instance [10, Corollary 9.7] or [4, Theorem 3.44].

A generalization of Birkhoff's result for quasivarieties will be needed in Section 9.2. Given a class of algebras $\mathsf{K}$, an algebra $\boldsymbol{A}$ is *subdirectly irreducible relative to* $\mathsf{K}$, or is *subdirectly* $\mathsf{K}$*-irreducible*, if for every subdirect embedding $\alpha : \boldsymbol{A} \to \prod_{i\in I} \boldsymbol{A}_i$, with $\{\boldsymbol{A}_i : i \in I\} \subseteq \mathsf{K}$, there exists $i \in I$ such that $\pi_i \circ \alpha : \boldsymbol{A} \to \boldsymbol{A}_i$ is an isomorphism. Quasivarieties can also be characterized in terms of their relative subdirectly irreducible elements—this fact is not so well-known as its particular case for varieties, but a proof can be found in [11, Corollary 6] or [35, Theorem 3.1.1]; in the latter reference the original result is credited to Mal'cev [41]. Consequently, if $\mathsf{K}$ is a quasivariety, then $\mathsf{K} = \mathbb{IP}_s(\mathsf{K}_{\text{r.s.i.}})$, where $\mathsf{K}_{\text{r.s.i.}}$ denotes a set of subdirectly $\mathsf{K}$-irreducible elements of $\mathsf{K}$.

By a (sentential) *logic* we understand a structural consequence relation on $\mathrm{Fm}_\mathcal{L}$, where $\mathrm{Fm}_\mathcal{L}$ denotes the universe of the free algebra of terms $\boldsymbol{Fm}$ generated by a denumerable set of variables Var over an algebraic language $\mathcal{L}$. It is common practice to denote by $\mathcal{S}$ an arbitrary logic, i.e., the set of all *rules of* $\mathcal{S}$, and by $\Gamma \vdash_\mathcal{S} \varphi$ or $\langle \Gamma, \varphi\rangle \in \vdash_\mathcal{S}$, with $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_\mathcal{L}$, one such particular rule. We abbreviate $\varnothing \vdash \varphi$ simply by $\vdash \varphi$, and the facts $\varphi \vdash_\mathcal{S} \psi$ and $\psi \vdash_\mathcal{S} \varphi$ by $\varphi \dashv\vdash_\mathcal{S} \psi$. The relation identifying such pairs of formulas is called the *interderivability relation*.

Let $\mathcal{S}$ be a logic. A logic $\mathcal{S}'$ (in the same language of $\mathcal{S}$) is an *extension of* $\mathcal{S}$, if $\vdash_\mathcal{S} \subseteq \vdash_{\mathcal{S}'}$, a fact which we will denote simply by $\mathcal{S} \leq \mathcal{S}'$. Now, let $\mathcal{L}'$ be a language such that $\mathcal{L} \subseteq \mathcal{L}'$. A logic $\mathcal{S}'$ in the language $\mathcal{L}'$ is an *expansion of* $\mathcal{S}$, if for every set of $\mathcal{L}$-formulas $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_\mathcal{L}$, $\Gamma \vdash_\mathcal{S} \varphi \Rightarrow \Gamma \vdash_{\mathcal{S}'} \varphi$; in case the converse implication also holds, that is for every set of $\mathcal{L}$-formulas $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_\mathcal{L}$ we have $\Gamma \vdash_\mathcal{S} \varphi \Leftrightarrow \Gamma \vdash_{\mathcal{S}'} \varphi$, the logic $\mathcal{S}'$ is called a *conservative expansion of* $\mathcal{S}$, and the logic $\mathcal{S}$ is called the $\mathcal{L}$*-fragment of* $\mathcal{S}'$.

The sets of formulas which are $\vdash_\mathcal{S}$-closed are called $\mathcal{S}$*-theories*. The set of all $\mathcal{S}$-theories is denoted by $\mathcal{T}h(\mathcal{S})$ and the least $\mathcal{S}$-theory, whose elements are called $\mathcal{S}$*-theorems*, is denoted by $\mathrm{Thm}_\mathcal{S}$. A logic $\mathcal{S}$ is called *inconsistent* if $\mathrm{Thm}_\mathcal{S} = \mathrm{Fm}_\mathcal{L}$. The notion of $\mathcal{S}$-theory turns out to be an instance of a more general notion, applicable to arbitrary algebras. Given an algebra $\boldsymbol{A}$, an $\mathcal{S}$*-filter of* $\boldsymbol{A}$ is a subset $F \subseteq A$ such that, for every homomorphism $h : \boldsymbol{Fm} \to \boldsymbol{A}$ and every $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_\mathcal{L}$, if $\Gamma \vdash_\mathcal{S} \varphi$ and $h(\Gamma) \subseteq F$, then $h(\varphi) \in F$. We denote the set of all $\mathcal{S}$-filters of $\boldsymbol{A}$ by $\mathcal{F}i_\mathcal{S}\boldsymbol{A}$. As claimed, $\mathcal{F}i_\mathcal{S}\boldsymbol{Fm} = \mathcal{T}h(\mathcal{S})$. The least $\mathcal{S}$-filter of $\boldsymbol{A}$ containing $X \subseteq A$ is denoted by $\mathrm{Fg}^{\boldsymbol{A}}_\mathcal{S}(X)$, and the least $\mathcal{S}$-theory containing $\Gamma \subseteq \mathrm{Fm}_\mathcal{L}$ is denoted by $\mathrm{Cn}_\mathcal{S}(\Gamma)$.

Although the Leibniz and Tarski congruences were defined for arbitrary sets and families of sets, respectively, we shall usually consider them over $\mathcal{S}$-filters and families of $\mathcal{S}$-filters, respectively. The next auxiliary lemma relates the notions of logical filter, Leibniz congruence, and subalgebra.

LEMMA 2.1. *Let $\mathcal{S}$ be a logic, $\boldsymbol{A}$ an algebra, and $\boldsymbol{B}$ a subalgebra of $\boldsymbol{A}$. For every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$,*

1. $F \cap B \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$;
2. $\boldsymbol{\Omega}^{\boldsymbol{A}}(F)|_{\boldsymbol{B}} \subseteq \boldsymbol{\Omega}^{\boldsymbol{B}}(F \cap B)$.

An *$\mathcal{L}$-equation* is a pair of formulas $\langle \varphi, \psi \rangle \in \mathrm{Fm}_{\mathcal{L}} \times \mathrm{Fm}_{\mathcal{L}}$, usually abbreviated as $\varphi \approx \psi$. We denote the set of all $\mathcal{L}$-equations by $\mathrm{Eq}_{\mathcal{L}}$. Given a class of algebras $\mathsf{K}$, the *equational consequence relation relative to* $\mathsf{K}$ is the relation $\vDash_{\mathsf{K}} \subseteq \mathcal{P}(\mathrm{Eq}_{\mathcal{L}}) \times \mathrm{Eq}_{\mathcal{L}}$ defined by

$$\Pi \vDash_{\mathsf{K}} \varphi \approx \psi \quad \text{iff} \quad \forall \boldsymbol{A} \in \mathsf{K}\ \forall h : \boldsymbol{Fm} \to \boldsymbol{A}$$
$$\forall \delta \approx \varepsilon \in \Pi \quad h(\delta) = h(\varepsilon) \ \Rightarrow\ h(\varphi) = h(\psi).$$

It is common practice to abbreviate $\vDash_{\{\boldsymbol{A}\}}$ simply by $\vDash_{\boldsymbol{A}}$.

A *matrix model* of $\mathcal{S}$ is a pair $\langle \boldsymbol{A}, F \rangle$ such that $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. We denote the class of all matrix models of a logic $\mathcal{S}$ by $\mathsf{Mod}(\mathcal{S})$. A matrix model $\langle \boldsymbol{A}, F \rangle \in \mathsf{Mod}(\mathcal{S})$ is called *reduced* if $\boldsymbol{\Omega}^{\boldsymbol{A}}(F) = id_{\boldsymbol{A}}$. We denote the class of all *reduced* matrix models by $\mathsf{Mod}^*(\mathcal{S})$.

Two classes of algebras are usually considered in AAL as naturally, and intrinsically, associated with a logic. These are obtained as follows:

$$\mathsf{Alg}^*(\mathcal{S}) := \{\boldsymbol{A} : \exists F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \text{ such that } \boldsymbol{\Omega}^{\boldsymbol{A}}(F) = id_{\boldsymbol{A}}\}, \tag{1}$$
$$\mathsf{Alg}(\mathcal{S}) := \{\boldsymbol{A} : \exists \mathcal{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \text{ such that } \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathcal{C}) = id_{\boldsymbol{A}}\}.$$

It turns out that $\mathsf{Alg}(\mathcal{S}) = \mathbb{P}_{\mathrm{S}}\mathsf{Alg}^*(\mathcal{S})$ [33, Theorem 2.23]. When one wishes to associate with a logic $\mathcal{S}$ an "algebraic counterpart" which is necessarily a variety, a third class of algebras is often considered, defined by

$$\mathbb{V}(\mathcal{S}) := \mathbb{V}\Big(\boldsymbol{Fm}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{Fm}}(\mathcal{T}h(\mathcal{S}))\Big).$$

It can be proved that $\mathbb{V}(\mathcal{S})$ is in fact the least variety containing $\mathsf{Alg}^*(\mathcal{S})$, i.e., $\mathbb{V}(\mathcal{S}) = \mathbb{V}\mathsf{Alg}^*(\mathcal{S})$, and it is called the *intrinsic variety of* $\mathcal{S}$.

**2.1. The Leibniz and Frege hierarchies.** The main classification of sentential logics in AAL is the so-called *Leibniz hierarchy*, characterizing logics by means of algebraic properties of the *Leibniz operator*.[4] For instance, the Leibniz operator (over the $\mathcal{S}$-filters of an arbitrary algebra $\boldsymbol{A}$) of an algebraizable logic $\mathcal{S}$ is an isomorphism between the lattice of $\mathcal{S}$-filters of $\boldsymbol{A}$ and the lattice of $\mathsf{Alg}^*(\mathcal{S})$-relative congruences of $\boldsymbol{A}$, which furthermore commutes with inverse images of homomorphisms. In [39], the non-algebraizability of $\mathcal{C}_1$ was established by exhibiting an algebra $\boldsymbol{A}$ where the Leibniz operator over the $\mathcal{C}_1$-filters of $\boldsymbol{A}$ was not injective.

We next introduce the main classes of logics belonging to the Leibniz hierarchy. However, since their characterizations in terms of the algebraic

[4] When we consider the map assigning to each subset $F \subseteq A$ its Leibniz congruence $\boldsymbol{\Omega}^{\boldsymbol{A}}(F)$, and restrict its domain to the set of $\mathcal{S}$-filters of $\boldsymbol{A}$, we refer to the map $\boldsymbol{\Omega}^{\boldsymbol{A}} : \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \to \mathrm{Co}\boldsymbol{A}$ as the *Leibniz operator on $\boldsymbol{A}$*.

properties of Leibniz operator will not be used in the sequel, we have chosen to define them here in a manner that will suit our purposes better.

DEFINITION 2.2. A logic $\mathcal{S}$ is:

(i) *protoalgebraic* if there exists a set of formulas $\boldsymbol{\rho}(x, y) \subseteq \mathrm{Fm}_{\mathcal{L}}$ satisfying

$$\varnothing \vdash_{\mathcal{S}} \boldsymbol{\rho}(x, x), \tag{R}$$

$$x, \boldsymbol{\rho}(x, y) \vdash_{\mathcal{S}} y, \tag{MP}$$

(ii) *equivalential* if there exists a set of formulas $\boldsymbol{\rho}(x, y) \subseteq \mathrm{Fm}_{\mathcal{L}}$ satisfying (R), (MP) and for every $n$-ary function symbol $f \in \mathcal{L}$,

$$\boldsymbol{\rho}(x_1, y_1) \cup \cdots \cup \boldsymbol{\rho}(x_n, y_n) \vdash_{\mathcal{S}} \boldsymbol{\rho}\big(f(x_1 \dots x_n), f(y_1 \dots y_n)\big), \tag{RP}$$

(iii) *truth-equational* if there exists a set of equations $\boldsymbol{\tau}(x) \subseteq \mathrm{Eq}_{\mathcal{L}}$ such that

$$\forall \langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}^*(\mathcal{S})\ \forall a \in A \quad a \in F \ \Leftrightarrow\ \forall \delta \approx \varepsilon \in \boldsymbol{\tau}(x)\ \delta^{\boldsymbol{A}}(a) = \varepsilon^{\boldsymbol{A}}(a),$$

(iv) *weakly algebraizable* if it is protoalgebraic and truth-equational,
(v) *algebraizable* if it is equivalential and truth-equational.

In case the set of congruence formulas in item (ii) is finite, the underlying logic is called *finitely* equivalential, and if the logic is furthermore truth-equational, it is called *finitely* algebraizable.

We shall make use of an equivalent formulation of truth-equationality also established by Raftery in [51, Proposition 22]. A logic $\mathcal{S}$ is truth-equational if and only if there exists a set of equations $\boldsymbol{\tau}(x) \subseteq \mathrm{Eq}_{\mathcal{L}}$ such that

$$\forall \langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}(\mathcal{S}) \quad F = \{a \in A : \boldsymbol{\tau}^{\boldsymbol{A}}(a) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)\},$$

where $\boldsymbol{\tau}^{\boldsymbol{A}}(a) = \{\delta^{\boldsymbol{A}}(a) \approx \varepsilon^{\boldsymbol{A}}(a) : \delta \approx \varepsilon \in \boldsymbol{\tau}(x)\}$. Notice that condition (iii) is the particular case of the condition above for reduced $\mathcal{S}$-models.

Parallel to the Leibniz hierarchy, the Frege hierarchy classifies sentential logics according to some replacement properties, which can also be expressed by the congruentiality of the Frege relation (either over the $\mathcal{S}$-theories of the underlying logic or over the $\mathcal{S}$-filters of arbitrary algebras). The *Frege relation of* $F \subseteq A$ *on* $\boldsymbol{A}$ (relative to $\mathcal{S}$) is defined by

$$\boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(F) := \left\{\langle a, b\rangle \in A \times A : \mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}(F, a) = \mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}(F, b)\right\}.$$

Notice that unlike the Leibniz congruence $\boldsymbol{\Omega}^{\boldsymbol{A}}(F)$, the equivalence relation $\boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(F)$ is not necessarily a congruence on $\boldsymbol{A}$. The relation $\boldsymbol{\Lambda}^{\boldsymbol{Fm}}_{\mathcal{S}}(\varnothing)$ is called the *interderivability relation on* $\boldsymbol{A}$, and traditionally $\langle \varphi, \psi\rangle \in \boldsymbol{\Lambda}^{\boldsymbol{Fm}}_{\mathcal{S}}(\varnothing)$ is abbreviated by $\varphi \dashv\vdash_{\mathcal{S}} \psi$, as we have seen already on page 482.

The Frege hierarchy comprises four classes of logics, which we next introduce.

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Definition 2.3. A logic $\mathcal{S}$ is:

(i) *self-extensional*, if $\varLambda^{\boldsymbol{Fm}}_{\mathcal{S}}(\varnothing) \in \mathrm{Co}\,\boldsymbol{Fm}$,
(ii) *Fregean*, if for every $T \in \mathcal{T}h(\mathcal{S})$, $\varLambda^{\boldsymbol{Fm}}_{\mathcal{S}}(T) \in \mathrm{Co}\,\boldsymbol{Fm}$,
(iii) *fully self-extensional*, if for every $\boldsymbol{A}$, $\varLambda^{\boldsymbol{A}}_{\mathcal{S}}(\varnothing) \in \mathrm{Co}\,\boldsymbol{A}$,
(iv) *fully Fregean*, if for every $\boldsymbol{A}$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, $\varLambda^{\boldsymbol{A}}_{\mathcal{S}}(F) \in \mathrm{Co}\,\boldsymbol{A}$.

**2.2. Parameterized sets of congruence formulas.** The notion of (parameterized) sets of congruence formulas is a cornerstone of the theory of protoalgebraic logics and will play a prominent role in the present work. In particular, the subtle difference between having parameters or not will be crucial in our study of axiomatic extensions of $\mathcal{C}_1$. We next compile some auxiliary facts concerning (parameterized) sets of congruence formulas which we will be used in the sequel.

Throughout the rest of the section, let $x, y \in \mathrm{Var}$ be two fixed distinct variables, and $\overline{z}$ an arbitrary possibly infinite sequence of variables (distinct from $x$ and $y$). In particular, $\Delta(x, y, \overline{z}) \subseteq \mathrm{Fm}_{\mathcal{L}}$ denotes a set of formulas in the variables $x, y$ and possibly variables $\overline{z}$. For ease of notation, we will use $c_1, c_2, c_3, \ldots$ to denote the elements of $\overline{c}$, and simply write $\overline{c} \in A$ if all the elements are in $A$.

Definition 2.4. Let $\mathcal{S}$ be a logic, $\boldsymbol{A}$ an algebra, and $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. A set of formulas $\Delta(x, y, \overline{z}) \subseteq \mathrm{Fm}_{\mathcal{L}}$ is a *set of parameterized congruence formulas for* $\mathcal{S}$, if for every $\boldsymbol{A}$, every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, and every $a, b \in A$,

$$\langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}(F) \quad \Leftrightarrow \quad \forall \overline{c} \in A \quad \Delta^{\boldsymbol{A}}(a, b, \overline{c}) \subseteq F.$$

In case $\Delta(x, y) \subseteq \mathrm{Fm}_{\mathcal{L}}$ has no parameters, it is called a *set of congruence formulas for* $\mathcal{S}$.

The *parameters* are the variables $\overline{z}$, which by no means need be in finite number. But if we pick a formula $\delta \in \Delta(x, y, \overline{z})$, then we can write $\delta(x, y, z_1, \ldots, z_n)$, with $z_1, \ldots, z_n$ occurring in $\overline{z}$. For the sake of notation easiness, when the context is understood we sometimes write simply $\Delta$ instead of $\Delta(x, y, \overline{z})$, or $\Delta(x, y)$.

We shall need yet another important notation. Given a set of formulas $\Delta(x, y, \overline{z}) \subseteq \mathrm{Fm}_{\mathcal{L}}$, let

$$\Delta\langle x, y\rangle := \bigcup \{\Delta(x, y, \overline{\xi}) : \overline{\xi} \in \mathrm{Fm}_{\mathcal{L}}\}.$$

The main characterization of a set of parameterized congruence formulas (in fact the original definition for the case without parameters [28, Definition I.10]; for the case with parameters, see [32, Exercise 6.28]) is the following:

Theorem 2.5. *A set of formulas* $\Delta(x, y, \overline{z}) \subseteq \mathrm{Fm}_{\mathcal{L}}$ *is a set of parameterized congruence formulas for* $\mathcal{S}$ *if and only if the following conditions hold*:

$$\varnothing \vdash_{\mathcal{S}} \Delta\langle x, x\rangle, \tag{p-R}$$
$$\Delta\langle x, y\rangle \vdash_{\mathcal{S}} \Delta\langle y, x\rangle, \tag{p-Sym}$$
$$\Delta\langle x, y\rangle \cup \Delta\langle y, z\rangle \vdash_{\mathcal{S}} \Delta\langle x, z\rangle, \tag{p-Trans}$$

$$x, \Delta\langle x, y\rangle \vdash_{\mathcal{S}} y, \tag{p-MP}$$

$$\Delta\langle x_1, y_1\rangle \cup \cdots \cup \Delta\langle x_n, y_n\rangle \vdash_{\mathcal{S}} \Delta\big\langle f(x_1, \dots, x_n), f(y_1, \dots, y_n)\big\rangle \tag{p-Re}$$

*for every n-ary function symbol* $f \in \mathcal{L}$.

Conditions (p-Sym) and (p-Trans) actually follow from the three remaining conditions (see [32, Corollary 6.61]), but their presence helps us understand the intuition behind the terminology "congruence formulas."

As it turns out, Definition 2.4 captures two classes of logics within the Leibniz hierarchy. Indeed, the existence of a set of parameterized congruence formulas characterizes protoalgebraic logics, while the existence of a set of congruence formulas *without* parameters characterizes equivalential logics (see [32, Theorem 6.57 and Definition 6.63]).

The next technical result will be used in Section 10 to prove that the logic $\mathcal{C}ilow$ is not equivalential. The auxiliary lemma is taken from Jansana's AAL lecture notes [37, Exercise 4.36, p. 89].

Lemma 2.6. *If* $\Delta(x, y, \overline{z})$ *and* $\Delta'(x, y, \overline{z'})$ *are two sets of parameterized congruence formulas for* $\mathcal{S}$, *then* $\Delta\langle x, y\rangle \dashv\vdash_{\mathcal{S}} \Delta'\langle x, y\rangle$.

Proposition 2.7. *Let* $\mathcal{S}$ *be an equivalential logic. If* $\Delta \subseteq \mathrm{Fm}_{\mathcal{L}}$ *is a set of parameterized congruence formulas for* $\mathcal{S}$, *then for every substitution* $\sigma : \boldsymbol{Fm} \to \boldsymbol{Fm}(x, y)$ *leaving* $x, y$ *unchanged and replacing any other variable* $z$ *with* $\sigma z \in \{x, y\}$, $\sigma\Delta \subseteq \mathrm{Fm}_{\mathcal{L}}(x, y)$ *is a set of congruence formulas for* $\mathcal{S}$.

Proof. Let $\Delta(x, y, \overline{z}) \subseteq \mathrm{Fm}_{\mathcal{L}}$ be a set of parameterized congruence formulas for $\mathcal{S}$ and $\Delta'(x, y)$ be a set of congruence formulas (without parameters) for $\mathcal{S}$. Notice that $\Delta'$ must exist by hypothesis. It follows by Lemma 2.6 that $\Delta\langle x, y\rangle \dashv\vdash_{\mathcal{S}} \Delta'(x, y)$. It follows by structurality that $\sigma\Delta\langle x, y\rangle \dashv\vdash_{\mathcal{S}} \Delta'(x, y)$, bearing in mind that $\sigma$ leaves $x, y$ unchanged and replaces any other variable $z$ with $\sigma z \in \{x, y\}$. Thus $\sigma\Delta\langle x, y\rangle \subseteq \mathrm{Fm}_{\mathcal{L}}(x, y)$ is also a set of congruence formulas (without parameters) for $\mathcal{S}$. ⊣

**2.3. Algebraic semantics.** The notion of algebraic semantics was originally introduced in [6] (assuming $\mathcal{S}$ is finitary, $\mathsf{K}$ a quasivariety, and $\boldsymbol{\tau}(x)$ finite), and was further investigated *per se* in [8]. We choose here to make explicit its dependence on the set of equations $\boldsymbol{\tau}$, following the more works [7, 51]. More recent developments on algebraic semantics can be found in [43].

Definition 2.8. Let $\mathcal{S}$ be a logic and $\boldsymbol{\tau}(x) \subseteq \mathrm{Eq}_{\mathcal{L}}$. A class of algebras $\mathsf{K}$ is a $\boldsymbol{\tau}$*-algebraic semantics for* $\mathcal{S}$ if for every $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$,

$$\Gamma \vdash_{\mathcal{S}} \varphi \quad \Leftrightarrow \quad \boldsymbol{\tau}(\Gamma) \vDash_{\mathsf{K}} \boldsymbol{\tau}(\varphi). \tag{ALG1}$$

Despite the fact that no characterization of Definition 2.8 in terms of the Leibniz operator is known, there is a close relation between the notion of algebraic semantics and the Leibniz hierarchy. Indeed, algebraizable logics were originally defined (apart from finitariness issues) as those logics $\mathcal{S}$ for which there exist a class of algebras $\mathsf{K}$, a set of equations in at most

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one variable $\boldsymbol{\tau}(x) \subseteq \mathrm{Eq}_{\mathcal{L}}$, and a set of formulas in at most two variables $\boldsymbol{\rho}(x, y) \subseteq \mathrm{Fm}_{\mathcal{L}}$, such that

$$\Gamma \vdash_{\mathcal{S}} \varphi \Leftrightarrow \boldsymbol{\tau}(\Gamma) \vDash_{\mathsf{K}} \boldsymbol{\tau}(\varphi), \tag{ALG1}$$

$$x \approx y \dashv\vDash_{\mathsf{K}} \boldsymbol{\tau}\big(\boldsymbol{\rho}(x, y)\big) \tag{ALG2}$$

for every $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$. It is thus clear by the definitions involved that every algebraizable logic has an algebraic semantics, which in this case is called an *equivalent* algebraic semantics.

The next result, whose proof can be found in [51, Corollary 6], will be used for establishing that $\mathcal{C}_1$ admits no algebraic semantics (Theorem 5.6). We state it here for the particular case of protoalgebraic logics.

Proposition 2.9. *If a protoalgebraic logic $\mathcal{S}$ has a $\boldsymbol{\tau}$-algebraic semantics, then $\boldsymbol{\tau}(T) \subseteq \boldsymbol{\Omega}^{\boldsymbol{Fm}}(T)$ for every $T \in \mathcal{T}h(\mathcal{S})$.*

**Part I. Da Costa's logic $\mathcal{C}_1$.**

In the first part of our work we deal exclusively with the logic $\mathcal{C}_1$ and the related family of logics $\mathcal{C}_n$, with $n \geq 1$. Our main goals are to classify $\mathcal{C}_1$ within the Leibniz and Frege hierarchies, collect some algebraic consequences of the fact that $\mathcal{C}_1$ admits a Deduction-Detachment Theorem (Theorem 3.10), present a set of parameterized congruence formulas for $\mathcal{C}_1$ (Theorem 4.3), and prove that $\mathcal{C}_1$ does not admit any algebraic semantics whatsoever in the sense of Blok and Pigozzi (Theorem 5.6).

**§3. The logic $\mathcal{C}_1$.** In this section we introduce the logic $\mathcal{C}_1$ alongside with the family of logics $\mathcal{C}_n$, with $n \geq 1$. We then proceed to classify $\mathcal{C}_1$ within the Leibniz and Frege hierarchies and state some algebraic consequences of the Deduction-Detachment Theorem for $\mathcal{C}_1$.

Throughout the present work we assume fixed the language

$$\mathcal{L} = \langle \wedge, \vee, \to, \neg \rangle,$$

where $\wedge, \vee, \to$ are binary operation symbols and $\neg$ is a unary operation symbol. We shall also consider two unary non-primitive connectives $\varphi^\circ := \neg(\varphi \wedge \neg\varphi)$ and ${\sim}\varphi := \neg\varphi \wedge \varphi^\circ$, and one binary non-primitive connective $\varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi)$, with $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}$. The traditional Hilbert-style axiomatics for $\mathcal{C}_1$ is presented in the next definition, as given in [21, pp. 498–499].

