Proceedings of the Royal Society of Edinburgh, **130A**, 397–418, *2000*

Perturbation of elliptic operators and complex dynamics of parabolic partial differential equations

M. Prizzi
Universität Rostock, Fachbereich Mathematik, Universitätsplatz 1, 18055 Rostock, Germany
(martino.prizzi@mathematik.uni-rostock.de)

(MS received 22 July 1998; accepted 1 March 1999)

Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain. Let

$$Lu := \sum_{i,j=i}^{N} \partial_i(g_{ij}(x)\partial_j u), \quad x \in \Omega$$

be a second-order strongly elliptic differential operator with smooth symmetric coefficients. Let B denote the Dirichlet or the Neumann boundary operator. We prove the existence of a smooth potential $a : \bar{\Omega} \to \mathbb{R}$ such that all sufficiently small vector fields on $\mathbb{R}^{N+1}$ can be realized on the centre manifold of the semilinear parabolic equation

$$\begin{aligned} u_t &= Lu + a(x)u + f(x, u, \nabla u), & t > 0, \quad & x \in \Omega, \\ Bu &= 0, & t > 0, \quad & x \in \partial\Omega, \end{aligned}$$

by an appropriate nonlinearity $f : (x, s, w) \in \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N \mapsto f(x, s, w) \in \mathbb{R}$.

For $N = 2$, n, $k \in \mathbb{N}$, we prove the existence of a smooth potential $a : \bar{\Omega} \to \mathbb{R}$ such that all sufficiently small k-jets of vector fields on $\mathbb{R}^n$ can be realized on the centre manifold of the semilinear parabolic equation

$$\begin{aligned} u_t &= Lu + a(x)u + f(x, u) \cdot \nabla u, & t > 0, \quad & x \in \Omega \\ Bu &= 0, & t > 0, \quad & x \in \partial\Omega, \end{aligned}$$

by an appropriate nonlinearity $f : (x, s) \in \bar{\Omega} \times \mathbb{R} \mapsto f(x, s) \in \mathbb{R}^2$ (here, '$\cdot$' denotes the scalar product in $\mathbb{R}^2$).

1. Introduction

Let $N \geqslant 2$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain of class $C^{2,\gamma}$ with $0 < \gamma < 1$. Let L be a differential operator of the form

$$Lu = \sum_{i,j=1}^{N} \partial_i(g_{ij}(x)\partial_j u). \tag{1.1}$$

We assume throughout that L is uniformly elliptic and its coefficient functions satisfy $g_{ij} \in C^{1,\gamma}(\bar{\Omega})$ and $g_{ij} = g_{ji}$ for i, $j = 1, \dots, N$. For $a \in C^{\gamma}(\bar{\Omega})$, define

$$L_a u := Lu + a(x)u. \tag{1.2}$$

https://doi.org/10.1017/S0308210500000226 Published online by Cambridge University Press

Consider the semilinear parabolic equations

$$\left.\begin{aligned} u_t &= L_a u + f(x,u,\nabla u), \quad t>0, \quad x\in\Omega, \\ u(x,t) &= 0, \qquad\qquad\qquad\quad t>0, \quad x\in\partial\Omega, \end{aligned}\right\} \tag{1.3}$$

and

$$\left.\begin{aligned} u_t &= L_a u + f(x,u,\nabla u), \quad t>0, \quad x\in\Omega, \\ \frac{\partial u}{\partial\nu}(x,t) &= 0, \qquad\qquad\qquad\quad t>0, \quad x\in\partial\Omega. \end{aligned}\right\} \tag{1.4}$$

Here, $f : (x,s,w) \in \bar{\Omega}\times\mathbb{R}\times\mathbb{R}^N \mapsto f(x,s,w)\in\mathbb{R}$ is some 'smooth' nonlinearity and $\partial u/\partial\nu$ denotes the conormal derivative $G(x)\nabla u\cdot\nu$, where $G(x) := (g_{ij}(x))_{ij}$ and ν is the outward normal to $\partial\Omega$. To study (1.3) and (1.4), we shall rewrite these problems in a more abstract way. Set

$$X := L^p(\Omega) \quad \text{for some } p, \quad 1<p<\infty.$$

The operator $-L_a$ with Dirichlet (respectively Neumann) boundary conditions on $\partial\Omega$ defines a sectorial operator A_a on X with domain $W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$ (respectively $W_N^{2,p}(\Omega)$, where $W_N^{2,p}(\Omega)$ is the space of all functions in $W^{2,p}(\Omega)$ that satisfy the Neumann condition on $\partial\Omega$ in the sense of traces). The operator A_0 generates the corresponding family X^α of fractional power spaces and, if $p>N$, one can fix α, $0<\alpha<1$, with $(N+p)/(2p)<\alpha<1$, so that $X^\alpha\subset C^1(\bar{\Omega})$ with continuous inclusion. If f is continuous on $\bar{\Omega}\times\mathbb{R}\times\mathbb{R}^N$ and locally Lipschitz continuous with respect to (s,w), then the formula

$$\hat{f}(y)(x) := f(x,y(x),\nabla y(x)), \quad y\in X^\alpha, \quad x\in\bar{\Omega}, \tag{1.5}$$

defines the locally Lipschitz continuous Nemitski operator

$$\hat{f} : X^\alpha \to X.$$

We can rewrite problems (1.3) and (1.4) in the form

$$\dot{y} = -A_a y + \hat{f}(y). \tag{1.6}$$

It is well known that the solutions of (1.6) define a local semiflow on X^α. It is known that for $N=1$ the dynamics of (1.6) is very simple, as all bounded solutions are convergent. On the other side, if the nonlinearity f is independent of gradient terms, then the local semiflow generated by (1.6) is gradient like, and so the dynamics is again rather simple and non-chaotic. The situation is completely different if $N\geqslant 2$ and if f depends explicitly on gradient terms. It has recently been proved that the dynamics of (1.6) can be very complicated, in fact even 'arbitrary'. A first result of this kind was given by Poláčik in [7]. More specifically, he proved that every finite jet of a vector field on $\mathbb{R}^n$ can be realized on the centre manifold of (1.6) by an appropriate nonlinearity f, provided the kernel of the operator L_a (with Dirichlet boundary conditions on $\partial\Omega$) has dimension n and the corresponding eigenfunctions satisfy a certain non-degeneracy condition (which we call the *Poláčik condition*). In this case, $n=N$ or $n=N+1$ and Poláčik also gave examples of operators satisfying this condition, both with $n=N$ (and Ω being the unit ball)

https://doi.org/10.1017/S0308210500000226 Published online by Cambridge University Press

and $n = N+1$ (with Ω being smooth and smoothly diffeomorphic to the unit ball), and with $L = \Delta$, the Laplace operator. In [15], Rybakowski showed that under the Poláčik condition actually all sufficiently smooth and sufficiently small vector fields v on $\mathbb{R}^n$ can be realized on the centre manifold of (1.6) by an appropriate nonlinearity f. The method of proof used in [15] (the Nash–Moser implicit mapping theorem) leads to a loss of derivatives: f is less smooth that v. In [10], Poláčik and Rybakowski, using a non-canonical imbedding of the centre manifold, proved that if L_a has analytic coefficients and the Poláčik condition holds, then a vector field realization result holds without loss of derivatives. They also showed that there are real analytic functions a on $\mathbb{R}^N$ such that the operator $\Delta u + a(x)u$ satisfies the Poláčik condition on a ball of $\mathbb{R}^N$ with $n = N + 1$. In [12], the present author improved the result of [10], removing the analyticity assumption for the coefficients of L_a and avoiding the use of non-canonical imbeddings of the centre manifold.

The above-mentioned vector field realizations lead to a restriction in the choice of the dimension N of the spatial domain Ω: to get realizability of any vector field of $\mathbb{R}^n$ we have to choose $n = N$ or $n = N+1$. Therefore, the question arises: what is the least possible space dimension that allows arbitrary dynamics in (1.6)? In [8], it was shown by Poláčik that every finite jet of a vector field on $\mathbb{R}^n$ can be realized on the centre manifold of (1.6) by an appropriate polynomial nonlinearity f and an appropriate two-dimensional domain (close to a square). As a consequence, a dense (in the C^1 topology) subset of vector fields can be realized, up to flow equivalence, on the centre manifold of (1.6). Let us mention in passing that it is often the case that density results are sufficient for detecting chaotic phenomena. In [8], the form of the nonlinearity f involves high powers of the gradient of the solution u. On the other hand, when modelling scientific phenomena by equations (1.3) and (1.4), one usually tries to make the convection terms (i.e. the terms depending on ∇u) as simple as possible. In [13], it is shown that arbitrary jets can be realized on the centre manifold of (1.6) even for functions f depending on the gradient in a linear fashion.

All the above realization results were proved only on very particular domains, diffeomorphic to a ball or close to a square, and for operators of the form $\Delta + a(x)$. One can ask if it is possible to extend such results to the case of arbitrary (sufficiently regular) domains and general second-order elliptic operators in divergence form.

A first affirmative answer to this question was given by Rybakowski and the present author in [14]. More specifically, they proved that the vector field realization result from [10] is valid for the Laplacian on an arbitrary bounded domain Ω of class $C^{2,\gamma}$, $0 < \gamma < 1$.

The goal of this paper is to extend all the above realization results to the case of a general second-order elliptic operator on an arbitrary spatial domain, both with Dirichlet and Neumann boundary conditions. In order to achieve this result, we prove a 'localization lemma' (lemma 4.1 below), which is the key that makes it possible to apply the techniques developed in [14] to the general situation.

For more realization results see the bibliography. In particular, the recent paper [2] proves realization of jets (and consequently of a dense subset of vector fields) in the class of spatially homogeneous equations, i.e. equations of the form $u_t = \Delta u + f(u, \nabla u)$. In this case, the open set Ω acts like a parameter and, since the nature of the problem itself excludes the possibility of perturbing the linear part by non-homogeneous potential functions $a(x)$, the methods developed in [14] and in the present paper can't be used to yield a jet realization result in the class of spatially homogeneous functions on arbitrary open domains.

