THE BULLETIN OF SYMBOLIC LOGIC
Volume 26, Number 2, June 2020

DOMINATION AND REGULARITY

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Abstract. We discuss the close relationship between structural theorems in (generalized) stability theory, and graph regularity theorems.

§1. Introduction and preliminaries. We point out analogies between domination theorems in model theory and graph regularity theorems in various "tame" contexts, showing that these are essentially the same theorems, modulo compactness and the pseudofinite yoga, and if one is not so concerned with optimal or explicit bounds.

The motivation comes partly from our joint works with Conant and Terry [5, 6], where "tame" regularity theorems in a group environment *are* obtained from structural theorems (sometimes new) concerning stable and *NIP* groups. The stable case extended work of Terry and Wolf [19] which dealt with the special case of vector spaces over $\mathbb{F}_p$ but with different methods.

We will give later precise statements of all theorems (as well as references to other works). But for now we give a heuristic introduction to the notions in this paper.

First on the graph-theoretic side we recall the regularity theorems which specialize the well known Szemerédi regularity theorem or lemma. We will focus on bi-partite graphs. Szemerédi regularity concerns *all* finite graphs (V, W, E) (where as usual the graph relation E is a subset of $V \times W$). It says that one can partition the vertex sets V, W into a small number of sets $V_1, \dots, V_n$, $W_1, \dots, W_m$ such that outside a small exceptional set of pairs (i,j), the induced subgraphs $(V_i, W_j, E \cap (V_i \times W_j))$ are almost regular, namely sufficiently large induced subgraphs have approximately the same density.

I am writing it first in this informal way, so as to convey the idea. But here is a precise statement of Szemerédi's regularity lemma: For every $\varepsilon > 0$, there is a natural number N_ε, such that for every finite graph (V, W, E), there are partitions $V = V_1 \cup \dots \cup V_n$ and $W = W_1 \cup \dots \cup W_m$ with $n, m \leq N_\varepsilon$, and also an exceptional set Σ of pairs (i,j) with $1 \leq i \leq n$ and $1 \leq j \leq m$, such that,

1. $|\bigcup_{(i,j)\in\Sigma} V_i \times W_j| \leq \varepsilon |V \times W|$,
2. For each $(i,j) \notin \Sigma$, the graph $(V_i, W_j, E \cap (V_i \times W_j))$ is ε-regular.

Received June 15, 2018.
2020 *Mathematics Subject Classification*. 03C45, 03C20.
Key words and phrases. Graph regularity, stability, NIP.

1079-8986/20/2602-0001
DOI:10.1017/bsl.2020.40

The ε-regularity of a finite graph (V,W,E) means that whenever $V' \subseteq V$, $W' \subseteq W$, $|V'| \geq \varepsilon|V|$, and $|W'| \geq \varepsilon|W|$, then the densities of $(V',W',E\cap(V'\times W'))$ and (V,W,E) differ by at most ε, where the density of (V,W,E) is $|E|/|V\times W|$, and likewise for $(V',W',E\cap(V'\times W'))$.

(We are omitting here a certain additional condition that the sizes of the V_i are almost the same, and likewise for the W_j.)

Tame versions of Szemerédi regularity place restrictions on the class of finite graphs (V,W,E) considered, and try to get stronger conclusions. The kind of restrictions are: omitting a certain collection of induced subgraphs, being uniformly definable in some nice structure, or being the collection of finite induced subgraphs of some given graph definable in a nice structure. The improvements in the conclusions typically replace almost regularity by the stronger condition of almost homogeneity (and sometimes outright homogeneity so giving a Ramsey-type theorem) and sometimes remove the need for the exceptional set.

Our use of the expression "homogeneity" here is rather nonstandard, and is in analogy with calling a subset Y of a set X homogeneous with respect to a given partition of the set of unordered pairs of X, if all unordered pairs of elements of Y are in the same member of the partition. So homogeneity of a graph (V,W,E) would mean either complete or empty, and ε-homogeneity means that either all but an ε-proportion of $V\times W$ is in the relation E, or all but an ε-proportion of $V\times W$ is not in the relation E.

On the model theory side, we work with theories T, or formulas $\phi(x,y)$ in a given theory, which are well-behaved in various senses, and we consider a Keisler measure μ on the x-sort (sometimes written μ_x) over a saturated model $\overline{M}$, possibly restricted to definable sets in the Boolean algebra generated by instances of $\phi(x,y)$. The domination statements have the form: there is a small model M_0, and a suitable space S of types over M_0, such that if μ_0 is the measure on S induced by μ then we have (generic) *domination* of the x sort X say by S via the tautological map $\pi : X \to S$ taking $a \in X$ to its type over M_0: for any suitable formula $\psi(x)$ over $\overline{M}$, there is a closed subset Σ_ψ of S of μ_0-measure 0, such that for each $p \in S\setminus\Sigma_\psi$, not both $\pi^{-1}(p)\cap\psi(\overline{M})$ and $\pi^{-1}(p)\cap\neg\psi(\overline{M})$ are "μ-wide". Here μ-wideness of an intersection of definable sets means that every finite subintersection has positive μ-measure.

This is actually closely related to a stationarity statement: μ is the unique nonforking extension of its restriction to M_0. And we also see an exceptional set Σ appearing, as in the graph regularity statement.

The work with Conant and Terry mentioned earlier is concerned with regularity (and structure) theorems in the context of finite groups G equipped with a distinguished subset A. These give rise to bipartitite graphs of the form (G,G,E) where $(a,b)\in E$ iff $ab \in A$. Under assumptions (k-stable, k-*NIP*) on the relation E, we obtained strong theorems on the structure of the set A and its translates, where local stable and *NIP* group theory played a major role. We refer the reader to the papers [5, 6], and we will not explicitly discuss these group results any further in the current paper.

We will go through three model-theoretic situations where there is a domination statement; smooth measures, generically stable measures in *NIP* theories, and ϕ-measures where $\phi(x,y)$ is stable. In each environment we will conclude more or less directly, via compactness, the relevant graph-regularity statement for suitable classes of finite graphs. These statements are already in the literature in various forms and we will give full references.

This paper is expository in the sense that we are discussing known theorems. On the other hand, we are giving rather different approaches and proofs, which do not appear already in the literature, so there is also an "original research" aspect or component to the paper. We will try to make the paper relatively self-contained, but we will have to assume familiarity with some basic model-theoretic methods and techniques, and we will also have to refer to the literature in various places.