Definition 3.1. The logic $\mathcal{C}_1$ is induced by the following Hilbert-style axioms and inference rule:

$$\vdash \varphi \to (\psi \to \varphi), \tag{Ax1}$$

$$\vdash (\varphi \to \psi) \to \Big(\big(\varphi \to (\psi \to \xi)\big) \to (\varphi \to \xi)\Big), \tag{Ax2}$$

$$\vdash (\varphi \wedge \psi) \to \varphi, \tag{Ax3}$$

$$\vdash (\varphi \wedge \psi) \to \psi, \tag{Ax4}$$

$$\vdash \varphi \to \big(\psi \to (\varphi \wedge \psi)\big), \tag{Ax5}$$

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$$\vdash \varphi \to (\varphi \vee \psi), \tag{Ax6}$$
$$\vdash \psi \to (\varphi \vee \psi), \tag{Ax7}$$
$$\vdash (\varphi \to \xi) \to \Big((\psi \to \xi) \to ((\varphi \vee \psi) \to \xi)\Big), \tag{Ax8}$$
$$\vdash \varphi \vee \neg\varphi, \tag{Ax9}$$
$$\vdash \neg\neg\varphi \to \varphi, \tag{Ax10}$$
$$\vdash \psi^\circ \to \Big((\varphi \to \psi) \to ((\varphi \to \neg\psi) \to \neg\varphi)\Big), \tag{Ax11}$$
$$\vdash (\varphi^\circ \wedge \psi^\circ) \to (\varphi \wedge \psi)^\circ, \tag{Ax12}$$
$$\vdash (\varphi^\circ \wedge \psi^\circ) \to (\varphi \vee \psi)^\circ, \tag{Ax13}$$
$$\vdash (\varphi^\circ \wedge \psi^\circ) \to (\varphi \to \psi)^\circ, \tag{Ax14}$$
$$\varphi, \varphi \to \psi \vdash \psi. \tag{MP}$$

By definition, $\mathcal{C}_1$ is obviously a finitary logic, also called a *deductive system* in the literature. The schemata (Ax1)–(Ax8) axiomatize positive intuitionistic propositional logic ($\mathcal{IL}^+$), and hence $\mathcal{C}_1$ is an (axiomatic) extension of $\mathcal{IL}^+$. Also, classical propositional logic ($\mathcal{CL}$) is an extension of $\mathcal{C}_1$, axiomatized relatively to it by the *principle of contradiction* $\neg(\varphi \wedge \neg\varphi)$.

The logic $\mathcal{C}_\omega$ is axiomatized by the schemata (Ax1)–(Ax10) and inference rule (MP). It is the weakest among all of da Costa's $\mathcal{C}$-systems. Let[5]

$$\varphi^{(1)} := \varphi^\circ \quad \text{and} \quad \varphi^{(n)} := \varphi^{(n-1)} \wedge (\varphi^{(n-1)})^\circ \text{ for } n > 1.$$

The logic $\mathcal{C}_n$, with $n \geq 1$, is the least extension of $\mathcal{C}_\omega$ closed under the following additional axiom schemata:

$$\vdash \psi^{(n)} \to \Big((\varphi \to \psi) \to ((\varphi \to \neg\psi) \to \neg\varphi)\Big), \tag{Ax11n}$$
$$\vdash (\varphi^{(n)} \wedge \psi^{(n)}) \to (\varphi \wedge \psi)^{(n)}, \tag{Ax12n}$$
$$\vdash (\varphi^{(n)} \wedge \psi^{(n)}) \to (\varphi \vee \psi)^{(n)}, \tag{Ax13n}$$
$$\vdash (\varphi^{(n)} \wedge \psi^{(n)}) \to (\varphi \to \psi)^{(n)}. \tag{Ax14n}$$

For $n = 1$, the logic just defined coincides with the logic $\mathcal{C}_1$ in Definition 3.1. Da Costa proved in [18, Théorème 7] (the proof is credited to A. I. Arruda) that his $\mathcal{C}$-systems form a hierarchy of proper extensions, i.e.,

$$\mathcal{C}_\omega < \cdots < \mathcal{C}_{n+1} < \mathcal{C}_n < \cdots < \mathcal{C}_1 .$$

The logic $\mathcal{C}_\omega$ is not the infimum of the family of logics $\{\mathcal{C}_n : n \geq 1\}$, although the notation may suggest it. For more details on this issue, see [15].

The following auxiliary facts about $\mathcal{C}_1$ will be used exhaustively in the sequel. See [26, Theorems 2.1.6, 2.1.8, 2.1.13, and 2.1.14].

[5]The definition of $\mathcal{C}_n$ here presented reproduces [26, Definition 2.2.1] instead of the original definition [18, p. 3792], because $\varphi^{\overbrace{\circ \cdots \circ}^{n \text{ times}}} \in \mathrm{Thm}_{\mathcal{C}_1}$, for $n \geq 2$, in light of Lemma 3.2.20.

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LEMMA 3.2.

1. $\vdash_{\mathcal{C}_1} \varphi \leftrightarrow \varphi$.
2. $\varphi \leftrightarrow \psi \vdash_{\mathcal{C}_1} \psi \leftrightarrow \varphi$.
3. $\varphi \leftrightarrow \psi, \psi \leftrightarrow \xi \vdash_{\mathcal{C}_1} \varphi \leftrightarrow \xi$.
4. $\varphi \leftrightarrow \psi, \xi \leftrightarrow \zeta \vdash_{\mathcal{C}_1} (\varphi \wedge \xi) \leftrightarrow (\psi \wedge \zeta)$.
5. $\varphi \leftrightarrow \psi, \xi \leftrightarrow \zeta \vdash_{\mathcal{C}_1} (\varphi \vee \xi) \leftrightarrow (\psi \vee \zeta)$.
6. $\varphi \leftrightarrow \psi, \xi \leftrightarrow \zeta \vdash_{\mathcal{C}_1} (\varphi \to \xi) \leftrightarrow (\psi \to \zeta)$.
7. $\psi^\circ, \varphi \to \psi \vdash_{\mathcal{C}_1} \neg\psi \to \neg\varphi$.
8. $\varphi^\circ, \psi^\circ, \varphi \leftrightarrow \psi \vdash_{\mathcal{C}_1} \neg\varphi \leftrightarrow \neg\psi$.
9. $\vdash_{\mathcal{C}_1} \varphi \to \varphi$.
10. $\varphi \to \psi, \psi \to \xi \vdash_{\mathcal{C}_1} \varphi \to \xi$.
11. $\varphi \to \psi \vdash_{\mathcal{C}_1} (\varphi \wedge \xi) \to (\psi \wedge \xi)$.
12. $\varphi \to \psi \vdash_{\mathcal{C}_1} (\varphi \vee \xi) \to (\psi \vee \xi)$.
13. $\varphi \to \psi \vdash_{\mathcal{C}_1} (\varphi \to \xi) \to (\psi \to \xi)$.
14. $\varphi \to \psi \vdash_{\mathcal{C}_1} (\xi \wedge \varphi) \to (\xi \wedge \psi)$.
15. $\varphi \to \psi \vdash_{\mathcal{C}_1} (\xi \vee \varphi) \to (\xi \vee \psi)$.
16. $\varphi \to \psi \vdash_{\mathcal{C}_1} (\xi \to \varphi) \to (\xi \to \psi)$.
17. $\varphi^\circ \vdash_{\mathcal{C}_1} (\neg\varphi)^\circ$.
18. $\vdash_{\mathcal{C}_1} (\varphi^\circ)^\circ$.

Several semantics have been put forward in the literature for the logic $\mathcal{C}_1$, namely, behavioral semantics [12, 13], possible-translations semantics [9], and the so-called *da Costa algebras* investigated in [14, 52] and generalizing the $\mathcal{C}_1$-algebras introduced by da Costa himself in [19]. Da Costa's proposal has the drawback (from an AAL point of view) of changing the underlying language of the class of algebras associated with $\mathcal{C}_1$. One further semantics for $\mathcal{C}_1$ can be found in the literature, developed by da Costa and Alves in [23], and which will be very useful in the sequel. We next introduce da Costa and Alves's two-valued semantics for the logic $\mathcal{C}_1$ and state the completeness result w.r.t. this non-truth-functional semantics.

DEFINITION 3.3. A bivaluation of $\mathcal{C}_1$ is a map $v : \boldsymbol{Fm} \to \{0, 1\}$ such that:

- $v(\varphi) = 0 \;\Rightarrow\; v(\neg\varphi) = 1$,
- $v(\neg\neg\varphi) = 1 \;\Rightarrow\; v(\varphi) = 1$,
- $v(\psi^\circ) = v(\varphi \to \psi) = v(\varphi \to \neg\psi) = 1 \;\Rightarrow\; v(\varphi) = 0$,
- $v(\varphi \to \psi) = 1 \;\Leftrightarrow\; v(\varphi) = 0$ or $v(\psi) = 1$,
- $v(\varphi \wedge \psi) = 1 \;\Leftrightarrow\; v(\varphi) = 1$ and $v(\psi) = 1$,
- $v(\varphi \vee \psi) = 1 \;\Leftrightarrow\; v(\varphi) = 1$ or $v(\psi) = 1$,
- $v(\varphi^\circ) = v(\psi^\circ) = 1 \;\Rightarrow\; v\big((\varphi \wedge \psi)^\circ\big) = v\big((\varphi \vee \psi)^\circ\big) = v\big((\varphi \to \psi)^\circ\big) = 1$.

A bivaluation of $\mathcal{C}_1$ is a model of a formula $\varphi \in \mathrm{Fm}_{\mathcal{L}}$ if $v(\varphi) = 1$, and it is a model of a set of formulas $\Gamma \subseteq \mathrm{Fm}_{\mathcal{L}}$ if $v(\gamma) = 1$, for every $\gamma \in \Gamma$.

THEOREM 3.4 (da Costa). *For every $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$, $\Gamma \vdash_{\mathcal{C}_1} \varphi$ if and only if every bivaluation v of $\mathcal{C}_1$ which is a model of $\Gamma$ is also a model of $\varphi$.*

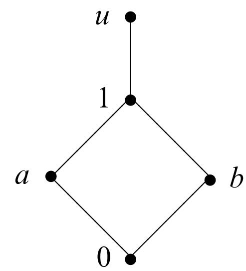


FIGURE 2. The algebra $\boldsymbol{A}$ of [39] and its $\mathcal{C}_1$-filters.

Given a bivaluation $v$ of $\mathcal{C}_1$, we have [23, Theorem 2]:

$$v(\sim\varphi) = 1 \;\Leftrightarrow\; v(\varphi) = 0.$$

In fact, the connective $\sim$ behaves like classical negation.

**3.1. Classification of $\mathcal{C}_1$ within the Leibniz and Frege hierarchies.** The classification of $\mathcal{C}_1$ (or more generally, of da Costa's $\mathcal{C}$-systems $\mathcal{C}_n$) within the Leibniz and Frege hierarchies is scattered along the literature, sometimes appearing as mere observations, and most often without explicit proofs. The notorious exception is the non-algebraizability of $\mathcal{C}_1$ established in [39]. We next compile these scattered results and fully classify the logic $\mathcal{C}_1$ within the Leibniz hierarchy, which since the non-algebraizability result of Lewin, Mikenberg, and Schwarze in 1991, has been enlarged with two further classes of logics (namely, the classes of weakly algebraizable [29] and truth-equational logics [51]).

The protoalgebraicity of $\mathcal{C}_1$ was left as an open question in [16, p. 81] (perhaps because the condition said to characterize protoalgebraicity in [16, p. 81] is in fact equivalent to weak algebraizability), but was later observed to hold in [9, p. 2]. Indeed, it follows by manipulation of axioms (Ax1), (Ax2), and rule (MP) in Definition 3.1, or at once by Lemma 3.2.11, that $\boldsymbol{\rho}(x, y) = \{x \to y\}$ complies with Definition 2.2 (i).

PROPOSITION 3.5. *$\mathcal{C}_1$ is protoalgebraic.*

Proposition 3.5 immediately prompts the question of whether $\mathcal{C}_1$ is furthermore equivalential. The answer this time is negative, as first observed in [36, pp. 425–426], and also mentioned in [32, Example 6.77]. The proof uses as a counterexample the five-element algebra of [39] depicted in Figure 2 (Table 1).

PROPOSITION 3.6. *$\mathcal{C}_1$ is not equivalential.*

PROOF. Consider the algebra $\boldsymbol{A} = \langle A, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, \to^{\boldsymbol{A}}, \neg^{\boldsymbol{A}}, 0^{\boldsymbol{A}}, 1^{\boldsymbol{A}}\rangle$ with universe $A = \{0, a, b, 1, u\}$, whose lattice operations are given by Figure 2 and with the truth-tables of $\neg^{\boldsymbol{A}}$ and $\to^{\boldsymbol{A}}$ given by Table 2, respectively. Consider moreover the subalgebra $\boldsymbol{B}$ with universe $B = \{0, a, b, 1\}$. It is clear that $\boldsymbol{A} \in \mathsf{Alg}^*(\mathcal{C}_1)$, since for instance $F := \{a, 1, u\}$ witnesses $\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}^*(\mathcal{C}_1)$. Now, fix $G := F \cap B = \{a, 1\} \in \mathcal{F}i_{\mathcal{C}_1}\boldsymbol{B}$. It is not difficult to check that $\langle a, 1\rangle \in \boldsymbol{\Omega}^{\boldsymbol{B}}(G)$. Then $\boldsymbol{\Omega}^{\boldsymbol{B}}(G) \neq id_{\boldsymbol{B}}$, and therefore $\langle \boldsymbol{B}, G\rangle \notin \mathsf{Mod}^*(\mathcal{C}_1)$. We

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| $F \in \mathcal{F}i_{\mathcal{C}_1}(\boldsymbol{A})$ | $\boldsymbol{\Omega}^{\boldsymbol{A}}(F)$ |
|---|---|
| $A$ | $A \times A$ |
| $\{1, u\}$ | $id_{\boldsymbol{A}}$ |
| $\{a, 1, u\}$ | $id_{\boldsymbol{A}}$ |
| $\{b, 1, u\}$ | $id_{\boldsymbol{A}}$ |

TABLE 1. The algebra $\boldsymbol{A}$ of [39] and the Leibniz congruences of its $\mathcal{C}_1$-filters.

| | $\neg^{\boldsymbol{A}}$ |
|---|---|
| $u$ | 1 |
| 1 | 0 |
| $a$ | $b$ |
| $b$ | $a$ |
| 0 | 1 |

| $\to^{\boldsymbol{A}}$ | $u$ | 1 | $a$ | $b$ | 0 |
|---|---|---|---|---|---|
| $u$ | $u$ | $u$ | $a$ | $b$ | 0 |
| 1 | $u$ | 1 | $a$ | $b$ | 0 |
| $a$ | $u$ | 1 | 1 | $b$ | $b$ |
| $b$ | $u$ | 1 | $a$ | 1 | $a$ |
| 0 | $u$ | 1 | 1 | 1 | 1 |

TABLE 2. Truth-tables of the connectives $\neg^{\boldsymbol{A}}$ and $\to^{\boldsymbol{A}}$.

conclude that the class $\mathsf{Mod}^*(\mathcal{C}_1)$ is not closed under submatrices, and hence $\mathcal{C}_1$ cannot be equivalential (see for instance [32, Theorem 6.73]). ⊣

We are left to see whether $\mathcal{C}_1$ is truth-equational. But, in fact, this was implicitly established in [39]—implicitly, simply because truth-equational logics had not yet been defined in 1991. Indeed, the five-element algebra $\boldsymbol{A}$ exhibited in [39] (and reproduced in Figure 2) is such that the Leibniz operator on $\boldsymbol{A}$ over the $\mathcal{C}_1$-filters is not injective. Recall that Raftery proved that the Leibniz operator is completely order-reflecting (and hence, injective) on arbitrary algebras for truth-equational logics [51]. It follows at once that:

PROPOSITION 3.7. *$\mathcal{C}_1$ is not truth-equational.*

Having in mind Definition 2.2(iv), it follows at once that $\mathcal{C}_1$ is not weakly algebraic. Indeed, Font's [32, Example 6.122.9] generalizes that "neither of the logic $\mathcal{C}_n$ is weakly algebraizable."

As a matter of fact, by direct inspection of Table 2, one sees that truth is not even *implicitly definable*[6] in the class $\mathsf{Mod}^*(\mathcal{C}_1)$, which is a weaker condition than equational definability of truth in the class $\mathsf{Mod}^*(\mathcal{C}_1)$.

Propositions 3.5–3.7 settle the classification of the logic $\mathcal{C}_1$ within the Leibniz hierarchy. As to the Frege hierarchy, the logic $\mathcal{C}_1$ falls outside its scope, as first shown by da Costa and Guillaume as early as in [25].

PROPOSITION 3.8. *$\mathcal{C}_1$ is not self-extensional.*

PROOF. For instance, $\varphi \to \varphi \dashv\vdash_{\mathcal{C}_1} \psi \to \psi$, but the matrix model $\langle \boldsymbol{A}, \{1, u\}\rangle \in \mathsf{Mod}(\mathcal{C}_1)$, with $\boldsymbol{A}$ given as in Figure 2, and any homomorphism $h : \boldsymbol{Fm} \to \boldsymbol{A}$ such that $h(\varphi) = u$ and $h(\psi) = 1$ witness $\neg(\varphi \to \varphi) \nvdash_{\mathcal{C}_1} \neg(\psi \to \psi)$. ⊣

[6]Truth is *implicitly definable* in a class of matrices M, if whenever $\langle \boldsymbol{A}, F\rangle, \langle \boldsymbol{A}, G\rangle \in M$, then $F = G$.

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It is worth mentioning that it was precisely the non-self-extensionality of $\mathcal{C}_1$ that motivated the first algebraic studies about this paraconsistent logic.

So far we have only seen one positive result about $\mathcal{C}_1$, namely that it is protoalgebraic. This fact will be the ground upon which Sections 3.2 and 4 will be built on.

**3.2. The Deduction-Detachment Theorem for $\mathcal{C}_1$.** Protoalgebraic logics are characterized by possessing a weak form of the Deduction-Detachment Theorem (DDT), the so-called Parameterized Local Deduction-Detachment Theorem (PLDDT). Some famous bridge theorems in AAL establish correspondences between stronger versions of the PLDDT and purely algebraic properties. For instance, for finitary logics, the LDDT (a non-parameterized version of the PLDDT) corresponds to the Filter Extension Property (FEP) of the class of matrix models of the underlying logic, while, again for finitary logics, the DDT (a non-local and non-parameterized version of the PLDDT) corresponds to the property of the lattice of finitely generated filters being dually Brouwerian.

It is easy to check that logic $\mathcal{C}_1$ admits a "classical" DDT witnessed by one single formula with no parameters. In fact, any finitary logic in the language $\mathcal{L}$ having (Ax1) and (Ax2) among its axioms and (MP) as its only inference rules satisfies the classical DDT (see [32, Theorem 3.72]).

PROPOSITION 3.9. *For every* $\Gamma \cup \{\varphi, \psi\} \in \mathrm{Fm}_{\mathcal{L}}$,

$$\Gamma, \varphi \vdash_{\mathcal{C}_1} \psi \quad \Leftrightarrow \quad \Gamma \vdash_{\mathcal{C}_1} \varphi \to \psi. \tag{DDT}$$

*That is,* $\{p \to q\}$ *is a DD set for the logic* $\mathcal{C}_1$.

As a consequence of Proposition 3.9, the class $\mathsf{Mod}(\mathcal{C}_1)$ has the FEP.[7] We compile this and other algebraic properties of the logic $\mathcal{C}_1$ in the next result, all of which are consequences of more general AAL results for finitary protoalgebraic logics having a DD set—see [32, Theorems 6.24 and 6.28 and Corollary 6.30].

THEOREM 3.10.

1. *For every* $\boldsymbol{A}$, *the join-semilattice of the finitely generated* $\mathcal{C}_1$*-filters of* $\boldsymbol{A}$ *is dually Brouwerian.*
2. *The logic* $\mathcal{C}_1$ *is filter-distributive.*
3. *The class* $\mathsf{Mod}(\mathcal{C}_1)$ *has the FEP.*

An observation which will be used in the sequel is that *every axiomatic extension of* $\mathcal{C}_1$ *enjoys the FEP.*

The DDT *lifts to arbitrary algebras* given an underlying protoalgebraic logic—see [32, Theorem 3.81]. Once applied to the logic $\mathcal{C}_1$, we obtain that for every $\boldsymbol{A}$ and every $X \cup \{a, b\} \subseteq A$,

$$b \in \mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{C}_1}(X, a) \quad \Leftrightarrow \quad a \to^{\boldsymbol{A}} b \in \mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{C}_1}(X).$$

[7] Given a logic $\mathcal{S}$, we say that the class $\mathsf{Mod}(\mathcal{S})$ *has the filter extension property (FEP)*, if for every matrix $\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}(\mathcal{S})$, every submatrix $\langle \boldsymbol{B}, G\rangle$ of it, and every $G' \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$ such that $G' \supseteq G$, there exists $F' \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ such that $F' \supseteq F$ and $G' = F' \cap B$.

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A consequence of this fact is a characterization of the Frege relation for the logic $\mathcal{C}_1$. We omit the easy proof.

PROPOSITION 3.11. *For every* $\boldsymbol{A}$, $F \in \mathcal{F}i_{\mathcal{C}_1}\boldsymbol{A}$, *and* $a, b \in A$,

$$\langle a, b\rangle \in \boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{C}_1}(F) \quad \Leftrightarrow \quad a \leftrightarrow^{\boldsymbol{A}} b \in F.$$

Proposition 3.11 shows us that the classical characterization of the Leibniz congruence of a $\mathcal{CL}$-filter defines the Frege relation of a $\mathcal{C}_1$-filter. This relation is *not* necessarily a congruence, for $\mathcal{C}_1$ is not (fully) self-extensional.

**§4. Parameterized congruence formulas for $\mathcal{C}_1$.** Our goal in this section is to provide a set of parameterized congruence formulas for the logic $\mathcal{C}_1$. In fact, we shall provide two such sets, namely in Proposition 4.1 and Theorem 4.3. While the former is a rather trivial one, the latter is fairly complex and will be needed in the sequel.

Since $\mathcal{C}_1$ is protoalgebraic, but not equivalential (recall Propositions 3.5 and 3.6), we know *a priori* that $\mathcal{C}_1$ admits a set of parameterized congruence formulas, but no set of congruence formulas without parameters. The first set of parameterized congruence formulas we provide for $\mathcal{C}_1$ is rather trivial, but will be useful in our analysis.

PROPOSITION 4.1. *The set*

$$\Delta(x, y, \overline{z}) := \{\xi(x, \overline{z}) \leftrightarrow \xi(y, \overline{z}) : \xi(w, \overline{z}) \in \mathrm{Fm}_{\mathcal{L}}\}$$

*is a set of parameterized congruence formulas for the logic* $\mathcal{C}_1$.

PROOF. Conditions (p-R), (p-Sym), and (p-Trans) follow by Lemma 3.2.1–3, respectively. Taking $\xi(w, \overline{z}) = w$, we have $x \leftrightarrow y = \xi(x, \overline{z}) \leftrightarrow \xi(y, \overline{z}) \in \Delta(x, y, \overline{z}) \subseteq \Delta\langle x, y\rangle$, so clearly $x, \Delta\langle x, y\rangle \vdash_{\mathcal{C}_1} y$, that is, $\Delta$ satisfies (p-MP). Finally, notice that $\bigcup_{i=1}^{n} \Delta\langle x_i, y_i\rangle \vdash_{\mathcal{C}_1} \Delta\big\langle f(x_1, \dots, x_n), f(y_1, \dots, y_n)\big\rangle$ holds by extensivity, for every $n$-ary function symbol $f \in \mathcal{L}$, given our choice of $\Delta$. Therefore, $\Delta$ satisfies (p-Re). It follows by Theorem 2.5 that $\Delta(x, y, \overline{z})$ is a set of parameterized congruence formulas for the logic $\mathcal{C}_1$. ⊣

The second set of parameterized congruence formulas for $\mathcal{C}_1$ we consider, on the opposite, is quite complex. In order to establish it we need an auxiliary lemma.

LEMMA 4.2.

1. $(\neg\psi \wedge \varphi^\circ) \to (\neg\psi \wedge \psi^\circ), (\neg\varphi \wedge \psi^\circ) \to (\neg\varphi \wedge \varphi^\circ), \varphi \leftrightarrow \psi \vdash_{\mathcal{C}_1} \neg\varphi \leftrightarrow \neg\psi$.
2. $\varphi \leftrightarrow \psi, \varphi^\circ \leftrightarrow \psi^\circ \vdash_{\mathcal{C}_1} \neg\varphi \leftrightarrow \neg\psi$.
3. $\varphi \leftrightarrow \psi, \neg\varphi \leftrightarrow \neg\psi \vdash_{\mathcal{C}_1} \varphi^\circ \leftrightarrow \psi^\circ$.
4. $\varphi \leftrightarrow \psi \vdash_{\mathcal{C}_1} (\neg\varphi \leftrightarrow \neg\psi) \leftrightarrow (\varphi^\circ \leftrightarrow \psi^\circ)$.

PROOF. The proof uses da Costa's completeness Theorem 3.4.

1. Let $v : \boldsymbol{Fm} \to \{0, 1\}$ be a bivaluation such that $v(\neg\varphi \leftrightarrow \neg\psi) = 0$ and assigning 1 to all the premises of our claim. Then, $v(\neg\varphi) \neq v(\neg\psi)$. Assume

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without loss of generality that $v(\neg\varphi) = 0$ and $v(\neg\psi) = 1$. Since $v(\varphi \leftrightarrow \psi) = 1$, either $v(\varphi) = v(\psi) = 0$ or $v(\varphi) = v(\psi) = 1$. In the first case, it follows that $v(\neg\varphi) = v(\neg\psi) = 1$, contradicting our assumption. So, $v(\varphi) = v(\psi) = 1$. Next, notice that since $v(\neg\varphi) = 0$, $v(\varphi^\circ) = 1$. Since moreover $v(\neg\psi) = 1$, it follows $v(\neg\psi \wedge \varphi^\circ) = 1$. Finally, notice that $\neg\psi \wedge \psi^\circ = {\sim}\psi$, and since $v(\psi) = 1$, it must be the case $v({\sim}\psi) = 0$. Thus $v\big((\neg\psi \wedge \varphi^\circ) \to (\neg\psi \wedge \psi^\circ)\big) = 0$, and we reach a contradiction.