The paper is organized as follows. In § 2 we introduce general conditions for realizability of jets and vector fields. In § 3 we briefly recall the perturbation results of [14]. In § 4 we state and prove the above-mentioned 'localization lemma'. Finally, in §§ 5 and 6 we show that the conditions introduced in § 2 actually are satisfied by any symmetric strongly elliptic second-order differential operator in divergence form on an arbitrary spatial domain, both with Dirichlet and Neumann boundary conditions.

2. Vector field and jet realizations

Let L_a be the differential operator defined by (1.1) and (1.2), with Dirichlet or Neumann boundary conditions, and let A_a be the corresponding sectorial operator in $X = L^p(\Omega)$ $(p > N)$. Since Ω is bounded and L_a is formally self-adjoint, the spectrum of A_a is pure point and consists of a sequence of real numbers tending to $+\infty$; every eigenvalue has the same finite geometric and algebraic multiplicity. Define

$$X_0 := \ker A_a$$

and suppose that

$$n := \dim X_0 \geqslant 1.$$

Let P be the $L^2(\Omega)$-orthogonal projection of X onto X_0. Fix an arbitrary L^2-orthonormal basis $\phi_1, \dots, \phi_n$ of X_0 and write

$$\phi(x) := (\phi_1(x), \dots, \phi_n(x)).$$

Note that the assignment

$$Q : \mathbb{R}^n \to X_0, \qquad Q\xi := \xi \cdot \phi = \sum_{i=1}^{n} \xi_i \phi_i$$

is a linear isomorphism. Finally, let X_- (respectively X_+) be the generalized eigenspace of all eigenvalues with negative (respectively positive) real part.

We first consider vector field realizations. Let Y_m be the set of all functions

$$f : (x, s, w) \in \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N \mapsto f(x, s, w) \in \mathbb{R}$$

such that, for all $0 \leqslant k \leqslant m$, the Fréchet derivative $\mathrm{D}^k_{(s,w)} f$ exists and is continuous and bounded on $\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N$. Y_m is a linear space which becomes a Banach space when endowed with the norm

$$|f|_m := \sup_{(x,s,w) \in \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N} \sup_{0 \leqslant k \leqslant m} |\mathrm{D}^k_{(s,w)} f(x, s, w)|_{\mathcal{L}^k((\mathbb{R} \times \mathbb{R}^N)^k, \mathbb{R})}.$$

For $f \in Y_m$, formula (1.5) defines the Nemitski operator $\hat{f} : X^\alpha \to X$ of class C^m_b.

https://doi.org/10.1017/S0308210500000226 Published online by Cambridge University Press

We can apply to (1.6) the standard theory of centre manifolds: we can find an open neighbourhood $\mathcal{U}_m$ in Y_m, $0 \in \mathcal{U}_m$, and a map

$$\Gamma : (f, \xi) \in \mathcal{U}_m \times \mathbb{R}^n \mapsto \Gamma_f(\xi) \in X_-^\alpha \oplus X_+$$

with the following properties.

(1) $\Gamma_0(\xi) \equiv 0$.

(2) Γ is of class C_b^m.

(3) The set

$$\mathcal{M}_f := \{Q\xi + \Gamma_f(\xi) \mid \xi \in \mathbb{R}^n\}$$

is the global centre manifold of (1.6).

(4) If $v_f : \mathbb{R}^n \to \mathbb{R}^n$ is defined by

$$v_f(\xi) := Q^{-1} P_0 \hat{f}(Q\xi + \Gamma_f(\xi)), \quad \xi \in \mathbb{R}^n,$$

then the flow on $\mathcal{M}_f$ is described by the ordinary differential equation (ODE) in $\mathbb{R}^n$

$$\dot{\xi} = v_f(\xi).$$

As we will see below, it turns out that if $n = N$ or $n = N+1$, then the vector field v_f is arbitrary in the following sense: given any sufficiently small map $v : \mathbb{R}^n \to \mathbb{R}^n$ of class C_b^m, there exists an appropriate nonlinearity $f \in Y_m$ such that $v_f = v$.

Let us recall the following fundamental concept.

DEFINITION 2.1. We say that the operator L_a satisfies the *Poláčik condition* on Ω with Dirichlet (or Neumann) boundary conditions if $\dim \ker A_a = N + 1$ and for some (hence every) basis $\phi_1, \dots, \phi_{N+1}$ of $\ker A_a$, $R(x) \neq 0$ for some $x \in \Omega$, where

$$R(\phi_1, \dots, \phi_{N+1})(x) := \det \begin{pmatrix} \phi_1(x) & \nabla\phi_1(x) \\ \vdots & \vdots \\ \phi_{N+1}(x) & \nabla\phi_{N+1}(x) \end{pmatrix}, \quad x \in \Omega.$$

REMARK 2.2. We have $n = N+1$ in case the Poláčik condition holds. One can also define a (weaker and less interesting) version of the Poláčik condition with $n = N$ (cf. [15]).

For $m \in \mathbb{N}_0$, let $C_b^m(\mathbb{R}^n, \mathbb{R}^n)$ be the set of all maps

$$v : \mathbb{R}^n \to \mathbb{R}^n$$

such that, for all $0 \leqslant k \leqslant m$, the Fréchet derivative $\mathrm{D}^k v$ exists and is continuous and bounded on $\mathbb{R}^n$.

$C_b^m(\mathbb{R}^n, \mathbb{R}^n)$ is a linear space which becomes a Banach space when endowed with the norm

$$|v|_m := \sup_{y \in \mathbb{R}^n} \sup_{0 \leqslant k \leqslant m} |\mathrm{D}^k v(y)|_{\mathcal{L}^k((\mathbb{R}^n)^k, \mathbb{R}^n)}.$$

The following result was proved in [12].

https://doi.org/10.1017/S0308210500000226 Published online by Cambridge University Press

THEOREM 2.3. *Assume L_a satisfies the Poláčik condition on Ω, with Dirichlet (or Neumann) boundary conditions. Then there exists $\eta_m > 0$ such that for every $v \in C_b^m(\mathbb{R}^n, \mathbb{R}^n)$, with $|v|_m \leqslant \eta_m$, there exists $f \in \mathcal{U}_m$ such that*

$$Q^{-1}P_0\hat{f}(Q\xi + \Gamma_f(\xi)) = v(\xi), \quad \xi \in \mathbb{R}^n.$$

In [10] Polàčik and Rybakowski proved that, if Ω is a ball in $\mathbb{R}^N$, then, for a suitable potential function a, the operator $\Delta + a$ satisfies the Polàčik condition on Ω with Dirichlet boundary condition. In [14], Rybakowski and the present author extended this result to the case of an arbitrary smooth bounded domain. In § 5 below we will prove that, for an arbitrary principal part L of the form (1.1), both with Dirichlet and Neumann boundary conditions on $\partial\Omega$, it is possible to construct a potential $a : \bar{\Omega} \to \mathbb{R}$ such that L_a satisfies the the Polàčik condition on Ω.

Next we consider jet realizations. Fix $k \in \mathbb{N}$ and arbitrary integers $q_1, \dots, q_k$ such that $1 \leqslant q_l \leqslant l$ for $l = 1, \dots, k$. Let $\mathcal{E} = \mathcal{E}(q_1, \dots, q_k)$ be the set of all functions $f : \bar{\Omega} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ of the form

$$f(x, y, s, z) = \sum_{l=1}^{k} a_l(x, y) s^{l-q_l} z^{q_l}, \quad ((x, y), s, z) \in \bar{\Omega} \times \mathbb{R} \times \mathbb{R}, \tag{2.1}$$

where $a_l \in H^2(\Omega)$ for $l = 1, \dots, k$. We can identify $\mathcal{E}$ with $(H^2(\Omega))^k$; with the norm induced by this identification, $\mathcal{E}$ becomes a Banach space whose topology is stronger than the topology of locally uniform convergence of all derivatives $D^h_{(s,z)} f(x, y, s, z)$, $h = 0, \dots, k + 1$ on $\bar{\Omega} \times \mathbb{R} \times \mathbb{R}$.

For $f \in \mathcal{E}$ and $\varpi \in \mathbb{R}^2$, define the function $f^\varpi : \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ by

$$f^\varpi(x, y, s, w) := f(x, y, s, \varpi \cdot w), \quad ((x, y), s, w) \in \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^2.$$

Formula (1.5) (with f replaced by f^ϖ) defines the Nemitski operator $\hat{f}^\varpi : X^\alpha \to X$ of class C^∞.

Again, we can apply to (1.6) the standard theory of centre manifolds. Let $B_1^n(0)$ be the unit ball centred at 0 in $\mathbb{R}^n$; we can find an open neighbourhood $\mathcal{U}$ in $\mathcal{E}$, $0 \in \mathcal{U}$, and a map

$$\Gamma : (f, \xi) \in \mathcal{U} \times B_1^n(0) \subset \mathcal{E} \times \mathbb{R}^n \mapsto \Gamma_f(\xi) \in X_-^\alpha \oplus X_+$$

with the following properties.

(1) $\Gamma(\xi) \equiv Q\xi$.

(2) Γ is of class C^{k+1}.

(3) The set
$$\mathcal{M}_f^{\mathrm{loc}} := \{Q\xi + \Gamma_f(\xi) \mid \xi \in B_1^n(0)\}$$
is a local invariant manifold of (1.6).