This paper is based partly on seminar talks the author gave at the Institut Henri Poincaré in spring 2018 during the trimester on Model theory, Combinatorics and Valued Fields. Thanks to the IHP for its hospitality and to the organizers of the trimester and the seminars. Thanks to Gabriel Conant and Caroline Terry for many discussions. For the record I would also like to thank Udi Hrushovski who already in 2012 pointed out to me (and our co-author) connections between the Lovasz–Szegedy paper [13] and our paper [11] (in particular generic compact domination). Thanks to Christopher Hawthorne who pointed out a correction needed in the proof of Proposition 4.1. And finally, thanks to the referee for some comments, especially the suggestions to add some explanations and details in a few places.

No additional background is needed on the combinatorial side, as all the relevant statements (rather than proofs) are transparent.

On the model theory side we will make use of Keisler measures in an *NIP* and (formula-by-formula) stable environment. But we will make precise a few things which are not made explicit in the literature although should be considered folklore.

Our model theory notation is standard. T denotes a complete theory in a language L and we will work in a very saturated or monster model $\overline{M}$ of T.

The book [17] is a useful reference for material on the *NIP* side, but we will usually refer to the original sources [9] and [10] for Keisler measures, and [11] for generically stable and smooth measures. Insofar as stability is concerned, [16] is a reference, although we take our definition of forking to be Shelah's.

For $\phi(x,y)$ an L-formula, by a ϕ-measure μ over M we mean a finitely additive probability measure on the Boolean algebra of ϕ-formulas over M, where by a ϕ-formula over M we mean a (finite) Boolean combination of instances $\phi(x,b)$ of $\phi(x,y)$ with $b \in M$, and instances $x = c$ of $x = z$ with $c \in M$. So as to be rigorous we should call this a Δ-measure over M where $\Delta = \{\phi(x,y), x = z\}$. A special case of a ϕ-measure over M, is a complete ϕ-type over M, where the values are either 1 (for in the type) or 0 (for not in the type). When we talk about a global measure or type, it means over the monster model $\overline{M}$.

As usual a ϕ-measure over M can be identified with a regular Borel probability measure on the space $S_\phi(M)$ of complete ϕ-types over M. Regularity of a Borel measure on a compact topological space means that the measure of a Borel set can be approximated from above and below by open and closed sets, respectively. In the present situation where the space is profinite, it implies, for example, that the measure of an open set is approximated by the measures of the clopen sets contained in it, and similarly the measure of a closed set is approximated by the measures of the clopen sets containing it. See the beginning of Section 4 of [10].

A ϕ-measure over M is said to be smooth if it has a unique extension to a ϕ-measure over any larger model.

A characteristic property of ϕ-measures when $\phi(x,y)$ is stable is the following (see also Lemma 1.7 of [12]):

Fact 1.1. Let $\phi(x,y)$ be stable (for T). Then any ϕ-measure, μ_x, over a model M is of the form $\sum_{i\in I}\alpha_i p_i$ where p_i is a complete ϕ-type over M, the α_i are positive real numbers, $\sum_{i\in I}\alpha_i = 1$, and where $\boldsymbol{I}$ is either $\{1,\dots,n\}$ for some n, or $\boldsymbol{I} = \mathbb{N}$.

Proof. We give a proof, for completeness, as this has not been made so explicit in earlier papers. We do induction on Shelah's ϕ-rank $R_\phi(-)$ from Section 3, Chapter 1 of [16] where its basic properties are given (and where really we mean Δ-rank where $\Delta = \{\phi(x,y), x = z\}$). Let $p_1,\dots,p_k$ be the finitely many complete ϕ-types of maximal ϕ-rank n say. Without loss of generality $p_1,\dots,p_r$ have positive μ-measures, (say $\alpha_1,\dots,\alpha_r$, respectively) and $p_{r+1},\dots,p_k$ have μ-measure 0.

Working in the space $S_\phi(M)$ let U be the complement of $\{p_1,\dots,p_r\}$, an open set whose μ_x-measure is $\beta = 1-(\alpha_1+\cdots+\alpha_r)$, which we can assume to be positive (otherwise already $\mu = \alpha_1 p_1 + \cdots + \alpha_r p_r$). By the remarks above on regular measures, we can find clopen $U_1 \subset U_2 \cdots \subset U_i \subset \cdots \subset U$, and positive reals $\beta_1 < \beta_2 < \cdots < \beta_i < \cdots$ such that $\mu(U_i) = \beta_i$ for all i and $lim_{i\to\infty}\beta_i = \beta$.

Now U_1 and each $U_{i+1}\setminus U_i$ are ϕ-definable sets of positive measure and with ϕ-rank $< n$. So we can apply induction, to write each of $\mu|U_1$, ..., $\mu|(U_{i+1}\setminus U_i)$, ...as a suitable $\sum_j \gamma_j q_j$. Summing all of these, after suitably scaling, and adding to $\alpha_1 p_1 + \cdots + \alpha_r p_r$ gives the required expression of μ. (Exact details are left to the interested reader.) ⊣

The following is not required, but included for completeness.

Corollary 1.2. *If $\phi(x,y)$ is stable and μ_x is a ϕ-measure over M. Then μ is smooth if and only if μ is a weighted sum of realized ϕ-types, i.e., of the form $\sum_{i\in I}\alpha_i tp_\phi(a_i/M)$ with $a_i \in M$, where again I is an initial segment of* $\mathbb{N}$.

Finally we discuss pseudofiniteness.

Definition 1.3. Let M be an L-structure and A an arbitrary (not necessarily definable) subset of a sort X in M. We say that A is *pseudofinite* in M if for any sentence σ in the language L together with an additional predicate symbol P on sort X, if $(M,A)\models\sigma$ then there is an L-structure M' and a finite subset A' of $X(M')$ such that $(M',A')\models\sigma$.

If M is 1-sorted and A is M itself then we say that M is pseudofinite.

From the definition, finite implies pseudofinite.

REMARK 1.4. Suppose A is definable by a formula $\phi(x,b)$ in the structure M. Then pseudofiniteness of A in M is equivalent to; for every L-formula $\psi(y) \in tp_M(b)$ there is an L-structure M' and $b' \in M'$ satisfying $\psi(y)$ such that $\phi(x,b')(M')$ is finite.

The following is routine.