2. Just notice that $\varphi^\circ \leftrightarrow \psi^\circ \vdash_{\mathfrak{C}_1} (\neg\psi \wedge \varphi^\circ) \to (\neg\psi \wedge \psi^\circ)$ and $\varphi^\circ \leftrightarrow \psi^\circ \vdash_{\mathfrak{C}_1} (\neg\varphi \wedge \psi^\circ) \to (\neg\varphi \wedge \varphi^\circ)$, applying Lemma 3.2.16 with $\xi = \neg\psi$ and $\xi = \neg\varphi$ respectively. The result now follows by item 1.

3. Let $v : \boldsymbol{Fm} \to \{0, 1\}$ be a bivaluation such that $v(\varphi^\circ \leftrightarrow \psi^\circ) = 0$ and $v(\varphi \leftrightarrow \psi) = v(\neg\varphi \leftrightarrow \neg\psi) = 1$. Then, $v(\varphi^\circ) \neq v(\psi^\circ)$. Assume without loss of generality that $v(\varphi^\circ) = 0$ and $v(\psi^\circ) = 1$. So it must be the case $v(\varphi) = 1$, otherwise $v(\varphi^\circ) = 1$. It follows by assumption that $v(\psi) = 1$ as well. Similarly, it must be the case $v(\neg\varphi) = 1$, otherwise $v(\varphi^\circ) = 1$. It follows by assumption that $v(\neg\psi) = 1$ as well. So, $v(\psi^\circ) = v(\varphi \to \psi) = v(\varphi \to \neg\psi) = 1$. It follows by definition of bivaluation that $v(\varphi) = 0$. Thus $v(\varphi^\circ) = 1$, and we reach an absurdity.

4. It follows by Proposition 3.9 over items 2 and 3. ⊣

Compare Lemma 3.2.8 with Lemma 4.2.2. The latter provides a weaker condition in order to establish $\neg\varphi \leftrightarrow \neg\psi$, and it will be used exhaustively throughout the rest of the work.

We still need a couple of auxiliary definitions before stating the main result of this section. For every $m \in \omega$ and every $\varphi \in \mathrm{Fm}_{\mathcal{L}}$, let[8]

$$\neg^m\varphi := \underbrace{\neg \cdots \neg}_{m \text{ times}} \varphi.$$

In particular, $\neg^0\varphi = \varphi$ and $\neg^1\varphi = \neg\varphi$. Let us also define

$$\begin{aligned}
\Phi_0(x, y) &= \{x \leftrightarrow y\},\\
\Phi_1(x, y, z_1) &= \{\neg^{m_1}(x *_1 z_1) \leftrightarrow \neg^{m_1}(y *_1 z_1) :\\
&\qquad *_1 \in \{\wedge, \vee, \to\}, m_1 \in \omega\},\\
\Phi_2(x, y, z_1, z_2) &= \{\neg^{m_2}\big(\neg^{m_1}(x *_1 z_1) *_2 z_2\big) \leftrightarrow\\
&\qquad \neg^{m_2}\big(\neg^{m_1}(y *_1 z_1) *_2 z_2\big) :\\
&\qquad *_1, *_2 \in \{\wedge, \vee, \to\}, m_1, m_2 \in \omega\},\\
&\ \ \vdots\\
\Phi_n(x, y, z_1, z_2, \dots, z_n) &= \{\neg^{m_n}\big(\cdots \neg^{m_2}(\neg^{m_1}(x *_1 z_1) *_2 z_2) \cdots *_n z_n\big) \leftrightarrow\\
&\qquad \neg^{m_n}\big(\cdots \neg^{m_2}(\neg^{m_1}(y *_1 z_1) *_2 z_2) \cdots *_n z_n\big) :\\
&\qquad *_1, \dots, *_n \in \{\wedge, \vee, \to\}, m_1, \dots, m_n \in \omega\},
\end{aligned}$$

[8] We adopt here the notation employed in [26, p. 16].

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and

$$\begin{aligned}
\Psi_0(x, y) &= \{x \leftrightarrow y\},\\
\Psi_1(x, y, z_1) &= \{\neg^{m_1}(z_1 *_1 x) \leftrightarrow \neg^{m_1}(z_1 *_1 y) :\\
&\qquad *_1 \in \{\wedge, \vee, \to\}, m_1 \in \omega\},\\
\Psi_2(x, y, z_1, z_2) &= \{\neg^{m_2}\big(\neg^{m_1}(z_1 * x) *_2 z_2\big) \leftrightarrow\\
&\qquad \neg^{m_2}\big(\neg^{m_1}(z_1 * y) *_2 z_2\big) :\\
&\qquad *_1, *_2 \in \{\wedge, \vee, \to\}, m_1, m_2 \in \omega\},\\
&\vdots\\
\Psi_n(x, y, z_1, z_2, \dots, z_n) &= \{\neg^{m_n}\big(\cdots \neg^{m_2}(\neg^{m_1}(z_1 *_1 x) *_2 z_2) \cdots *_n z_n\big) \leftrightarrow\\
&\qquad \neg^{m_n}\big(\cdots \neg^{m_2}(\neg^{m_1}(z_1 *_1 y) *_2 z_2) \cdots *_n z_n\big) :\\
&\qquad *_1, \dots, *_n \in \{\wedge, \vee, \to\}, m_1, \dots, m_n \in \omega\}.
\end{aligned}$$

We are now ready to state the main result of this section.

THEOREM 4.3. *For every* $\boldsymbol{A}$, $F \in \mathcal{F}i_{\mathcal{C}_1}\boldsymbol{A}$, *and* $a, b \in A$,

$$\begin{aligned}
\langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}(F) \Leftrightarrow \forall n \in \omega\ \forall m \in \omega\ \forall c_1, \dots, c_n \in A,&\\
\Phi_n^{\boldsymbol{A}}(\neg^m a, \neg^m b, c_1, \dots, c_n) \subseteq F,&\\
\Psi_n^{\boldsymbol{A}}(\neg^m a, \neg^m b, c_1, \dots, c_n) \subseteq F.&
\end{aligned}$$

PROOF. For the sake of simplicity, define the relation $R \subseteq A \times A$ by

$$\begin{aligned}
\langle a, b\rangle \in R \Leftrightarrow \forall n \in \omega\ \forall m \in \omega\ \forall c_1, \dots, c_n \in A,&\\
\Phi_n^{\boldsymbol{A}}(\neg^m a, \neg^m b, c_1, \dots, c_n) \subseteq F,&\\
\Psi_n^{\boldsymbol{A}}(\neg^m a, \neg^m b, c_1, \dots, c_n) \subseteq F.&
\end{aligned}$$

Suppose that $\langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. It follows by Proposition 4.1 that $aRb$. Conversely, we claim that $R$ is a congruence relation on $\boldsymbol{A}$ compatible with $F$. It should be clear that $R$ is an equivalence relation on $A$, given Lemma 3.2.1–3. Notice next that the relation $R$ is compatible with the connective $\neg$, since by construction of $R$ we are ranging $\neg^m a$ and $\neg^m b$ over $m \in \omega$. We are left to prove that $R$ is also compatible with the binary language operations of $\mathcal{C}_1$. Let $\langle a_1, b_1\rangle \in R$ and $\langle a_2, b_2\rangle \in R$. We claim that for every $c_1, \dots, c_n \in A$,

$$\bigcup_{n\in\omega}\bigcup_{m\in\omega} \Phi_n^{\boldsymbol{A}}(\neg^m(a_1 * a_2), \neg^m(b_1 * b_2), c_1, \dots, c_n) \subseteq F,$$

with $* \in \{\wedge, \vee, \to\}$. Let $n \in \omega$, $c_1, \dots, c_n \in A$, $*, *_1, \dots, *_n \in \{\wedge, \vee, \to\}$ and $m, m_1, \dots, m_n \in \omega$. Consider $\Phi_{n+1}(a_1, b_1, a_2, c_1, \dots, c_n)$ with $m' = 0$, $m_1' = m, m_2' = m_1, \dots, m_{n+1}' = m_n$. Since $\Phi_{n+1}(a_1, b_1, a_2, c_1, \dots, c_n) \subseteq F$ by the assumption $\langle a_1, b_1\rangle \in R$, we have

$$\begin{aligned}
&\neg^{m_n}(\cdots \neg^{m_1}(\neg^m(a_1 * a_2) *_1 c_1) \cdots *_n c_n) \leftrightarrow\\
&\neg^{m_n}(\cdots \neg^{m_1}(\neg^m(b_1 * a_2) *_1 c_1) \cdots *_n c_n) \in F.
\end{aligned}$$

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Next, consider $\Psi_{n+1}(a_2, b_2, b_1, c_1, \dots, c_n)$ with $m' = 0, m'_1 = m, m'_2 = m_1, \dots, m'_{n+1} = m_n$. Since $\Psi_{n+1}(a_2, b_2, b_1, c_1, \dots, c_n) \subseteq F$ by the assumption $\langle a_2, b_2 \rangle \in R$, we have

$$\neg^{m_n}(\cdots \neg^{m_1}(\neg^m(b_1 * a_2) *_1 c_1) \cdots *_n c_n) \leftrightarrow \neg^{m_n}(\cdots \neg^{m_1}(\neg^m(b_1 * b_2) *_1 c_1) \cdots *_n c_n) \in F.$$

It then follows by transitivity (Lemma 3.2.3) that

$$\neg^{m_n}(\cdots \neg^{m_1}(\neg^m(a_1 * a_2) *_1 c_1) \cdots *_n c_n) \leftrightarrow \neg^{m_n}(\cdots \neg^{m_1}(\neg^m(b_1 * b_2) *_1 c_1) \cdots *_n c_n) \in F.$$

Thus, $\Phi_n\big(\neg^m(a_1 * a_2), \neg^m(b_1 * b_2), c_1, \dots, c_n\big) \subseteq F$.

Similarly, one proves that for every $c_1, \dots, c_n \in A$,

$$\bigcup_{n\in\omega} \bigcup_{m\in\omega} \Psi_n^{\boldsymbol{A}}(\neg^m(a_1 * a_2), \neg^m(b_1 * b_2), c_1, \dots, c_n) \subseteq F,$$

with $* \in \{\wedge, \vee, \to\}$. Finally, in order to see that $R$ is compatible with $F$, let $\langle a, b \rangle \in R$ and assume $a \in F$. Since by assumption $\Phi_0(a, b) \subseteq F$, it follows that $a \to b \in F$, and hence by (MP) that $b \in F$. Thus, $R \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. ⊣

In other words, Theorem 4.3 tells us that the set

$$\Delta(x, y, \overline{z}) = \bigcup_{n\in\omega} \bigcup_{m\in\omega} \Phi_n(\neg^m x, \neg^m y, z_1, \dots, z_n) \cup \bigcup_{n\in\omega} \bigcup_{m\in\omega} \Psi_n(\neg^m x, \neg^m y, z_1, \dots, z_n)$$

is a set of parameterized congruence formulas for the logic $\mathcal{C}_1$. Although the set $\Delta(x, y, \overline{z})$ above may seem of little more practical usage than the one given in Proposition 4.1, it will be the key to proving Theorem 9.1 later on.

## §5. Algebraic semantics for $\mathcal{C}_1$.

As we have mentioned in the Introduction, several semantical approaches to the logic $\mathcal{C}_1$ have been put forward in the literature, namely, behavioral semantics [12, 13], possible-translations semantics [9], da Costa algebras [14, 52], $\mathcal{C}_1$-algebras [19], and a two-valued semantics [22, 23], this latter already introduced in Definition 3.3. From an AAL perspective, however, there are still two natural candidates for a semantical approach left unstudied—the class $\mathsf{Alg}^*(\mathcal{C}_1)$ and Blok and Pigozzi's notion of algebraic semantics. In this section we will investigate both these approaches, and conclude that neither provides a meaningful algebraic counterpart to the logic $\mathcal{C}_1$.

### 5.1. The class $\mathsf{Alg}^*(\mathcal{C}_1)$.

The first (negative) result on the algebraization of $\mathcal{C}_1$ was established by Mortensen in [44], by proving that the only congruence on the formula algebra compatible with the set of $\mathcal{C}_1$-theorems is the identity relation. This fact can be re-written as:

Proposition 5.1 (Mortensen). $\boldsymbol{\Omega}^{\boldsymbol{Fm}}(\mathrm{Thm}_{\mathcal{C}_1}) = id_{\boldsymbol{Fm}}$.

In light of Proposition 5.1, it follows by Theorem 4.3 that for every $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}$,

$$\begin{aligned} \varphi = \psi \Leftrightarrow \forall n \in \omega\ \forall m \in \omega\ \forall \xi_1, \dots, \xi_n \in \mathrm{Fm}_{\mathcal{L}}, \\ \Phi_n^{\boldsymbol{Fm}}(\neg^m \varphi, \neg^m \psi, \xi_1, \dots, \xi_n) \subseteq \mathrm{Thm}_{\mathcal{C}_1}, \\ \Psi_n^{\boldsymbol{Fm}}(\neg^m \varphi, \neg^m \psi, \xi_1, \dots, \xi_n) \subseteq \mathrm{Thm}_{\mathcal{C}_1}. \end{aligned}$$

Having in mind (1) on page 483, Mortensen's result can be re-stated as follows:

Proposition 5.2. $\boldsymbol{Fm} \in \mathsf{Alg}^*(\mathcal{C}_1)$.

Proposition 5.2 has a most crucial consequence upon the intrinsic variety of the logic $\mathcal{C}_1$.

Corollary 5.3. *The intrinsic variety of* $\mathcal{C}_1$ *is the class of all* $\mathcal{L}$*-algebras. That is,* $\mathbb{V}(\mathcal{C}_1) = \{\boldsymbol{A} : \boldsymbol{A}$ *is an* $\mathcal{L}$*-algebra*$\}$.

Proof. Since $\boldsymbol{Fm}$ is the absolutely free $\mathcal{L}$-algebra over the (denumerable) set of variables Var, it is well-known that $\mathbb{V}(\boldsymbol{Fm})$ is the class of all $\mathcal{L}$-algebras. Moreover $\mathbb{V}(\mathcal{C}_1) = \mathbb{V}\big(\boldsymbol{Fm}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{Fm}}(\mathcal{T}h(\mathcal{C}_1))\big) = \mathbb{V}(\boldsymbol{Fm})$, bearing in mind that $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{Fm}}\big(\mathcal{T}h(\mathcal{C}_1)\big) = \boldsymbol{\Omega}^{\boldsymbol{Fm}}(\mathrm{Thm}_{\mathcal{C}_1}) = id_{\boldsymbol{Fm}}$ by protoalgebraicity of $\mathcal{C}_1$ and Proposition 5.1, respectively. ⊣

Corollary 5.3 sets aside the possibility of associating with $\mathcal{C}_1$ its intrinsic variety as the class of algebras one could canonically associate with it.

Notice that since $\mathcal{C}_1$ is an extension of $\mathcal{C}_n$, we also have $\boldsymbol{Fm} \in \mathsf{Alg}^*(\mathcal{C}_1) \subseteq \mathsf{Alg}^*(\mathcal{C}_n)$, for every $n \geq 1$. As such, Corollary 5.3 also holds for the logics $\mathcal{C}_n$, with $n \geq 1$. Consequently, the intrinsic varieties of all da Costa's $\mathcal{C}$-systems coincide. It is an open problem whether the quasivarieties $\mathbb{Q}\mathsf{Alg}^*(\mathcal{C}_n)$, with $n \geq 1$, also coincide, or even if the classes $\mathsf{Alg}^*(\mathcal{C}_n)$, with $n \geq 1$, happen to coincide.

Given the fact that the intrinsic variety of a logic is generated by its algebraic counterpart (see page 483), it follows by Corollary 5.3 that $\mathsf{Alg}^*(\mathcal{C}_1)$ generates the variety of all $\mathcal{L}$-algebras.

We now proceed to prove that $\mathsf{Alg}^*(\mathcal{C}_1)$ is not a quasivariety. In order to see that, we borrow the next example from [54, p. 590], and prove that $\mathsf{Alg}^*(\mathcal{C}_1)$ is not closed under subalgebras. Notice that the proof of Theorem 3.6 provides a counterexample for the fact that $\mathsf{Mod}^*(\mathcal{C}_1)$ is not closed under submatrices, but the subalgebra $\boldsymbol{B}$ there presented still belongs to $\mathsf{Alg}^*(\mathcal{C}_1)$, because $\boldsymbol{\Omega}^{\boldsymbol{B}}(\{1\}) = id_{\boldsymbol{B}}$ and $\{1\} \in \mathcal{F}i_{\mathcal{C}_1}\boldsymbol{B}$.

Proposition 5.4. $\mathsf{Alg}^*(\mathcal{C}_1)$ *is not a quasivariety.*

Proof. We prove that $\mathsf{Alg}^*(\mathcal{C}_1)$ is not closed under subalgebras. Consider the algebra $\boldsymbol{A}$ defined by the truth-tables given in Table 3 and fix $F := \{0, 1, 2, 3, 4\}$. First, one must check that $F \in \mathcal{F}i_{\mathcal{C}_1}\boldsymbol{A}$ and furthermore that $\boldsymbol{\Omega}^{\boldsymbol{A}}(F) = id_{\boldsymbol{A}}$ (we leave the details to the reader). Thus, $\boldsymbol{A} \in \mathsf{Alg}^*(\mathcal{C}_1)$. Now, consider the subalgebra $\boldsymbol{B} \leq \boldsymbol{A}$ with universe $B = \{0, 4, 5\}$. We have $\mathrm{Sg}^{\boldsymbol{A}}_{\mathcal{L}}(B) = B$ and $\{0, 4\} = F \cap B \in \mathcal{F}i_{\mathcal{C}_1}\boldsymbol{B}$. In fact $\mathcal{F}i_{\mathcal{C}_1}\boldsymbol{B} = \{F \cap B, B\}$, because the singletons $\{0\}, \{4\}, \{5\}$ are not $\mathcal{C}_1$-filters, and neither are

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| $\to^{A}$ | 0 | 1 | 2 | 3 | 4 | 5 | $\neg^{A}$ |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 5 | 5 |
| 1 | 0 | 0 | 0 | 0 | 0 | 5 | 2 |
| 2 | 0 | 0 | 0 | 0 | 0 | 5 | 3 |
| 3 | 0 | 0 | 0 | 0 | 0 | 5 | 4 |
| 4 | 0 | 0 | 0 | 0 | 0 | 5 | 5 |
| 5 | 0 | 0 | 0 | 0 | 0 | 0 | 4 |

| $\wedge^{A}$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 | 5 |
| 1 | 1 | 1 | 4 | 3 | 4 | 5 |
| 2 | 2 | 3 | 2 | 4 | 4 | 5 |
| 3 | 3 | 3 | 3 | 3 | 4 | 5 |
| 4 | 4 | 4 | 4 | 4 | 4 | 5 |
| 5 | 5 | 5 | 5 | 5 | 5 | 5 |

| $\vee^{A}$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 0 | 1 | 1 | 1 |
| 2 | 0 | 0 | 2 | 0 | 2 | 2 |
| 3 | 0 | 1 | 2 | 3 | 3 | 3 |
| 4 | 0 | 1 | 2 | 3 | 4 | 4 |
| 5 | 0 | 1 | 2 | 3 | 4 | 5 |

TABLE 3. Truth-tables of the algebra $\boldsymbol{A}$ of [54, p. 590].

the subsets $\{0,5\}$ and $\{4,5\}$. We now claim that $\langle 0,4\rangle \in \boldsymbol{\Omega}^{\boldsymbol{B}}(F\cap B)$. We shall make use of Theorem 4.3. Notice that $0 \leftrightarrow 4 = 0 \in F \cap B$, and since $\neg 0 = \neg 4$, it follows that $\neg^m 0 = \neg^m 4$, for every $m \in \omega$, and therefore $\neg^m 0 \leftrightarrow \neg^m 4 = 0 \in F\cap B$. Furthermore, by looking at the truth tables of $\boldsymbol{A}$ restricted to the subuniverse $B$, one sees that for every $*_1, \dots, *_n \in \{\wedge, \vee, \to\}$, every $m_1, \dots, m_n \in \omega$, and every $c_1, \dots, c_n \in A$,

$$\neg^{m_n}(\cdots(\neg^{m_2}(\neg^{m_1}(0 *_1 c_1) *_2 c_2)\cdots) *_n c_n) =$$
$$\neg^{m_n}(\cdots(\neg^{m_2}(\neg^{m_1}(4 *_1 c_1) *_2 c_2)\cdots) *_n c_n),$$

and

$$\neg^{m_n}(\cdots(\neg^{m_2}(\neg^{m_1}(c_1 *_1 0) *_2 c_2)\cdots) *_n c_n) =$$
$$\neg^{m_n}(\cdots(\neg^{m_2}(\neg^{m_1}(c_1 *_1 4) *_2 c_2)\cdots) *_n c_n).$$

Therefore,

$$\neg^{m_n}(\cdots(\neg^{m_2}(\neg^{m_1}(0 *_1 c_1) *_2 c_2)\cdots) *_n c_n) \leftrightarrow$$
$$\neg^{m_n}(\cdots(\neg^{m_2}(\neg^{m_1}(4 *_1 c_1) *_2 c_2)\cdots) *_n c_n) = 0 \in F \cap B,$$

and

$$\neg^{m_n}(\cdots(\neg^{m_2}(\neg^{m_1}(c_1 *_1 0) *_2 c_2)\cdots) *_n c_n) \leftrightarrow$$
$$\neg^{m_n}(\cdots(\neg^{m_2}(\neg^{m_1}(c_1 *_1 4) *_2 c_2)\cdots) *_n c_n) = 0 \in F \cap B.$$

It follows by Theorem 4.3 that $\langle 0,4\rangle \in \boldsymbol{\Omega}^{\boldsymbol{B}}(F\cap B)$. Thus, $\boldsymbol{\Omega}^{\boldsymbol{B}}(F\cap B) \neq id_{\boldsymbol{B}}$. We conclude that $\boldsymbol{B} \notin \mathsf{Alg}^*(\mathcal{C}_1)$ and hence $\mathsf{Alg}^*(\mathcal{C}_1)$ is not closed under subalgebras. ⊣

COROLLARY 5.5. $\mathsf{Alg}^*(\mathcal{C}_1) \subsetneq \mathbb{V}(\mathcal{C}_1)$.

PROOF. Since $\mathsf{Alg}^*(\mathcal{C}_1)$ is not a quasivariety by Proposition 5.4, it follows at once that $\mathsf{Alg}^*(\mathcal{C}_1) \subsetneq \mathbb{Q}\mathsf{Alg}^*(\mathcal{C}_1) \subseteq \mathbb{V}(\mathcal{C}_1)$. ⊣

In summary, the class $\mathsf{Alg}^*(\mathcal{C}_1)$ seems to be of little interest from an algebraic point of view. Indeed, it is not a quasivariety (Proposition 5.4), the formula algebra $\boldsymbol{Fm}$ belongs to $\mathsf{Alg}^*(\mathcal{C}_1)$ (Proposition 5.2), and the variety generated by $\mathsf{Alg}^*(\mathcal{C}_1)$ is the class of all $\mathcal{L}$-algebras (Corollary 5.3).

**5.2. Algebraic semantics for $\mathcal{C}_1$.** Our next goal is to prove that $\mathcal{C}_1$ admits no algebraic semantics in the sense of Blok and Pigozzi, as originally defined in [6] and introduced in Definition 2.8. We shall appeal to a recent result established in [43] characterizing non-trivial protoalgebraic logics with an algebraic semantics and (once again) to Mortensen's result.

THEOREM 5.6. *The logic $\mathcal{C}_1$ has no algebraic semantics.*

PROOF. Let $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}$ be two logically equivalent formulas. That is, for every $\xi(x, \overline{z}) \in \mathrm{Fm}_{\mathcal{L}}$ we have $\xi(\varphi, \overline{z}) \dashv\vdash_{\mathcal{C}_1} \xi(\psi, \overline{z})$. It trivially follows that $\xi(\varphi, \overline{z}) \in \mathrm{Thm}_{\mathcal{C}_1}$ if and only if $\xi(\psi, \overline{z}) \in \mathrm{Thm}_{\mathcal{C}_1}$. Therefore, $\langle \varphi, \psi \rangle \in \boldsymbol{\Omega}^{\boldsymbol{Fm}}(\mathrm{Thm}_{\mathcal{C}_1})$—see for instance [32, Corollary 4.24]. But then $\varphi = \psi$, in light of Mortensen's Proposition 5.1. We conclude by [43, Theorem 9.3] (which states that a non-trivial protoalgebraic logic admits an algebraic semantics if and only if there are two distinct logically equivalent formulas) that $\mathcal{C}_1$ admits no algebraic semantics. ⊣

The non-algebraizability of da Costa's logic $\mathcal{C}_1$ established in [39] tells us that there exists no triple $\langle \mathsf{K}, \boldsymbol{\tau}, \boldsymbol{\rho} \rangle$ satisfying conditions (ALG1) and (ALG2) for the logic $\mathcal{C}_1$; in other words, $\mathcal{C}_1$ fails to possess an equivalent algebraic semantics. Theorem 5.6 now tells us that there exists no pair $\langle \mathsf{K}, \boldsymbol{\tau} \rangle$ satisfying condition (ALG1) for the logic $\mathcal{C}_1$; in other words, $\mathcal{C}_1$ fails to possess an algebraic semantics. Theorem 5.6 definitely leaves out of the picture any attempt to associate with $\mathcal{C}_1$ a canonical class of algebras $\mathsf{K}$, whose $\mathsf{K}$-relative congruences would algebraically translate the logical properties of $\mathcal{C}_1$ according to Definition 2.8.

Having an algebraic semantics is a property preserved by extensions [8, Theorem 2.15]—the cited result is stated for finitary logics, but for our purposes it suffices. Therefore, an immediate consequence of Theorem 5.6 is the following:

COROLLARY 5.7. *For every $n \geq 1$, the logic $\mathcal{C}_n$ has no algebraic semantics.*

On the other hand, the property of having an algebraic semantics is not necessarily preserved by expansions, and $\mathcal{C}_1$ provides one such counterexample. Indeed, $\mathcal{IL}^+$ possesses an (equivalent) algebraic semantics—the variety of generalized Heyting algebras, also known as relatively pseudo-complemented lattices—while $\mathcal{C}_1$ does not.

It is worth mentioning that a (protoalgebraic) logic without any algebraic semantics is not a novel phenomenon in the literature. Indeed, [8, Section 2.3] investigates a necessary condition for a logic to possess an algebraic semantics, and then builds a very simple logic (with one single axiom schema and the inference rule of *Modus Ponens*), which fails to satisfy such condition. In fact, this logic has been exhaustively studied from an AAL perspective in [31]. Another *ad-hoc* example of a logic without any algebraic semantics appears in [51, Example 3]. For the sake of completeness, the negation fragments of $\mathcal{CL}$ and $\mathcal{IL}$ also fail to admit any algebraic semantics, as observed in [8, p. 170], as well as the logic $\mathcal{PW}$ having a single binary connective [51, Example 4]. The novelty of Theorem 5.6 is the fact that such pathological behaviour is here exhibited by a well-established logic, rather than a built-up counterexample, or a mono-connective logic. As a matter of fact, the existence of an interesting logic without an algebraic semantics was posed as an open problem right from the very introduction of this notion by Blok and Pigozzi [6, p. 18]: "It is an open question if any interesting deductive systems fail to have an algebraic semantics."