(4) If $v_f : B_1^n(0) \to \mathbb{R}^n$ is defined by
$$v_f(\xi) := Q^{-1}P_0\hat{f}^\varpi(Q\xi + \Gamma_f(\xi)), \quad \xi \in B_1^n(0),$$

https://doi.org/10.1017/S0308210500000226 Published online by Cambridge University Press

then the local flow on $\mathcal{M}_f^{\mathrm{loc}}$ is described by the ODE in $B_1^n(0)$

$$\dot{\xi} = v_f(\xi).$$

Define the map

$$\Psi : \mathcal{U} \subset \mathcal{E} \to C_b^k(B_1^n(0), \mathbb{R}^n),$$
$$\Psi(f)(\xi) := Q^{-1} \circ P_0 \circ \hat{f}^{\varpi} \circ (Q + \Gamma_f)(\xi), \quad \xi \in B_1^n(0).$$

Simple computations shows that Ψ is of class C^1 and

$$D\Psi(0)f(\xi) = (Q^{-1} \circ P_0 \circ \hat{f}^{\varpi})(Q\xi).$$

Let $J_0^k(\mathbb{R}^n)$ denote the set of all k-jets on $\mathbb{R}^n$ mapping 0 into itself. Equivalently, $h \in J_0^k(\mathbb{R}^n)$ if and only if h is a polynomial on $\mathbb{R}^n$ of order $\leqslant k$ with $h(0) = 0$. We say that a jet h can be realized in (1.6) by the nonlinearity f if the kth order Taylor polynomial of the vector field v_f is equal to h. We introduce the linear bounded operator

$$T^k : C_b^k(B_1^n(0), \mathbb{R}^n) \to J_0^k,$$
$$(T^k v)(\xi) = \sum_{i=0}^{k} \frac{1}{i!} D^i v(0)\xi^i, \quad v \in C_b^k(B_1^n(0), \mathbb{R}^n), \quad \xi \in \mathbb{R}^n.$$

If $D(T^k \circ \Psi)(0)$ is surjective onto J_0^k, then the classical surjective mapping theorem implies that all sufficiently small k-jets $h \in J_0^k(\mathbb{R}^n)$ can be realized in (1.6) by an appropriate nonlinearity $f \in \mathcal{E}$.

It can be proved that $D(T^k \circ \Psi)(0)$ is surjective, provided a certain algebraic independence condition is satisfied by the basis $\phi_1, \ldots, \phi_n$ of $\ker A_a$ (see [13]). In order to state this condition, we introduce the following notations. Given $\gamma, \beta \in \mathbb{N}_0^n$, we say that $\gamma \leqslant \beta$ if and only if $\gamma_i \leqslant \beta_i$, $i = 1, \ldots, n$. If $\gamma \in \mathbb{R}_0^n$, $\varpi \in \mathbb{R}^2$, set $\phi^\gamma := \phi_1^{\gamma_1} \cdots \phi_n^{\gamma_n}$, $\phi_{i\varpi} := \varpi \cdot \nabla \phi_i$ and $\phi_\varpi^\gamma := \phi_{1\varpi}^{\gamma_1} \cdots \phi_{n\varpi}^{\gamma_n}$. Moreover, set

$$\epsilon_j := (\underbrace{0, \ldots, 0, 1}_{j}, 0, \ldots, 0) \in \mathbb{N}_0^n.$$

With these notations, the algebraic independence condition reads as follows.

(IC) For every $l = 1, \ldots, k$ and for every q, $1 \leqslant q \leqslant l$, the functions

$$\left\{ \sum_{\gamma \leqslant \beta,\, |\gamma| = q} \frac{1}{\gamma!(\beta - \gamma)!} \phi^{\beta - \gamma + \epsilon_j} \phi_\varpi^\gamma \right\}_{j=1,\ldots,n,\, |\beta| = l}$$

are linearly independent.

The following result was proved in [13].

THEOREM 2.4. *Let n and $k \in \mathbb{N}$. Assume* $\dim \ker A_a = n$ *and assume there is an $L^2(\Omega)$-orthonormal basis $\phi_1, \ldots, \phi_n$ of* $\ker A_a$ *and a vector $\varpi \in \mathbb{R}^2$ such that (IC) is satisfied up to the order k. Then there is an open neighbourhood $\mathcal{B}$ of 0 in $J_0^k(\mathbb{R}^n)$ such that every jet $h \in \mathcal{B}$ can be realized in (1.6) by a a nonlinearity $f \in \mathcal{E}$.*

https://doi.org/10.1017/S0308210500000226 Published online by Cambridge University Press

Remark 2.5. As in [8], theorem 2.4 can be strengthened to obtain realizability of C^m-families of jets; this implies that a dense set of vector fields in $\mathbb{R}^n$ can be realized, up to flow equivalence, in equation (1.6) by an appropriate nonlinearity f of the form (2.1).

Remark 2.6. Choosing $q_l = 1$ for all $l = 1, \ldots, k$, we obtain a jet realization result for nonlinearities which are polynomials in u and which are linear functions of ∇u.

In [13], Rybakowski and the present author showed that, given n, $k \in \mathbb{N}$, there exists a smooth bounded domain Ω and a potential $a : \Omega \to \mathbb{R}$ such that the operator $\Delta + a(x, y)$ with Dirichlet boundary condition has an n-dimensional kernel spanned by eigenfunctions $\phi_1, \ldots, \phi_n$ satisfying (IC) up to the order k with $\varpi = (0, 1)$. In § 6 below we will prove that, for an arbitrary principal part L of the form (1.1) on an arbitrary domain Ω, and both with Dirichlet and Neumann boundary conditions, given n, $k \in \mathbb{N}$, it is possible to construct a potential $a : \bar{\Omega} \to \mathbb{R}$ and a vector $\varpi \in \mathbb{R}^2$ such that $\dim\ker A_a = n$ and (IC) is satisfied up to the order k by an appropriate basis of $\ker A_a$.

3. Perturbation and convergence of eigenfunctions

This and the next sections are devoted to the construction of potential functions a such that the corresponding operators L_a have the properties described in § 2. First, we recall two general results on perturbation and convergence of eigenvalues and eigenfunctions of self-adjoint operators in Hilbert spaces. The reader is referred to [14] for a detailed discussion.

Consider a second-order elliptic differential operator L of the form (1.1) on an open bounded smooth domain $\Omega \subset \mathbb{R}^N$, and let $a \in C^\gamma(\bar{\Omega})$. Moreover, let D be an open bounded domain with $\bar{D} \subset \Omega$. Define the following sequence of differential operators on Ω:

$$\begin{aligned} L_k u &= L_a u + \beta_k b_k(x) u, \quad && x \in \Omega, \\ u(x) &= 0, && x \in \partial\Omega, \end{aligned}$$

or

$$\begin{aligned} L_k u &= L_a u + \beta_k b_k(x) u, \quad && x \in \Omega, \\ \frac{\partial u}{\partial \nu}(x) &= 0, && x \in \partial\Omega. \end{aligned}$$

Here, β_k, $k \in \mathbb{N}$, are positive real numbers and b_k, $k \in \mathbb{N}$, are (coefficient) functions. It was proved in [14] that under appropriate hypotheses on β_k and b_k, the eigenvalues of L_k converge, as $k \to \infty$, to the eigenvalues of the following 'limit' differential operator L_∞ on D:

$$\begin{aligned} L_\infty u &= L_a u, \quad && x \in D, \\ u(x) &= 0, && x \in \partial D. \end{aligned}$$

H^1 convergence of the corresponding eigenfunctions was also proved. The hypotheses are, essentially, that $\beta_k b_k(x)$ is very small on D but very large outside of D. To

https://doi.org/10.1017/S0308210500000226 Published online by Cambridge University Press

give a unified treatment for the different boundary conditions, it is more convenient to work, not with differential operators, but rather with the corresponding bilinear forms or even with certain abstract bilinear forms as we shall now explain.

In what follows, all vector spaces are over the reals.

DEFINITION 3.1. Let V be a vector space and $a : V \times V \to \mathbb{R}$ be symmetric bilinear form on V. If $\lambda \in \mathbb{R}$, $u \in V \setminus \{0\}$ satisfy

$$a(u,v) = \lambda\langle u,v\rangle \quad \text{for all } v \in V,$$

then we say that λ is *a proper value of* a and u is *a proper vector of* a, *corresponding to* λ. The dimension of the span of all proper vectors of a corresponding to λ is called *the multiplicity of* λ. If the set of proper values of a is countably infinite and if each proper value has finite multiplicity, then *the repeated sequence of the proper values of* a is the uniquely determined non-decreasing sequence $(\lambda_n)_{n\in\mathbb{N}}$ which contains exactly the proper values of a and the number of occurrences of each proper value in this sequence is equal to its multiplicity.

The following theorem was proved in [14].

THEOREM 3.2. *Assume the following hypotheses.*

(1) $\Omega \subset \mathbb{R}^N$ *is a bounded domain and* $D \subset \mathbb{R}^N$ *is a Lipschitz domain with* $\bar{D} \subset \Omega$. *Given a function* u *defined on* D, $u\tilde{}$ *denotes the trivial extension of* u *to* Ω.

(2) b, $b_k : \bar{\Omega} \to \mathbb{R}$, $k \in \mathbb{N}$, *are continuous functions and* β_k, $k \in \mathbb{N}$, *are positive real numbers. Moreover,* $b(x) > 0$ *for* $x \in \Omega \setminus D$, $b_k \to b$ *uniformly on* $\bar{\Omega}$, $\beta_k \to \infty$,

$$\inf_{x\in\Omega,\, k\in\mathbb{N}}\{\beta_k b_k(x)\} > -\infty \quad \textit{and} \quad \sup_{x\in D}\{\beta_k|b_k(x)|\} \to 0.$$

(3) V *is a closed linear subspace of* $H^1(\Omega)$ *such that whenever* $u \in H_0^1(D)$, *then* $u\tilde{} \in V$. V *is endowed with the scalar product of* $H^1(\Omega)$.

