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# Rational, Log Canonical, Du Bois Singularities: On the Conjectures of Kollár and Steenbrink*

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**Abstract.** Let $X$ be a proper complex variety with Du Bois singularities. Then $H^i(X, \mathbb{C}) \to H^i(X, \mathcal{O}_X)$ is surjective for all $i$. This property makes this class of singularities behave well with regard to Kodaira type vanishing theorems. Steenbrink conjectured that rational singularities are Du Bois and Kollár conjectured that log canonical singularities are Du Bois. Kollár also conjectured that under some reasonable extra conditions Du Bois singularities are log canonical. In this article Steenbrink's conjecture is proved in its full generality, Kollár's first conjecture is proved under some extra conditions and Kollár's second conjecture is proved under a set of reasonable conditions, and shown that these conditions cannot be relaxed.

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**Key words:** singularities, rational, log canonical, Du Bois.

## 0. Introduction

Following P. Deligne's work on mixed Hodge structures, it was shown by Ph. Du Bois that for every complex scheme $X$ one can define a filtered complex $\underline{\Omega}^{\cdot}_X$ that 'resolves' $\mathbb{C}_X$, its associated graded quotients, $\mathrm{Gr}^p_{\mathrm{filt}}\,\underline{\Omega}^{\cdot}_X$, have $\mathcal{O}_X$-linear differentials and coherent cohomology sheaves and the analogue of the Hodge–de Rham spectral sequence exists and degenerates at the $E_1$ term [DB81].

Du Bois singularities were introduced by J. H. M. Steenbrink. They have the property that the zeroth graded piece of Du Bois' complex, $\mathrm{Gr}^0_{\mathrm{filt}}\,\underline{\Omega}^{\cdot}_X$, 'resolves' $\mathcal{O}_X$. In particular, let $(X, x)$ be a normal isolated singularity, $f\colon Y \to X$ a resolution of sngularities such that $E = f^{-1}(0)$ is a divisor with only simple normal crossings and $Y \setminus E \simeq X \setminus \{x\}$. Then $(X, x)$ is Du Bois if and only if $R^i f_*\mathcal{O}_Y \simeq R^i f_*\mathcal{O}_E$. for all $i > 0$.

Cohomological properties of Du Bois singularities are often similar to those of smooth points. For instance, every proper, flat degeneration $f$ over the unit disk $\Delta$ is cohomologically insignificant provided $f^{-1}(0)$ has Du Bois singularities [S81].

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As a simple consequence of the degeneration of the Hodge–de Rham spectral sequence, one easily sees that if $X$ is proper with Du Bois singularities, then $H^i(X,\mathbb{C}) \to H^i(X,\mathcal{O}_X)$ is surjective for all $i$. According to J. Kollár's Principle [K87, 23; CKM88, 8.1; K95, 9.12] this surjectivity property makes these singularities the natural class for Kodaira type vanishing theorems [K95, 9.12, 12.3, 12.10].

Therefore it is interesting to know what kind of singularities have the Du Bois property. After classifying normal Gorenstein two-dimensional Du Bois singularities Steenbrink made the following conjecture:

CONJECTURE S. *Rational singularities are Du Bois.*

At the same time he proved the conjecture for isolated singularities (cf. [S83]).

Conjecture S was recently confirmed for projective varieties by Kollár [K95, Ch. 12]. This would have completely solved the problem if one were able to prove that a variety with rational singularities can be embedded into a projective variety with only rational singularities. Unfortunately, that is still unknown. The first result of this article is that in fact Steenbrink's conjecture is true in general.

THEOREM S. *Conjecture S holds.*

The proof is somewhat similar to the one in [ibid.]. There are two main additional ideas: First, one observes that the degeneration of the Hodge–de Rham spectral sequence implies another surjectivity (in fact a weaker one) than the one used by Kollár. The second idea is that this surjectivity allows one to pass to local cohomology, where the other necessary ingredient of the proof holds. For details, see the argument in Section 2.

Kollár has also made a conjecture regarding Du Bois singularities [K92,1.13]:

CONJECTURE K. *Log canonical singularities are Du Bois.*

This has only been confirmed for isolated singularities. S. Ishii showed that a normal isolated Gorenstein singularity is Du Bois if and only if it is log canonical [I85] (see definition at the end of the Introduction). More generally it was shown by K. Watanabe and Ishii that an isolated log canonical singularity is Du Bois [W86], [I86b]. The second result of this paper is another step toward Kollár's conjecture.