**Part II. From $\mathcal{C}_1$ to its extensions.**

The second part of our work may be seen as a bridge between the logic $\mathcal{C}_1$ and the extensions of $\mathcal{C}_1$ we intend to study. Indeed, while the results of Section 6 still hold, in particular, for the logic $\mathcal{C}_1$, from Section 7 onwards the results will depend on conditions which $\mathcal{C}_1$ fails to satisfy. We were led to such conditions while investigating the same results for several extensions of $\mathcal{C}_1$. The reason for this abstraction is twofold. On the one hand, our new assumptions allow us to establish stronger results than those seen for the logic $\mathcal{C}_1$. On the other hand, we avoid repeating the same results for each extension of $\mathcal{C}_1$ under study, while unifying them under a general setting.

Ultimately, the goal of this second part is to arrive at Theorems 8.3 and 8.7 concerning the non-trivial algebra subdirectly irreducible relative to $\mathsf{Alg}^*(\mathcal{S})$, where $\mathcal{S}$ is an extension of $\mathcal{C}_1$ satisfying the aforementioned conditions. These results will allow us in Part III to characterize the quasivarieties $\mathsf{Alg}^*(\mathcal{S})$ and the intrinsic variety $\mathbb{V}(\mathcal{S})$, for some known extensions $\mathcal{S}$ of $\mathcal{C}_1$.

**§6. The subalgebra $\boldsymbol{A}^\circ$.** In this section we study a special subalgebra whose $\mathcal{C}_1$-filters satisfy "classical" properties. Most notably, the quotient of such subalgebra over the Leibniz congruence of any of its $\mathcal{C}_1$-filters is a Boolean Algebra (Theorem 6.6). The results of this section lay the groundwork for Sections 7 and 8.

Throughout the present section let $\mathcal{S}$ be an extension of $\mathcal{C}_1$, extended itself by $\mathcal{CL}$, that is, $\mathcal{C}_1 \leq \mathcal{S} \leq \mathcal{CL}$. Let also $\boldsymbol{A}$ be an arbitrary fixed algebra. Consider the subset $A^\circ := \left\{a \in A : a^\circ \in \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\right\} \subseteq A$ and let $\boldsymbol{A}^\circ$ be the subalgebra of $\boldsymbol{A}$ generated by the subuniverse $\mathrm{Sg}^{\boldsymbol{A}}_{\mathcal{L}}(A^\circ)$.

Let us start by observing that the subset $A^\circ$ is in fact a subuniverse of $\boldsymbol{A}$.

Lemma 6.1. *For every $\boldsymbol{A}$, $A^\circ = \mathrm{Sg}^{\boldsymbol{A}}_{\mathcal{L}}(A^\circ)$.*

Proof. Let $a, b \in A^\circ$. That is, $a^\circ, b^\circ \in \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. It follows by axioms (Ax12), (Ax13), and (Ax14) and Lemma 3.2.19 that $(a \wedge b)^\circ, (a \vee b)^\circ, (a \to b)^\circ, (\neg a)^\circ \in \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, respectively. That is, $a \wedge b, a \vee b, a \to b, \neg a \in A^\circ$. Thus, $A^\circ$ is closed under the language operations. ⊣

In light of Lemma 6.1, from here on, we shall denote the subuniverse $\mathrm{Sg}^{\boldsymbol{A}}_{\mathcal{L}}(A^\circ)$ simply by $A^\circ$. Another useful observation is that $a^\circ \in A^\circ$, for every $a \in A$. We register this fact for future reference.

Lemma 6.2. *For every $\boldsymbol{A}$ and $a \in A$, $a^\circ \in A^\circ$.*

Proof. Just notice that $(a^\circ)^\circ \in \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, by Lemma 3.2.20. ⊣

Our first goal is to show that, by restricting ourselves to the subuniverse $A^\circ$, the Leibniz congruence on $\boldsymbol{A}^\circ$ of $\mathcal{S}$-filters of $\boldsymbol{A}^\circ$ can be "classically" characterized. For ease of notation we shall drop the superscript $^{\boldsymbol{A}}$ on $a^{\circ \boldsymbol{A}}$.

Proposition 6.3. *For every $\boldsymbol{A}$, $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}^\circ$ and $a, b \in A^\circ$,*

$$\langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}^\circ}(F) \quad \Leftrightarrow \quad a \leftrightarrow^{\boldsymbol{A}^\circ} b \in F.$$

Proof. Let $\boldsymbol{A}$ arbitrary and $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}^\circ$. Consider the relation $R \subseteq A^\circ \times A^\circ$ defined by $xR\,y$ iff $x \leftrightarrow^{\boldsymbol{A}^\circ} y \in F$. Suppose $\langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}^\circ}(F)$. It follows

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by Proposition 4.1 that $a \leftrightarrow^{A^\circ} b \in F$. Thus, $a R b$. Conversely, we claim that $R$ is a congruence relation on $A^\circ$ compatible with $F$. First, it should be clear from Lemma 3.2.1–3 that $R$ is an equivalence relation. Next, it follows by Lemma 3.2.4–6 that $R$ is compatible with the language operations $\wedge, \vee, \to$, respectively. To see that it is also compatible with the operation $\neg$, let $x, y \in A^\circ$ such that $x R y$. Then, $x^\circ, y^\circ \in \bigcap \mathcal{F}i_{\mathcal{S}} A \subseteq F$. It follows by Lemma 3.2.8 that $\neg x \leftrightarrow^{A^\circ} \neg y \in F$. Finally, it is clear by (MP) that $R$ is compatible with $F$. Thus, $R \subseteq \boldsymbol{\Omega}^{A^\circ}(F)$. ⊣

Corollary 6.4. *For every $\boldsymbol{A}$ and $F \in \mathcal{F}i_{\mathcal{S}} \boldsymbol{A}^\circ$,*

$$\boldsymbol{\Omega}^{A^\circ}(F) = \boldsymbol{\Lambda}_{\mathcal{S}}^{A^\circ}(F).$$

Proof. Given Proposition 3.11, the result follows immediately by Proposition 6.3. ⊣

Corollary 6.4 reinforces the "classical" flavour of the subalgebra $\boldsymbol{A}^\circ$. Recall that the Leibniz congruence and Frege relation coincide on classical filters, for arbitrary algebras.

As it turns out, not only is the Leibniz congruence on $\boldsymbol{A}^\circ$ of an $\mathcal{S}$-filter classically defined (Proposition 6.3), but it is actually a congruence relative to BA.[9] (Theorem 6.6) In order to prove it, we compile some auxiliary facts, whose proofs we leave for the reader—item 3 below is proved in [15, Proposition 4.2].

Lemma 6.5.

1. $\vdash_{\mathcal{C}_1} (\varphi \wedge \psi) \vee \xi \to (\varphi \vee \psi) \wedge (\psi \vee \xi)$.
2. $\vdash_{\mathcal{C}_1} (\varphi \vee \psi) \wedge (\psi \vee \xi) \to (\varphi \wedge \psi) \vee \xi$.
3. $\varphi \vdash_{\mathcal{C}_1} \varphi \leftrightarrow (\psi \vee \neg\psi)$.
4. $\vdash_{\mathcal{C}_1} (\psi \wedge \neg\psi) \leftrightarrow \neg\neg(\psi \wedge \neg\psi)$.
5. $\varphi, \varphi^\circ, \psi^\circ \vdash_{\mathcal{C}_1} (\psi \wedge \neg\psi) \leftrightarrow \neg\varphi$.

Theorem 6.6. *For every $\boldsymbol{A}$ and $F \in \mathcal{F}i_{\mathcal{S}} \boldsymbol{A}^\circ$, $\boldsymbol{A}^\circ / \boldsymbol{\Omega}^{A^\circ}(F) \in$ BA.*

Proof. Let $\boldsymbol{A}$ arbitrary and $F \in \mathcal{F}i_{\mathcal{S}} \boldsymbol{A}^\circ$. Fix $\boldsymbol{B} := \boldsymbol{A}^\circ / \boldsymbol{\Omega}^{A^\circ}(F)$. We must prove that $\langle \boldsymbol{B}, \wedge^{\boldsymbol{B}}, \vee^{\boldsymbol{B}}, \to^{\boldsymbol{B}}, \neg^{\boldsymbol{B}} \rangle$ is a distributive, bounded, and complemented lattice. Let $\pi : \boldsymbol{A}^\circ \to \boldsymbol{B}$ be the natural map. For every $a, b \in A^\circ$, define the relation $\leq^{\boldsymbol{B}} \subseteq B \times B$ by

$$\pi(a) \leq^{\boldsymbol{B}} \pi(b) \quad \Leftrightarrow \quad a \to^{A^\circ} b \in F.$$

It is easy to see that $\leq^{\boldsymbol{B}}$ is reflexive and transitive, by Lemma 3.2.11 and 12. Furthermore, if $\pi(a) \leq^{\boldsymbol{B}} \pi(b)$ and $\pi(b) \leq^{\boldsymbol{B}} \pi(a)$, that is, $a \leftrightarrow^{A^\circ} b \in F$, it follows by Proposition 6.3 that $\langle a, b \rangle \in \boldsymbol{\Omega}^{A^\circ}(F)$, and hence $\pi(a) = \pi(b)$. So, $\leq^{\boldsymbol{B}}$ is a partial order on $\boldsymbol{B}$.

It follows by (Ax3), (Ax4), and (Ax5) that $\wedge^{\boldsymbol{B}}$ is the infimum induced by the order $\leq^{\boldsymbol{B}}$, and by (Ax6), (Ax7), and (Ax8) that $\vee^{\boldsymbol{B}}$ is the supremum induced by the order $\leq^{\boldsymbol{B}}$. So, $\langle \boldsymbol{B}, \wedge^{\boldsymbol{B}}, \vee^{\boldsymbol{B}} \rangle$ is indeed a lattice. Moreover, it follows by Lemma 6.5.1 and 2 that it is a distributive lattice.

[9] BA denotes the class of all Boolean algebras.

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We next prove that $\boldsymbol{B}$ is limited. Let $a \in F \subseteq A^\circ$ and $b \in A^\circ$. It follows by (Ax1) and (MP) that $b \to^{\boldsymbol{A}^\circ} a \in F$, that is, $\pi(b) \leq^{\boldsymbol{B}} \pi(a)$. So, $\pi(a)$ is the top element of $B$ w.r.t. the order $\leq^{\boldsymbol{B}}$. In particular, $\pi(\neg^{\boldsymbol{A}^\circ} b) \leq^{\boldsymbol{B}} \pi(a)$, i.e., $\neg^{\boldsymbol{A}^\circ} b \to^{\boldsymbol{A}^\circ} a \in F$. Since $a \in A^\circ$, we have $a^\circ \in \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \subseteq F$. It follows by Lemma 3.2.7 that $\neg^{\boldsymbol{A}^\circ} a \to^{\boldsymbol{A}^\circ} \neg^{\boldsymbol{A}^\circ}\neg^{\boldsymbol{A}^\circ} b \in F$. But $\neg^{\boldsymbol{A}^\circ}\neg^{\boldsymbol{A}^\circ} b \to^{\boldsymbol{A}^\circ} b \in F$ by (Ax10), so by transitivity $\neg^{\boldsymbol{A}^\circ} a \to^{\boldsymbol{A}^\circ} b \in F$, that is, $\pi(\neg^{\boldsymbol{A}^\circ} a) \leq^{\boldsymbol{B}} \pi(b)$. Therefore, $\neg^{\boldsymbol{B}}\pi(a)$ is the bottom element of $B$ w.r.t. the order $\leq^{\boldsymbol{B}}$.

We are left to prove that $\boldsymbol{B}$ is complemented. Let $a \in F \subseteq A^\circ$ and $b \in A^\circ$. It follows by Lemma 6.5.3 that $a \leftrightarrow^{\boldsymbol{A}^\circ} (b \vee^{\boldsymbol{A}^\circ} \neg^{\boldsymbol{A}^\circ} b) \in F$, and hence

$$\pi(a) = \pi(b \vee^{\boldsymbol{A}^\circ} \neg^{\boldsymbol{A}^\circ} b) = \pi(b) \vee^{\boldsymbol{B}} \neg^{\boldsymbol{B}}\pi(b).$$

Moreover, since $a^\circ, b^\circ \in \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \subseteq F$ and $a \in F$, it follows by Lemma 6.5.5 that $(b \wedge^{\boldsymbol{A}^\circ} \neg^{\boldsymbol{A}^\circ} b) \leftrightarrow^{\boldsymbol{A}^\circ} \neg^{\boldsymbol{A}^\circ} a \in F$, and hence

$$\pi(b) \wedge^{\boldsymbol{B}} \neg^{\boldsymbol{B}}\pi(b) = \pi(b \wedge^{\boldsymbol{A}^\circ} \neg^{\boldsymbol{A}^\circ} b) = \pi(\neg^{\boldsymbol{A}^\circ} a) = \neg^{\boldsymbol{B}}\pi(a).$$

Since we have seen already $\pi(a)$ and $\neg^{\boldsymbol{B}}\pi(a)$ to be the top and bottom elements of $B$, respectively, it follows that $\boldsymbol{B}$ is complemented. ⊣

**§7. A sufficient condition for $\boldsymbol{A}^\circ \in \mathsf{BA}$.** Our goal in this section is to isolate a general condition which will allow us to prove stronger algebraic results than those seen so far for the logic $\mathcal{C}_1$.

Throughout the present section, let $\mathcal{S}$ be such that $\mathcal{C}_1 \leq \mathcal{S} \leq \mathcal{CL}$ and satisfying the following condition: For every $\boldsymbol{A}$, $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and $a, b \in A^\circ$,

$$\langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}(F) \quad \Leftrightarrow \quad a \leftrightarrow^{\boldsymbol{A}} b \in F. \tag{$\star$}$$

To put it in words, by restricting ourselves to the subuniverse $A^\circ$, the Leibniz congruence on $\boldsymbol{A}$ of $\mathcal{S}$-filters of $\boldsymbol{A}$ can be "classically" characterized. Comparing condition ($\star$) with Proposition 6.3, we see that the former is applicable to $\mathcal{S}$-filters of $\boldsymbol{A}$ while the latter is applicable to $\mathcal{S}$-filters of the subalgebra $\boldsymbol{A}^\circ$.

Notice that for every $a, b \in A^\circ$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, we have $a^\circ, b^\circ \in \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \subseteq F$, and hence $a^\circ \leftrightarrow b^\circ \in F$. Therefore, condition ($\star$) is equivalent to the following condition: For every $\boldsymbol{A}$, $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and $a, b \in A^\circ$,

$$\langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}(F) \quad \Leftrightarrow \quad a \leftrightarrow^{\boldsymbol{A}} b \in F \text{ and } a^\circ \leftrightarrow^{\boldsymbol{A}} b^\circ \in F,$$

and having in mind Lemma 4.2.2, it is also equivalent to the following condition: For every $\boldsymbol{A}$, $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and $a, b \in A^\circ$,

$$\langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}(F) \quad \Leftrightarrow \quad a \leftrightarrow^{\boldsymbol{A}} b \in F \text{ and } \neg a \leftrightarrow^{\boldsymbol{A}} \neg b \in F. \tag{$\star\star$}$$

Observe that, as claimed, condition ($\star$) does *not* hold for $\mathcal{C}_1$. Indeed, consider the algebra $\boldsymbol{A}$ depicted in Figure 2 and fix $F_0 := \bigcap \mathcal{F}i_{\mathcal{C}_1}\boldsymbol{A} = \{1, u\}$. On the one hand, $a^\circ = 1^\circ = 1 \in F_0$ and $a \leftrightarrow 1 \in F_0$. On the other hand, $\langle a, 1\rangle \notin \boldsymbol{\Omega}^{\boldsymbol{A}}(F_0) = id_A$.

In light of Lemma 6.2, a particular case of condition ($\star$) arises when dealing with $a^\circ, b^\circ \in A$.

COROLLARY 7.1. *For every* $\boldsymbol{A}$, $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ *and* $a, b \in A$,

$$\langle a^\circ, b^\circ\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}(F) \quad\Leftrightarrow\quad a^\circ \leftrightarrow b^\circ \in F.$$

The next result is a very important consequence of condition $(\star)$—in fact, it is equivalent to it.

THEOREM 7.2. *For every* $\boldsymbol{A}$ *and* $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$,

$$\boldsymbol{\Omega}^{\boldsymbol{A}}(F)|_{A^\circ} = \boldsymbol{\Omega}^{\boldsymbol{A}^\circ}(F \cap A^\circ).$$

PROOF. The inclusion $\boldsymbol{\Omega}^{\boldsymbol{A}}(F)|_{A^\circ} \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}^\circ}(F \cap A^\circ)$ holds in general by Lemma 2.1.2. As for the converse inclusion, let $\langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}^\circ}(F \cap A^\circ)$. Then $a \leftrightarrow^{\boldsymbol{A}^\circ} b \in F \cap A^\circ$ by Proposition 4.1, and therefore $a \leftrightarrow^{\boldsymbol{A}} b \in F$ because $\boldsymbol{A}^\circ \leq \boldsymbol{A}$. Since $a, b \in A^\circ$, the result now follows by condition $(\star)$. ⊣

Notice that Theorem 7.2 holds for every equivalential extension of $\mathcal{C}_1$. Indeed, it is known (see [32, Lemma 6.72]) that given an equivalential logic $\mathcal{S}$, we have $\boldsymbol{\Omega}^{\boldsymbol{A}}(F)|_{B} = \boldsymbol{\Omega}^{\boldsymbol{B}}(F \cap B)$, for every algebra $\boldsymbol{A}$, $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, and every subalgebra $\boldsymbol{B} \leq \boldsymbol{A}$. Theorem 7.2 is simply the particular case for the subalgebra $\boldsymbol{A}^\circ$.

Theorems 6.6 and 7.2 taken together have one major consequence. As it turns out, the subalgebra $\boldsymbol{A}^\circ$ of an algebra $\boldsymbol{A} \in \mathsf{Alg}^*(\mathcal{S})$ is a Boolean algebra.

COROLLARY 7.3. *If* $\boldsymbol{A} \in \mathsf{Alg}^*(\mathcal{S})$, *then* $\boldsymbol{A}^\circ \in \mathsf{BA}$.

PROOF. Let $\boldsymbol{A} \in \mathsf{Alg}^*(\mathcal{S})$. Then there exists $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ such that $\boldsymbol{\Omega}^{\boldsymbol{A}}(F) = id_A$. It follows by Theorem 7.2 that $\boldsymbol{\Omega}^{\boldsymbol{A}^\circ}(F \cap A^\circ) = \boldsymbol{\Omega}^{\boldsymbol{A}}(F)|_{A^\circ} = id_A|_{A^\circ} = id_{A^\circ}$. Since $F \cap A^\circ \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}^\circ \subseteq \mathcal{F}i_{\mathcal{C}_1}\boldsymbol{A}^\circ$, it follows by Theorem 6.6 that $\boldsymbol{A}^\circ \cong \boldsymbol{A}^\circ/\boldsymbol{\Omega}^{\boldsymbol{A}^\circ}(F \cap A^\circ) \in \mathsf{BA}$. ⊣

Corollary 7.3 does *not* hold for the logic $\mathcal{C}_1$. Indeed, consider the algebra $\boldsymbol{A}$ whose truth-tables are given as in Table 3. We have $A^\circ = \{0, 4, 5\}$. Since $|A^\circ| = 3$, necessarily $\boldsymbol{A}^\circ \notin \mathsf{BA}$.

The next two consequences of Corollary 7.3 concern the $\mathcal{S}$-filters of the subalgebra $\boldsymbol{A}^\circ$, and in particular its least element. Given a lattice $\langle \boldsymbol{A}, \wedge, \vee\rangle$, let Filt$\boldsymbol{A}$ denote the set of lattice filters of $\boldsymbol{A}$ w.r.t. the order $a \leq^{\boldsymbol{A}} b$ iff $a \wedge^{\boldsymbol{A}} b = a$ iff $a \vee^{\boldsymbol{A}} b = b$, for every $a, b \in A$. Recall that, given $\boldsymbol{A} \in \mathsf{BA}$, we have $\mathcal{F}i_{\mathcal{CL}}\boldsymbol{A} = $ Filt$\boldsymbol{A}$.

COROLLARY 7.4. *For every* $\boldsymbol{A} \in \mathsf{Alg}^*(\mathcal{S})$, $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}^\circ = \mathcal{F}i_{\mathcal{CL}}\boldsymbol{A}^\circ$.

PROOF. Since $\mathcal{S} \leq \mathcal{CL}$, the inclusion $\mathcal{F}i_{\mathcal{CL}}\boldsymbol{A}^\circ \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}^\circ$ is clear. Conversely, given axioms (Ax4) and (Ax5), we have $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}^\circ \subseteq$ Filt$\boldsymbol{A}^\circ$. Furthermore, since $\boldsymbol{A}^\circ \in \mathsf{BA}$ by Corollary 7.3, it is well-known that Filt$\boldsymbol{A}^\circ = \mathcal{F}i_{\mathcal{CL}}\boldsymbol{A}^\circ$. ⊣

COROLLARY 7.5. *If* $\boldsymbol{A} \in \mathsf{Alg}^*(\mathcal{S})$, *then*

$$\bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \cap A^\circ = \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}^\circ = \{1\},$$

*where* $1$ *is the top element of* $\boldsymbol{A}^\circ$.

PROOF. It follows by Corollary 7.4 that $\bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}^\circ = \bigcap \mathcal{F}i_{\mathcal{CL}}\boldsymbol{A}^\circ = \bigcap$ Filt$\boldsymbol{A}^\circ = \{1\}$, where 1 is the top element of $\boldsymbol{A}^\circ$, having in mind

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Corollary 7.3. So, $1 \in \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}^{\circ}$. Next, since $\bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \cap A^{\circ}$ is an $\mathcal{S}$-filter of $\boldsymbol{A}^{\circ}$, we have $\bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}^{\circ} \subseteq \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \cap A^{\circ}$. So, $1 \in \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \cap A^{\circ}$. Finally, let $a \in \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \cap A^{\circ}$. Since both $a, 1 \in \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \cap A^{\circ}$, we have $a \leftrightarrow^{\boldsymbol{A}^{\circ}} 1 \in \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \cap A^{\circ}$. Therefore $\langle a, 1\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}^{\circ}}\big(\bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \cap A^{\circ}\big)$, by Proposition 6.3. But $\boldsymbol{\Omega}^{\boldsymbol{A}^{\circ}}\big(\bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \cap A^{\circ}\big) = \boldsymbol{\Omega}^{\boldsymbol{A}}\big(\bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\big)\lfloor_{A^{\circ}} = id_A\lfloor_{A^{\circ}} = id_{A^{\circ}}$, using Theorem 7.2 and the fact $\boldsymbol{A} \in \mathsf{Alg}^*(\mathcal{S})$. Thus $a = 1$, and therefore $\bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \cap A^{\circ} \subseteq \{1\}$. ⊣

**§8. Subdirectly irreducible algebras relative to $\mathsf{Alg}^*(\mathcal{S})$.** We are now ready to investigate the subdirectly irreducible algebras in $\mathsf{Alg}^*(\mathcal{S})$ relative to $\mathsf{Alg}^*(\mathcal{S})$, and to prove the two main results of Part II (Theorems 8.3 and 8.7), namely:

1. Let $\mathcal{C}_1 \leq \mathcal{S} \leq \mathcal{CL}$ satisfy condition $(\star)$. If $\boldsymbol{A} \in \mathsf{Alg}^*(\mathcal{S})$ is a non-trivial algebra subdirectly irreducible relative to $\mathsf{Alg}^*(\mathcal{S})$, then $\boldsymbol{A}^{\circ} \cong \mathbf{2}$.
2. Let $\mathcal{C}_1 \leq \mathcal{S} \leq \mathcal{CL}$ be finitely equivalential with set of congruence formulas $\boldsymbol{\rho}(x, y) = \{x \leftrightarrow y, \neg x \leftrightarrow \neg y\}$. If $\boldsymbol{A} \in \mathsf{Alg}^*(\mathcal{S})$ is a non-trivial algebra subdirectly irreducible relative to $\mathsf{Alg}^*(\mathcal{S})$, then $|A| \leq 3$.

We start by discarding the case of a trivial (Boolean) algebra in Corollary 7.3. Unless otherwise stated, we continue to assume that the underlying logic $\mathcal{S}$ satisfies condition $(\star)$.