LEMMA 1.5. *For M an L-structure and A a subset of a sort X in M, the following are equivalent*:

1. *A is pseudofinite in M,*
2. *Let Σ be the set of $L(P)$-sentences which are true of every (M',A') where M' is an L-structure, and A' is a finite subset of the interpretation of the sort X in M'. Then $(M,A) \models \Sigma$.*
3. *(M,A) is elementarily equivalent to some ultraproduct of $L(P)$-structures (M',A') where A' is finite.*

We will now talk about the standard model $\mathbb{V}$ of set theory and saturated elementary extensions $\mathbb{V}^*$ of $\mathbb{V}$. It doesn't really make so much sense, but really we work with some small fragment of set theory including the natural numbers, the reals and all arithmetic operations on them together with cardinality maps for finite sets. The reader can work out for himself or herself the appropriate rigorous statements.

PROPOSITION 1.6. *Suppose (M,A) is pseudofinite. Then there is a (saturated if you wish) elementary extension $\mathbb{V}^*$ of $\mathbb{V}$ and some (M^*,A^*) in $\mathbb{V}^*$ such that*

1. *(M^*,A^*) is elementarily equivalent to (M,A),*
2. *A^* is finite in the sense of $\mathbb{V}^*$, and*
3. *Whenever ψ is a formula of set theory which is true in $\mathbb{V}^*$ of (M^*,A^*) then there is (M,A) (in the standard model), such that ψ is true of (M,A) and A is finite.*

   Moreover suppose that (M,A) is a model of the common theory of (M_n,A_n) for $n<\omega$ where A_n is finite and of increasing size, and A is infinite, then (M^*,A^*) can be chosen to satisfy also
4. *Whenever ψ is a formula of set theory true of (M^*,A^*) in $\mathbb{V}^*$ then ψ is true of infinitely many (M_n,A_n) in $\mathbb{V}$.*

PROOF. This is a compactness argument. Consider the complete diagram of $\mathbb{V}$ together with set of formulas $\psi(y,z)$ true of every (M,A) in $\mathbb{V}$ where M is an L-structure and A a finite subset of the appropriate sort, as well as the formulas expressing that (y,z) is elementarily equivalent (in $L(P)$) to (M,A). It is finitely satisfiable (in $\mathbb{V}$), so has a (saturated if you wish) model. The moreover statement is also clear. ⊣

REMARK 1.7. Typically we take $\mathbb{V}^*$ to be saturated so (M^*,A^*) will be appropriately saturated, so isomorphic to (M,A) assuming the latter was already saturated (assuming some set theory and appropriate choices of degree of saturation).

Given $\mathbb{V}^*$ and (M^*, A^*) as in Proposition 1.6, as A^* is finite in the sense of $\mathbb{V}^*$, every internal subset $\boldsymbol{Z}$ of A^* has a finite cardinality in the sense of $\mathbb{V}^*$ (i.e., $|Z| \in \mathbb{N}^*$) and we obtain the nonstandard normalized counting measure μ^* on the Boolean algebra of internal subsets of A^* which takes $\boldsymbol{Z}$ to $|Z|/|A^*|$, a number in $[0,1]^*$. For $\boldsymbol{Z}$ a definable (in M^*) subset of the ambient sort X in which A^* lives, define $\mu^*(Z) = \mu^*(Z \cap A^*)$. So μ^* gives a "nonstandard" Keisler measure on the sort X in M^*, in the sense that the values of μ^* are in the nonstandard unit interval (as well as finite additivity, etc.). We define μ to be the standard part of μ^* (restricted to definable sets) and we see that that μ is a Keisler measure on the sort X in the L-structure M^*, which we call the *pseudofinite* Keisler measure on X given by A^* (and the ambient structure $\mathbb{V}^*$).

The following is important (and well-known). It can be proved by an adaptation of the material in Section 2.2 of [4]. In any case we follow the notation and context of Proposition 1.6 and the above construction.

PROPOSITION 1.8. *Assuming $Th(M^*)$ is NIP, then the pseudofinite Keisler measure on the sort X is generically stable, namely definable over and finitely satisfiable in some small model M_0.*

In the light of the proposition above, this may be the right time to remind the reader of the notions of forking, definability, and finite satisfiability, in the context of Keisler measures.

A Keisler measure over a model M *does not fork over* $A \subseteq M$ if every formula with positive measure does not fork over A.

When T is *NIP*, $M = \overline{M}$ is the monster model, and $A = M_0$ is a model (small elementary substructure of M), then μ does not fork over M_0 iff μ is $Aut(\overline{M}/M_0)$-invariant.

A global Keisler measure μ is *definable over* a (small) model (elementary substructure of $\overline{M}$) M_0 if it is $Aut(\overline{M}/M_0)$-invariant, and for each L-formula $\phi(x,y)$, the map taking $tp(b/M_0)$ to $\mu(\phi(x,b))$ is continuous. μ is *finitely satisfiable in* M_0 if each formula over $\overline{M}$ with positive μ-measure is satisfied by an element (or tuple) from M_0.

§2. The distal case.

Thedistal regularity theorem [2] is an attractive generalization of a result of Fox et al. [7] giving a strong regularity theorem for the class of finite subgraphs of a semialgebraic graph.

Our treatment here is related to that of Simon [18], but we make more explicit the connection with compact domination.

The relevant structural theorem concerns arbitrary smooth Keisler measures. Recall that a Keisler measure μ_x over M is smooth if it has a unique extension over any elementary extension N containing M. A global Keisler measure μ_x is said to be smooth over a small submodel M_0 if μ is the unique extension over $\overline{M}$ of $\mu|M_0$, where remember $\overline{M}$ denotes the monster model.

Here is the domination theorem for smooth measures, which is basically just a restatement of smoothness. The "tautological map " from a sort $\boldsymbol{X}$ to

https://doi.org/10.1017/bsl.2020.40 Published online by Cambridge University Press

a type space $S_X(M_0)$ was discussed in the previous section and is precisely the map taking $a \in X$ to $tp(a/M_0)$.