(4) $\|\cdot\|_D$ (*respectively* $\|\cdot\|$) *denotes the* $H^1(D)$- (*respectively* $H^1(\Omega)$-) *norm,* $|\cdot|_D$ (*respectively* $|\cdot|$) *denotes the* $L^2(D)$- (*respectively the* $L^2(\Omega)$-) *norm and* $\langle\cdot,\cdot\rangle_D$ (*respectively* $\langle\cdot,\cdot\rangle$) *denotes the* $L^2(D)$- (*respectively* $L^2(\Omega)$-) *scalar product.*

(5) $a : V \times V \to \mathbb{R}$ *is a symmetric bilinear form and there are constants* d, C, $\alpha \in \mathbb{R}$, $\alpha > 0$, *such that, for all* u, $v \in V$,

$$|a(u,v)| \leqslant C\|u\|\|v\|,$$
$$a(u,u) \geqslant \alpha\|u\|^2 - d|u|^2.$$

Let $a_\infty : H_0^1(D) \times H_0^1(D) \to \mathbb{R}$ *be the restriction of* a *to* $H_0^1(D)$. *For* $k \in \mathbb{N}$, *let* $(\lambda_n^k)_{n\in\mathbb{N}}$ *be the repeated sequence of proper values of the symmetric bilinear form* $a_k : V \times V \to \mathbb{R}$ *defined by*

$$a_k(u,v) = a(u,v) + \beta_k \int_\Omega b_k(x)u(x)v(x)\,\mathrm{d}x$$

https://doi.org/10.1017/S0308210500000226 Published online by Cambridge University Press

and $(u_n^k)_{n\in\mathbb{N}}$ *be an* $L^2(\Omega)$*-orthonormal sequence of corresponding proper vectors of* a_k*. Moreover, let* $(\mu_n)_{n\in\mathbb{N}}$ *be the repeated sequence of proper values of* a_∞*.*

Then there is an increasing function $\phi:\mathbb{N}\to\mathbb{N}$ *and a sequence* $(v_n)_{n\in\mathbb{N}}$ *in* $H_0^1(D)$ *such that for every* $n\in\mathbb{N}$*,* v_n *is a proper vector of* a_∞ *corresponding to* μ_n*, the subsequence*

$$(\lambda_n^{\phi(k)})_{k\in\mathbb{N}} \quad \textit{of} \quad (\lambda_n^k)_{k\in\mathbb{N}}$$

converges to μ_n *and the subsequence*

$$(u_n^{\phi(k)})_{k\in\mathbb{N}} \quad \textit{of} \quad (u_n^k)_{k\in\mathbb{N}}$$

converges to $v_n\tilde{}$ *in* V*, as* $k\to\infty$*.*

As a consequence of this theorem, we will get that, if, for some $D\subset\Omega$ and for some $a\in C^\gamma(\bar{\Omega})$, the operator L_a restricted to D, with Dirichlet boundary condition on ∂D, has an n-dimensional kernel, and the corresponding eigenfunctions satisfy the Poláčik condition (or the algebraic independence condition), then, for sufficiently large k, the operators L_k defined above on Ω have n eigenvalues closed to zero and the corresponding eigenfunctions satisfy the Poláčik condition (or the algebraic independence condition). The next result will show that we can further perturb the operator L_k and get an n-dimensional kernel with corresponding eigenfunctions still satisfying the Poláčik condition (or the algebraic independence condition).

We use the following notation. If X is a normed space and $r>0$, then $\mathrm{B}_r(c)$ denotes the open ball in X of radius r centred at c. Moreover, $\mathrm{B}_r:=\mathrm{B}_r(0)$. Given normed spaces X and Y, we denote by $\mathcal{L}(X,Y)$ the space of all bounded linear operators from X to Y, endowed with the operator norm. Given a real Hilbert space H, $\mathcal{L}_{\mathrm{sym}}(H,H)$ is the (closed) linear subspace of $\mathcal{L}(H,H)$ consisting of all symmetric operators.

By $\mathcal{S}_p$ we denote the (finite-dimensional) space of all real symmetric $p\times p$ matrices, endowed with an arbitrary norm. The spectrum of A is denoted by $\operatorname{spec} A$.

Let H be an infinite-dimensional real Hilbert space, and let $A:\operatorname{dom}A\to H$ be linear, symmetric, bounded below and with compact resolvent. Then it follows that the spectrum of A is a countable set of real eigenvalues of finite multiplicity. This set is bounded below. We can therefore uniquely define a non-decreasing sequence $(\lambda_n)_{n\in\mathbb{N}}$ which contains exactly the eigenvalues of A, each one repeated according to its multiplicity. We call $(\lambda_n)_{n\in\mathbb{N}}$ *the repeated sequence of eigenvalues of* A.

Definition 3.3. We say that the triple $(H,\mathcal{G},A)$ *is of type* $[p,M,\eta,\theta]$ if and only if the following properties hold.

(1) $\mathcal{G}$ is a closed linear subspace of $\mathcal{L}_{\mathrm{sym}}(H,H)$.

(2) p is a positive integer, M, η and θ are positive reals.

(3) Let $(\lambda_n)_{n\in\mathbb{N}}$ be the repeated sequence of the eigenvalues of A. There exist real numbers γ_1 and γ_2 and $l\in\mathbb{N}_0$ such that, setting $\lambda_0=-\infty$,

$$0<\gamma_2-\gamma_1<M,$$
$$\lambda_l<\gamma_1-4\eta<\gamma_1<\lambda_{l+1}\leqslant\lambda_{l+p}<\gamma_2<\gamma_2+4\eta<\lambda_{l+p+1}.$$

(4) There exists an H-orthonormal set of vectors ϕ_j, $j = 1, \ldots, p$, in $\operatorname{dom} A$ such that $A\phi_j = \lambda_{l+j}\phi_j$, $j = 1, \ldots, p$, and such that the operator $T : \mathcal{G} \to \mathcal{S}_p$,

$$B \mapsto (\langle B\phi_i, \phi_j \rangle)_{ij},$$

is such that

$$T(\mathrm{B}_1) \supset \mathrm{B}_\theta,$$

i.e. the image of the unit ball (at zero) in $\mathcal{G}$ contains the θ-ball (at zero) in $\mathcal{S}_p$.

The following theorem was proved in [14].

Theorem 3.4. *For every $(p, M, \eta, \theta) \in \mathbb{N} \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$, there exists a positive number $\alpha_0 = \alpha_0(p, M, \eta, \theta)$ with the following property. Whenever the triple $(H, \mathcal{G}, A)$ is of type $[p, M, \eta, \theta]$, l, γ_1 and γ_2 are as in definition 3.3 (with respect to the triple $(H, \mathcal{G}, A)$), $0 < \alpha \leqslant \alpha_0$ and $(\mu_1, \ldots, \mu_p) \in \mathbb{R}^p$ is non-decreasing with $|\mu_j - \lambda_{l+j}| < \alpha$ for $j = 1, \ldots, p$, and if $\mathcal{D}$ is an arbitrary linear dense subspace of $\mathcal{G}$, then there exists a $B \in \mathcal{D}$ with $|B| < (1/2)\theta\alpha$, such that, if $(\lambda_n(B))_{n \in \mathbb{N}}$ denotes the repeated sequence of eigenvalues of $A + B$ and $\lambda_0(B) := -\infty$, then*

$$\lambda_l(B) < \gamma_1 - 3\eta < \gamma_1 - \eta < \lambda_{l+1}(B) \leqslant \lambda_{l+p}(B) < \gamma_2 + \eta < \gamma_2 + 3\eta < \lambda_{l+p+1}(B) \tag{3.1}$$

and

$$\lambda_{l+j}(B) = \mu_j, \quad j = 1, \ldots, p.$$

More details about the results described in this section can be found in [14].

4. Localization

Let $N \geqslant 2$ and $\Omega \subset \mathbb{R}^N$ be an open bounded domain with $C^{2,\gamma}$ boundary. Let L be a second-order strongly elliptic differential operator of the form (1.1). As we have seen, the problem of realization of vector fields and jets in scalar parabolic partial differential equations (PDEs) reduces to the problem of constructing a potential function a in such a way that the operator L_a with Dirichlet or Neumann boundary conditions has a high-dimension kernel, spanned by eigenfunctions satisfying certain non-degeneracy conditions. As a first step towards this direction, we will prove a sort of 'localization lemma'. The content of this lemma is essentially the following. We can always find some small subdomain $D \subset \Omega$ and some potential a on Ω in such a way that the operator L_a restricted to D with Dirichlet condition on ∂D satisfies the above-mentioned properties, provided we are able to construct a potential a_0 on some other open bounded domain S in such a way that the operator $\Delta + a_0$ on S with Dirichlet condition on S satisfies the same properties.

Lemma 4.1. *Let Ω and L be as above. Moreover, let $S \subset \mathbb{R}^N$ be another open bounded domain; assume S has $C^{2,\gamma}$ boundary. Let us suppose there exists a function $a_0 \in C^\gamma(\bar{S})$ such that the operator $\Delta + a_0(x)$ on S with Dirichlet boundary condition on ∂S has an n-dimensional kernel, spanned by $L^2(S)$-orthonormal eigenfunctions $\phi_1, \ldots, \phi_n$, and that the set of functions*

$$\{\phi_i\phi_j,\ 1 \leqslant i \leqslant j \leqslant n\}$$

https://doi.org/10.1017/S0308210500000226 Published online by Cambridge University Press

is linearly independent. Then, for every $\epsilon > 0$*, there exist an invertible affine transformation* $W_\epsilon : \mathbb{R}^N \to \mathbb{R}^N$*, an open bounded domain* $D_\epsilon \subset\subset \Omega$ *and a potential* $a_\epsilon \in C^\gamma(\bar{\Omega})$*, with the following properties.*

(1) $D_\epsilon = W_\epsilon(S)$.

(2) *The operator* L_{a_ϵ} *restricted to* D_ϵ*, with Dirichlet boundary condition on* ∂D_ϵ *has an* n*-dimensional kernel spanned by the* $L^2(D_\epsilon)$*-orthonormal functions* $\psi^\epsilon_1, \dots, \psi^\epsilon_n$.

(3) $\|(\det DW_\epsilon)^{1/2}\psi^\epsilon_i(W_\epsilon(\cdot)) - \phi_i(\cdot)\|_{H^1(S)} < \epsilon$, $i = 1, \dots, n$.

Proof. First, observe that, since S has a $C^{2,\gamma}$ boundary, we can assume that $a_0 \in C^\gamma(\mathbb{R}^N)$.

We indicate by λ_m, $m \in \mathbb{N}$, the repeated sequence of the eigenvalues of the operator $\Delta + a_0(x)$ on S with Dirichlet boundary condition on ∂S; in the hypothesis, we have assumed that this operator has an n-dimensional kernel, so there is an $l \geqslant 1$ such that $\lambda_l < \lambda_{l+1} = \cdots = \lambda_{l+n} = 0 < \lambda_{l+n+1}$.

We proceed in several steps.