Lemma 8.1. *If $\boldsymbol{A} \in \mathsf{Alg}^*(\mathcal{S})$ is non-trivial, then $\boldsymbol{A}^{\circ}$ is non-trivial.*

Proof. If $A = A^{\circ}$, then the result follows immediately by assumption. Assume therefore that there exists $a \in A - A^{\circ}$. Suppose for the sake of contradiction that $A^{\circ} = \{b\}$, for some $b \in A$. Then, $\bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}^{\circ} = \{b\}$. Fix $F := \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. We first claim that we can assume, without loss of generality, that $a \in F$. Indeed notice that $b^{\circ} \in A^{\circ}$, so necessarily $b^{\circ} = b \in \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}^{\circ} \subseteq F$. Moreover $a \to^{\boldsymbol{A}} b \in F$, by (Ax1) and (MP). It follows by Lemma 3.2.7 that $\neg^{\boldsymbol{A}} b \to^{\boldsymbol{A}} \neg^{\boldsymbol{A}} a \in F$. But, $\neg^{\boldsymbol{A}} b = \neg^{\boldsymbol{A}^{\circ}} b = b$, because $\boldsymbol{A}^{\circ} \leq \boldsymbol{A}$ and $A^{\circ} = \{b\}$, respectively. It follows by (MP) that $\neg^{\boldsymbol{A}} a \in F$. Now, if by chance $\neg^{\boldsymbol{A}} a = b$, then $\neg^{\boldsymbol{A}}\neg^{\boldsymbol{A}} a = \neg^{\boldsymbol{A}} b = \neg^{\boldsymbol{A}^{\circ}} b = b \in F$, and therefore by (Ax10) and (MP) we have $a \in F$. So, either $\neg^{\boldsymbol{A}} a \neq b$ and $\neg^{\boldsymbol{A}} a \in F$, or $\neg^{\boldsymbol{A}} a = b$ and $a \in F$. In any case, there exists $c \in A - A^{\circ}$ such that $c \in F$. Assume therefore that $a \in F$. We next claim that $\langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. On the one hand, since both $a, b \in F$, we have $a \leftrightarrow^{\boldsymbol{A}} b \in F$. On the other hand, notice that $a^{\circ} = b^{\circ} = b \in F$, because $A^{\circ} = \{b\}$. It follows by condition $(\star)$ that $\langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. But $\boldsymbol{\Omega}^{\boldsymbol{A}}(F) = id_A$, because $\boldsymbol{A} \in \mathsf{Alg}^*(\mathcal{S})$. Thus $a = b$, and we reach an absurdity. ⊣

Lemma 8.2. *For every $\boldsymbol{A} \in \mathsf{Alg}^*(\mathcal{S})$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}^{\circ}$, there exists $G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ such that $\boldsymbol{\Omega}^{\boldsymbol{A}}(G)\lfloor_{A^{\circ}} = \boldsymbol{\Omega}^{\boldsymbol{A}^{\circ}}(F)$.*

Proof. Consider the $\mathcal{L}$-matrix $\langle \boldsymbol{A}, \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\rangle$ and its $\mathcal{L}$-submatrix $\big\langle \boldsymbol{A}^{\circ}, \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \cap A^{\circ}\big\rangle$. Since $\bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \cap A^{\circ} = \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}^{\circ}$ by Corollary 7.5, and the class $\mathsf{Mod}(\mathcal{S})$ has the FEP by the remark following Theorem 3.10.3, we conclude that for every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}^{\circ}$ there exists $G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$

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such that $F = G \cap A^\circ$. It now follows by Theorem 7.2 that $\boldsymbol{\Omega}^{\boldsymbol{A}}(G){\restriction_{A^\circ}} = \boldsymbol{\Omega}^{\boldsymbol{A}^\circ}(G \cap A^\circ) = \boldsymbol{\Omega}^{\boldsymbol{A}^\circ}(F)$. ⊣

We are now able to identify, up to isomorphism, the (Boolean) subalgebra $\boldsymbol{A}^\circ$ of any non-trivial algebra $\boldsymbol{A} \in \mathsf{Alg}^*(\mathcal{S})$ subdirectly irreducible relative to $\mathsf{Alg}^*(\mathcal{S})$.

THEOREM 8.3. *If* $\boldsymbol{A} \in \mathsf{Alg}^*(\mathcal{S})$ *is a non-trivial algebra subdirectly irreducible relative to* $\mathsf{Alg}^*(\mathcal{S})$, *then* $\boldsymbol{A}^\circ \cong \mathbf{2}$.

PROOF. Let $\boldsymbol{A} \in \mathsf{Alg}^*(\mathcal{S})$ be a non-trivial algebra subdirectly irreducible relative to $\mathsf{Alg}^*(\mathcal{S})$. Fix $\theta := \min \mathrm{Co}_{\mathsf{Alg}^*(\mathcal{S})}\boldsymbol{A} - \{id_A\}$ and $\vartheta := \theta{\restriction_{A^\circ}}$. We claim that $\vartheta = \min \mathrm{Co}_{\mathsf{Alg}^*(\mathcal{S})}\boldsymbol{A}^\circ - \{id_{A^\circ}\}$. First, since $\theta \in \mathrm{Co}_{\mathsf{Alg}^*(\mathcal{S})}\boldsymbol{A}$, there exists $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ such that $\theta = \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. So, $\vartheta = \boldsymbol{\Omega}^{\boldsymbol{A}}(F){\restriction_{A^\circ}} = \boldsymbol{\Omega}^{\boldsymbol{A}^\circ}(F \cap A^\circ) \in \mathrm{Co}_{\mathsf{Alg}^*(\mathcal{S})}\boldsymbol{A}^\circ$, using Theorem 7.2. Next, let $\alpha \in \mathrm{Co}_{\mathsf{Alg}^*(\mathcal{S})}\boldsymbol{A}^\circ - \{id_{A^\circ}\}$. On the one hand, there exists $G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}^\circ$ such that $\alpha = \boldsymbol{\Omega}^{\boldsymbol{A}^\circ}(G)$. On the other hand, by Lemma 8.2 there exists $H \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ such that $\alpha = \boldsymbol{\Omega}^{\boldsymbol{A}}(H){\restriction_{A^\circ}}$. Since $\boldsymbol{\Omega}^{\boldsymbol{A}}(H) \in \mathrm{Co}_{\mathsf{Alg}^*(\mathcal{S})}\boldsymbol{A}$ and $\boldsymbol{\Omega}^{\boldsymbol{A}}(H) \neq id_A$ (otherwise $\alpha = id_{A^\circ}$), we must have $\theta \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(H)$. Hence $\vartheta = \theta{\restriction_{A^\circ}} \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(H){\restriction_{A^\circ}} = \alpha$. Thus, $\vartheta = \min \mathrm{Co}_{\mathsf{Alg}^*(\mathcal{S})}\boldsymbol{A}^\circ - \{id_{A^\circ}\}$, as claimed.

We now claim that $\boldsymbol{A}^\circ$ is subdirectly irreducible. We know that $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}^\circ = \mathcal{F}i_{\mathcal{CL}}\boldsymbol{A}^\circ$, by Corollary 7.4. Therefore, $\vartheta = \boldsymbol{\Omega}^{\boldsymbol{A}^\circ}(F \cap A^\circ) \in \mathrm{Co}_{\mathsf{BA}}\boldsymbol{A}^\circ$. Moreover, since $\mathsf{BA} \subseteq \mathsf{Alg}^*(\mathcal{S})$, it also holds $\vartheta = \min \mathrm{Co}_{\mathsf{BA}}\boldsymbol{A}^\circ - \{id_{A^\circ}\}$. Finally, since $\mathsf{BA}$ is closed under $\mathbb{H}$, $\mathrm{Co}_{\mathsf{BA}}\boldsymbol{A}^\circ = \mathrm{Co}\,\boldsymbol{A}^\circ$, and hence $\vartheta = \min \mathrm{Co}\,\boldsymbol{A}^\circ - \{id_{A^\circ}\}$. We conclude that $\boldsymbol{A}^\circ$ is subdirectly irreducible—see [10, Theorem 8.4]. Since $\boldsymbol{A}^\circ \in \mathsf{BA}$ is non-trivial by Lemma 8.1, it can only be the case $\boldsymbol{A}^\circ \cong \mathbf{2}$. ⊣

Having determined the cardinality of the subuniverse $A^\circ$, given $\boldsymbol{A} \in \mathsf{Alg}^*(\mathcal{S})$ non-trivial subdirectly irreducible relative to $\mathsf{Alg}^*(\mathcal{S})$, we now wish to determine the cardinality of the universe $A$ itself. In fact, we want to prove that $|A| \leq 3$. For this purpose we will need to strengthen our assumption over $\mathcal{S}$. The next auxiliary results however still holds under our current assumption, that is, $\mathcal{S}$ satisfying ($\star$). In light of Corollary 7.3, let 1 denote the top element of $\boldsymbol{A}^\circ$ and let $0 = \neg 1$.

LEMMA 8.4. *Let* $\boldsymbol{A} \in \mathsf{Alg}^*(\mathcal{S})$ *be a non-trivial subdirectly irreducible algebra relative to* $\mathsf{Alg}^*(\mathcal{S})$. *For every* $a \in A^\circ$, $a^\circ = 1$; *for every* $a \in A - A^\circ$, $a^\circ = 0$.

PROOF. First, it should be clear that for every $a \in A^\circ$, we have $\neg^{\boldsymbol{A}}(a \wedge^{\boldsymbol{A}} \neg^{\boldsymbol{A}} a) = \neg^{\boldsymbol{A}^\circ}(a \wedge^{\boldsymbol{A}^\circ} \neg^{\boldsymbol{A}^\circ} a) = \neg^{\boldsymbol{A}^\circ} 0 = 1$, because $\boldsymbol{A}^\circ \leq \boldsymbol{A}$ and $\boldsymbol{A}^\circ \in \mathsf{BA}$. Next, let $a \in A - A^\circ$. We know by Theorem 8.3 that $\boldsymbol{A}^\circ \cong \mathbf{2}$. But $a^\circ \in A^\circ = \{0, 1\}$, using Lemma 6.2. If $a^\circ = 1 \in \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ (recall Corollary 7.5), then $a \in A^\circ$ and we reach an absurdity. Necessarily, $a^\circ = 0$. ⊣

COROLLARY 8.5. *Let* $\boldsymbol{A} \in \mathsf{Alg}^*(\mathcal{S})$ *be a non-trivial subdirectly irreducible algebra relative to* $\mathsf{Alg}^*(\mathcal{S})$.

1. $\bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} = A - \{0\}$.
2. $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A} = \{\bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}, A\}$.
3. $\mathrm{Co}_{\mathsf{Alg}^*(\mathcal{S})}\boldsymbol{A} = \{id_A, A \times A\}$.

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PROOF. 1. Let $a \in A - A^\circ$. Then $a^\circ = 0$, by Lemma 8.4. So, $\neg a^\circ = 1 \in \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Since $\neg\varphi^\circ = \neg\neg(\varphi \wedge \neg\varphi)$, it follows by (Ax10) that $a \wedge \neg a \in \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, and then by (Ax3) that $a \in \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Thus, $A - A^\circ \subseteq \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Since $A^\circ = \{0, 1\}$ and $1 \in \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, it can only be the case $\bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} = A - \{0\}$. Indeed, if $0 \in \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, then $A = \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, so $id_A = \boldsymbol{\Omega}^{\boldsymbol{A}}\big(\bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\big) = \boldsymbol{\Omega}^{\boldsymbol{A}}(A) = A \times A$, using the fact $\boldsymbol{A} \in \mathsf{Alg}^*(\mathcal{S})$, and it would follow that $A$ is trivial.

2. It follows immediately by 1.

3. Every $\theta \in \mathrm{Co}_{\mathsf{Alg}^*(\mathcal{S})}\boldsymbol{A}$ is of the form $\boldsymbol{\Omega}^{\boldsymbol{A}}(F)$ with $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, so the result follows by 2, having in mind that $\boldsymbol{\Omega}^{\boldsymbol{A}}\big(\bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\big) = id_A$ and $\boldsymbol{\Omega}^{\boldsymbol{A}}(A) = A \times A$. ⊣

For the last result of the section let $\mathcal{S}$ be a finitary and finitely equivalential extension of $\mathcal{C}_1$, extended itself by $\mathcal{CL}$, with (finite) set of congruence formulas $\boldsymbol{\rho}(x, y) = \{x \leftrightarrow y, \neg x \leftrightarrow \neg y\}$. Let us first see that our new assumption is stronger than condition $(\star)$.

LEMMA 8.6. *If $\mathcal{S}$ is finitely equivalential with a set of congruence formulas $\boldsymbol{\rho}(x, y) = \{x \leftrightarrow y, \neg x \leftrightarrow \neg y\}$, then $\mathcal{S}$ satisfies $(\star)$.*

PROOF. Our hypothesis tells us that for every algebra $\boldsymbol{A}$, $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and $a, b \in A$,

$$\langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}(F) \quad \Leftrightarrow \quad a \leftrightarrow^{\boldsymbol{A}} b \in F \text{ and } \neg a \leftrightarrow^{\boldsymbol{A}} \neg b \in F.$$

But this clearly implies $(\star\star)$—which is the particular case where $a, b \in A^\circ$—and we have seen it already to be equivalent to $(\star)$. ⊣

THEOREM 8.7. *If $\boldsymbol{A} \in \mathsf{Alg}^*(\mathcal{S})$ is a non-trivial algebra subdirectly irreducible relative to $\mathsf{Alg}^*(\mathcal{S})$, then $|A| \leq 3$.*

PROOF. Let $\boldsymbol{A} \in \mathsf{Alg}^*(\mathcal{S})$ be a non-trivial algebra subdirectly irreducible relative to $\mathsf{Alg}^*(\mathcal{S})$. Then $|A^\circ| = 2$, by Theorem 8.3. Assume for the sake of contradiction that there exist $a, b \in A - A^\circ$, with $a \neq b$. Then $a^\circ = b^\circ = 0$ by Lemma 8.4, and therefore $\neg a^\circ = \neg b^\circ = 1 \in \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Since $\neg\varphi^\circ = \neg\neg(\varphi \wedge \neg\varphi)$, it follows by (Ax10) that $a \wedge \neg a \in \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and $b \wedge \neg b \in \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, and then by (Ax3) and (Ax4) that $a, \neg a, b, \neg b \in \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Clearly then $a \leftrightarrow b \in \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and $\neg a \leftrightarrow \neg b \in \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Since by hypothesis $\mathcal{S}$ is finitely equivalential with set of congruence formulas $\boldsymbol{\rho}(x, y) = \{x \leftrightarrow y, \neg x \leftrightarrow \neg y\}$, it follows that $\langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}\big(\bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\big) = id_A$, using the fact $\boldsymbol{A} \in \mathsf{Alg}^*(\mathcal{S})$. We reach a contradiction. Thus $|A - A^\circ| \leq 1$, and hence $|A| \leq 3$. ⊣

A careful look at the proof of Theorem 8.7 show us why condition $(\star)$ does not suffice. We need to extend $(\star)$ to elements $a, b \in A - A^\circ$. That is: for every $\boldsymbol{A}$, $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and $a, b \in A - A^\circ$,

$$\langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}(F) \quad \Leftrightarrow \quad a \leftrightarrow^{\boldsymbol{A}} b \in F \text{ and } \neg a \leftrightarrow^{\boldsymbol{A}} \neg b \in F.$$

An important consequence of our stronger assumption on $\mathcal{S}$ (that is, $\mathcal{S}$ is finitary and finitely equivalential) is that the class of algebraic reducts of $\mathcal{S}$ is a quasivariety—see [32, Corollary 6.80]. Quasivarieties are fully

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determined by their relative subdirectly irreducible algebras. In Part III, we will characterize the quasivariety $\mathsf{Alg}^*(\mathcal{S})$ in terms of the subdirectly irreducible algebras in $\mathsf{Alg}^*(\mathcal{S})$ relative to $\mathsf{Alg}^*(\mathcal{S})$, for the equivalential extensions of $\mathcal{C}_1$ considered in the literature $\mathcal{S} = \mathbf{P^1}, \mathbf{P^2}, \mathbf{P^3}$. In light of Theorem 8.7, we will do this by determining the truth-tables of the three-element algebras in $\mathsf{Alg}^*(\mathcal{S})$ subdirectly irreducible relative to $\mathsf{Alg}^*(\mathcal{S})$.

**Part III. Axiomatic extensions of $\mathcal{C}_1$.**

In this last part we extend our AAL study of $\mathcal{C}_1$ to some of its paraconsistent extensions considered in the literature. To settle the scope of the logics under study, we shall focus ourselves on the so-called $d\mathcal{C}$-systems, according to the terminology of [16, Section 3.8] or [17, Section 5.1]. In particular, we shall be mainly interested in the logics $\mathcal{C}ilo$, $\mathcal{C}ilow$, $\mathcal{C}ibv$, $\mathbf{P^1}$, $\mathbf{P^2}$, and $\mathbf{P^3}$—recall Figure 1 for the inclusion relations among these logics.

In the sequel we will only classify sentential logics within the Leibniz hierarchy, leaving aside the Frege hierarchy. The reason for this omission is that every extension of $\mathcal{C}_1$ (weaker than $\mathcal{CL}$) is not self-extensional—see [16, Corollary 3.65]—and therefore falls outside the Frege hierarchy.

**§9. The logic $\mathcal{C}ilo$.** The goals of this section are to introduce the logic $\mathcal{C}ilo$, prove that it satisfies condition ($\star$) (Corollary 9.2), and classify it within the Leibniz hierarchy (Propositions 9.6 and 9.9). We finish with a brief discussion about the class $\mathsf{Alg}^*(\mathcal{C}ilo)$ in order to clarify a couple of incorrect claims in the literature.

We first consider an extension of $\mathcal{C}_1$ introduced by da Costa, Béziau, and Bueno in [24]. We follow the presentation of [16], as well as the notation adopted there—the reason for choosing the notation $\mathcal{C}ilo$ of [16] instead of the original notation $\mathcal{C}_1^+$ of [24] is so that it does not collide with the notation of the strong version of $\mathcal{C}_1$ [1]. Let us define the logic $\mathcal{C}ilo$ as the logic axiomatized by (Ax1)–(Ax11) plus (MP), together with the following axioms:

$$\vdash (\varphi^\circ \vee \psi^\circ) \to (\varphi \wedge \psi)^\circ. \tag{Ax12a}$$

$$\vdash (\varphi^\circ \vee \psi^\circ) \to (\varphi \vee \psi)^\circ. \tag{Ax13a}$$

$$\vdash (\varphi^\circ \vee \psi^\circ) \to (\varphi \to \psi)^\circ. \tag{Ax14a}$$

Clearly the logic $\mathcal{C}ilo$ is an extension of $\mathcal{C}_1$, that is $\mathcal{C}_1 \leq \mathcal{C}ilo$, and therefore Lemma 3.2 still holds for the consequence relation $\vdash_{\mathcal{C}ilo}$. In particular,

$$\varphi^\circ \vdash_{\mathcal{C}ilo} (\neg\varphi)^\circ.$$

Although the axiomatization of $\mathcal{C}ilo$ resembles that of $\mathcal{C}_1$, the logic $\mathcal{C}ilo$ is stronger enough to obtain better results than those seen for $\mathcal{C}_1$, at least from an AAL point of view. By "stronger enough," we mean $\mathcal{C}ilo$ satisfies ($\star$).

To begin with, let us stress that $\mathcal{F}i_{\mathcal{C}ilo}\boldsymbol{A} \subseteq \mathcal{F}i_{\mathcal{C}_1}\boldsymbol{A}$, for an arbitrary algebra $\boldsymbol{A}$, and therefore the results seen so far for $\mathcal{C}_1$-filters are also applicable to $\mathcal{C}ilo$-filters. In particular, and having in mind that sets of parameterized

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congruence formulas are preserved by extensions, for every algebra $\boldsymbol{A}$, $F \in \mathcal{F}i_{\mathcal{C}ilo}\boldsymbol{A}$, and $a, b \in A$,

$$\begin{aligned} \langle a,b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}(F) \quad \Leftrightarrow \quad & \forall n \in \omega\ \forall m \in \omega\ \forall c_1,\dots,c_n \in A, \\ & \Phi_n^{\boldsymbol{A}}(\neg^m a, \neg^m b, c_1,\dots,c_n) \subseteq F, \\ & \Psi_n^{\boldsymbol{A}}(\neg^m a, \neg^m b, c_1,\dots,c_n) \subseteq F. \end{aligned}$$

We now prove that $\mathcal{C}ilo$ satisfies ($\star$). The proof makes use of the (rather complex) set of parameterized congruence formulas given in Theorem 4.3. It is interesting to observe where the axioms (Ax12a)–(Ax14a) play its role, and why the axioms (Ax12)–(Ax14) do not suffice for our purposes.

THEOREM 9.1. *Let $\boldsymbol{A}$ be arbitrary, $F \in \mathcal{F}i_{\mathcal{C}ilo}\boldsymbol{A}$, and $a, b \in A$. If $a^\circ, b^\circ \in F$, then*

$$\langle a,b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}(F) \quad \Leftrightarrow \quad a \leftrightarrow b \in F.$$

PROOF. If $\langle a,b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$, then $a \leftrightarrow b \in F$ by Proposition 4.1. Conversely, assume $a \leftrightarrow b \in F$. We claim that for every $c_1,\dots,c_n \in A$,

$$\bigcup_{n\in\omega}\bigcup_{m\in\omega} \Phi_n^{\boldsymbol{A}}(\neg^m a, \neg^m b, c_1,\dots,c_n) \subseteq F.$$

The proof goes by induction on $n \in \omega$.

**Basis:** We must prove that for every $m \in \omega$, $\neg^m a \leftrightarrow \neg^m b \in F$. The proof goes by induction on $m \in \omega$. Assume first $m = 0$. It follows by assumption that $a \leftrightarrow b \in F$. Assume now that $m > 0$. On the one hand, we have $\neg^m a \leftrightarrow \neg^m b \in F$, by inductive hypothesis. On the other hand, since $a^\circ, b^\circ \in F$, it follows by $m$ applications of Lemma 3.2.19 that $(\neg^m a)^\circ, (\neg^m b)^\circ \in F$. So it follows by Lemma 3.2.8 that $\neg^{m+1} a \leftrightarrow \neg^{m+1} b \in F$. Thus, $\bigcup_{m\in\omega} \Phi_0(\neg^m a, \neg^m b) \subseteq F$.

**Step:** Let $m, m_1, \dots, m_{n+1} \in \omega$, $c_1, \dots, c_{n+1} \in A$, and $*_1, \dots, *_{n+1} \in \{\wedge, \vee, \to\}$.

- $\Phi_{n+1}(\neg^m a, \neg^m b, c_1, c_2, \dots, c_{n+1})$ with $m_{n+1} = 0$: It follows by inductive hypothesis that $\Phi_n(\neg^m a, \neg^m b, c_1, \dots, c_n) \subseteq F$. Therefore,

$$\begin{aligned} &\neg^{m_n}(\cdots \neg^{m_2}(\neg^{m_1}(\neg^m a *_1 c_1) *_2 c_2) \cdots *_n c_n) \leftrightarrow \\ &\neg^{m_n}(\cdots \neg^{m_2}(\neg^{m_1}(\neg^m b *_1 c_1) *_2 c_2) \cdots *_n c_n) \in F. \end{aligned}$$

  Since moreover $c_{n+1} \leftrightarrow c_{n+1} \in F$ by Lemma 3.2.1, it follows by Lemma 3.2.4–6 that

$$\begin{aligned} &\neg^{m_n}(\cdots \neg^{m_2}(\neg^{m_1}(\neg^m a *_1 c_1) *_2 c_2) \cdots *_n c_n) *_{n+1} c_{n+1} \leftrightarrow \\ &\neg^{m_n}(\cdots \neg^{m_2}(\neg^{m_1}(\neg^m b *_1 c_1) *_2 c_2) \cdots *_n c_n) *_{n+1} c_{n+1} \in F. \end{aligned}$$

  Thus, $\Phi_{n+1}(\neg^m a, \neg^m b, c_1, c_2, \dots, c_{n+1}) \subseteq F$.
- $\Phi_{n+1}(\neg^m a, \neg^m b, c_1, c_2, \dots, c_{n+1})$ with $m_{n+1} > 0$: Since $a^\circ \in F$ by hypothesis, it follows by $m$ applications of Lemma 3.2.19 that $(\neg^m a)^\circ \in F$. It then follows by (Ax12a)–(Ax14a) that $(\neg^m a *_1 c_1)^\circ \in F$; then by $m_1$ applications of Lemma 3.2.19 that $\big(\neg^{m_1}(\neg^m a *_1 c_1)\big)^\circ \in F$; and

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again by (Ax12a)–(Ax14a) that $\big(\neg^{m_1}(\neg^m a *_1 c_1) *_2 c_2\big)^\circ \in F$; and so on, until we obtain

$$\big(\neg^{m_n}\big(\cdots\neg^{m_2}\big(\neg^{m_1}(\neg^m a *_1 c_1) *_2 c_2\big)\cdots *_n c_n\big) *_{n+1} c_{n+1}\big)^\circ \in F. \tag{2}$$

Similarly,

$$\big(\neg^{m_n}\big(\cdots\neg^{m_2}\big(\neg^{m_1}(\neg^m b *_1 c_1) *_2 c_2\big)\cdots *_n c_n\big) *_{n+1} c_{n+1}\big)^\circ \in F. \tag{3}$$

Moreover, it follows by inductive hypothesis that

$$\neg^{m_n}\big(\cdots\neg^{m_2}\big(\neg^{m_1}(\neg^m a *_1 c_1) *_2 c_2\big)\cdots *_n c_n\big) \leftrightarrow \\ \neg^{m_n}\big(\cdots\neg^{m_2}\big(\neg^{m_1}(\neg^m b *_1 c_1) *_2 c_2\big)\cdots *_n c_n\big) \in F.$$

So by Lemma 3.2.4–6,

$$\neg^{m_n}\big(\cdots\neg^{m_2}\big(\neg^{m_1}(\neg^m a *_1 c_1) *_2 c_2\big)\cdots *_n c_n\big) *_{n+1} c_{n+1} \leftrightarrow \\ \neg^{m_n}\big(\cdots\neg^{m_2}\big(\neg^{m_1}(\neg^m b *_1 c_1) *_2 c_2\big)\cdots *_n c_n\big) *_{n+1} c_{n+1} \in F. \tag{4}$$

It finally follows by (2), (3), (4) and Lemma 3.2.8 that

$$\neg\big(\neg^{m_n}\big(\cdots\neg^{m_2}\big(\neg^{m_1}(\neg^m a *_1 c_1) *_2 c_2\big)\cdots *_n c_n\big) *_{n+1} c_{n+1}\big) \leftrightarrow \\ \neg\big(\neg^{m_n}\big(\cdots\neg^{m_2}\big(\neg^{m_1}(\neg^m b *_1 c_1) *_2 c_2\big)\cdots *_n c_n\big) *_{n+1} c_{n+1}\big) \in F. \tag{5}$$

Having arrived here, by (2) and Lemma 3.2.19 we have

$$\Big(\neg\big(\neg^{m_n}\big(\cdots\neg^{m_2}\big(\neg^{m_1}(\neg^m a *_1 c_1) *_2 c_2\big)\cdots *_n c_n\big) *_{n+1} c_{n+1}\big)\Big)^\circ \in F. \tag{6}$$

Similarly, by (3) and Lemma 3.2.19 we have

$$\Big(\neg\big(\neg^{m_n}\big(\cdots\neg^{m_2}\big(\neg^{m_1}(\neg^m b *_1 c_1) *_2 c_2\big)\cdots *_n c_n\big) *_{n+1} c_{n+1}\big)\Big)^\circ \in F. \tag{7}$$

This time it follows by (5)–(7) and Lemma 3.2.8 that

$$\neg^2\big(\neg^{m_n}\big(\cdots\big(\neg^{m_2}\big(\neg^{m_1}(\neg^m a *_1 c_1) *_2 c_2\big)\cdots\big) *_n c_n\big) *_{n+1} c_{n+1}\big) \leftrightarrow \\ \neg^2\big(\neg^{m_n}\big(\cdots\big(\neg^{m_2}\big(\neg^{m_1}(\neg^m b *_1 c_1) *_2 c_2\big)\cdots\big) *_n c_n\big) *_{n+1} c_{n+1}\big) \in F.$$

By repeating this process $m_{n+1}$ times, we eventually obtain

$$\neg^{m_{n+1}}\big(\neg^{m_n}\big(\cdots\neg^{m_2}\big(\neg^{m_1}(\neg^m a *_1 c_1) *_2 c_2\big)\cdots *_n c_n\big) *_{n+1} c_{n+1}\big) \leftrightarrow \\ \neg^{m_{n+1}}\big(\neg^{m_n}\big(\cdots\neg^{m_2}\big(\neg^{m_1}(\neg^m b *_1 c_1) *_2 c_2\big)\cdots *_n c_n\big) *_{n+1} c_{n+1}\big) \in F.$$

Thus, $\Phi_{n+1}(\neg^m a, \neg^m b, c_1, c_2, \dots, c_{n+1}) \subseteq F$.