Proposition 2.1. *Fix a complete theory T, a sort $\boldsymbol{X}$, and a saturated model $\overline{M}$ and Keisler measure μ on $\boldsymbol{X}$ over $\overline{M}$. Let M_0 be a small elementary submodel of $\overline{M}$ and $\pi : X \to S_X(M_0)$ the tautological map. Suppose μ is smooth over M_0. Then for every definable (with parameters from $\overline{M}$) subset $\boldsymbol{Y}$ of $\boldsymbol{X}$ there is a closed subset Σ of $S_X(M_0)$ of μ_0-measure 0 such that for every $p \in S_X(M_0)$ either $\pi^{-1}(p) \subseteq Y$ or $\pi^{-1}(p) \cap Y = \emptyset$.*

Proof. Otherwise the (closed) set Σ of $p \in S_X(M_0)$ such that $p(x)$ is consistent with each of $x \in Y$ and $x \notin Y$, has μ_0-measure $= \alpha > 0$. Let $(\mu_0)_\Sigma$ be the localization of μ_0 to Σ. Then $(\mu_0)_\Sigma$ has two different extensions to a measure over $\overline{M}$, one giving Y measure 1 and one giving it measure 0. It follows that μ_0 itself has two different extensions to $\overline{M}$, contradicting smoothness. ⊣

More explanation. I am adding here a few details concerning the proof above. First why is Σ closed? Consider a formula $\phi(x,b)$ with parameters b which are not necessarily in M_0. Consider now the set of $p(x) \in S_X(M_0)$ which are consistent with the formula $\phi(x,b)$. Let $q(y)$ be $tp(b/M_0)$. Consider the expression $\exists y(q(y) \wedge \theta(x,y))$. Then this expression is equivalent in the saturated model $\overline{M}$ to a partial type $\Gamma(x)$ over M_0, and the closed set determined by $\Gamma(x)$ is precisely the set of $p(x) \in S_X(M_0)$ consistent with $\theta(x,b)$. This argument explains why Σ is closed.

Secondly, the "localization $(\mu_0)_\Sigma$ is defined as: for any Borel subset $\boldsymbol{B}$ of $S_X(M_0)$, $(\mu_0)_\Sigma(B) = \mu(B \cap \Sigma)/\mu(\Sigma)$.

The last part of the proof, namely that $(\mu_0)_\Sigma$ has the two different extensions, goes precisely as in the proof of the Claim in the proof of Theorem 5.4 in [11].

The following strong regularity (or Ramsey-type) statement is a simple compactness argument applied to Proposition 2.1.

Corollary 2.2. *Let (V,W,R) be a bipartite graph definable in a structure M. Let μ be a smooth Keisler measure on V over M, and ν an arbitrary Keisler measure on W over M. Let $\varepsilon > 0$. Then there are partitions $V = V_1 \cup \cdots \cup V_n$ and $W = W_1 \cup \cdots \cup W_m$ of V, W respectively into definable sets, and a set Σ of pairs $(i,j) \in [n] \times [m]$ of indices such that*

1. $(\mu \times \nu)(\cup_{(i,j)\in\Sigma}(V_i \times W_j)) < \varepsilon$ *and*
2. *for $(i,j) \notin \Sigma$, $V_i \times W_j$ is homogeneous for $\boldsymbol{R}$, namely $V_i \times W_j$ is either contained in $\boldsymbol{R}$ or disjoint from $\boldsymbol{R}$.*

Proof. We may assume M to be saturated. Let M_0 be a small elementary submodel of M such that μ is smooth over M_0 and $R(x,y)$ is definable over M_0. We make use of Proposition 2.1 with $X = V$.

Fix $\varepsilon > 0$. For any b, Let E_b the closed μ_0-measure 0 subset of $S_V(M_0)$ outside of which each fiber of π is either contained in or disjoint from $R(x,b)$. Clearly E_b depends only on $q = tp(b/M_0)$ and so we write it as E_q. Let Z_q be an M_0-definable set containing E_q and with μ_0-measure $< \varepsilon$. By

https://doi.org/10.1017/bsl.2020.40 Published online by Cambridge University Press

compactness we can partition $V \setminus Z_q$ into M_0-definable sets $V_{q,1}, \dots, V_{q,n_q}$ such that for each i, $\pi^{-1}(V_{q,i})$ is either contained in $R(x,b)$ (for some/all b realizing q) or disjoint from $R(x,b)$ (for some/all b realizing q). We can now, by compactness, replace q by a formula (or M_0-definable set) W_q in q, so that for each $i \leq n_q$ either $V_{q,i}$ is contained in $R(x,b)$ for all $b \in W_q$, or $V_{q,i}$ is disjoint from $R(x,b)$ for all $b \in W_q$.

Doing this for each q and applying compactness gives us a partition $W_{q_1}, \dots, W_{q_m}$ of W into M_0-definable sets, and for each $j = 1, \dots, m$ a partition $V = V_{q_j,1} \cup V_{q_j,2} \cup \dots \cup V_{q_j,n_{q_j}} \cup Z_{q_j}$, such that $\mu_0(Z_{q_j}) = 0$ for all j, and for all j, i,

(*) $\pi^{-1}(V_{q_j,i})$ is either contained in $R(x,b)$ for all $b \in W_{q_j}$ or is disjoint from $R(x,b)$ for all $b \in W_{q_j}$.

Let $V_1, \dots, V_t$ be a common refinement of this finite collection of partitions of V. We claim that this partition, together with the partition $W_{q_1}, \dots, W_{q_m}$ of W is as required. We have to identify the exceptional set Σ of pairs. So let $\Sigma = \{(i, q_j) : V_i \subseteq Z_{q_j}\}$. For each q_j, $\cup\{V_i \times W_{q_j} : V_i \subseteq Z_{q_j}\} = Z_{q_j} \times W_{q_j}$ which has $\mu \times \nu$ measure $< \varepsilon\nu(W_{q_j})$. So summing over the q_j we get $(\mu \times \nu)(\cup_{(i,q_j)\in\Sigma}(V_i \times W_{q_j})) < \varepsilon$. And for $(i, q_j) \notin \Sigma$, V_i will be contained in $V_{q_j,i}$ for some i, so by (*) $V_i \times W_{q_j}$ is either contained in or disjoint from $\boldsymbol{R}$. ⊣

The notion of a *distal* first order theory T was introduced by Pierre Simon in his thesis (see [17]). One of the characterizations of distality is that T has *NIP* and every generically stable measure is smooth. Among distal theories are o-minimal theories (such as *RCF*), the theory of $\mathbb{Q}_p$, and $Th(\mathbb{Z}, +, <)$.

So here is the distal regularity theorem, stated for suitable families of finite graphs.

Proposition 2.3. *Let $\mathcal{G} = (G_i : i \in I)$ be a family of finite (bipartite) graphs $G = (V, W, R)$ such that one of the following happens:*

(i) *The graphs are uniformly definable in some model M of a distal theory T,*
(ii) *For some model M of some distal theory T, there is a graph (V, W, R) definable in M such that $\mathcal{G}$ is the family of finite (induced) subgraphs of (V, W, E), or*
(iii) *Every model (V, W, R) of the common theory of the G_i's is interpretable in a model of some distal theory.*

THEN, for any ε there is N_ε, such that for every $(V, W, R) \in \mathcal{G}$, there are partitions $V = V_1 \cup \dots \cup V_n$, and $W = W_1 \cup \dots \cup W_m$, with $n, m < N_\varepsilon$ such that for some "exceptional" set Σ of pairs (i, j) (with $1 \leq i \leq n$ and $1 \leq j \leq m$),

(**) the cardinality of $\cup_{(i,j)\in\Sigma}(V_i \times W_j)$ is $< \varepsilon|V||W|$, and for all $(i, j) \notin \Sigma$, $V_i \times W_j$ is either contained in $\boldsymbol{R}$ or disjoint from $\boldsymbol{R}$.