THEOREM K. *Let $X$ be a complex variety with log canonical Cohen–Macaulay singularities. Let $\Sigma = \operatorname{Sing} X$ be the set of singular points of $X$ and let $\Sigma_r$ denote the smallest closed subset of $X$ such that $X \setminus \Sigma_r$ has rational singularities. Assume that either* $\dim \Sigma + \dim \Sigma_r + 1 < \dim X$ *or $\Sigma$ has Du Bois singularities. Then $X$ has Du Bois singularities.*



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*Remark*. Unfortunately, this still does not solve even the three-dimensional case completely.* However, it gives some new results already in that case: It implies that a log canonical Cohen–Macaulay threefold with only finitely many nonrational points has Du Bois singularities. Similarly a log canonical Cohen–Macaulay threefold whose singular set is a curve with only ordinary singularities is Du Bois. In higher dimensions there are many more new cases: for instance a $d$-dimensional log canonical Cohen–Macaulay variety whose singular set is a curve is Du Bois as soon as $d \geqslant 4$, a log canonical Cohen–Macaulay variety whose singular set is smooth is always Du Bois.

On the other hand, one would also like to know how far Du Bois singularities are from being log canonical. Simple examples show that there are Du Bois singularities whose canonical divisor is not $\mathbb{Q}$-Cartier and there are Du Bois singularities whose canonical divisor is $\mathbb{Q}$-Cartier, but the singularity fails to be log canonical cf. [W86, 4.13; I86b, 2.5; I86a, 3.3]. One fact to keep in mind is that rational singularities are not necessarily log canonical, in particular there are two-dimensional rational singularities (and therefore Du Bois) that are not log canonical, but have a $\mathbb{Q}$-Cartier canonical divisor.

In light of these facts the following seems to be the best one can hope for in this direction.

THEOREM K′. *Let $U$ be a normal variety and assume that $K_U$ is Cartier and $U$ has Du Bois singularities. Then $U$ is log canonical.*

This was also conjectured by Kollár.

DEFINITIONS AND NOTATION. Throughout the article the groundfield will always be $\mathbb{C}$, the field of complex numbers. A *complex scheme* (resp. *complex variety*) will mean a separated scheme (resp. variety) of finite type over $\mathbb{C}$.

A divisor $D$ is called $\mathbb{Q}$-Cartier if $mD$ is Cartier for some $m > 0$. A normal variety $X$ is said to have *log canonical* (resp. *log terminal*, *canonical*) *singularities* if $K_X$ is $\mathbb{Q}$-Cartier and for any resolution of singularities $f\colon Y \to X$, with the collection of exceptional prime divisors $\{E_i\}$, there exist $a_i \in \mathbb{Q}$, $a_i \geqslant -1$ (resp. $a_i > -1$, $a_i \geqslant 0$) such that $K_Y \equiv f^*K_X + \Sigma a_i E_i$ (cf. [CKM88]). The *index* of a normal variety $X$ with $K_X$ $\mathbb{Q}$-Cartier is the smallest positive integer $m$ such that $mK_X$ is Cartier. Note that for a normal variety $X$ with $K_X$ $\mathbb{Q}$-Cartier, there exists locally an *index 1 cover*, i.e., a finite surjective morphism $\sigma\colon X' \to X$ such that $X'$ has index 1 [R87, 3.6].

A singularity is called *Gorenstein* (resp. *Cohen–Macaulay*) if its local ring is a Gorenstein (resp. Cohen–Macaulay) ring. A variety is *Gorenstein* (resp. *Cohen–Macaulay*) if it admits only Gorenstein (resp. Cohen–Macaulay) singularities. Let

* Recently, I found that the results in this article give basis to a rather easy proof of the three-dimensional Cohen–Macaulay case. This is included in the next article.

$X$ be a normal variety and $f\colon Y \to X$ a resolution of singularities. $X$ is said to have *rational singularities* if $R^i f_* \mathcal{O}_Y = 0$ for all $i > 0$.

Let $X$ be a complex scheme of dimension $n$. $D_{\text{filt}}(X)$ denotes the derived category of filtered complexes of $\mathcal{O}_X$-modules with differentials of order $\leqslant 1$ and $D_{\text{filt, coh}}(X)$ the subcategory of $D_{\text{filt}}(X)$ of complexes $K^\cdot$, such that for all $i$, the cohomology sheaves of $\operatorname{Gr}^i_{\text{filt}} K^\cdot$ are coherent (cf. DB81, [GNPP88]). $D(X)$ and $D_{\text{coh}}(X)$ denotes the derived categories with the same definition except that the complexes are assumed to have the trivial filtration. The superscripts $+, -, b$ carry the usual meaning (bounded below, bounded above, bounded). $C(X)$ is the category of complexes of $\mathcal{O}_X$-modules with differentials of order $\leqslant 1$ and for $u \in \operatorname{Mor}(C(X))$, $M(u) \in \operatorname{Ob}(C(X))$ denotes the mapping cone of $u$ (cf. [H66]). The isomorphism in these categories is denoted by $\simeq_{\text{qis}}$. If $K^\cdot$ is a complex in any of the above defined categories, then $h^i(K^\cdot)$ denotes the $i$th cohomolgy sheaf of $K^\cdot$. In particular every sheaf is naturally a complex with $h^i = 0$ for $i \neq 0$.