We conclude that for every $c_1, \dots, c_n \in A$,

$$\bigcup_{n\in\omega}\bigcup_{m\in\omega}\Phi_n^{\boldsymbol{A}}(\neg^m a, \neg^m b, c_1, \dots, c_n) \subseteq F.$$

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Similarly, one proves that for every $c_1, \dots, c_n \in A$,

$$\bigcup_{n\in\omega}\bigcup_{m\in\omega} \Psi_n^{\boldsymbol{A}}(\neg^m a, \neg^m b, c_1, \dots, c_n) \subseteq F.$$

Since $F \in \mathcal{F}i_{\mathcal{C}ilo}\boldsymbol{A} \subseteq \mathcal{F}i_{\mathfrak{C}_1}\boldsymbol{A}$, it follows by Theorem 4.3 that $\langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. ⊣

Corollary 9.2. *The logic $\mathcal{C}ilo$ satisfies* ($\star$).

Proof. Let $\boldsymbol{A}$ be arbitrary, $F \in \mathcal{F}i_{\mathcal{C}ilo}\boldsymbol{A}$, and $a, b \in A^\circ$. Then $a^\circ, b^\circ \in \bigcap \mathcal{F}i_{\mathcal{C}ilo}\boldsymbol{A} \subseteq F$. It follows by Theorem 9.1 that

$$\langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}(F) \quad\Leftrightarrow\quad a \leftrightarrow b \in F.$$

⊣

As a consequence of Corollary 9.2, all the results seen in Section 7 hold for the logic $\mathcal{C}ilo$.

A particular case of Theorem 9.1 arises when dealing with $\sim a, \sim b \in A$. In order to see it, we first prove an auxiliary lemma.

Lemma 9.3. $\vdash_{\mathcal{C}ilo} (\sim\varphi)^\circ$.

Proof. Since $\sim\varphi = \neg\varphi \wedge \varphi^\circ$ and $\vdash_{\mathcal{C}ilo} (\varphi^\circ)^\circ$ by Lemma 3.2.20, it follows by (Ax12a) that $\vdash_{\mathcal{C}ilo} (\neg\varphi \wedge \varphi^\circ)^\circ$, that is $\vdash_{\mathcal{C}ilo} (\sim\varphi)^\circ$. ⊣

Corollary 9.4. *For every $\boldsymbol{A}$, $F \in \mathcal{F}i_{\mathcal{C}ilo}\boldsymbol{A}$, and $a, b \in A$,*

$$\langle \sim a, \sim b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}(F) \quad\Leftrightarrow\quad \sim a \leftrightarrow \sim b \in F.$$

**9.1. Classification of $\mathcal{C}ilo$ within the Leibniz hierarchy.** We now address the classification of $\mathcal{C}ilo$ within the Leibniz hierarchy and relate $\mathcal{C}ilo$ with the strong version of $\mathfrak{C}_1$.

Since protoalgebraicity is preserved by extensions, the next result should come with no surprise.

Proposition 9.5. *$\mathcal{C}ilo$ is protoalgebraic.*

Proposition 9.5 prompts the question of whether $\mathcal{C}ilo$ is equivalential. The problem this time is harder to grasp than it was for the logic $\mathfrak{C}_1$, for we lack a counterexample of a reduced $\mathcal{C}ilo$-model with a non-reduced $\mathcal{C}ilo$-submodel. The strategy to prove that $\mathcal{C}ilo$ is not equivalential will make use of another extension, $\mathcal{C}ilow$, yet to appear, and hence we will have to wait until Proposition 10.3 in order to see in full detail the proof. Nevertheless, we state the next result here, where it belongs naturally.

Proposition 9.6. *$\mathcal{C}ilo$ is not equivalential.*

Proposition 9.6 corrects a claim on [42, p. 183] stating: "The logic $\mathcal{C}ilo$, for instance, is (finitely) equivalential."

We are left to address the truth-equationality of $\mathcal{C}ilo$. Unlike $\mathfrak{C}_1$, the logic $\mathcal{C}ilo$ has its filters equationally definable, hence being the first truth-equational logic we have come across in our study.

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Lemma 9.7. $\xi, (\xi \wedge \varphi) \leftrightarrow (\xi \wedge \psi) \vdash_{\mathcal{C}_1} \varphi \leftrightarrow \psi$.

Proposition 9.8. *Cilo is truth-equational, witnessed by the set of truth equations* $\boldsymbol{\tau}(x) = \{(x^\circ)^\circ \wedge x \approx (x^\circ)^\circ \wedge (x \to x)\}$.

Proof. Let $\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}(\mathcal{C}ilo)$. Assume first $a \in F$. Since both $a \in F$ and $a \to a \in F$ by assumption and Lemma 3.2.11, respectively, it follows by manipulation of (Ax1) and (MP) that $a \leftrightarrow (a \to a) \in F$. It then follows by Lemma 3.2.16 that $\big((a^\circ)^\circ \wedge a\big) \leftrightarrow \big((a^\circ)^\circ \wedge (a \to a)\big) \in F$. Moreover, since $\big((a^\circ)^\circ\big)^\circ \in F$ by Lemma 3.2.20 (taking $\varphi^\circ$ instead of $\varphi$), it follows by (Ax12a) that $\big((a^\circ)^\circ \wedge a\big)^\circ \in F$ and $\big((a^\circ)^\circ \wedge (a \to a)\big)^\circ \in F$. It finally follows by Theorem 9.1 that $\boldsymbol{\tau}^{\boldsymbol{A}}(a) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. Conversely, assume $\boldsymbol{\tau}^{\boldsymbol{A}}(a) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. Then, $\big((a^\circ)^\circ \wedge a\big) \leftrightarrow \big((a^\circ)^\circ \wedge (a \to a)\big) \in F$ by Proposition 4.1. Since $(a^\circ)^\circ \in F$ by Lemma 3.2.20, it follows by Lemma 9.7 that $a \leftrightarrow (a \to a) \in F$. Next, since $a \to a \in F$ by Lemma 3.2.11, it follows by (MP) that $a \in F$. We conclude that $F = \{a \in A : \boldsymbol{\tau}^{\boldsymbol{A}}(a) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)\}$, which proves the result. ⊣

Bearing in mind Definition 2.2, and putting together Propositions 9.5 and 9.8, it follows at once that:

Proposition 9.9. *Cilo is weakly algebraizable.*

In light of Proposition 9.6, the logic $\mathcal{C}ilo$ is *not* algebraizable. There are few examples of weakly algebraizable, but not algebraizable, logics in the literature. Namely, the logic of ortholattices $\mathcal{S}_{\mathrm{OL}}$, the logic of Andréka and Nemeti $\mathcal{S}_{\mathrm{AN}}$, and some assertional logics of protoregular classes—the latter being in fact *regularly* weakly algebraizable. All these examples, as well as the corresponding proofs of their weak algebraizability, can be found in [29, Section 6].

**9.2. The class $\mathsf{Alg}^*(\mathcal{C}ilo)$.** The class $\mathsf{Alg}^*(\mathcal{C}ilo)$ enjoys much nicer properties than $\mathsf{Alg}^*(\mathcal{C}_1)$. Perhaps most notably, Mortensen's result (recall Proposition 5.2) is no longer valid. We next prove this and other results concerning the class $\mathsf{Alg}^*(\mathcal{C}ilo)$.

A consequence of Proposition 9.8 is that $\mathsf{Alg}^*(\mathcal{C}ilo)$ is an algebraic semantics for the logic $\mathcal{C}ilo$. Indeed, every truth-equational logic $\mathcal{S}$ admits $\mathsf{Alg}(\mathcal{S})$ as an algebraic semantics—see [51, Corollary 26]. This contrasts with the situation seen for the logic $\mathcal{C}_1$—recall Theorem 5.6.

Since $\mathcal{C}ilo$ satisfies ($\star$) by Theorem 9.1, it follows by Corollary 7.3 that $\boldsymbol{A}^\circ \in \mathsf{BA}$ whenever $\boldsymbol{A} \in \mathsf{Alg}^*(\mathcal{C}ilo)$.

Proposition 9.10. *If $\boldsymbol{A} \in \mathsf{Alg}^*(\mathcal{C}ilo)$ then $\boldsymbol{A}^\circ \in \mathsf{BA}$.*

The converse of Proposition 9.10 is false. Consider once again the algebra $\boldsymbol{A}$ depicted in Figure 2. We have $\boldsymbol{A}^\circ \cong \mathbf{2} \in \mathsf{BA}$, but $\boldsymbol{A} \notin \mathsf{Alg}^*(\mathcal{C}ilo)$—this last fact is easily checked by noticing that no non-trivial $\mathcal{C}_1$-filter in Table 2 is closed under axioms (Ax12a)–(Ax14a).

As already claimed, unlike the logic $\mathcal{C}_1$, Mortensen's result no longer holds for the logic $\mathcal{C}ilo$.

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PROPOSITION 9.11. $\boldsymbol{Fm} \notin \mathsf{Alg}^*(\mathcal{C}ilo)$.

PROOF. Suppose for the sake of contradiction that $\boldsymbol{Fm} \in \mathsf{Alg}^*(\mathcal{C}ilo)$. Since $\mathcal{C}ilo$ is a non-trivial logic with theorems, it follows by the observation after Theorem 5.6 that $\mathcal{C}ilo$ admits no algebraic semantics, and we reach a contradiction. ⊣

As a consequence of Proposition 9.11, there exists a non-trivial congruence on $\boldsymbol{Fm}$ compatible with the $\mathcal{C}ilo$-theorems—just consider $\boldsymbol{\Omega}^{\boldsymbol{Fm}}(\mathrm{Thm}_{\mathcal{C}ilo}) \neq id_{\boldsymbol{Fm}}$. The existence of such a congruence has already been stated in the literature—originally in [24, Theorem 3.21], but also in [16, Fact 3.81] and [9, p. 3]. However, from the existence of a non-trivial congruence on $\boldsymbol{Fm}$ (compatible with $\mathrm{Thm}_{\mathcal{C}ilo}$) it does not follow that $\mathcal{C}ilo$ is algebraizable nor even equivalential, as mentioned in [16, p. 79] and [42, p. 118].

**§10. The logic $\mathcal{C}ilow$.** The goals of this section are to introduce the logic $\mathcal{C}ilow$, classify it within the Leibniz hierarchy, and use its classification (namely, the fact that it is not equivalential) to establish that $\mathcal{C}ilo$ is not equivalential (although formally stated in Proposition 9.6, it is an immediate consequence of Proposition 10.3).

Consider the axiom:

$$\vdash (\neg\varphi)^{\circ}. \tag{Ax15}$$

Let us call the logic $\mathcal{C}_1 + (\mathrm{Ax15})$ by $\mathcal{C}ilaw$[10] and the logic $\mathcal{C}ilo + (\mathrm{Ax15})$ by $\mathcal{C}ilow$.[11] Unlike the logic $\mathcal{C}ilo$, the extensions $\mathcal{C}ilaw$ and $\mathcal{C}ilow$ have not been considered in the literature. Our main purpose in considering them is to help us establish that $\mathcal{C}ilo$ is not equivalential.

Putting together Lemma 3.2.8 and (Ax15), we obtain:

LEMMA 10.1. $\neg\varphi \leftrightarrow \neg\psi \vdash_{\mathcal{C}ilaw} \neg\neg\varphi \leftrightarrow \neg\neg\psi$.

The classification of the logic $\mathcal{C}ilaw$ within the Leibniz hierarchy coincides with that of $\mathcal{C}_1$. In fact, the exact same counterexample used in the proofs of Propositions 3.6 and 3.7 (that is, the algebra $\boldsymbol{A}$ in Figure 2) can also be used for the logic $\mathcal{C}ilaw$. We need only to check that the subsets $\{1, u\}$ and $\{a, 1, u\}$ are $\mathcal{C}ilaw$-filters of $\boldsymbol{A}$. But this is easy, because $(\neg 0)^{\circ} = (\neg a)^{\circ} = (\neg b)^{\circ} = (\neg 1)^{\circ} = (\neg u)^{\circ} = 1$. Therefore both $\{1, u\}$ and $\{a, 1, u\}$ satisfy (Ax15). Therefore, $\mathcal{C}ilaw$ is neither equivalential nor truth-equational.

As for the logic $\mathcal{C}ilow$, since it extends $\mathcal{C}ilo$ it follows at once by Proposition 9.9 that it is weakly algebraizable.

We now move on to the main result of this section which is the non-equivalentiality of $\mathcal{C}ilow$. The next auxiliary lemma makes use of Proposition 2.7 seen in the Preliminaries.

[10] Recall that the logic $\mathcal{C}_1$ is denoted by $\mathcal{C}ila$ in [16].

[11] Following the terminology suggested in [16, p. 68]: "and add **w** do the name of a logic containing (cw)," where (cw) is precisely (Ax15).

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LEMMA 10.2. *If* $\mathcal{C}ilow$ *were equivalential, then* $\boldsymbol{\rho}(x, y) := \{x \leftrightarrow y, \neg x \leftrightarrow \neg y\}$ *would be a set of congruence formulas for* $\mathcal{C}ilow$.

PROOF. Suppose that $\mathcal{C}ilow$ is equivalential. Let $\sigma : \boldsymbol{Fm} \to \boldsymbol{Fm}(x, y)$ such that $\sigma(x) = x$, $\sigma(y) = y$, and $\sigma(z) = \neg x$, for every $z \in \mathrm{Var}\backslash\{x, y\}$. It follows by Proposition 2.7 and Theorem 4.3 that $\sigma\Delta$ is a set of congruence formulas for $\mathcal{C}ilow$, where

$$\Delta(x, y, \overline{z}) = \bigcup_{n\in\omega}\bigcup_{m\in\omega} \Phi_n(\neg^m x, \neg^m y, z_1, \dots, z_n) \cup \bigcup_{n\in\omega}\bigcup_{m\in\omega} \Psi_n(\neg^m x, \neg^m y, z_1, \dots, z_n).$$

Now, let $m, n \in \omega$ and $\varphi \in \Phi_n(\neg^m x, \neg^m y, z_1, \dots, z_n)$. That is,

$$\varphi = \neg^{m_n}(\cdots(\neg^{m_2}(\neg^{m_1}(\neg^m x *_1 z_1) *_2 z_2)\cdots) *_n z_n) \leftrightarrow \neg^{m_n}(\cdots(\neg^{m_2}(\neg^{m_1}(\neg^m y *_1 z_1) *_2 z_2)\cdots) *_n z_n),$$

for some $*_1, \dots, *_n \in \{\wedge, \vee, \to\}$ and some $m_1, \dots, m_n \in \omega$. We claim that

$$x \leftrightarrow y, \neg x \leftrightarrow \neg y \vdash_{\mathcal{C}ilow} \sigma\varphi.$$

The proof of our claim goes by induction on $n \in \omega$.
**Basis:n = 0**:
Then $\sigma\varphi = \neg^m x \leftrightarrow \neg^m y$, for some $m \in \omega$. If $m = 0$ or $m = 1$, then clearly $x \leftrightarrow y, \neg x \leftrightarrow \neg y \vdash_{\mathcal{C}ilow} \sigma\varphi$ holds by extensivity. Assume $m > 1$. It follows by $m - 1$ applications of Lemma 10.1 that $\neg x \leftrightarrow \neg y \vdash_{\mathcal{C}ilow} \neg^m x \leftrightarrow \neg^m y$. Therefore, $x \leftrightarrow y, \neg x \leftrightarrow \neg y \vdash_{\mathcal{C}ilow} \sigma\varphi$.
**Step:n > 0**:
Then

$$\sigma\varphi = \neg^{m_{n+1}}\big(\neg^{m_n}(\cdots\neg^{m_2}(\neg^{m_1}(\neg^m x *_1 \neg x) *_2 \neg x)\cdots *_n \neg x) *_{n+1} \neg x\big) \leftrightarrow \neg^{m_{n+1}}\big(\neg^{m_n}(\cdots\neg^{m_2}(\neg^{m_1}(\neg^m y *_1 \neg x) *_2 \neg x)\cdots *_n \neg x) *_{n+1} \neg x\big),$$

for some $*_1, \dots, *_{n+1} \in \{\wedge, \vee, \to\}$ and $m_1, \dots, m_{n+1} \in \omega$. It follows by inductive hypothesis that

$$x \leftrightarrow y, \neg x \leftrightarrow \neg y \vdash_{\mathcal{C}ilow} \neg^{m_n}(\cdots\neg^{m_2}(\neg^{m_1}(\neg^m x *_1 \neg x) *_2 \neg x)\cdots *_n \neg x) \leftrightarrow \neg^{m_n}(\cdots\neg^{m_2}(\neg^{m_1}(\neg^m y *_1 \neg x) *_2 \neg x)\cdots *_n \neg x).$$

Now, it follows by Lemma 3.2.13–15 (depending on $*_{n+1} \in \{\wedge, \vee, \to\}$, respectively) that

$$x \leftrightarrow y, \neg x \leftrightarrow \neg y \vdash_{\mathcal{C}ilow} \neg^{m_n}(\cdots\neg^{m_2}(\neg^{m_1}(\neg^m x *_1 \neg x) *_2 \neg x)\cdots *_n \neg x) *_{n+1} \neg x \leftrightarrow \neg^{m_n}(\cdots\neg^{m_2}(\neg^{m_1}(\neg^m y *_1 \neg x) *_2 \neg x)\cdots *_n \neg x) *_{n+1} \neg x. \tag{8}$$

If $m_{n+1} = 0$, then we are done. If $m_{n+1} > 0$, then since $\vdash_{\mathcal{C}ilow} (\neg x)^\circ$ by (Ax15), it follows by (Ax12a)–(Ax14a) (depending on $*_{n+1} \in \{\wedge, \vee, \to\}$, respectively) that

$$\vdash_{\mathcal{C}ilow} \big(\neg^{m_n}(\cdots\neg^{m_2}(\neg^{m_1}(\neg^m x *_1 \neg x) *_2 \neg x)\cdots *_n \neg x) *_{n+1} \neg x\big)^\circ \tag{9}$$

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| $\to^{A}$ | 0 | b | a | 1 | $\neg^{A}$ |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 |
| b | 0 | b | 1 | 1 | 1 |
| a | 0 | 1 | a | 1 | 1 |
| 1 | 0 | 1 | 1 | 1 | 0 |

| $\wedge^{A}$ | 0 | b | a | 1 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| b | 0 | b | b | 1 |
| a | 0 | b | a | 1 |
| 1 | 0 | 1 | 1 | 1 |

| $\vee^{A}$ | 0 | b | a | 1 |
|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 1 |
| b | 1 | b | a | 1 |
| a | 1 | a | a | 1 |
| 1 | 1 | 1 | 1 | 1 |

TABLE 4. Truth-tables of the algebra $A$.

and similarly

$$\vdash_{\mathcal{C}ilow} \left(\neg^{m_n}(\cdots\neg^{m_2}(\neg^{m_1}(\neg^{m} y *_1 \neg x) *_2 \neg x)\cdots *_n \neg x) *_{n+1} \neg x\right)^{\circ}. \qquad (10)$$

It follows by (8)–(10) and Lemma 3.2.8 that

$$\begin{aligned} & x \leftrightarrow y, \neg x \leftrightarrow \neg y \\ \vdash_{\mathcal{C}ilow} & \neg\left(\neg^{m_n}(\cdots\neg^{m_2}(\neg^{m_1}(\neg^{m} x *_1 \neg x) *_2 \neg x)\cdots *_n \neg x) *_{n+1} \neg x\right) \leftrightarrow \\ & \neg\left(\neg^{m_n}(\cdots\neg^{m_2}(\neg^{m_1}(\neg^{m} y *_1 \neg x) *_2 \neg x)\cdots *_n \neg x) *_{n+1} \neg x\right). \end{aligned}$$

It then follows by $m_{n+1} - 1$ applications of Lemma 10.1 that

$$\begin{aligned} & x \leftrightarrow y, \neg x \leftrightarrow \neg y \\ \vdash_{\mathcal{C}ilow} & \neg^{m_{n+1}}\left(\neg^{m_n}(\cdots\neg^{m_2}(\neg^{m_1}(\neg^{m} x *_1 \neg x) *_2 \neg x)\cdots *_n \neg x) *_{n+1} \neg x\right) \leftrightarrow \\ & \neg^{m_{n+1}}\left(\neg^{m_n}(\cdots\neg^{m_2}(\neg^{m_1}(\neg^{m} y *_1 \neg x) *_2 \neg x)\cdots *_n \neg x) *_{n+1} \neg x\right). \end{aligned}$$

That is, $x \leftrightarrow y, \neg x \leftrightarrow \neg y \vdash_{\mathcal{C}ilow} \sigma\varphi$.

We conclude that $x \leftrightarrow y, \neg x \leftrightarrow \neg y \vdash_{\mathcal{C}ilow} \sigma\varphi$, as claimed. Similarly, one proves that $x \leftrightarrow y \vdash_{\mathcal{C}ilow} \sigma\psi$, with $\psi \in \Psi_n(\neg^m x, \neg^m y, z_1, \dots, z_n)$. Thus $x \leftrightarrow y, \neg x \leftrightarrow \neg y \vdash_{\mathcal{C}ilow} \sigma\Delta$, which proves the result. ⊣

PROPOSITION 10.3. *The logic $\mathcal{C}ilow$ is not equivalential.*

PROOF. Suppose for the sake of contradiction that $\mathcal{C}ilow$ is equivalential. It follows by Lemma 10.2 that $\boldsymbol{\rho}(x, y) := \{x \leftrightarrow y, \neg x \leftrightarrow \neg y\}$ is a set of congruence formulas for $\mathcal{C}ilow$. Consider the algebra $A$ whose truth-tables are given in Table 4 and fix $F := \{b, a, 1\}$. First of all, one must check that $F \in \mathcal{F}i_{\mathcal{C}ilow} A$. Next, on the one hand $a \leftrightarrow^{A} b = 1 \in F$ and $\neg^{A} a \leftrightarrow^{A} \neg^{A} b = 1 \in F$. On the other hand, $\neg^{A}(a \to^{A} b) \leftrightarrow^{A} \neg^{A}(b \to^{A} b) = 0 \leftrightarrow^{A} 1 = 0 \notin F$. Thus $\boldsymbol{\rho}^{A}(a, b) \subseteq F$, but $\langle a, b\rangle \notin \boldsymbol{\Omega}^{A}(F)$. We reach a contradiction. ⊣

Consequently, $\mathcal{C}ilow$ is not algebraizable.

Still concerning the logic $\mathcal{C}ilow$, it is interesting to observe that the Leibniz congruence of a $\mathcal{C}ilow$-filter is "classically" defined for negated elements. Indeed, we have seen this to be the case in Theorem 9.1 for all "well-behaved" elements. But given axiom (Ax15), all negated elements are now "well-behaved."

COROLLARY 10.4. *For every $A$, $F \in \mathcal{F}i_{\mathcal{C}ilow} A$, and $a, b \in A$,*

$$\langle \neg a, \neg b\rangle \in \boldsymbol{\Omega}^{A}(F) \quad \Leftrightarrow \quad \neg a \leftrightarrow \neg b \in F.$$

Compare Corollary 10.4 with Corollary 9.4, seen for the logic $\mathcal{C}ilo$.

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As a final note, observe that by considering axiom

$$\vdash \varphi \to \neg\neg\varphi, \tag{Ax16}$$

one can define the logics $\mathcal{C}ilae$ and $\mathcal{C}iloe$[12] as the extensions of $\mathcal{C}_1$ and $\mathcal{C}ilo$ by (Ax16), respectively. Although we will not study these logics in detail here, we shall nevertheless include them among the extensions of $\mathcal{C}_1$ in Figure 1.

**§11. The logics $\mathcal{C}ibv$, $\mathbf{P}^1$, and $\mathbf{P}^2$.** The goals of this section are to introduce the logics $\mathcal{C}ibv$, $\mathbf{P}^1$, and $\mathbf{P}^2$, classify $\mathcal{C}ibv$ within the Leibniz hierarchy (Theorem 11.5)—the classifications of $\mathbf{P}^1$ and $\mathbf{P}^2$ are known—and characterize the non-trivial algebras in $\mathsf{Alg}^*(\mathcal{S})$ subdirectly irreducible relative to $\mathsf{Alg}^*(\mathcal{S})$, with $\mathcal{S} = \mathbf{P}^1, \mathbf{P}^2$ (Theorems 11.9 and 11.10, respectively). These last characterizations will allow us to determine the quasivarieties $\mathsf{Alg}^*(\mathbf{P}^1)$ and $\mathsf{Alg}^*(\mathbf{P}^2)$ (Propositions 11.11 and 11.13), as well as the varieties $\mathbb{V}(\mathbf{P}^1)$ and $\mathbb{V}(\mathbf{P}^2)$ (Corollaries 11.12 and 11.14).

So far we have only seen non-equivalential axiomatic extensions of $\mathcal{C}_1$. In this section we shall consider three equivalential extensions of $\mathcal{C}ilo$. Following the terminology of [16, p. 68], let us consider the logic $\mathcal{C}ibv$ as the logic axiomatized by (Ax1)–(Ax11) and (MP), together with the following axioms:

$$\vdash (\varphi \wedge \psi)^\circ, \tag{Ax12b}$$

$$\vdash (\varphi \vee \psi)^\circ, \tag{Ax13b}$$

$$\vdash (\varphi \to \psi)^\circ. \tag{Ax14b}$$

The logic $\mathbf{P}^1$ (also called $\mathcal{C}ibvw$ in [16, p. 68] and $\mathrm{C}_{0.1}$ in [45, p. 301])[13] is the extension of $\mathcal{C}ibv$ by axiom (Ax15). The logic $\mathbf{P}^2$ (also called $\mathcal{C}ibve$ in [16, p. 68] and $\mathrm{C}_{0.2}$ in [45, p. 301]) is the extension of $\mathcal{C}ibv$ by axiom (Ax16).

The logic $\mathcal{C}ibv$ is the first equivalential (in fact, algebraizable) logic we have come across so far.