Step 1. Take $\bar{x} \in S$ and $x_0 \in \Omega$. Let $G_0 := G(x_0)$, where $G(x) := (g_{ij}(x))_{ij}$. G_0 is a symmetric positive definite $N \times N$ matrix, so we can take an invertible $N \times N$ matrix Q such that $G_0 = QQ^T$. We define the affine transformation

$$Z : \mathbb{R}^N \to \mathbb{R}^N,$$
$$x \mapsto x_0 + Q(x - \bar{x})$$

and we set $D_1 := Z(S)$. Finally, we define

$$\tilde{a} : D_1 \to \mathbb{R},$$
$$\tilde{a}(x) := a_0(Z^{-1}(x)).$$

The operator $\Delta + a_0(x)$ on S with Dirichlet boundary condition on ∂S has the same repeated sequence of eigenvalues of the operator $\operatorname{div}(G_0\nabla) + \tilde{a}(x)$ on D_1 with Dirichlet boundary condition on ∂D_1. In particular, this last operator has an n-dimensional kernel spanned by the $L^2(D_1)$-orthonormal functions

$$\tilde{\phi}_i(x) := (\det Q)^{-1/2}\phi_i(Z^{-1}(x)), \quad i = 1, \dots, n.$$

Obviously, the set of functions

$$\{\tilde{\phi}_i\tilde{\phi}_j,\ 1 \leqslant i \leqslant j \leqslant n\}$$

is linearly independent.

Step 2. For $\rho \geqslant 0$ sufficiently small, we consider the differential operators

$$L_\rho := \operatorname{div}(G(x_0 + \rho(x - x_0))\nabla) + \tilde{a}(x), \quad x \in D_1,$$

on D_1 with Dirichlet boundary condition on ∂D_1; note $L_0 = \operatorname{div}(G_0\nabla) + \tilde{a}(x)$. We indicate by λ^ρ_m, $m \in \mathbb{N}$, the repeated sequence of eigenvalues of L_ρ.

https://doi.org/10.1017/S0308210500000226 Published online by Cambridge University Press

Let A_ρ be the abstract self-adjoint closed operator in $L^2(D_1)$ defined by L_ρ; since the boundary of D_1 is of class $C^{2,\gamma}$ and the coefficients g_{ij} are in $C^{1,\gamma}$, it follows that, for all ρ, the domain of A_ρ is $H^2(D_1) \cap H_0^1(D_1)$. Moreover, the map

$$\rho \mapsto A_\rho,$$
$$[0, \rho_0[\to \mathcal{L}(H^2(D_1) \cap H_0^1(D_1), L^2(D_1))$$

is continuous. This implies that $\lambda_m^\rho \to \lambda_m$ as $\rho \to 0$ for all m; then we can find numbers $\gamma_1, \gamma_2 \in \mathbb{R}$, $M, \eta \in \mathbb{R}_+$, such that $0 < \gamma_2 - \gamma_1 < M$ and, for all sufficiently small ρ,

$$\lambda_l^\rho < \gamma_1 - 4\eta < \gamma_1 < \lambda_{l+1}^\rho \leqslant \cdots \leqslant \lambda_{l+n}^\rho < \gamma_2 < \gamma_2 + 4\eta < \lambda_{l+n+1}^\rho;$$

in particular, the set

$$\{\lambda_{l+1}^\rho, \dots, \lambda_{l+n}^\rho\}$$

is a spectral set of A_ρ and we can consider the corresponding spectral projection P_ρ and the corresponding spectral invariant subspace X_ρ. By the general formula

$$P_\rho = \frac{1}{2\pi \mathrm{i}} \int_\Gamma (\zeta - A_\rho)^{-1} \, \mathrm{d}\zeta,$$

it follows that the map

$$\rho \mapsto P_\rho,$$
$$[0, \rho_0[\to \mathcal{L}(L^2(D_1), H^2(D_1) \cap H_0^1(D_1))$$

is continuous. By using the spectral projection P_ρ, together with the Grahm–Schmidt orthonormalization algorithm, we can find, for all ρ, an $L^2(D_1)$-orthonormal basis $\tau_1^\rho, \dots, \tau_n^\rho$ of X_ρ, with

$$\tau_i^\rho \to \tilde{\phi}_i \quad \text{as } \rho \to 0$$

in $H^2(D_1) \cap H_0^1(D_1)$ for all $i = 1, \dots, n$. In order to apply theorem 3.4, we need a *basis of eigenfunctions*; to overcome this difficulty, we proceed in the following way. For all $\rho > 0$, we can find an orthogonal $n \times n$ matrix $R^\rho = (r_{ij}^\rho)_{ij}$ such that the functions

$$\chi_i^\rho := \sum_{j=1}^n r_{ij}^\rho \tau_j^\rho, \quad i = 1, \dots n,$$

are an $L^2(D_1)$-orthonormal basis of eigenfunctions of X_ρ, with

$$A_\rho \chi_i^\rho = \lambda_i^\rho \chi_i^\rho, \quad i = 1, \dots, n.$$

By compactness, we can find a sequence $(\rho_k)_{k \in \mathbb{N}}$, with $\rho_k \to 0$ as $k \to \infty$, and an orthogonal matrix $R = (r_{ij})_{ij}$, such that

$$R^{\rho_k} \to R \quad \text{as } k \to \infty.$$

It follows that, for all $i = 1, \dots, n$,

$$\chi_i^{\rho_k} \to \sum_{j=1}^n r_{ij} \tilde{\phi}_j =: \chi_i \quad \text{as } k \to \infty$$

https://doi.org/10.1017/S0308210500000226 Published online by Cambridge University Press

in $H^2(D_1) \cap H_0^1(D_1)$. Of course, $\chi_1, \dots, \chi_n$ are an orthonormal basis of the n-dimensional kernel of $L_0 = \operatorname{div}(G_0 \nabla) + \tilde{a}(x)$. Moreover, we claim that the set of functions

$$\{\chi_i \chi_j,\ 1 \leqslant i \leqslant j \leqslant n\}$$

is still linearly independent. In fact, let

$$\sum_{1 \leqslant i \leqslant j \leqslant n} a_{ij} \chi_i \chi_j \equiv 0, \quad a_{ij} \in \mathbb{R}, \quad 1 \leqslant i \leqslant j \leqslant n.$$

Define

$$\begin{aligned} \tilde{a}_{ii} &:= a_{ii}, & 1 \leqslant i \leqslant n, \\ \tilde{a}_{ij} &:= \tfrac{1}{2} a_{ij}, & 1 \leqslant i < j \leqslant n, \\ \tilde{a}_{ij} &:= \tfrac{1}{2} a_{ji}, & 1 \leqslant j < i \leqslant n. \end{aligned}$$

The matrix $(\tilde{a}_{ij})_{ij}$ is symmetric. Next, define

$$\tilde{b}_{ij} := \sum_{h,k=1}^{n} r_{hi} \tilde{a}_{hk} r_{kj}, \quad i, j = 1, \dots, n.$$

Since the matrix $(r_{ij})_{ij}$ is orthogonal, the matrix $(\tilde{b}_{ij})_{ij}$ is also symmetric. Moreover, $\tilde{b}_{ij} = 0$ for $i, j = 1, \dots, n$, if and only if $\tilde{a}_{ij} = 0$ for $i,\ j = 1, \dots, n$. Finally, define

$$\begin{aligned} b_{ii} &:= \tilde{b}_{ii}, & 1 \leqslant i \leqslant n, \\ b_{ij} &:= 2\tilde{b}_{ij}, & 1 \leqslant i < j \leqslant n, \\ b_{ij} &:= 0, & 1 \leqslant j < i \leqslant n. \end{aligned}$$

Then we have

$$\begin{aligned} 0 \equiv & \sum_{1 \leqslant i \leqslant j \leqslant n} a_{ij} \chi_i \chi_j \\ &= \sum_{i,j=1}^{n} \tilde{a}_{ij} \chi_i \chi_j \\ &= \sum_{i,j=1}^{n} \tilde{a}_{ij} \biggl(\sum_{h=1}^{n} r_{ih} \tilde{\phi}_h \biggr) \biggl(\sum_{k=1}^{n} r_{jk} \tilde{\phi}_k \biggr) \\ &= \sum_{h,k=1}^{n} \biggl(\sum_{i,j=1}^{n} r_{ih} \tilde{a}_{ij} r_{jk} \biggr) \tilde{\phi}_h \tilde{\phi}_k \\ &= \sum_{h,k=1}^{n} \tilde{b}_{hk} \tilde{\phi}_h \tilde{\phi}_k \\ &= \sum_{1 \leqslant h \leqslant k \leqslant n} b_{hk} \tilde{\phi}_h \tilde{\phi}_k. \end{aligned}$$

Since the set of functions

$$\{\tilde{\phi}_i \tilde{\phi}_j,\ 1 \leqslant i \leqslant j \leqslant n\}$$

https://doi.org/10.1017/S0308210500000226 Published online by Cambridge University Press

is linearly independent, it follows that $b_{ij} = 0$ for $1 \leqslant i \leqslant j \leqslant n$ and, consequently, also $a_{ij} = 0$ for $1 \leqslant i \leqslant j \leqslant n$. This proves the claim.

For $c \in C^0(\bar{D}_1)$, let $B_c \in \mathcal{L}_{\text{sym}}(L^2(D_1), L^2(D_1))$ be the map

$$(B_c u)(x) = c(x)u(x), \quad u \in L^2(D_1), \quad x \in D_1.$$

Note that

$$|B_c|_{\mathcal{L}(L^2(D_1),L^2(D_1))} = |c|_{C^0(\bar{D}_1)}. \tag{4.1}$$

Let $\mathcal{G}$ be the set of all B_c with $c \in C^0(\bar{D}_1)$. It follows that $\mathcal{G}$ is a closed linear subspace of $\mathcal{L}_{\text{sym}}(L^2(D_1), L^2(D_1))$. Now, since the functions $\{\chi_i\chi_j,\ 1 \leqslant i \leqslant j \leqslant n\}$ are linearly independent, it is easy to see that the operator $T : \mathcal{G} \to \mathcal{S}_p$,

$$B \mapsto (\langle B\chi_i, \chi_j\rangle)_{ij}$$

is surjective. By the open mapping theorem there is a $\theta > 0$ such that

$$T(\mathrm{B}_1) \supset \mathrm{B}_\theta.$$

For $k \in \mathbb{N}$ let $T_k : \mathcal{G} \to \mathcal{S}_p$ be the map

$$B \mapsto (\langle B\chi_i^{\rho_k}, \chi_j^{\rho_k}\rangle)_{ij}.$$

Then $T_k \to T$ in $\mathcal{L}(\mathcal{G}, \mathcal{S}_p)$, so it is easy to see that

$$T_k(\mathrm{B}_1) \supset \mathrm{B}_\theta \quad \text{for } k \text{ large enough.}$$