Proof. Context (ii) is the one dealt with in [2] and which generalizes [7]. Note that Context (i) would be vacuous when T is o-minimal as we have finite bounds on the cardinalities of uniformly definable finite sets, but for the p-adics and/or Presburger, it is nonvacuous.

The proof of the proposition uses Propositions 1.6 and 1.8 (with a possible variant in Context (iii)). We focus here on Context (i). Suppose the conclusion fails. So for some fixed ε, no finite N works. So we can find $N_1 < N_2 < \cdots$, and counterexamples G_{N_1} for N_1, G_{N_2} for N_2 , etc. in $\mathcal{G}$ with the cardinalities of the vertex sets increasing. G_{N_i} being a "counterexample" for N_i means the obvious thing: there do NOT exist partitions of the vertex sets of G_{N_i} into at most N_i parts and an exceptional set of pairs of indices such that (**) in the statement of the proposition holds (for ε, the partitions, and the exceptional set of pairs of indices).

Proposition 1.6 gives us a (saturated) model M^* of T in some $\mathbb{V}^*$ and definable $G^* = (V^*, W^*, R^*)$ in M^* such that V^*, W^* are finite in the sense of $\mathbb{V}^*$ and the moreover clause 4 holds. Let μ^*, ν^* be the nonstandard normalized counting measures on V^* and W^*, given by the construction following Remark 1.7, and let μ and ν be the corresponding pseudofinite Keisler measures. By Proposition 1.8 μ is generically stable, so smooth as $Th(M^*)$ is distal. Apply Corollary 2.2 to (V^*, W^*, R^*) with say $\varepsilon/2$, to get (definable) partitions $V_1^*, \dots, V_n^*$ of $\boldsymbol{V}$ and $W_1^*, \dots, W_m^*$ of $\boldsymbol{W}$, and an exceptional set Σ of pairs (i,j) satisfying the conclusions of Corollary 2.2. Choose $\varepsilon/2 < \delta < \varepsilon$. We can express the existence of the partitions, that $(\mu^* \times \nu^*)(\cup_{(i,j)\in E}(V_i^* \times W_j^*)) < \delta$ and that for $(i,j) \notin \Sigma$, $V_i^* \times W_j^*$ is homogeneous for R^*, by the truth of a formula ψ of set theory for M^*, G^* in $\mathbb{V}^*$. (There is nothing really more to say here. We have just to quantify over the V_i^*, W_j^*, and Σ. The interested reader is invited to write down the formula ψ.)

The moreover clause of Proposition 1.6 tells that ψ is true for infinitely many of the (M, G_{N_k}) in $\mathbb{V}$, and for $N_k > n,m$, we get a contradiction. $\dashv$

§3. The NIP case. The regularity lemma for finite graphs (V, W, R) where the relation R is "k-*NIP*" (or has *VC*-dimension bounded by k) has a nice and elementary direct proof in [3] (in the greater generality of hypergraphs). Combinatorial sources are [1], [13], and [8] (where the latter also deals with the hypergraph version). However we want to again deduce this k-*NIP* regularity from a domination statement, so we work in the context where such statements are currently available, namely inside a *NIP* theory.

We first recall the notion "μ-wide". If μ_x is a (say global) Keisler measure and $\Gamma(x)$ is a partial type over a small set, we say that $\Gamma(x)$ is μ-wide if every finite conjunction of formulas in Γ has μ measure strictly greater than zero.

Again we start with the (generic) domination theorem for generically stable measures.

PROPOSITION 3.1. *Suppose T is NIP. Let μ be a global generically stable measure on the definable set (or sort) X and assume that μ does not fork over M_0 (so is definable over M_0). Let $\pi : X \to S_X(M_0)$ be the tautological map, and let μ_0 be the induced measure on $S_X(M_0)$. Then for every definable (with parameters from $\overline{M}$) subset Y of X, there is a closed set $\Sigma \subseteq S_X(M_0)$ of μ_0-measure 0, such that for each $p \in S_X(M_0) \setminus \Sigma$, either $p \cup \{x \in Y\}$ is not μ-wide, or $p \cup \{x \notin Y\}$ is not μ-wide (and maybe both are not μ-wide).*

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Proof. We deduce this formally from the basic results in [11]. First by Proposition 3.3 of [11], μ is the unique global nonforking extension of its restriction to M_0. Let $\mathcal{P}$ be the space of global complete types $p(x)$ which do not fork over M_0, and let π' be the restriction map from $\mathcal{P}$ to $S_X(M_0)$. Then by Theorem 5.4 of [11], $\mathcal{P}$ is dominated by $(S_X(M_0), \pi', \mu)$ in the sense that for any formula $\phi(x)$ over $\overline{M}$ the set Σ of $p \in S_X(M_0)$ such that $\pi'^{-1}(p)$ intersects both (the clopen determined by) $\phi(x)$ and (the clopen determined by) $\neg\phi(x)$, has μ_0 measure 0. Note that Σ is closed, as in the Explanation following the proof of Proposition 2.1. Recall that $\pi : X \to S_X(M_0)$ takes $a \in X(\overline{M})$ to $tp(a/M_0)$. Now suppose that $p(x) \in S_X(M_0) \setminus \Sigma$. If $p(x) \cup \{\phi(x)\}$ is μ-wide, then as μ does not fork over M_0, $p(x) \cup \{\phi(x)\}$ does not fork over M_0, so extends to some $p' \in \mathcal{P}$. Likewise if $p(x) \cup \{\neg\phi(x)\}$ is μ-wide, it extends to some $p'' \in \mathcal{P}$. As $p \notin \Sigma$, we conclude that not both $p(x) \cup \{\phi(x)\}$ and $p(x) \cup \{\neg\phi(x)\}$ are μ-wide. ⊣

A simple compactness argument applied to Proposition 3.1 again gives a strong regularity theorem.