PROPOSITION 11.1. *The logic $\mathcal{C}ibv$ is equivalential, witnessed by the set of congruence formulas* $\boldsymbol{\rho}(x, y) = \{\neg^m x \leftrightarrow \neg^m y : m \in \omega\}$.

PROOF. We prove that for every $\boldsymbol{A}$ and every $F \in \mathcal{F}i_{\mathcal{C}ibv}\boldsymbol{A}$,

$$\langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}(F) \quad \Leftrightarrow \quad \forall m \in \omega \quad \neg^m a \leftrightarrow \neg^m b \in F.$$

Fix $\boldsymbol{A}$ and $F \in \mathcal{F}i_{\mathcal{C}ibv}\boldsymbol{A}$. Define the relation $R \subseteq A \times A$ by $\langle a, b\rangle \in R \Leftrightarrow \forall m \in \omega \quad \neg^m a \leftrightarrow \neg^m b \in F$. Assume that $\langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. It follows by Proposition 4.1 that $a R b$. Thus, $\boldsymbol{\Omega}^{\boldsymbol{A}}(F) \subseteq R$. Conversely, we claim that $R$ is a congruence relation on $\boldsymbol{A}$ compatible with $F$. It should be clear that it is an equivalence relation on $A$ by Lemma 3.2.1–3. It is also clear by definition of $R$ that it is compatible with the connective $\neg$. We are left to prove

[12] Following the notation introduced in [16, p. 67].

[13] Sette's paraconsistent logic $\mathbf{P}^1$ was originally defined in [53] within the language $\mathcal{L}' = \langle \to, \neg \rangle$, with the connectives $\wedge, \vee$ being defined in terms of the primitive connectives.

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that $R$ is compatible with the binary language operations of $\mathcal{L}$. The proof goes by induction on $m \in \omega$. **Basis:** Take $m = 0$. The compatibility with the language operations $\wedge, \vee, \rightarrow$ follows by Lemma 3.2.4–6, respectively. **Step:** Let $m > 0$. Let $a_1 R b_1$ and $a_2 R b_2$. It follows by axioms (Ax12b)–(Ax14b) that $(a_1 * a_2)^\circ \in F$ and $(b_1 * b_2)^\circ \in F$, with $* \in \{\wedge, \vee, \rightarrow\}$. It then follows by Lemma 3.2.19 (applied $m$ times) that $\left(\neg^m(a_1 * a_2)\right)^\circ \in F$ and $\left(\neg^m(b_1 * b_2)\right)^\circ \in F$, with $* \in \{\wedge, \vee, \rightarrow\}$. Moreover, it follows by inductive hypothesis that $\neg^m(a_1 * a_2) \leftrightarrow \neg^m(b_1 * b_2)$, with $* \in \{\wedge, \vee, \rightarrow\}$. Hence, it follows by Lemma 3.2.8 that $\neg^{m+1}(a_1 * a_2) \leftrightarrow \neg^{m+1}(b_1 * b_2) \in F$, with $* \in \{\wedge, \vee, \rightarrow\}$. We conclude that $R$ is compatible with the language operations $\wedge, \vee, \rightarrow$. Finally, taking $m = 0$, it follows by (MP) that $R$ is compatible with $F$. Thus, $R \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. ⊣

We now proceed to prove that $\mathcal{C}ibv$ is *not* finitely equivalential.

LEMMA 11.2. *If $\mathcal{C}ibv$ were finitely equivalential, then there would exist $k \in \omega$ such that $\{\neg^m x \leftrightarrow \neg^m y : m \leq k\}$ is a set of congruence formulas for $\mathcal{C}ibv$.*

PROOF. Assume $\mathcal{C}ibv$ is finitely equivalential. Since $\mathcal{C}ibv$ is finitary, it follows by Proposition 11.1 and [32, Proposition 6.65.6] that there exists a finite subset of $\{\neg^m x \leftrightarrow \neg^m y : m \in \omega\}$, say $\{\neg^i x \leftrightarrow \neg^i y : i \in I, I \text{ finite}\}$, which is still a set of congruence formulas for $\mathcal{C}ibv$. Let $k = \max I$. Notice that

$$\{\neg^i x \leftrightarrow \neg^i y : i \in I\} \dashv\vdash_{\mathcal{C}ibv} \{\neg^m x \leftrightarrow \neg^m y : m \leq k\}.$$

Thus $\{\neg^m x \leftrightarrow \neg^m y : m \leq k\}$ is a set of congruence formulas for $\mathcal{C}ibv$. ⊣

PROPOSITION 11.3. *The logic $\mathcal{C}ibv$ is not finitely equivalential.*

PROOF. Suppose for the sake of contradiction that $\mathcal{C}ibv$ is finitely equivalential. Consider the algebra $\boldsymbol{A}_n$ with universe $A_n = \{0, \frac{1}{n-1}, \dots, \frac{n-2}{n-1}, 1\}$. Define the operations $\neg^{\boldsymbol{A}_n}, \wedge^{\boldsymbol{A}_n}, \vee^{\boldsymbol{A}_n}, \rightarrow^{\boldsymbol{A}_n}$ as follows:

$$\neg^{\boldsymbol{A}_n} x = \begin{cases} 1, & \text{if } x = 0, \\ x + \frac{1}{n-1}, & \text{if } x \in \{\frac{1}{n-1}, \dots, \frac{n-2}{n-1}\}, \\ 0, & \text{if } x = 1, \end{cases}$$

$$x \wedge^{\boldsymbol{A}_n} y = \begin{cases} 0, & \text{if } x = 0 \text{ or } y = 0, \\ 1, & \text{otherwise}, \end{cases}$$

$$x \vee^{\boldsymbol{A}_n} y = \begin{cases} 0, & \text{if } x = 0 \text{ and } y = 0, \\ 1, & \text{otherwise}, \end{cases}$$

$$x \rightarrow^{\boldsymbol{A}_n} y = \begin{cases} 0, & \text{if } x \neq 0 \text{ and } y = 0, \\ 1, & \text{otherwise}. \end{cases}$$

For the case $n = 4$, the truth-tables of $\boldsymbol{A}_4$ are given in Table 5. Under our assumption, it follows by Lemma 11.2 that there exists $k \in \omega$ such

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| $\to^{A_4}$ | 0 | 1/3 | 2/3 | 1 | $\neg^{A_4}$ |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 |
| 1/3 | 0 | 1 | 1 | 1 | 2/3 |
| 2/3 | 0 | 1 | 1 | 1 | 1 |
| 1 | 0 | 1 | 1 | 1 | 0 |

| $\wedge^{A_4}$ | 0 | 1/3 | 2/3 | 1 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1/3 | 0 | 1 | 1 | 1 |
| 2/3 | 0 | 1 | 1 | 1 |
| 1 | 0 | 1 | 1 | 1 |

| $\vee^{A_4}$ | 0 | 1/3 | 2/3 | 1 |
|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 1 |
| 2/3 | 1 | 1 | 1 | 1 |
| 1/3 | 1 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 | 1 |

TABLE 5. Truth-tables of the algebra $\boldsymbol{A}_4$.

that $\boldsymbol{\rho}(x, y) := \{\neg^m x \leftrightarrow \neg^m y : m \leq k\}$ is a set of congruence formulas for $\mathcal{C}ibv$. Let us consider the algebra $\boldsymbol{A}_{k+3}$ and fix $F := \{\frac{1}{k+2}, \dots, \frac{k+1}{k+2}, 1\}$. First of all, notice that $F \in \mathcal{F}i_{\mathcal{C}ibv}\boldsymbol{A}$. Next, observe that $\neg^m \frac{1}{k+2} \leftrightarrow \neg^m \frac{2}{k+2} = 1 \in F$, for every $m \leq k$, but $\neg^{k+1}\frac{1}{k+2} \leftrightarrow \neg^{k+1}\frac{2}{k+2} = 1 \leftrightarrow 0 = 0 \notin F$. Thus $\boldsymbol{\rho}^{\boldsymbol{A}}\big(\frac{1}{k+2}, \frac{2}{k+2}\big) \subseteq F$, but $\langle \frac{1}{k+2}, \frac{2}{k+2}\rangle \notin \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. We reach a contradiction. ⊣

Regarding the family $\boldsymbol{A}_n$, with $n \in \omega$, observe that $A_2 = \mathbb{2}$ and $A_3 = \boldsymbol{S}$, where $\boldsymbol{S}$ is Sette's algebra.

The (few) known examples of equivalential logics which are not finitely equivalential are Herrmann's logic, Dellunde's logic, and the local logic associated with the least normal modal system. All these examples can be found in [32, Examples 3.42, 3.53, and 6.67.3]. The logic $\mathcal{C}ibv$ provides one further example to this list.

Since truth-equationality is preserved by extensions, it follows at once by Proposition 9.8 that $\mathcal{C}ibv$ has its filters equationally definable by $\boldsymbol{\tau}(x) = \big\{(x^\circ)^\circ \wedge x \approx (x^\circ)^\circ \wedge (x \to x)\big\}$. However, in the case of $\mathcal{C}ibv$ one can identify a simpler set of truth-equations.

PROPOSITION 11.4. *The logic $\mathcal{C}ibv$ is truth-equational, witnessed by the set of truth-equations $\boldsymbol{\tau}(x) = \{(x \to x) \to x \approx x \to x\}$.*

PROOF. Let $\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}(\mathcal{C}ibv)$. Assume $a \in F$. Since $a \to a \in F$ by Lemma 3.2.11, it follows by some easy manipulation of (Ax1) and (MP) that $\big((a \to a) \to a\big) \leftrightarrow (a \to a) \in F$. Moreover, since both $\big((a \to a) \to a\big)^\circ \in F$ and $(a \to a)^\circ \in F$ by (Ax14b), it follows by Theorem 9.1 that $\boldsymbol{\tau}^{\boldsymbol{A}}(a) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. Conversely, assume $\boldsymbol{\tau}^{\boldsymbol{A}}(a) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. Then, $\big((a \to a) \to a\big) \leftrightarrow (a \to a) \in F$. Since $a \to a \in F$ by Lemma 3.2.11, it follows by (MP) that $(a \to a) \to a$, and again by (MP) that $a \in F$. We conclude that $F = \{a \in A : \boldsymbol{\tau}^{\boldsymbol{A}}(a) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)\}$, which proves the result. ⊣

The algebraizability of $\mathcal{C}ibv$ is a straightforward consequence of Propositions 11.1 and 11.4. However, in light of Proposition 11.3, the logic $\mathcal{C}ibv$ is *not* finitely algebraizable.

THEOREM 11.5. *The logic $\mathcal{C}ibv$ is algebraizable, with set of congruence formulas $\boldsymbol{\rho}(x, y) = \{\neg^m x \leftrightarrow \neg^m y : m \in \omega\}$ and set of truth equations $\boldsymbol{\tau}(x) = \{(x \to x) \to x \approx x \to x\}$.*

Since algebraizability is preserved by extensions, it follows by Theorem 11.5 that both $\mathbf{P}^1$ and $\mathbf{P}^2$ are algebraizable. However, as we next prove, these logics admit *finite* sets of congruence formulas.

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| | $\neg^{\boldsymbol{A}}$ | $\wedge^{\boldsymbol{A}}$ | 0 | a | 1 | $\vee^{\boldsymbol{A}}$ | 0 | $a$ | 1 | $\to^{\boldsymbol{A}}$ | 0 | $a$ | 1 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 1 | 1 | 1 |
| $a$ | $a$ or 1 | $a$ | 0 | $a$ or 1 | 1 | $a$ | 1 | $a$ or 1 | 1 | $a$ | 0 | $a$ or 1 | 1 |
| 0 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 |

TABLE 6. The truth-tables of Theorem 11.8.

PROPOSITION 11.6. *The logics* $\mathbf{P}^1$ *and* $\mathbf{P}^2$ *are finitely equivalential, with a set of congruence formulas* $\boldsymbol{\rho}(x, y) = \{x \leftrightarrow y, \neg x \leftrightarrow \neg y\}$.

PROOF. Let $\mathcal{S}$ be any of $\mathbf{P}^1, \mathbf{P}^2$. We prove that for every $\boldsymbol{A}$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$,

$$\langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}(F) \quad \Leftrightarrow \quad a \leftrightarrow b \in F \text{ and } \neg a \leftrightarrow \neg b \in F.$$

Fix $\boldsymbol{A}$ and $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Define the relation $R \subseteq A \times A$ by $\langle a, b\rangle \in R$ iff $a \leftrightarrow b \in F$ and $\neg a \leftrightarrow \neg b \in F$. Assume that $\langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. It follows by Proposition 4.1 that $a R b$. Thus, $\boldsymbol{\Omega}^{\boldsymbol{A}}(F) \subseteq R$. Conversely, we claim that $R$ is a congruence relation on $\boldsymbol{A}$ compatible with $F$. It should be clear that it is an equivalence relation on $A$ by Lemma 3.2.1–3. Let $a_1 R b_1$ and $a_2 R b_2$. On the one hand, it follows by Lemma 3.2.4–6 that $(a_1 * a_2) \leftrightarrow (b_1 * b_2) \in F$, with $* \in \{\wedge, \vee, \to\}$. Moreover, $\neg a \leftrightarrow \neg b \in F$ by assumption. On the other hand, it follows by (Ax12b)–(Ax14b) that $(a_1 * a_2)^\circ \in F$ and $(b_1 * b_2)^\circ \in F$, with $* \in \{\wedge, \vee, \to\}$. It then follows by Lemma 3.2.8 that $\neg(a_1 * b_1) \leftrightarrow \neg(a_2 * b_2) \in F$, with $* \in \{\wedge, \vee, \to\}$. Now if $\mathcal{S} = \mathbf{P}^1$, then it follows by (Ax15) that $(\neg a_1)^\circ \in F$ and $(\neg b_1)^\circ \in F$. Since $\neg a_1 \leftrightarrow \neg b_1 \in F$ by assumption, it follows once again by Lemma 3.2.8 that $\neg\neg a_1 \leftrightarrow \neg\neg b_1 \in F$. If $\mathcal{S} = \mathbf{P}^2$, then it follows by (Ax10) and (Ax16) that $a_1 \leftrightarrow \neg\neg a_1 \in F$ and $b_1 \leftrightarrow \neg\neg b_1 \in F$. Since $a_1 \leftrightarrow b_1 \in F$ by assumption, it follows by transitivity (Lemma 3.2.3) that $\neg\neg a_1 \leftrightarrow \neg\neg b_1 \in F$. We conclude that $R$ is compatible with the language operations. Finally, it follows by (MP) that $R$ is compatible with $F$. Thus, $R \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. ⊣

THEOREM 11.7. *The logics* $\mathbf{P}^1$ *and* $\mathbf{P}^2$ *are finitely algebraizable, with a set of congruence formulas* $\boldsymbol{\rho}(x, y) = \{x \leftrightarrow y, \neg x \leftrightarrow \neg y\}$ *and a set of truth equations* $\boldsymbol{\tau}(x) = \{(x \to x) \to x \approx x \to x\}$.

The algebraizability of $\mathbf{P}^1$ was first established in [39, Theorem 2.1]. The algebraizability of $\mathbf{P}^2$ is contained in [16, Fact 3.82] or [17, Theorem 135], although it is unclear whether the relations there considered are in fact congruence relations—see the remarks on page 479.

For the next result let $\mathcal{S}$ be a finitary and finitely equivalential extension of $\mathcal{C}ilo$, extended itself by $\mathcal{CL}$, with (finite) set of congruence formulas $\boldsymbol{\rho}(x, y) = \{x \leftrightarrow y, \neg x \leftrightarrow \neg y\}$. This encompasses both $\mathbf{P}^1$ and $\mathbf{P}^2$.

THEOREM 11.8. *If* $\boldsymbol{A} \in \mathsf{Alg}^*(\mathcal{S})$ *is a subdirectly irreducible algebra relative to* $\mathsf{Alg}^*(\mathcal{S})$, *then the truth tables of the* $\mathcal{L}$*-connectives on* $\boldsymbol{A}$ *agree with those in Table 6.*

| | $\neg^{\boldsymbol{T}}$ |
|---|---|
| 1 | 0 |
| a | 1 |
| 0 | 1 |

| $\wedge^{\boldsymbol{T}}$ | 0 | a | 1 |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| a | 0 | 1 | 1 |
| 1 | 0 | 1 | 1 |

| $\vee^{\boldsymbol{T}}$ | 0 | a | 1 |
|---|---|---|---|
| 0 | 0 | 1 | 1 |
| a | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 |

| $\to^{\boldsymbol{T}}$ | 0 | a | 1 |
|---|---|---|---|
| 0 | 1 | 1 | 1 |
| a | 0 | 1 | 1 |
| 1 | 0 | 1 | 1 |

TABLE 7. The truth-tables of the algebra $\boldsymbol{T}$ [45, p. 301].

PROOF. Let $\boldsymbol{A}$ comply with the assumption. It follows by Theorem 8.7 that $|A| \leq 3$, say $A = \{0, 1, a\}$. We proceed to construct the truth table of $\boldsymbol{A}$. Fix $F \coloneqq \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} = \{1, a\}$, having in mind Corollary 8.5.1.

i. The truth table of the subuniverse $A^\circ = \{0, 1\}$ is obvious, given the fact that $\boldsymbol{A}^\circ \cong \mathbb{2}$ by Theorem 8.3.
ii. Let $* \in \{\wedge, \vee, \to\}$. Notice that since $0^\circ = 1^\circ = 1 \in F$, it follows by (Ax12a)–(Ax14a) that $(a * 0)^\circ, (0 * a)^\circ, (a * 1)^\circ, (1 * a)^\circ \in F$. But clearly all these elements are in $A^\circ$. Therefore, $(a * 0)^\circ, (0 * a)^\circ, (a * 1)^\circ, (1 * a)^\circ \in F \cap A^\circ = \{1\}$, using Corollary 7.5. It follows by Lemma 8.4 that $a * 0, 0 * a, a * 1, 1 * a \in \{0, 1\}$. We now reason by cases until we obtain the (incomplete) truth-tables of Table 7.
   a. If $1 \wedge a = 0$, then since both $a, 1 \in F$, it would follow by (Ax5) and (MP) that $0 \in F$, reaching a contradiction. Thus, $1 \wedge a = 1$. *Mutatis mutandis* for $a \wedge 1 = 1$.
   b. If $0 \wedge a = 1$, then it would follow by (Ax3) and (MP) that $0 \in F$, reaching a contradiction. Thus, $0 \wedge a = 0$. *Mutatis mutandis* for $a \wedge 0 = 0$, using (Ax4).
   c. Let $x \in \{0, 1\}$. If $x \vee a = 0$, then since $a \in F$, it would follow by (Ax7) that $0 \in F$, reaching a contradiction. Thus, $x \vee a = 1$. *Mutatis mutandis* for $a \vee x = 1$, using (Ax6).
   d. If $a \to 1 = 0$, then since $1 \in F$, it would follow by (Ax1) and (MP) that $0 \in F$, reaching a contradiction. Thus, $a \to 1 = 1$.
   e. If $a \to 0 = 1$, then since $a \in F$, it would follow by (MP) that $0 \in F$, reaching a contradiction. Thus, $a \to 0 = 0$.
   f. If $1 \to a = 0$, then since $a \in F$, it would follow by (Ax1) and (MP) that $0 \in F$, reaching a contradiction. Thus, $1 \to a = 1$. *Mutatis mutandis* for $0 \to a = 1$.
iii. Let $* \in \{\wedge, \vee, \to\}$. We are left to prove that $\neg a \neq 0$ and $a * a \neq 0$. If $\neg a = 0$, then $a^\circ = 1$, contradicting Lemma 8.4. Finally, since $a \in F$, notice that we also have $a * a \in F$. Thus, $a * a \neq 0$. ⊣

This does not necessarily mean that there exist $2^4$ non-trivial algebras in $\mathsf{Alg}^*(\mathcal{S})$ subdirectly irreducible relative to $\mathsf{Alg}^*(\mathcal{S})$. For instance, any truth-table agreeing with Table 6 and such that $a \wedge a = a$ and $\neg a = a$ will not correspond to a non-trivial algebra in $\mathsf{Alg}^*(\mathcal{S})$ subdirectly irreducible relative to $\mathsf{Alg}^*(\mathcal{S})$, otherwise $a^\circ = \neg(a \wedge \neg a) = \neg(a \wedge a) = \neg a = a$, contradicting Lemma 8.4. It does mean however that there exist at most 12 non-trivial algebras in $\mathsf{Alg}^*(\mathcal{S})$ with three elements subdirectly irreducible relative to

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| | $\neg^{S}$ |
|---|---|
| 1 | 0 |
| a | 1 |
| 0 | 1 |

| $\wedge^{S}$ | 0 | a | 1 |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| a | 0 | 1 | 1 |
| 1 | 0 | 1 | 1 |

| $\vee^{S}$ | 0 | a | 1 |
|---|---|---|---|
| 0 | 0 | 1 | 1 |
| a | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 |

| $\to^{S}$ | 0 | a | 1 |
|---|---|---|---|
| 0 | 1 | 1 | 1 |
| a | 0 | 1 | 1 |
| 1 | 0 | 1 | 1 |

TABLE 8. The truth-tables of the algebra $\boldsymbol{S}$ [53, pp. 176 and 179] and [50, p. 101].

$\mathsf{Alg}^*(\mathcal{S})$. We shall see three of them in greater detail—namely, the algebras $\boldsymbol{S}$, $\boldsymbol{T}$, and $\boldsymbol{R}$.

Given Proposition 11.6, we can apply the general results of Section 8 together with Theorem 11.8 to the logics $\mathbf{P}^1$ and $\mathbf{P}^2$.

THEOREM 11.9. *The only non-trivial algebras in* $\mathsf{Alg}^*(\mathbf{P}^1)$ *subdirectly irreducible relative to* $\mathsf{Alg}^*(\mathbf{P}^1)$ *are* $\mathbb{2}$, $\boldsymbol{S}$.

PROOF. Let $\boldsymbol{A}$ comply with the assumption. The proof goes exactly as in Theorem 11.8, with two further items:

iv. Let $* \in \{\wedge, \vee, \to\}$. On the one hand, notice that $(a * a)^\circ \in F \cap A^\circ = \{1\}$, using (Ax12b)–(Ax14b). It follows by Lemma 8.4 that $a * a \in \{0, 1\}$. On the other hand, since $a \in F$, we have $a * a \in F = \{a, 1\}$. It can only be the case then that $a * a = 1$.
v. Notice that $(\neg a)^\circ \in F \cap A^\circ = \{1\}$, using (Ax15). It follows by Lemma 8.4 that $\neg a \in \{0, 1\}$. But we had seen already in item iii (see the proof of Theorem 11.8) that $\neg a \neq 0$. Thus, $\neg a = 1$.

We conclude that $\boldsymbol{A} \cong \boldsymbol{S}$, whose truth-tables are depicted in Table 8. ⊣

Theorem 11.9 generalizes [40, Theorem 9], which states that the only subdirectly irreducible algebras in $\mathsf{Alg}^*(\mathbf{P}^1)$ are $\mathbb{2}$ and $\boldsymbol{S}$.

THEOREM 11.10. *The only non-trivial algebras in* $\mathsf{Alg}^*(\mathbf{P}^2)$ *subdirectly irreducible relative to* $\mathsf{Alg}^*(\mathbf{P}^2)$ *are* $\mathbb{2}$, $\boldsymbol{T}$.

PROOF. Let $\boldsymbol{A}$ comply with the assumption. The proof goes exactly as in Theorem 11.8, with two further items:

iv. Let $* \in \{\wedge, \vee, \to\}$. On the one hand, notice that $(a * a)^\circ \in F \cap A^\circ = \{1\}$, using (Ax12b)–(Ax14b). It follows by Lemma 8.4 that $a * a \in \{0, 1\}$. On the other hand, since $a \in F$, we have $a * a \in F = \{a, 1\}$. It can only be the case then that $a * a = 1$.
v. We know by item iii (see the proof of Theorem 11.8) that $\neg a \neq 0$. Suppose for the sake of contradiction that $\neg a = 1$. Since $1 \to a = 1 \in F$ and $a \to \neg\neg a$ by (Ax16), it follows by transitivity (Lemma 3.2.3) that $1 \to \neg\neg a \in F$, that is $1 \to \neg 1 = 1 \to 0 = 0 \in F$. We reach a contradiction. Thus, $\neg a = a$.

We conclude that $\boldsymbol{A} \cong \boldsymbol{T}$, whose truth-tables are depicted in Table 8. ⊣

Theorems 11.9 and 11.10 allow us to characterize the classes of algebraic reducts of $\mathbf{P}^1$ and $\mathbf{P}^2$, as well as their intrinsic varieties. In the case of

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$\mathbf{P}^1$, the result is already known and even has a lattice characterization, corresponding to the so-called Sette's algebras.

PROPOSITION 11.11. $\mathsf{Alg}^*(\mathbf{P}^1) = \mathbb{IP}_{\mathrm{S}}(2, \boldsymbol{S}) = \mathbb{Q}(\boldsymbol{S}) = \mathsf{SA}$.

PROOF. It is easy to check that $2, \boldsymbol{S} \in \mathsf{Alg}^*(\mathbf{P}^1)$. Therefore $\mathbb{IP}_{\mathrm{S}}(2, \boldsymbol{S}) \subseteq \mathsf{Alg}^*(\mathbf{P}^1)$, because $\mathsf{Alg}^*(\mathbf{P}^1)$ is closed under $\mathbb{IP}_{\mathrm{S}}$. Conversely, let $\boldsymbol{A} \in \mathsf{Alg}^*(\mathbf{P}^1)$. Since $\mathsf{Alg}^*(\mathbf{P}^1)$ is a quasivariety, because $\mathbf{P}^1$ is finitary and finitely equivalential [32, Corollary 6.80], it follows by Theorem 11.9 that $\boldsymbol{A} \in \mathbb{IP}_{\mathrm{S}}(2, \boldsymbol{S})$. Thus, $\mathsf{Alg}^*(\mathbf{P}^1) = \mathbb{IP}_{\mathrm{S}}(2, \boldsymbol{S})$. The inclusion $\mathbb{Q}(\boldsymbol{S}) \subseteq \mathsf{Alg}^*(\mathbf{P}^1)$ follows once again from the fact that $\mathsf{Alg}^*(\mathbf{P}^1)$ is a quasivariety. To prove the converse inclusion notice that since $\boldsymbol{S}$ is subdirectly irreducible relative to $\mathsf{Alg}^*(\mathbf{P}^1)$, it follows by Theorem 8.3 that $\boldsymbol{S}^\circ \cong 2$, and hence $2 \in \mathbb{S}(\boldsymbol{S}) \subseteq \mathbb{Q}(\boldsymbol{S})$. Finally, $\mathbb{Q}(\boldsymbol{S}) = \mathsf{SA}$ by definition of Sette algebra given in [50, p. 106]. ⊣

The identity $\mathsf{Alg}^*(\mathbf{P}^1) = \mathbb{Q}(\boldsymbol{S})$ was first established in [38, Corollary 2.2], while the identity $\mathbb{Q}(\boldsymbol{S}) = \mathsf{SA}$ was established in [50, Theorem 4.3], together with an axiomatization of Sette's algebras.