Moreover, we have

$$\lambda_l^{\rho_k} < \gamma_1 - 4\eta < \gamma_1 < \lambda_{l+1}^{\rho_k} \leqslant \lambda_{l+n}^{\rho_k} < \gamma_2 < \gamma_2 + 4\eta < \lambda_{l+n+1}^{\rho_k} \quad \text{for } k \text{ large enough.} \tag{4.2}$$

Set $p := n$ and let $\alpha_0 = \alpha_0(p, M, \eta, \theta)$ be as in theorem 3.4. For all large k, there is an $\alpha_k > 0$ such that $|\lambda_{l+j}^{\rho_k}| < \alpha_k < \alpha_0$ for $j = 1, \dots, n$ and $\alpha_k \to 0$ as $k \to 0$. Thus, by theorem 3.4 (with $A := A_{\rho_k}$, $\mu_j := 0$, $\lambda_{l+j} := \lambda_{l+j}^{\rho_k}$ for $j = 1, \dots, n$ and $\mathcal{D}$ equal to the set of all B_c where c is a $C^\gamma(\mathbb{R}^N)$ function), there exists, for each large k, a $C^\gamma(\mathbb{R}^N)$ function $c_k : \mathbb{R}^N \to \mathbb{R}$ such that $|c_k|_{C^0(\bar{D}_1)} < (1/2)\theta\alpha_k$ and such that if $(\hat{\lambda}_n^{\rho_k})_{n\in\mathbb{N}}$ denotes the repeated sequence of eigenvalues of $A_{\rho_k} + B_{c_k}$, then

$$\hat{\lambda}_l^{\rho_k} < -3\eta < -\eta < \hat{\lambda}_{l+1}^{\rho_k} \leqslant \hat{\lambda}_{l+n}^{\rho_k} < \eta < 3\eta < \hat{\lambda}_{l+n}^{\rho_k} \tag{4.3}$$

and

$$\hat{\lambda}_{l+j}^{\rho_k} = 0, \quad j = 1, \dots, n. \tag{4.4}$$

Moreover, since

$$A_{\rho_k} + B_{c_k} \to A_0 \quad \text{as } k \to \infty$$

in $\mathcal{L}(H^2(D_1) \cap H_0^1(D_1), L^2(D_1))$, by using again the spectral projection P_{ρ_k} on the kernel of $A_{\rho_k} + B_{c_k}$, together with the Grahm–Schmidt $L^2(D_1)$-orthonormalization algorithm, we can find an $L^2(D_1)$-orthonormal basis $\tilde{\phi}_1^{\rho_k}, \dots, \tilde{\phi}_n^{\rho_k}$ of $\ker(A_{\rho_k} + B_{c_k})$ with

$$\tilde{\phi}_i^{\rho_k} \to \tilde{\phi}_i \quad \text{as } k \to \infty$$

in $H^2(D_1) \cap H_0^1(D_1)$ for all $i = 1, \dots, n$.

https://doi.org/10.1017/S0308210500000226 Published online by Cambridge University Press

Summarizing, we have found a sequence of positive numbers ρ_k, $\rho_k \to 0$ as $k \to \infty$, and a sequence of potentials $c_\kappa \in C^\gamma(\mathbb{R}^N)$, $c_k \to 0$ in $C^0(\bar{D}_1)$ as $k \to \infty$, such that, for all (sufficiently large) k, the operator

$$L_{\rho_k} + c_k(x) = \operatorname{div}(G(x_0 + \rho_k(x - x_0))\nabla) + \tilde{a}(x) + c_k(x), \quad x \in D_1,$$

on D_1, with Dirichlet boundary condition on ∂D_1, has an n-dimensional kernel spanned by $L^2(D_1)$-orthonormal functions $\tilde{\phi}_1^{\rho_k}, \dots, \tilde{\phi}_n^{\rho_k}$, with

$$\tilde{\phi}_i^{\rho_k} \to \tilde{\phi}_i \quad \text{as } k \to \infty$$

in $H^2(D_1) \cap H_0^1(D_1)$ for $i = 1, \dots, n$.

STEP 3. For all $\rho > 0$, we define the homothety

$$O_\rho : \mathbb{R}^N \to \mathbb{R}^N, \qquad x \mapsto x_0 + \rho(x - x_0)$$

and we define

$$D_\rho := O_\rho(D_1) = \{y \in \mathbb{R}^N \mid y = x_0 + \rho(x - x_0),\, x \in D_1\}.$$

If ρ is sufficiently small, then $\bar{D}_\rho \subset \Omega$. So, for sufficiently large k, we can consider the operator

$$\begin{aligned}\operatorname{div}(G(x)\nabla) &+ (\rho_k)^{-2}\tilde{a}(x_0 + (\rho_k)^{-1}(x - x_0)) + (\rho_k)^{-2}c_k(x_0 + (\rho_k)^{-1}(x - x_0)) \\ &= \operatorname{div}(G(x)\nabla) + (\rho_k)^{-2}\tilde{a}((O_{\rho_k})^{-1}(x)) + (\rho_k)^{-2}c_k((O_{\rho_k})^{-1}(x)) \qquad (4.5)\end{aligned}$$

on D_{ρ_k} with Dirichlet boundary condition on ∂D_{ρ_k}. This operator has the same repeated sequence of eigenvalues of the operator

$$(\rho_k)^{-2} \operatorname{div}(G(x_0 + \rho_k(x - x_0))\nabla) + (\rho_k)^{-2}\tilde{a}(x) + (\rho_k)^{-2}c_k(x) \qquad (4.6)$$

on D_1 with Dirichlet boundary condition on ∂D_1. In particular, the operator (4.5) has an n-dimensional kernel spanned by the $L^2(D_{\rho_k})$-orthonormal functions

$$\begin{aligned}\psi_i^{\rho_k}(x) &:= (\rho_k)^{-N/2}\tilde{\phi}_i^{\rho_k}(x_0 + (\rho_k)^{-1}(x - x_0)) \\ &= (\rho_k)^{-N/2}\tilde{\phi}_i^{\rho_k}((O_{\rho_k})^{-1}(x)), \quad i = 1, \dots, n.\end{aligned}$$

Now we define $W_k := O_{\rho_k} \circ Z$, $D_k := W_k(S) = D_{\rho_k}$ and

$$\begin{aligned}a_k(x) &:= (\rho^k)^{-2}\tilde{a}((O_{\rho_k})^{-1}(x)) + (\rho^k)^{-2}c_k((O_{\rho_k})^{-1}(x)) \\ &= (\rho^k)^{-2}a_0((W_k)^{-1}(x)) + (\rho^k)^{-2}c_k((O_{\rho_k})^{-1}(x)).\end{aligned}$$

We finally estimate, for $i = 1, \dots, n$,

$$\begin{aligned}&\|(\det DW_k)^{1/2}\psi_i^{\rho_k}(W_k(\cdot)) - \phi_i(\cdot)\|_{H^1(S)} \\ &\quad = \|(\det Q)^{1/2}(\rho_k)^{N/2}\psi_i^{\rho_k}(W_k(\cdot)) - \phi_i(\cdot)\|_{H^1(S)} \\ &\quad = \|(\det Q)^{1/2}\tilde{\phi}_i^{\rho_k}((O_{\rho_k}^{-1} \circ W_k(\cdot)) - \phi_i(\cdot)\|_{H^1(S)} \\ &\quad = \|(\det Q)^{1/2}\tilde{\phi}_i^{\rho_k}(Z(\cdot)) - \phi_i(\cdot)\|_{H^1(S)} \\ &\quad = (\det Q)^{1/2}\|\tilde{\phi}_i^{\rho_k}(Z(\cdot)) - \tilde{\phi}_i(Z(\cdot))\|_{H^1(S)} \to 0\end{aligned}$$

as $k \to \infty$.

https://doi.org/10.1017/S0308210500000226 Published online by Cambridge University Press

Now, with fixed $\epsilon > 0$, we choose a sufficiently large k and we set $W_\epsilon := W_k$, $D_\epsilon := D_k$ and $a_\epsilon := a_k$ and we have concluded. □

5. The Poláčik condition

Let $\Omega \subset \mathbb{R}^N$ be an open bounded connected set with $C^{2,\gamma}$ boundary. Let L be a differential operator of the form (1.1). The next result shows that we can always construct a potential a such that L_a satisfies the Poláčik condition on Ω.

THEOREM 5.1. *Let Ω and L be as above. Then, both for Dirichlet and Neumann boundary conditions on $\partial\Omega$, there exists a potential $a \in C^\gamma(\bar{\Omega})$ such that the operator L_a satisfies the Poláčik condition on Ω.*

Proof. Our starting point is the existence (established in [10]) of a function a_0 such that the operator $\Delta + a_0(x)$ satisfies the Poláčik condition on the unit ball $B \subset \mathbb{R}^N$ with Dirichlet boundary conditions on ∂B. In this case there is a basis of $\ker \Delta + a_0(x)$ given by functions

$$\phi_i(x) = \frac{w(|x|)}{|x|} x_i, \quad x \in B, \quad i = 1, \dots, N,$$

and

$$\phi_{N+1}(x) = v(|x|), \quad x \in B,$$

where $w,\ v : \mathbb{R} \to \mathbb{R}$ are analytic functions such that

$$w(0) = 0, \quad w'(0) \neq 0, \quad v(0) \neq 0, \quad v'(0) = 0. \tag{5.1}$$

We claim that

the functions $\phi_i\phi_j$, $1 \leqslant i \leqslant j \leqslant N+1$, are linearly independent.