Corollary 3.2. *Suppose T is NIP and (V, W, R) is a graph definable in a model M of T. Let μ be a generically stable measure on $\boldsymbol{V}$ over M and ν any Keisler measure on $\boldsymbol{W}$ over M. Fix $\varepsilon > 0$. Then there are partitions $V = V_1 \cup \cdots \cup V_n$ and $W = W_1 \cup \cdots \cup W_m$ of V, W into definable sets, and an exceptional set Σ of pairs (i,j) of indices such that*

1. *The $\mu \times \nu$ measure of $\cup_{(i,j)\in\Sigma} V_i \times W_j$ is $< \varepsilon$, and*
2. *For any $(i,j) \notin \Sigma$, either $(\mu \otimes \nu)((V_i \times W_j) \cap R) < \varepsilon\mu(V_i)\nu(W_j)$ or $(\mu \otimes \nu)((V_i \times W_j) \setminus R) < \varepsilon\mu(V_i)(\nu(W_j)$.*

Proof. We follow the proof of Corollary 2.2, but with ε-homogeneous in place of homogeneous (using definability over M_0 of μ), and paying slightly more attention to the exceptional set. We will sometimes use an expression such as "for some/all b realizing q, ..." to mean that the truth of the ... is independent of which b is chosen.

Again assume M to be saturated, and suppose μ does not fork over M_0. Fix $\varepsilon > 0$. We use Proposition 3.1 with $X = V$. For each $q \in S_W(M_0)$ we find closed $E_q \subseteq S_V(M_0)$ of μ_0-measure 0, such that for each $p \in S_V(M_0) \setminus E_q$ and some/any b realizing q, at most one of $p(x) \cup \{R(x,b)\}$, $p(x) \cup \{\neg R(x,b)\}$ is μ-wide. Let Z_q be an M_0-definable set containing E_q and of μ_0-measure $< \varepsilon/2$. By compactness we can partition $V \setminus Z_q$ into M_0-definable sets $V_{q_1}, \ldots, V_{q,n_q}$ such that for each i, either $\mu(V_{q,i} \cap R(x,b)) = 0$ for some/all b realizing q, or $\mu(V_{q_i} \setminus R(x,b)) = 0$ for some/all b realizing q. Following a request from the referee, we will discuss briefly how this compactness argument goes, although a similar argument was already used in the proof of Corollary 2.2. Fix some type $p(x) \in S_V(M_0) \setminus Z_q$. Without loss of generality, $p(x) \cup \{R(x,b)\}$ is not μ-wide (for some/any b realizing q). By definition of μ-wideness, there is a formula $\psi_p(x) \in p(x)$ such that $\mu(\psi_p(x) \wedge R(x,b)) = 0$ for some/any b realizing q. The ψ_p (as p varies) cover $S_V(M_0) \setminus Z_q$, so we can choose a finite subcover, and then modify it to make

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the formulas disjoint. Of course this is all dependent on the choice of q, so we get our partition of $V \setminus Z_q$ into the $V_{q,i}$ as claimed.

We may assume that $\mu(V_{q,i}) > 0$ for each i (otherwise just add it to Z_q).

Now we use definability of μ over M_0 to find an M_0-definable set W_q containing q such that for each $i = 1, \dots, n_q$ exactly one of the following holds:

$(\mathrm{i})_q$ for all $b \in W_q$, $\mu(V_{q,i} \cap R(x,b)) < (\varepsilon^2/2)\mu(V_{q,i})$.

$(\mathrm{ii})_q$ for all $b \in W_q$, $\mu(V_{q,i} \setminus R(x,b)) < (\varepsilon^2/2)\mu(V_{q,i})$.

By compactness we can find $q_1, \dots, q_m$ such that $W_{q_1}, \dots, W_{q_m}$ partition W. Again we find a common refinement $V_1, \dots, V_r$ of the finitely many partitions $V_{q_j,1}, \dots, V_{q_j,n_j}, Z_{q_j}$ of V.

Then $V = V_1 \cup \dots \cup V_r$ and $W = W_{q_1} \cup \dots \cup W_{q_m}$ will be the desired partitions. We have to check that it works.

We have to identify the exceptional set of pairs of indices.

To that avail, let us fix some q_i and call it $\boldsymbol{q}$, and we focus on the subgraph $(V, W_q, R|(V \times W_q))$. Let $I = \{i : 1 \le i \le n_q : \text{and } (\mathrm{i})_q \text{ above holds}\}$. Let J be the rest of the indices i between 1 and n_q, namely where $(\mathrm{ii})_q$ holds.

Let $B \subseteq V \times W_q$ be $\cup_{i \in I}((V_{q,i} \times W_q) \cap R) \cup \cup_{i \in J}((V_{q,i} \times W_q) \setminus R)$. It is then clear that the following holds.

Claim 1. $(\mu \otimes \nu)(B) < (\varepsilon^2/2)\nu(W_q)$.

Let $\Sigma_{q,1} = \{i \in [r] : V_i \subseteq Z_q\}$, and $\Sigma_{q,2} = \{i \in [r] : (\mu \otimes \nu)((V_i \times W_q) \cap B) \ge \varepsilon\mu(V_i)\nu(W_q)\}$, and let $\Sigma_q = \Sigma_{q,1} \sqcup \Sigma_{q,2}$.

Claim 2. $\sum_{i \in \Sigma_q} (\mu \otimes \nu)(V_i \times W_q) < \varepsilon \nu(W_q)$.

Proof of Claim 2. Note that

$$\sum_{i \in \Sigma_{q,1}} (\mu \otimes \nu)(V_i \times W_q) \le (\mu \otimes \nu)(Z_q \times W_q) < (\varepsilon/2)\nu(W_q)$$

So it suffices to prove that

$$\sum_{i \in \Sigma_{q,2}} (\mu \otimes \nu)(V_i \times W_q) < (\varepsilon/2)\nu(W_q).$$

If not then by the definition of $\Sigma_{q,2}$,

$$\begin{aligned}(\mu \otimes \nu)(B) &\ge \sum_{i \in \Sigma_{q,2}} (\mu \otimes \nu)((V_i \times W_q) \cap B) \ge \sum_{i \in \Sigma_{q,2}} \varepsilon\mu(V_i)\nu(W_q) \\ &= \varepsilon \sum_{i \in \Sigma_{q,2}} (\mu \otimes \nu)(V_i \times W_q) \ge (\varepsilon^2/2)\nu(W_q),\end{aligned}$$

which contradicts Claim 1. This completes the proof of Claim 2.

Now suppose $t \notin \Sigma_q$.

Case (i). $V_t \subseteq V_{q,i}$ for some $i \in I$.

So $(V_t \times W_q) \cap B = (V_t \cap W_q) \cap R$ and has $\mu \otimes \nu$ measure $< \varepsilon\mu(V_t)\nu(W_q)$.