Interestingly enough, Proposition 11.11 and Pynko's [50, Corollary 5.8] provide us with a lattice characterization of the intrinsic variety of $\mathbf{P}^1$. Quasi-Sette algebras are introduced in [50, Section 5], and following [50, Definition 5.1] we shall denote the variety of all quasi-Sette algebras by QSA.

COROLLARY 11.12. $\mathbb{V}(\mathbf{P}^1) = \mathbb{V}(\boldsymbol{S}) = \mathsf{QSA}$.

PROOF. Since $\mathbb{V}(\mathbf{P}^1) = \mathbb{V}\mathsf{Alg}^*(\mathbf{P}^1)$, the identity $\mathbb{V}(\mathbf{P}^1) = \mathbb{V}(\boldsymbol{S})$ follows immediately by Proposition 11.11. The identity $\mathbb{V}(\boldsymbol{S}) = \mathsf{QSA}$ was proved in [50, Corollary 5.8]. ⊣

PROPOSITION 11.13. $\mathsf{Alg}^*(\mathbf{P}^2) = \mathbb{IP}_{\mathrm{S}}(2, \boldsymbol{T}) = \mathbb{Q}(\boldsymbol{T})$.

PROOF. It is easy to check that $2, \boldsymbol{T} \in \mathsf{Alg}^*(\mathbf{P}^2)$. Therefore $\mathbb{IP}_{\mathrm{S}}(2, \boldsymbol{T}) \subseteq \mathsf{Alg}^*(\mathbf{P}^2)$, because $\mathsf{Alg}^*(\mathbf{P}^2)$ is closed under $\mathbb{IP}_{\mathrm{S}}$. Conversely, let $\boldsymbol{A} \in \mathsf{Alg}^*(\mathbf{P}^2)$. Since $\mathsf{Alg}^*(\mathbf{P}^2)$ is a quasivariety, because $\mathbf{P}^1$ is finitary and finitely equivalential [32, Corollary 6.80], it follows by Theorem 11.9 that $\boldsymbol{A} \in \mathbb{IP}_{\mathrm{S}}(2, \boldsymbol{T})$. Thus, $\mathsf{Alg}^*(\mathbf{P}^2) = \mathbb{IP}_{\mathrm{S}}(2, \boldsymbol{T})$. The inclusion $\mathbb{Q}(\boldsymbol{T}) \subseteq \mathsf{Alg}^*(\mathbf{P}^2)$ follows once again from the fact that $\mathsf{Alg}^*(\mathbf{P}^2)$ is a quasivariety. To prove the converse inclusion notice that since $\boldsymbol{T}$ is subdirectly irreducible relative to $\mathsf{Alg}^*(\mathbf{P}^2)$, it follows by Theorem 8.3 that $\boldsymbol{T}^\circ \cong 2$, and hence $2 \in \mathbb{S}(\boldsymbol{T}) \subseteq \mathbb{Q}(\boldsymbol{T})$. ⊣

COROLLARY 11.14. $\mathbb{V}(\mathbf{P}^2) = \mathbb{V}(\boldsymbol{T})$.

**§12. The logic $\mathbf{P}^3$.** In this last section we introduce the logic $\mathbf{P}^3$, prove that it coincides with the logic *Cilor* also covered in the literature (Proposition 12.2), and classify it within the Leibniz hierarchy (Theorem 12.7). Similarly to the previous section, we then characterize the non-trivial algebras in $\mathsf{Alg}^*(\mathbf{P}^3)$ subdirectly irreducible relative to $\mathsf{Alg}^*(\mathbf{P}^3)$ (Theorem 12.8) in order to determine the quasivariety $\mathsf{Alg}^*(\mathbf{P}^3)$ (Proposition 12.9) and the variety $\mathbb{V}(\mathbf{P}^3)$ (Corollary 12.10).

Let us define the logic $\mathcal{C}ilor$[14] as the (axiomatic) extension of $\mathcal{C}ilo$ by the following axioms:

$$\vdash (\varphi \wedge \psi)^\circ \to (\varphi^\circ \vee \psi^\circ). \tag{Ax12c}$$

$$\vdash (\varphi \vee \psi)^\circ \to (\varphi^\circ \vee \psi^\circ). \tag{Ax13c}$$

$$\vdash (\varphi \to \psi)^\circ \to (\varphi^\circ \vee \psi^\circ). \tag{Ax14c}$$

The logic $\mathbf{P}^3$ is the extension of $\mathcal{C}ilor$ by the axiom (Ax15).

Before proceeding with the classification of $\mathcal{C}ilor$ and $\mathbf{P}^3$ within the Leibniz hierarchy, let us see that these two logics in fact coincide.

LEMMA 12.1.

1. $\vdash_{\mathcal{C}_1} (\varphi \wedge \psi) \wedge \xi \leftrightarrow \varphi \wedge (\psi \wedge \xi)$.
2. $\vdash_{\mathcal{C}ilor} \big((\varphi \wedge \psi) \wedge \xi\big)^\circ \leftrightarrow \big(\varphi \wedge (\psi \wedge \xi)\big)^\circ$.
3. $\vdash_{\mathcal{C}ilor} \neg\big((\varphi \wedge \psi) \wedge \xi\big) \leftrightarrow \neg\big(\varphi \wedge (\psi \wedge \xi)\big)$.
4. $\vdash_{\mathcal{C}ilor} (\varphi \wedge \neg\varphi)^\circ$.
5. $\vdash_{\mathcal{C}ilor} (\neg\varphi)^\circ$.

PROOF. 1. It follows easily by Definition 3.3 and Theorem 3.4.

2. Notice that $\vdash_{\mathcal{C}ilor} \big((\varphi \wedge \psi) \wedge \xi\big)^\circ \leftrightarrow \big((\varphi^\circ \vee \psi^\circ) \vee \xi^\circ\big)$ by (Ax12) and (Ax12c). Notice also that $\vdash_{\mathcal{C}_1} \big((\varphi^\circ \vee \psi^\circ) \vee \xi^\circ\big) \leftrightarrow \big(\varphi^\circ \vee (\psi^\circ \vee \xi^\circ)\big)$ by 1. Finally, notice that $\vdash_{\mathcal{C}ilor} \big(\varphi^\circ \vee (\psi^\circ \vee \xi^\circ)\big) \leftrightarrow \big(\varphi \wedge (\psi \wedge \xi)\big)^\circ$, again by (Ax12) and (Ax12c). The result now follows by transitivity (Lemma 3.2.3).

3. It follows by 1 and 2, together with Lemma 4.2.2.

4. Fix $\xi = \varphi \wedge \neg\varphi$. Notice that $\xi^\circ = \neg((\varphi \wedge \neg\varphi) \wedge \varphi^\circ)$. Having in mind 3, we have $\vdash_{\mathcal{C}ilor} \neg((\varphi \wedge \neg\varphi) \wedge \varphi^\circ) \leftrightarrow \neg(\varphi \wedge \sim\varphi)$. But $\vdash_{\mathcal{C}_1} \neg(\varphi \wedge \sim\varphi)$, fact which can be easily checked using Definition 3.3 da Costa's completeness Theorem 3.4. Therefore, $\vdash_{\mathcal{C}ilor} \neg((\varphi \wedge \neg\varphi) \wedge \varphi^\circ)$. That is, $\vdash_{\mathcal{C}ilor} \xi^\circ$.

5. It follows by (Ax12c) that $\vdash_{\mathcal{C}ilor} (\varphi \wedge \neg\varphi)^\circ \to \varphi^\circ \vee (\neg\varphi)^\circ$. Therefore by 4 and (MP), $\vdash_{\mathcal{C}ilor} \varphi^\circ \vee (\neg\varphi)^\circ$. Finally, since $\vdash_{\mathcal{C}ilor} \varphi^\circ \to (\neg\varphi)^\circ$ by Lemma 3.2.19 and $\vdash_{\mathcal{C}ilor} (\neg\varphi)^\circ \to (\neg\varphi)^\circ$ by Lemma 3.2.11, it follows by (Ax8) and (MP) that $\vdash_{\mathcal{C}ilor} (\neg\varphi)^\circ$. ⊣

Since $\mathbf{P}^3 = \mathcal{C}ilor + (\text{Ax15})$ by definition, and $\mathcal{C}ilor$ satisfies axiom (Ax15) by Lemma 12.1.5, both logics coincide.

PROPOSITION 12.2. $\mathcal{C}ilor = \mathbf{P}^3$.

Having considered the logic $\mathcal{C}ilor + (\text{Ax15})$, it is also natural to consider the logic $\mathcal{C}ilor + (\text{Ax16})$. However, in this case we are in the presence of classical logic.

PROPOSITION 12.3. $\mathcal{C}ilor + (\text{Ax16}) = \mathcal{CL}$.

PROOF. Let $\mathcal{S} = \mathcal{C}ilor + (\text{Ax16})$. We claim that $\vdash_{\mathcal{S}} \varphi^\circ$, for every $\varphi \in \text{Fm}_{\mathcal{L}}$. On the one hand, since $\mathcal{C}ilor$ satisfies axiom (Ax15) by Lemma 12.1.5,

[14] Following the terminology suggested in [16, p. 67]: "and so on, *mutatis mutandis*, for $\mathcal{C}ilo$."

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we have $\vdash_{\mathcal{S}} (\neg\varphi)^\circ$. Therefore, $\vdash_{\mathcal{S}} (\varphi \wedge \neg\varphi) \to (\neg\varphi)^\circ$. On the other hand, it follows by Lemma 3.2.13, axiom (Ax16) and the fact $\vdash_{\mathcal{C}_1} \psi \wedge \delta \leftrightarrow \delta \wedge \psi$, that $\vdash_{\mathcal{S}} (\varphi \wedge \neg\varphi) \to (\neg\varphi \wedge \neg\neg\varphi)$. Moreover, since $(\neg\varphi)^\circ = \neg(\neg\varphi \wedge \neg\neg\varphi)$, it follows again by (Ax16) that $\vdash_{\mathcal{S}} (\neg\varphi \wedge \neg\neg\varphi) \to \neg(\neg\varphi)^\circ$. Therefore, it follows by transitivity (Lemma 3.2.3) that $\vdash_{\mathcal{S}} (\varphi \wedge \neg\varphi) \to \neg(\neg\varphi)^\circ$. Finally, we have $\vdash_{\mathcal{S}} \big((\neg\varphi)^\circ\big)^\circ$ by Lemma 3.2.20. Now, fix $\xi := \varphi \wedge \neg\varphi$. We have by (Ax11),

$$\vdash_{\mathcal{S}} \xi^\circ \to \Big((\xi \to (\neg\varphi)^\circ) \to \big((\xi \to \neg(\neg\varphi)^\circ) \to \neg\xi\big)\Big).$$

It follows by successive applications of (MP) that $\vdash_{\mathcal{S}} \neg\xi$. That is $\vdash_{\mathcal{S}} \varphi^\circ$, as claimed. We conclude that $\mathcal{S} = \mathcal{CL}$—see [26, Theorem 2.1.5]. ⊣

Other than $\mathcal{C}_1$'s axioms, the above proof makes use only of axioms (Ax15) and (Ax16).

Corollary 12.4. $\mathcal{C}_1 + (\text{Ax15}) + (\text{Ax16}) = \mathcal{CL}$.

Similarly to the logics $\mathbf{P}^1$ and $\mathbf{P}^2$, the logic $\mathbf{P}^3$ is also finitely equivalential.

Proposition 12.5. *The logic $\mathbf{P}^3$ is finitely equivalential, witnessed by the set of congruence formulas $\boldsymbol{\rho}(x, y) = \{x \leftrightarrow y, x^\circ \leftrightarrow y^\circ\}$.*

Proof. We prove that for every $\boldsymbol{A}$ and every $F \in \mathcal{F}i_{\mathbf{P}^3}\boldsymbol{A}$,

$$\langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}(F) \quad \Leftrightarrow \quad a \leftrightarrow^{\boldsymbol{A}} b \text{ and } a^\circ \leftrightarrow^{\boldsymbol{A}} b^\circ \in F.$$

Let $\boldsymbol{A}$ arbitrary and $F \in \mathcal{F}i_{\mathbf{P}^3}\boldsymbol{A}$. Define the relation $R \subseteq A \times A$ by $\langle a, b\rangle \in R$ iff $a \leftrightarrow b \in F$ and $a^\circ \leftrightarrow b^\circ \in F$. Assume that $\langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. It follows by Proposition 4.1 that $a R b$. Thus, $\boldsymbol{\Omega}^{\boldsymbol{A}}(F) \subseteq R$. Conversely, we claim that $R$ is a congruence relation on $\boldsymbol{A}$ compatible with $F$. It should be clear that it is an equivalence relation on $A$ by Lemma 3.2.1–3. Let $a_1 R b_1$ and $a_2 R b_2$. On the one hand, it follows by Lemma 3.2.4–6 that $(a_1 * b_1) \leftrightarrow (a_2 * b_2) \in F$, with $* \in \{\wedge, \vee, \to\}$. Moreover, since both $a_1 \leftrightarrow b_1, a_1^\circ \leftrightarrow b_1^\circ \in F$ by assumption, it follows by Lemma 4.2.2 that $\neg a_1 \leftrightarrow \neg b_1 \in F$. On the other hand, we have by axioms (Ax12c)–(Ax14c) that $(a_1 * a_2)^\circ \to a_1^\circ \vee a_2^\circ \in F$, with $* \in \{\wedge, \vee, \to\}$. Since by assumption $a_1^\circ \leftrightarrow b_1^\circ \in F$ and $a_2^\circ \leftrightarrow b_2^\circ \in F$, it follows by Lemma 3.2.5 that $(a_1^\circ \vee a_2^\circ) \to (b_1^\circ \vee b_2^\circ) \in F$, $* \in \{\wedge, \vee, \to\}$. Moreover, it follows by axioms (Ax12)–(Ax14) that $(b_1^\circ \vee b_2^\circ) \to (b_1 * b_2)^\circ \in F$, with $* \in \{\wedge, \vee, \to\}$. Thus by transitivity $(a_1 * a_2)^\circ \to (b_1 * b_2)^\circ \in F$. Similarly, one proves that $(b_1 * b_2)^\circ \to (a_1 * a_2)^\circ \in F$. Thus, $(a_1 * a_2)^\circ \leftrightarrow (b_1 * b_2)^\circ \in F$. Finally, since $(\neg a_1)^\circ, (\neg b_1)^\circ$ by Lemma 12.1.5, it is clear that $(\neg a_1)^\circ \leftrightarrow (\neg b_1)^\circ \in F$. We conclude that $R$ is compatible with the language operations. Furthermore, it follows by (MP) that $R$ is compatible with $F$. Thus, $R \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. ⊣

In light of Lemma 4.2.2, it follows that $\boldsymbol{\rho}(x, y) = \{x \leftrightarrow y, \neg x \leftrightarrow \neg y\}$ is also a set of congruence formulas for $\mathbf{P}^3$.

Truth-equationality follows by the fact that $\mathcal{C}ilo \leq \mathbf{P}^3$. But similarly to $\mathcal{C}ibv$, we can point out another set of truth-equations for $\mathbf{P}^3$ simpler than the one presented for the logic $\mathcal{C}ilo$ in Proposition 9.8. Indeed, $\mathbf{P}^3$ admits the same set of truth-equations than classical logic $\mathcal{CL}$.

| | $\neg^{\boldsymbol{T}}$ |
|---|---|
| 1 | 0 |
| a | a |
| 0 | 1 |

| $\wedge^{\boldsymbol{T}}$ | 0 | a | 1 |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| a | 0 | 1 | 1 |
| 1 | 0 | 1 | 1 |

| $\vee^{\boldsymbol{T}}$ | 0 | a | 1 |
|---|---|---|---|
| 0 | 0 | 1 | 1 |
| a | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 |

| $\to^{\boldsymbol{T}}$ | 0 | a | 1 |
|---|---|---|---|
| 0 | 1 | 1 | 1 |
| a | 0 | 1 | 1 |
| 1 | 0 | 1 | 1 |

TABLE 9. The truth-tables of the algebra $\boldsymbol{R}$ [16, p. 68].

PROPOSITION 12.6. *The logic* $\mathbf{P}^3$ *is truth-equational, witnessed by the set of truth equations* $\boldsymbol{\tau}(x) = \{x \approx x \to x\}$.

PROOF. Let $\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}(\mathbf{P}^3)$. Assume $a \in F$. Since moreover $a \to a \in F$ by Lemma 3.2.11, we have $a \leftrightarrow (a \to a) \in F$. On the other hand, we have $(a \to a)^\circ \to a^\circ \in F$ by (Ax14c) and $a^\circ \to (a \to a)^\circ \in F$ by (Ax14a). Therefore, $a^\circ \leftrightarrow (a \to a)^\circ \in F$. Thus, $\boldsymbol{\tau}(a) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$ by Proposition 12.5. Conversely, assume $\boldsymbol{\tau}(a) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. Then, $a \leftrightarrow (a \to a) \in F$. Since $a \to a \in F$, it follows by (MP) that $a \in F$. We conclude that $F = \{a \in A : \boldsymbol{\tau}^{\boldsymbol{A}}(a) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)\}$, which proves the result. ⊣

We are now able to classify $\mathbf{P}^3$ within the Leibniz hierarchy.

THEOREM 12.7. *The logic* $\mathbf{P}^3$ *is finitely algebraizable, with a set of congruence formulas* $\boldsymbol{\rho}(x, y) = \{x \leftrightarrow y, \neg x \leftrightarrow \neg y\}$ *and aaa set of truth equations* $\boldsymbol{\tau}(x) = \{x \approx x \to x\}$.

The algebraizability of $\mathbf{P}^3$ is contained in [16, Fact 3.82] or [17, Theorem 135], again assuming that the relations there considered are in fact congruence relations—see the remarks on page 479. However, the truth-set of Theorem 12.7 is new—the one presented in the cited results is $\boldsymbol{\tau}(x) = \{(x \to x) \to x \approx x \to x\}$.

Proposition 12.5 makes it possible to apply once again the general results of Section 8 together with Theorem 11.8 to the logic $\mathbf{P}^3$.

THEOREM 12.8. *The only non-trivial algebras in* $\mathsf{Alg}^*(\mathbf{P}^3)$ *subdirectly irreducible relative to* $\mathsf{Alg}^*(\mathbf{P}^3)$ *are* $\mathbb{2}$ *and* $\boldsymbol{R}$.

PROOF. Let $\boldsymbol{A}$ comply with the assumption. The proof goes exactly as in Theorem 11.8, with two further items which go as follows:

iv. Let $* \in \{\wedge, \vee, \to\}$. On the one hand, notice that $(a * a)^\circ \in A^\circ = \{0, 1\}$. But if $(a * a)^\circ = 1$, then $a^\circ = 1$ by (Ax12c)–(Ax14c), contradicting Lemma 8.4. So, $(a * a)^\circ = 0$. It can only be the case then that $a * a = a$, again by Lemma 8.4.

v. We have two cases:
   a. If $\neg a = 1$, then $\boldsymbol{A} \cong \boldsymbol{R}$, whose truth-tables are depicted in Table 9.
   b. If $\neg a = a$, then $a^\circ = \neg(a \wedge \neg a) = \neg(a \wedge a) = \neg a = a$, contradicting Lemma 8.4. ⊣

PROPOSITION 12.9. $\mathsf{Alg}^*(\mathbf{P}^3) = \mathbb{IP}_{\mathrm{S}}(\mathbb{2}, \boldsymbol{R}) = \mathbb{Q}(\boldsymbol{R})$.

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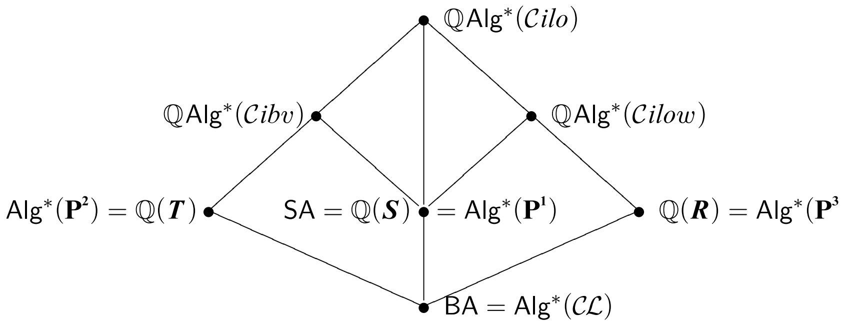


FIGURE 3. Quasivarieties under study.

PROOF. It is easy to check that $\mathbb{2}, \boldsymbol{R} \in \mathsf{Alg}^*(\mathbf{P^3})$. Therefore $\mathbb{IP}_S(\mathbb{2}, \boldsymbol{R}) \subseteq \mathsf{Alg}^*(\mathbf{P^3})$, because $\mathsf{Alg}^*(\mathbf{P^3})$ is closed under $\mathbb{IP}_S$. Conversely, let $\boldsymbol{A} \in \mathsf{Alg}^*(\mathbf{P^3})$. Since $\mathsf{Alg}^*(\mathbf{P^3})$ is a quasivariety, because $\mathbf{P^3}$ is finitary and finitely equivalential [32, Corollary 6.80], it follows by Theorem 12.8 that $\boldsymbol{A} \in \mathbb{IP}_S(\mathbb{2}, \boldsymbol{R})$. Thus, $\mathsf{Alg}^*(\mathbf{P^3}) = \mathbb{IP}_S(\mathbb{2}, \boldsymbol{R})$. The inclusion $\mathbb{Q}(\boldsymbol{R}) \subseteq \mathsf{Alg}^*(\mathbf{P^3})$ follows once again from the fact that $\mathsf{Alg}^*(\mathbf{P^3})$ is a quasivariety. To prove the converse inclusion notice that since $\boldsymbol{R}$ is subdirectly irreducible relative to $\mathsf{Alg}^*(\mathbf{P^3})$, it follows by Theorem 8.3 that $\boldsymbol{R}^\circ \cong \mathbb{2}$, and hence $\mathbb{2} \in \mathbb{S}(\boldsymbol{R}) \subseteq \mathbb{Q}(\boldsymbol{R})$. ⊣

COROLLARY 12.10. $\mathbb{V}(\mathbf{P^3}) = \mathbb{V}(\boldsymbol{R})$.

The relations between the quasivarieties studied in the second part of this work are depicted in Figure 3. Since $\mathcal{C}ibv$ is not *finitely* algebraizable (recall Proposition 11.3), we must consider the class $\mathbb{Q}\mathsf{Alg}^*(\mathcal{C}ibv)$, unlike the classes of algebraic reducts of the remaining algebraizable logics studied.

Our last result sums up the completeness results for the logics $\mathbf{P^1}$, $\mathbf{P^2}$, and $\mathbf{P^3}$. These are all known—see [53, Proposition 9] for 1, [45, Theorem 5.3][15] for 2, and [16, Theorem 3.69][16] for 3. We state it for the sake of completeness and also to provide a unified AAL proof for the three logics.[17]

PROPOSITION 12.11. *Consider the algebras whose truth-tables are given in Tables 7–9.*

1. $\mathbf{P^1} = \mathrm{Log}\langle \boldsymbol{S}, \{a, 1\}\rangle$.
2. $\mathbf{P^2} = \mathrm{Log}\langle \boldsymbol{R}, \{a, 1\}\rangle$.
3. $\mathbf{P^3} = \mathrm{Log}\langle \boldsymbol{T}, \{a, 1\}\rangle$.

[15] As observed in the proof of [16, Theorem 3.69], the logic $\mathbf{P^2}$ in [45] is (mistakenly) defined with the truth set $\{1\}$ rather than $\{a, 1\}$, compromising soundness; since the logic $\mathbf{P^2}$ is semantically defined in [45], the cited completeness result is obtained by proving it to be axiomatizable relative to $\mathcal{C}_1$ by the axioms (Ax12b)–(Ax14b) and (Ax16).

[16] The completeness result is stated, but the proof actually skips the logic $\mathbf{P^3}$.

[17] In rigor, we have not introduced the notation Log in the Preliminaries, but we trust the reader is acquainted with it, as well as with the fact that every class $\mathsf{M}$ of logical matrices induces a logic, denoted by $\mathrm{Log}(\mathsf{M})$.

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Proof. Fix $\mathcal{S} := \mathrm{Log}\langle \boldsymbol{A}, \{a, 1\}\rangle$, with $\boldsymbol{A} = \boldsymbol{S}, \boldsymbol{R}, \boldsymbol{T}$, and let $\mathcal{S}' = \mathbf{P}^1, \mathbf{P}^2, \mathbf{P}^3$, respectively. Notice that $\{\boldsymbol{A}\}$ is a $\boldsymbol{\tau}$-algebraic semantics for $\mathcal{S}$, with $\boldsymbol{\tau}(x) := \{(x \to x) \to x \approx x \to x\}$, because $\{a, 1\}$ is precisely the subset of elements of $A$ satisfying the equation $(x \to x) \to x \approx x \to x$ on $\boldsymbol{A}$. Bearing in mind that $\mathcal{S}$ is finitary (for it is defined by a finite matrix), it follows by [8, Proposition 2.2] that $\mathbb{Q}(\boldsymbol{A})$ is also a $\boldsymbol{\tau}$-algebraic semantics for $\mathcal{S}$. But $\mathbb{Q}(\boldsymbol{A}) = \mathsf{Alg}^*(\mathcal{S}')$, by Propositions 11.11, 11.13, and 12.9 (depending on $\boldsymbol{A} = \boldsymbol{S}, \boldsymbol{R}, \boldsymbol{T}$). Therefore both $\mathcal{S}$ and $\mathcal{S}'$ are the $\boldsymbol{\tau}$-assertional logic w.r.t. $\mathbb{Q}(\boldsymbol{A})$, and hence must coincide. ⊣

As future lines of research, one can point out finding lattice characterizations of $\mathbf{P}^2$-algebras and $\mathbf{P}^3$-algebras, analogous to that of Sette's algebras SA for $\mathbf{P}^1$-algebras; and lattice characterizations of the intrinsic varieties $\mathbb{V}(\mathbf{P}^2) = \mathbb{V}(\boldsymbol{T})$ and $\mathbb{V}(\mathbf{P}^3) = \mathbb{V}(\boldsymbol{R})$, analogous to that of quasi-Sette's algebras QSA for $\mathbb{V}(\mathbf{P}^1) = \mathbb{V}(\boldsymbol{S})$.

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