The right derived functor of an additive functor $F$, if exists, is denoted by $RF$ and $R^i F$ stands for $h^i \circ RF$. In particular, $\mathbb{H}^i$ denotes $R^i\Gamma$ and $\mathbb{H}^i_Z$ denotes $R^i\Gamma_Z$ where $\Gamma$ is the functor of global sections and $\Gamma_Z$ is the functor of global sections with support in the closed subset $Z$. Finally $\omega^\cdot_X = f^!\mathbb{C}$ is the dualizing complex of $X$ where $f\colon X \to \operatorname{Spec}\mathbb{C}$ is the natural morphism (cf. [H66]). Note that if $X$ has Gorenstein singularities, then $\omega^\cdot_X \simeq_{\text{qis}} \omega_X[n]$ and $\omega_X$ is a line bundle.

The dimension of the empty set is $-\infty$.

## 1. Du Bois Singularities

The actual construction of Du Bois' complex will not be used in this article. Therefore it is not repeated here. Instead the interested reader is referred to the original article. Note also that a simplified construction was later obtained by [GNPP88] via the general theory of cubic resolutions. An easily accessible introduction can be found in [S85].

The basic results regarding $\underline{\Omega}^\cdot_X$ that are essential in the sequel are summarized in the following theorem.

THEOREM 1.1 [DB81, 3.2, 3.10, 4.5, 4.11], [GNPP88, III.1.12, III.1.17, V.3.5]. *For every complex scheme $X$ there exists an $\underline{\Omega}^\cdot_X \in \operatorname{Ob}(D_{\text{filt}}(X))$ with the following properties.*

(1.1.1) *It is functorial, i.e., if $\phi\colon Y \to X$ is a morphism of complex schemes, then there exists a natural map $\phi^*$ of filtered complexes $\phi^*\colon \underline{\Omega}^\cdot_X \to R\phi_*\underline{\Omega}^\cdot_Y$. Furthermore, $\underline{\Omega}^\cdot_X \in \operatorname{Ob}(D^b_{\text{filt, coh}}(X))$ and if $\phi$ is proper, then $\phi^*$ is a morphism in $D^b_{\text{filt, coh}}(X)$.*

(1.1.2) *Let $\Omega^\cdot_X$ be the usual De Rham complex of Kähler differentials considered with the 'filtration bête'. Then there exists a natural map of filtered complexes $\Omega^\cdot_X \to \underline{\Omega}^\cdot_X$ and if $X$ is smooth, it is a quasi-isomorphism.*



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(1.1.3) *Let $U \subseteq X$ be an open subscheme of $X$. Then* $\underline{\Omega}^{\cdot}_X|_U \simeq_{\text{qis}} \underline{\Omega}^{\cdot}_U$.

(1.1.4) *If $X$ is proper, then there exists a spectral sequence degenerating at $E_1$ and abutting to the singular cohomology of $X$:* $E_1^{pq} = \mathbb{H}^q(X, \mathrm{Gr}^p_{\text{filt}} \underline{\Omega}^{\cdot}_X[p]) \Rightarrow H^{p+q}(X, \mathbb{C})$.

(1.1.5) $\dim \operatorname{Supp} h^i(\mathrm{Gr}^p_{\text{filt}} \underline{\Omega}^{\cdot}_X) \leqslant \dim X - i + p$.

(1.1.6) *Let* $\underline{\Omega}^0_X = \mathrm{Gr}^0_{\text{filt}} \underline{\Omega}^{\cdot}_X$. *Let* $\Sigma = \operatorname{Sing} X$ *be the singular set of $X$ and $f\colon Y \to X$ a resolution of singularities such that it is an isomorphism outside $\Sigma$ and $E = f^{-1}(\Sigma)$ is a divisor with normal crossings. Then there exists a distinguished triangle,* $\underline{\Omega}^0_X \to \underline{\Omega}^0_\Sigma \oplus Rf_*\mathcal{O}_Y \to Rf_*\mathcal{O}_E \xrightarrow{+1}$, *where the morphisms are those of* (1.1.1) *and* (1.1.2).

*Remark* 1.1.7. Let $f\colon Y \to X$ be a resolution of singularities of $X$. Then by (1.1.1) and (1.1.2) the natural morphism $\mathcal{O}_X \to Rf_*\mathcal{O}_Y$. factors through $\underline{\Omega}^0_X$ and if $X$ is proper, then by (1.1.4) $H^i(X, \mathbb{C}) \to \mathbb{H}^i(X, \underline{\Omega}^0_X)$ is surjective for all $i$.