Case (ii). $V_t \subseteq V_{q,i}$ for some $i \in J$.

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Likewise we have that $(\mu \otimes \nu)((V_t \times W_q) \setminus R) < \varepsilon\mu(V_t)\nu(W_q)$.

Now let the global exceptional set $\Sigma = \{(i, q_j) : i \in \Sigma_{q_j} : i = 1, \dots, r, j = 1, \dots, m\}$, and we see from Claim 2 (as well as the Case (i), Case (ii) discussion above) that that the conclusions (i) and (ii) of Corollary 3.2 are satisfied. ⊣

The application to families of finite graphs is almost identical to Proposition 2.3, with a similar proof, but we state it anyway.

PROPOSITION 3.3. *Let $\mathcal{G} = (G_i : i \in I)$ be a family of finite (bipartite) graphs $G = (V, W, R)$ such that one of the following happens*:

(i) *The graphs are uniformly definable in some model M of an NIP theory T,*
(ii) *For some model M of some NIP theory T, there is a graph (V, W, R) definable in M such that $\mathcal{G}$ is the family of finite (induced) subgraphs of (V, W, E), or*
(iii) *Every model (V, W, R) of the common theory of the G_i's is interpretable in a model of some NIP theory.*

THEN, for any ε there is N_ε, such that for every $(V, W, R) \in \mathcal{G}$, there are partitions $V = V_1 \cup \cdots \cup V_n$, and $W = W_1 \cup \cdots \cup W_m$, with $n, m < N_\varepsilon$ such that for some some "exceptional" set Σ of pairs (i,j) (with $1 \leq i \leq n$ and $1 \leq j \leq m$),

(a) *the cardinality of $\cup_{(i,j) \in \Sigma} V_i \times W_j$ is $< \varepsilon|V||W|$, and*
(b) *for all $(i,j) \notin \Sigma$, either $|(V_i \times W_j) \cap R| < \varepsilon|V_i||W_j|$ or $|(V_i \times W_j) \setminus R| < \varepsilon|V_i||W_j|$*

REMARK 3.4.

(i) Note that Corollary 3.2 depends only on the Keisler measure μ satisfying the generic domination statement over M_0 in Proposition 3.1, as well as being definable over M_0 (i.e., full generic stability of μ and *NIP*-ness of T are not needed).
(ii) Likewise, if $R(x, y)$ is $\phi(x, y)$, assuming the domination statement and definability for a ϕ-measure μ, Corollary 3.2 will hold.

§4. The stable case. The stable regularity theorem concerns finite graphs (V, W, R) where the edge relation $R(x, y)$ is k-stable, and gives a partition into almost homogeneous subgraphs but without any exceptional set. The original statement and proof are in [15] and involve finite combinatorics in the presence of the Shelah 2-rank and give optimal bounds. A pseudofinite proof making use of local stability theory was given in [14]. The proof we present here is a simplification of the latter. There is no explicit use of any local ranks, other than ingredients in the proof of Fact 1.1.

We first discuss the methods and relationship with the previous proofs. Fix a complete theory T and a stable formula $\phi(x, y)$, where x is of sort X. Let μ be a ϕ-measure on X over a saturated model say $\overline{M}$. Let M_0 be a small model such that μ does not fork over M_0 (i.e., every ϕ-formula over M_0 with positive measure does not fork over M_0). Now every complete ϕ-type p over

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M_0 has a unique nonforking extension over $\overline{M}$ (i.e., to a complete ϕ-type over $\overline{M}$). It follows that for each $p \in S_\phi(M_0)$ and any $b \in \overline{M}$ at most one of $p \cup \{\phi(x,b)\}$, $p \cup \{\neg\phi(x,b)\}$ is μ-wide. So we have domination of X by $S_\phi(M_0)$ (but with no exceptional sets). So we can run the proof of Corollary 3.2, but note that it nevertheless gives a possibly nonempty set of exceptional pairs (i,j) in the regularity statement. So more is needed, and this is precisely Fact 1.1.

Remember that a bipartite graph (V, W, R) (or rather the edge relation $\boldsymbol{R}$ on this graph) is k-stable if there do not exist $a_1,\dots,a_k \in V$, $b_1,\dots,b_k \in W$ such that $R(a_i,b_j)$ iff $i \le j$. Given an L-structure M and L-formula $\phi(x,y)$ we get a corresponding bipartite graph (X, Y, R) (where $\boldsymbol{X}$ is the x-sort in M, $\boldsymbol{Y}$ the y-sort in M and $\boldsymbol{R}$ the interpretation of $\phi(x,y)$ in M). The formula $\phi(x,y)$ is stable for $Th(M)$ iff (X, Y, R) is k-stable for some finite k. ϕ^* is the same formula as ϕ except the roles of variable and parameter variable are interchanged.

We first give the strong regularity theorem (analogues of Corollaries 2.2 and 3.2) for arbitrary graphs where the edge relation is stable.

Proposition 4.1. *Let (V, W, R) be a graph definable in some structure M where the relation $R(x,y)$ is stable. Identify $\boldsymbol{R}$ with the L-formula defining it. Let μ_x be a Keisler measure on $\boldsymbol{V}$ over M. Then for each $\varepsilon > 0$ there are partitions $V = V_1 \cup \dots \cup V_n$ of V and $W = W_1 \cup \dots \cup W_m$ of W into definable sets such that*

1. *for each i,j, either for all $b \in W_j$, $\mu(V_i \setminus R(x,b)) \le \varepsilon\mu(V_i)$, or for all $b \in W_j$, $\mu(V_i \cap R(x,b)) \le \varepsilon\mu(V_i)$.*
2. *Each V_i can defined by an R-formula, and each W_j by a R^*-formula.*

Proof. By Fact 1.1, $\mu|R = \sum_{i\in I}\alpha_i p_i$, for some countable I (which we may assume to be ω or a proper initial segment of ω), complete $\boldsymbol{R}$-types p_i over M, and α_i with $0 < \alpha_i \le 1$ such that $\sum_i \alpha_i = 1$. The p_i are assumed to be distinct.

Note that $\mu(p_i) = \alpha_i > 0$ for $i \in I$. Fix $\varepsilon > 0$. For each $i \in I$, let V_i be a formula (a clopen) in p_i such that $\mu(V_i) < \alpha_i/(1-\varepsilon)$. Let B be $S_R(M) \setminus \{p_i : i \in I\}$. So B is Borel and $\mu(B) = 0$. Let $\delta = (\alpha_0/(1-\varepsilon)) - \mu(V_0)$, and let $U \supseteq B$ be open such that $\mu(U) < \delta$. Then U together with the V_i form an open cover of $S_R(M_0)$, so let $U, V_0, \dots, V_n$ be a finite subcover. We may assume that $V_0, \dots, V_n$ are disjoint (and we still have that $V_j \in p_j$ and $\mu(V_j) < \alpha_j/(1-\varepsilon)$).