In fact, let ρ_{ij}, $1 \leqslant i \leqslant j \leqslant N+1$, be real numbers with

$$\sum_{1\leqslant i\leqslant j\leqslant N+1} \rho_{ij}\phi_i\phi_j \equiv 0.$$

Evaluating this expression at $x = 0$ and using (5.1), we obtain $\rho_{N+1,N+1} = 0$. Thus

$$\frac{w(|x|)^2}{|x|^2} \sum_{1\leqslant i\leqslant j\leqslant N} \rho_{ij}x_ix_j \equiv -\frac{w(|x|)v(|x|)}{|x|} \sum_{1\leqslant i\leqslant N} \rho_{i,N+1}x_i \quad \text{for } x \neq 0.$$

Since

$$\frac{w(|x|)^2}{|x|^2} \neq 0 \quad \text{and} \quad \frac{w(|x|)v(|x|)}{|x|} \neq 0 \quad \text{for } |x| \text{ small},$$

it follows that

$$\left|\sum_{1\leqslant i\leqslant N} \rho_{i,N+1}x_i\right| = o(|x|) \quad \text{for } x \to 0.$$

However, this implies that $\rho_{i,N+1} = 0$ for $i = 1, \dots, N$. Hence

$$\sum_{1\leqslant i\leqslant j\leqslant N} \rho_{ij}x_ix_j \equiv 0,$$

https://doi.org/10.1017/S0308210500000226 Published online by Cambridge University Press

which immediately implies that $\rho_{ij} = 0$ for $1 \leqslant i \leqslant j \leqslant N$. The claim is proved.

Now we can apply lemma 4.1, with $S = B$, $n = N + 1$ and a_0 given by the construction in [10]. Following the terminology of lemma 4.1, we claim that, if we choose a sufficiently small ϵ, then the corresponding operator L_{a_ϵ} restricted to $D_\epsilon = W_\epsilon(S)$ satisfies the Poláčik condition on D_ϵ with Dirichlet boundary condition on ∂D_ϵ. First, we observe that

$$\begin{pmatrix} \psi_1^\epsilon(W_\epsilon(x)) & \nabla_x \psi_1^\epsilon(W_\epsilon(x)) \\ \vdots & \vdots \\ \psi_{N+1}^\epsilon(W) & \nabla_x \psi_{N+1}^\epsilon(W_\epsilon(x)) \end{pmatrix} = \begin{pmatrix} \psi_1^\epsilon(W_\epsilon(x)) & (\nabla\psi_1^\epsilon)(W_\epsilon(x)) \\ \vdots & \vdots \\ \psi_{N+1}^\epsilon(W_\epsilon(x)) & (\nabla\psi_{N+1}^\epsilon)(W_\epsilon(x)) \end{pmatrix} \circ \begin{pmatrix} 1 & 0 \\ 0 & DW_\epsilon(x) \end{pmatrix}.$$

Since $DW_\epsilon(x)$ is constant and invertible, we have that, if $x \in D_\epsilon = W_\epsilon(S)$, then

$$\left.\begin{array}{c} R(\psi_1^\epsilon, \dots, \psi_{N+1}^\epsilon)(x) \neq 0 \\ \text{if and only if} \\ R(\psi_1^\epsilon(W_\epsilon(\cdot)), \dots, \psi_{N+1}^\epsilon(W_\epsilon(\cdot)))(W_\epsilon^{-1}x) \neq 0. \end{array}\right\} \tag{5.2}$$

Let

$$U_0 := \{x \in S \mid R(\phi_1, \dots, \phi_{N+1})(x) \neq 0\}.$$

Since U_0 is open and, as ϵ tends to zero,

$$(\det DW_\epsilon)^{1/2} \psi_i^\epsilon(W_\epsilon(\cdot)) \to \phi_i(\cdot)$$

in $H^1(S)$ for $i = 1, \dots, N + 1$, it follows that, for sufficiently small ϵ and for some $x \in U_0$,

$$R(\psi_1^\epsilon(W_\epsilon(\cdot)), \dots, \psi_{N+1}^\epsilon(W_\epsilon(\cdot)))(x) \neq 0$$

and hence, by (5.2),

$$R(\psi_1^\epsilon, \dots, \psi_{N+1}^\epsilon)(W_\epsilon(x)) \neq 0.$$

This proves the claim. A similar argument shows that, if ϵ is sufficiently small, then the functions $\psi_i^\epsilon \psi_j^\epsilon$, $1 \leqslant i \leqslant j \leqslant N + 1$, are linearly independent.

Summarizing, we have obtained the following intermediate result. We have found an open set $D \subset \Omega$ and a function $\bar{a} \in C^\gamma(\bar{\Omega})$ such that the operator $L_{\bar{a}}$ restricted to D satisfies the Poláčik condition on D, with Dirichlet boundary condition on ∂D. Moreover,

$$\left.\begin{array}{c} \text{for every basis } \psi_1, \dots, \psi_{N+1} \text{ of the kernel of } L_{\bar{a}} \text{ on } D \\ \text{with Dirichlet boundary condition on } \partial D, \\ \text{the functions } \psi_i \psi_j,\ 1 \leqslant i \leqslant j \leqslant N+1 \\ \text{are linearly independent.} \end{array}\right\} \tag{5.3}$$

Now we proceed as in the proof of theorem 4.4 in [14]. Let $H := L^2(\Omega)$, $V := H_0^1(\Omega)$ if we are working with Dirichlet boundary condition on $\partial\Omega$, $V := H^1(\Omega)$

https://doi.org/10.1017/S0308210500000226 Published online by Cambridge University Press

if we are working with Neumann boundary condition on $\partial\Omega$. If $d \in C^0(\bar{\Omega})$, define $g_d : V \times V \to \mathbb{R}$ by

$$g_d(u,v) = \int_\Omega G(x)\nabla u \cdot \nabla v \,\mathrm{d}x + \int_\Omega d(x)uv\,\mathrm{d}x,$$

where $G(x) := (g_{ij}(x))_{i,j}$. If $d \in C^\gamma(\bar{\Omega})$, regularity theory of PDEs implies that, both for Dirichlet and Neumann boundary conditions, λ is an eigenvalue of L_d and u is a corresponding eigenfunction if and only if λ is a proper value of g_d and u is a corresponding proper vector. (In fact, every proper vector of g_d lies in $C^{2,\gamma}(\bar{\Omega})$.) Let $b \in C^0(\bar{\Omega})$ be such that $b(x) = 0$ for $x \in \bar{D}$ and $b(x) > 0$ for $x \notin D$. Furthermore, let $(\beta_k)_{k\in\mathbb{N}}$ be an arbitrary sequence of positive numbers tending to ∞. Finally, for $k \in \mathbb{N}$, let $b_k \in C^\gamma(\bar{\Omega})$ be a function such that

$$\sup_{x\in\Omega} |b_k(x) - b(x)| < \frac{1}{k}\beta_k.$$

Let $L_k := L_{\bar{a}+\beta_k b_k}$, $g_k := g_{\bar{a}+\beta_k b_k}$ and let g_∞ be the restriction of g_a to $H_0^1(D)$. We are now in a position to apply theorem 3.2: for $k \in \mathbb{N}$ let $(\lambda_n^k)_{n\in\mathbb{N}}$ be the repeated sequence of proper values of g_k and $(u_n^k)_{n\in\mathbb{N}}$ be an H-orthonormal sequence of corresponding proper vectors of g_k. Moreover, let $(\mu_n)_{n\in\mathbb{N}}$ be the repeated sequence of proper values of g_∞.

Then, using theorem 3.2 and passing to a subsequence if necessary, we may assume that there is a sequence $(v_n)_{n\in\mathbb{N}}$ in $H_0^1(D)$ such that for every $n \in \mathbb{N}$, v_n is a proper vector of g_∞ corresponding to μ_n, $(\lambda_n^k)_{k\in\mathbb{N}}$ converges to μ_n and $(u_n^k)_{k\in\mathbb{N}}$ converges to $v_n\tilde{}$ in V, as $k \to \infty$. Set $p = N + 1$. There are numbers γ_1, $\gamma_2 \in \mathbb{R}$, M, $\eta \in \mathbb{R}_+$ and $l \in \mathbb{N}$, such that

$$0 < \gamma_2 - \gamma_1 < M,$$
$$\mu_l < \gamma_1 - 4\eta < \gamma_1 < 0 = \mu_{l+1} = \mu_{l+p} < \gamma_2 < \gamma_2 + 4\eta < \mu_{l+p+1}.$$

For $c \in C^0(\bar{\Omega})$ let $B_c \in \mathcal{L}_{\mathrm{sym}}(H,H)$ be the map

$$(B_c u)(x) = c(x)u(x), \quad u \in H, \quad x \in \Omega.$$

Note that

$$|B_c|_{\mathcal{L}(H,H)} = |c|_{C^0(\bar{\Omega})}. \tag{5.4}$$

Let $\mathcal{G}$ be the set of all B_c with $c \in C^0(\bar{\Omega})$. It follows that $\mathcal{G}$ is a closed linear subspace of $\mathcal{L}_{\mathrm{sym}}(H,H)$. Now (5.3) easily implies that the operator $T : \mathcal{G} \to \mathcal{S}_p$,

$$B \mapsto (\langle B(v_{l+i}\tilde{}), v_{l+j}\tilde{}\rangle)_{ij}$$

is surjective. By the open mapping theorem there is a $\theta > 0$ such that

$$T(\mathrm{B}_1) \supset \mathrm{B}_\theta.$$

For $k \in \mathbb{N}$, let $T_k : \mathcal{G} \to \mathcal{S}_p$ be the map

$$B \mapsto (\langle Bu_{l+i}^k, u_{l+j}^k\rangle)_{ij}.$$

https://doi.org/10.1017/S0308210500000226 Published online by Cambridge University Press

Then $T_k \to T$ in $\mathcal{L}(\mathcal{G}, \mathcal{S}_p)$, so it is easy to see that

$$T_k(\mathrm{B}_1) \supset \mathrm{B}_\theta \quad \text{for } k \text{ large enough.}$$

Moreover, we have

$$\lambda_l^k < \gamma_1 - 4\eta < \gamma_1 < \lambda_{l+1}^k \leqslant \lambda_{l+p}^k < \gamma_2 < \gamma_2 + 4\eta < \lambda_{l+p+1}^k \quad \text{for } k \text{ large enough.} \tag{5.5}$$

Let $\alpha_0 = \alpha_0(p, M, \eta, \theta)$ be as in theorem 3.4. For all large k, there is an $\alpha_k > 0$ such that $|\lambda_{l+j}^k| < \alpha_k < \alpha_0$ for $j = 1, \dots, p$ and $\alpha_k \to 0$ as $k \to 0$. Thus, by theorem 3.4 (with $A := L_k$, $\mu_j := 0$, $\lambda_{l+j} := \lambda_{l+j}^k$ for $j = 1, \dots, p$ and $\mathcal{D}$ equal to the set of all B_c where c is a $C^\gamma(\bar{\Omega})$ function), there exists, for each large k, a function $c_k \in C^\gamma(\bar{\Omega})$ such that $|c_k|_{C^0(\bar{\Omega})} < (\frac{1}{2})\theta\alpha_k$ and such that if $(\hat{\lambda}_n^k)_{n\in\mathbb{N}}$ denotes the repeated sequence of eigenvalues of $L_{\bar{a}+c_k+\beta_k b_k}$ and $(\hat{u}_n^k)_{n\in\mathbb{N}}$ is an H-orthogonal sequence of the corresponding eigenfunctions, then

$$\hat{\lambda}_l^k < \gamma_1 - 3\eta < \gamma_1 - \eta < \hat{\lambda}_{l+1}^k \leqslant \hat{\lambda}_{l+p}^k < \gamma_2 + \eta < \gamma_2 + 3\eta < \hat{\lambda}_{l+p+1}^k \tag{5.6}$$

and

$$\hat{\lambda}_{l+j}^k = 0, \quad j = 1, \dots, p. \tag{5.7}$$

Now the assumptions of theorem 3.2 are satisfied, with b_k replaced by $(1/\beta_k)c_k + b_k$. Therefore, using theorem 3.2 again and passing to a subsequence if necessary, we may assume that there is a sequence $(\hat{v}_n)_{n\in\mathbb{N}}$ in $H_0^1(D)$ such that for every $n \in \mathbb{N}$, $\hat{v}_n$ is a proper vector of g_∞ corresponding to μ_n, $(\hat{\lambda}_n^k)_{k\in\mathbb{N}}$ converges to μ_n and $(\hat{u}_n^k)_{k\in\mathbb{N}}$ converges to $\hat{v}_n\tilde{}$ in V, as $k \to \infty$.

Finally, $H^1(\Omega)$-convergence of u_n^k to $\hat{v}_n\tilde{}$ as k tends to infinity implies easily that, for sufficiently large k, the operator L_a, $a = \bar{a} + c_k + \beta_k b_k$, satisfies the Poláčik condition on Ω. The theorem is proved. □

6. The algebraic independence condition

Let $\Omega \subset \mathbb{R}^2$ be an open bounded domain with a $C^{2,\gamma}$ boundary and let L be a differential operator of the form (1.1). In this section we want to prove that, for both Dirichlet and Neumann boundary conditions on $\partial\Omega$, we can construct a potential $a \in C^\gamma(\bar{\Omega})$ such that the operator L_a has a kernel of a prescribed dimension n, spanned by eigenfunctions satisfying the algebraic independence condition (IC) in § 2 up to a prescribed order k, with an appropriate $\varpi \in \mathbb{R}^2$.

THEOREM 6.1. *Let Ω and L be as above and let n, $k \in \mathbb{N}$. Then, for both Dirichlet and Neumann boundary conditions on $\partial\Omega$, there exists a potential $a \in C^\gamma(\bar{\Omega})$ with the following properties.*

(1) *The operator L_a has an n-dimensional kernel.*

(2) *There exists a vector $\varpi \in \mathbb{R}^2$ and an $L^2(\Omega)$-orthonormal basis $u_1, \dots, u_n$ of the kernel of L such that the algebraic independence condition (IC) in § 2 is satisfied up to the order k.*

Proof. As in the proof of theorem 5.1, our starting point is the existence (established in [13]) of such a potential for a suitable smooth bounded domain when $L = \Delta$, with Dirichlet boundary conditions, and with $\varpi = (0,1)$. So we can always take a bounded smooth domain S and a smooth potential $a_0 : \bar{S} \to \mathbb{R}$ such that the following hold.

(1) The operator $\Delta + a_0(x,y)$ on S with Dirichlet boundary condition on ∂S has an n-dimensional kernel.

(2) There is an $L^2(S)$-orthonormal basis $\phi_1, \dots, \phi_n$ of the kernel of $\Delta + a_0(x,y)$ such that (IC) is satisfied up to the order k with $\varpi = (0,1)$, i.e. for every $l = 1, \dots, k$ and every q, $1 \leqslant q \leqslant l$, the functions

$$\left\{ \sum_{\gamma \leqslant \beta,\, |\gamma| = q} \frac{1}{\gamma!(\beta-\gamma)!} \phi^{\beta-\gamma+\epsilon_j} \phi_y^{\gamma} \right\}_{j=1,\dots,n,\, |\beta| = l}$$

are linearly independent.

Moreover, the functions $\phi_i \phi_j$, $1 \leqslant i \leqslant j \leqslant n$ are linearly independent. Now, as in the proof of theorem 5.1, we apply lemma 4.1; following the therminology of lemma 4.1, we obtain that, if we choose a sufficiently small ϵ, then, for some smooth potential a_ϵ, the kernel of L_{a_ϵ} on D_ϵ with Dirichlet condition on ∂D_ϵ is spanned by $L^2(D_\epsilon)$-orthonormal functions $\psi_1^{\epsilon}, \dots, \psi_n^{\epsilon}$ such that for every $l = 1, \dots, k$, and for every q, $1 \leqslant q \leqslant l$, the functions

$$\left\{ \sum_{\gamma \leqslant \beta,\, |\gamma| = q} \frac{1}{\gamma!(\beta-\gamma)!} \psi^{\epsilon}(W_\epsilon(\cdot))^{\beta-\gamma+\epsilon_j} \psi^{\epsilon}(W_\epsilon(\cdot))_y^{\gamma} \right\}_{j=1,\dots,n,\, |\beta| = l}$$

are linearly independent on S. Since, for $i = 1, \dots, n$,

$$\psi_i^{\epsilon}(W_\epsilon(\cdot))_y = (\nabla \psi_i^{\epsilon})(W_\epsilon(\cdot)) \cdot \varpi_\epsilon,$$

where ϖ_ϵ is the second column of the (constant) matrix $DW_\epsilon(\cdot)$, we reach that, for every $l = 1, \dots, k$ and for every q, $1 \leqslant q \leqslant l$, the functions

$$\left\{ \sum_{\gamma \leqslant \beta,\, |\gamma| = q} \frac{1}{\gamma!(\beta-\gamma)!} \psi^{\epsilon}(\cdot)^{\beta-\gamma+\epsilon_j} \psi_{\varpi}^{\epsilon}(\cdot)^{\gamma} \right\}_{j=1,\dots,n,\, |\beta| = l}$$

are linearly independent on D_ϵ. Moreover, the functions $\psi_i^{\epsilon} \psi_j^{\epsilon}$, $1 \leqslant i \leqslant j \leqslant n$, are linearly independent on D_ϵ. Finally, we conclude, arguing exactly as in the proof of theorem 5.1, applying theorems 3.2 and 3.4. □

Remark 6.2. The present result generalizes naturally to any space dimension $N \geqslant 2$ (see [8]).

Acknowledgments

This research is part of my PhD thesis (1997, SISSA, Trieste, Italy) under the direction of Professor K. Rybakowski, to whom I am grateful for suggesting the problem to me and for all our stimulating and fruitful discussions.

References

1 E. N. Dancer. On the existence of two-dimensional invariant tori for scalar parabolic equations with time periodic coefficients. *Annali Scuola Norm. Sup. Pisa* **18** (1991), 455–471.

2 E. N. Dancer and P. Poláčik. Realization of vector fields and dynamics of spatially homogeneous parabolic equations. *Mem. Am. Math. Soc.* **668** (1999), 82pp.

3 T. Faria and L. Magalhães. Realisation of ordinary differential equations by retarded functional differential equations in neighborhoods of equilibrium points. *Proc. R. Soc. Edinb.* A **125** (1995), 759–770.

4 B. Fiedler and P. Poláčik. Complicated dynamics of scalar reaction-diffusion equations with a nonlocal term. *Proc. R. Soc. Edinb.* A **115** (1990), 167–192.

5 B. Fiedler and B. Sandstede. Dynamics of periodically forced parabolic equations on the circle. *Ergodic Theory Dynamical Systems* **12** (1992), 559–571.

6 J. K. Hale. Flows on centre manifolds for scalar functional differential equations. *Proc. R. Soc. Edinb.* A **101** (1985), 193–201.

7 P. Poláčik. Complicated dynamics in scalar semilinear parabolic equations in higher space dimension. *J. Diff. Eqns* **89** (1991), 244–271.

8 P. Poláčik. High-dimensional ω-limit sets and chaos in scalar parabolic equations. *J. Diff. Eqns* **119** (1995), 24–53.

9 P. Poláčik. Reaction-diffusion equations and realization of gradient vector fields. *Proc. Equadiff.* (In the press.)

10 P. Poláčik and K. P. Rybakowski. Imbedding vector fields in scalar parabolic Dirichlet BVPs. *Annali Scuola Norm. Sup. Pisa* **22** (1995), 737–749.

11 P. Poláčik and K. P. Rybakowski. Nonconvergent bounded trajectories in semilinear heat equations. *J. Diff. Eqns* **124** (1995), 472–494.

12 M. Prizzi. Realizing vector fields without loss of derivatives. *Annali Scuola Norm. Sup. Pisa* **27** (1998), 289–307.

13 M. Prizzi and K. P. Rybakowski. Complicated dynamics of parabolic equations with simple gradient dependence. *Trans. Am. Math. Soc.* **350** (1998), 3119–3130.

14 M. Prizzi and K. P. Rybakowski. Inverse problems and chaotic dynamics of parabolic equations on arbitrary spatial domains. *J. Diff. Eqns* **142** (1998), 17–53.

15 K. P. Rybakowski. Realization of arbitrary vector fields on center manifolds of parabolic Dirichlet BVPs. *J. Diff. Eqns* **114** (1994), 199-221.

16 K. P. Rybakowski. Realization of arbitrary vector fields on invariant manifolds of delay equations. *J. Diff. Eqns* **114** (1994), 222–231.

17 K. P. Rybakowski. The center manifold technique and complex dynamics of parabolic equations. In *Topological methods in differential equations and inclusions*, NATO ASI Series (ed. A. Granas and M. Frigon), vol. 472, pp. 411–446 (Dordrecht: Kluwer, 1995).

(*Issued 17 March 2000*)