Let V_0' be the complement of $(V_1 \cup \dots \cup V_n)$ in $S_R(M)$. So V_0' is clopen and $p_0 \in V_0' \subseteq U \cup V_0$. Moreover by the choice of U we have that $\mu(V_0') < \alpha_0/(1-\varepsilon)$.

Thus, if we replace V_0 by V_0', we get that $V_0, \dots, V_n$ are clopen sets partitioning $S_R(M)$ (in other words R-definable sets which partition V), with $p_i \in V_i$, and $\mu(V_i \setminus p_i) < \varepsilon\mu(V_i)$.

Now we partition W using the R-definitions of $p_0,\dots,p_n$: For each $i = 0,\dots,n$ the R-definition of p_i is a R^*-formula $\psi_i(y)$ over M (with the property that for all $b \in M$, $R(x,b) \in p_i$ iff $M \models \psi_i(b)$). For each subset J of $\{1,\dots,n\}$, let W_J be the set defined by $\wedge_{i\in J}\psi_i(y) \wedge \wedge_{i\notin J}\neg\psi_i(y)$. So the W_J partition W

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into R^*-definable sets. $V = V_0 \cup \cdots \cup V_n$ and $W = \cup_J W_J$ will be the desired partitions of V, W.

We have to check that the conclusions hold. Note that for each $i \in \{0,\dots,n\}$ and $J \subseteq \{1,\dots,n\}$, we have either

(a) for all $b \in W_J$, $R(x,b) \in p_i$, or
(b) for all $b \in W_J$, $\neg R(x,b) \in p_i$.

In case (a), for all $b \in W_J$, $\mu(V_i \setminus R(x,b)) \le \mu(V_i \setminus \{p_i\}) < \varepsilon\mu(V_i)$. In case (b) for all $b \in W_J$, $\mu(V_i \cap R(x,b)) \le \mu(V_i \setminus \{p_i\}) < \varepsilon\mu(V_i)$.

So we have the desired partition and the proof is complete. ⊣

The stable regularity lemma (or a suitable version) for families of finite graphs now follows as earlier:

Corollary 4.2. *Fix k and let $\mathcal{G}$ be the family of finite graphs (V,W,R) where the relation* ***R*** *is k-stable. Then for any $\varepsilon > 0$, there is N such that for each $(V,W,R) \in \mathcal{G}$ there are partitions $V = V_1 \cup \cdots \cup V_n$ and $W = W_1 \cup \cdots \cup W_m$ with $n,m < N$ such that for each i and j, either $|(V_i \times W_j) \cap R| \le \varepsilon|V_i||W_j|$ (so the induced graph on V_i, W_j is almost empty) or $|(V_i \times W_j) \setminus R| \le \varepsilon|V_i||W_j|$ (so the induced graph on V_i, W_j is almost complete).*

Acknowledgment. The author was supported by NSF grants DMS-1360702, DMS-1665035, and DMS-1760212.

REFERENCES

[1] N. Alon, E. Fischer, and I. Newman, *Efficient testing of bipartite graphs for forbidden induced subgraphs*. ***SIAM Journal on Computing***, vol. 37 (2007), pp. 959–976.

[2] A. Chernikov and S. Starchenko, *Regularity lemma for distal structures*. ***Journal of the European Mathematical Society***, vol. 20 (2018), pp. 2437–2466.

[3] ———, *Definable regularity lemmas for NIP hypergraphs*, preprint, 2016.

[4] G. Conant and A. Pillay, *Pseudofinite groups and VC-dimension*, preprint, 2018.

[5] G. Conant, A. Pillay, and C. Terry, *A group version of stable regularity*. ***Proceedings of Cambridge Philosophical Society***, vol. 24 (2018), pp. 1–9.

[6] ———, *Structure and regularity for subsets of groups with finite VC-dimension*. ***Journal of the European Mathematical Society***, to appear.

[7] J. Fox, M. Gromov, V. Lafforgue, A. Naor, and J. Pach, *Overlap properties of geometric expanders*. ***Journal für die reine und angewandte Mathematik***, vol. 671 (2012), pp. 49–83.

[8] J. Fox, J. Pach, and A. Suk, *Erdos-Hajnal conjecture for graphs with bounded VC-dimension*. ***Discrete and Computational Geometry***, vol. 61 (2019), pp. 809–829.

[9] E. Hrushovski, Y. Peterzil, and A. Pillay, *Groups, measures, and the NIP*. ***Journal of the American Mathematical Society***, vol. 21 (2008), pp. 563–595.

[10] E. Hrushovski and A. Pillay, *ON NIP and invariant measures*. ***Journal of the European Mathematical Society***, vol. 13 (2011), pp. 1005–10061.

[11] E. Hrushovski, A. Pillay, and P. Simon, *Generically stable and smooth measures inNIPtheories*. ***Transactions of the American Mathematical Society***, vol. 365 (2013), pp. 2341–2366.

[12] H. J. Keisler, *Measures and forking*. ***Annals of Pure and Applied Logic***, vol. 45 (1987), pp. 119–169.

[13] L. Lovasz and B. Szegedy, *Regularity partitions and the topology of graphons*, ***An Irregular Mind: Szemerédi is 70*** (S. Bárány, editor), Springer, New York, 2010, pp. 415–446.

[14] M. Malliaris and A. Pillay, *The stable regularity lemma, revisited*. ***Proceedings of the American Mathematical Society***, vol. 244 (2016), pp. 1761–1765.

[15] M. Malliaris and S. Shelah, *Regularity lemmas for stable graphs*. ***Transactions of the American Mathematical Society***, vol. 366 (2014), pp. 1551–1585.

[16] A. Pillay, ***Geometric Stability Theory***, Oxford University Press, Oxford, 1996.

[17] P. Simon, ***A Guide to NIP Theories***, Lecture Notes in Logic, vol. 44, ASL-Cambridge University Press, Cambridge, 2015.

[18] ———, *A note on "Regularity theorem for distal structures"*. ***Proceedings of the American Mathematical Society***, vol. 144 (2016), pp. 3573–3578.

[19] C. Terry and J. Wolf, *Stable arithmetic regularity in the finite field model*. ***Bulletin of the London Mathematical Society***, vol. 51 (2019), pp. 70–88.

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