DEFINITION 1.2 [S83]. $X$ is said to have *Du Bois singularities* if $\mathcal{O}_X \to \underline{\Omega}^0_X$ is a quasi-isomorphism. (I.e., $h^0(\underline{\Omega}^0_X) \simeq \mathcal{O}_X$ and $h^i(\underline{\Omega}^0_X) = 0$ for all $i \neq 0$.) In particular, if $X$ is proper and has Du Bois singularities, then $H^i(X, \mathbb{C}) \to H^i(X, \mathcal{O}_X)$ is surjective for all $i$.

## 2. Rational Singularities – The Key Lemma

2.1. Let $X$ be a complex scheme that one would like to prove to have Du Bois singularities. Let $F^{\cdot}$ be a complex such that $\mathcal{O}_X \to \underline{\Omega}^0_X \to F^{\cdot} \xrightarrow{+1}$ forms a distinguished triangle and let $\Sigma_{DB} = \cup \operatorname{Supp} h^i(F^{\cdot})$ the union of the supports of the cohomology sheaves of $F^{\cdot}$. Then $\Sigma_{DB}$ is the *non-Du Bois locus* of $U$. By taking general hyperplane sections, as in [K95, 12.8], one may assume that $\dim \Sigma_{DB} \leqslant 0$. Therefore as long as the assumptions on $X$ are invariant under taking hyperplane sections, one can restrict to the case when the possibly non-Du Bois locus is at most a set of finite points.

The following is the key step in the proof of both Theorem S and Theorem K.

LEMMA 2.2. *Let $U$ be a complex scheme with a finite set of points, $P$, such that $U \setminus P$ has only Du Bois singularities and assume that* $H^i_P(U, \mathcal{O}_U) \to \mathbb{H}^i_P(U, \underline{\Omega}^0_U)$ *is injective for all* $i = 0, \ldots, \dim U$. *Then $U$ has Du Bois singularities.*

*Proof.* Since the statement is local, one may assume that $U$ is affine. Let $F^{\cdot}$ be the complex defined in (2.1). By assumption $P$ contains $\Sigma_{DB} = \cup \operatorname{Supp} h^i(F^{\cdot})$, the non-Du Bois locus of $U$.

Next let $X$ be a projective closure of $U$, and let $Q = X \setminus U$ and $Z = P \cup Q$. Then $X \setminus Z \simeq U \setminus P$ has only Du Bois singularities, i.e., $\mathcal{O}_{X \setminus Z} \simeq_{\text{qis}} \underline{\Omega}^0_{X \setminus Z}$.



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Now by (1.1.4) the composition $H^i(X, \mathbb{C}) \to H^i(X, \mathcal{O}_X) \to \mathbb{H}^i(X, \underline{\Omega}^0_X)$ is surjective for all $i$. Then in the commutative diagram

$$\begin{array}{ccccccc}
H^{i-1}(X\backslash Z, \mathcal{O}_{X\backslash Z}) & \longrightarrow & H^i_Z(X, \mathcal{O}_X) & \longrightarrow & H^i(X, \mathcal{O}_X) & \longrightarrow & H^i(X\backslash Z, \mathcal{O}_{X\backslash Z}) \\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} \\
\mathbb{H}^{i-1}(X\backslash Z, \underline{\Omega}^0_{X\backslash Z}) & \longrightarrow & \mathbb{H}^i_Z(X, \underline{\Omega}^0_X) & \longrightarrow & \mathbb{H}^i(X, \underline{\Omega}^0_X) & \longrightarrow & \mathbb{H}^i(X\backslash Z, \underline{\Omega}^0_{X\backslash Z})
\end{array}$$

the rows are exact, $\alpha$ and $\delta$ are isomorphisms, and $\gamma$ is surjective. Hence $\beta$ is surjective by the 5-lemma.

Observe that since $\dim P \leqslant 0$, $P \cap Q = \emptyset$, and then

$$H^i_Z(X, \mathcal{O}_X) \simeq H^i_P(X, \mathcal{O}_X) \oplus H^i_Q(X, \mathcal{O}_X),$$

$$\mathbb{H}^i_Z(X, \underline{\Omega}^0_X) \simeq \mathbb{H}^i_P(X, \underline{\Omega}^0_X) \oplus \mathbb{H}^i_Q(X, \underline{\Omega}^0_X)$$

and by excision $H^i_P(X, \mathcal{O}_X) \simeq H^i_P(U, \mathcal{O}_U)$ and $\mathbb{H}^i_P(X, \underline{\Omega}^0_X) \simeq \mathbb{H}^i_P(U, \underline{\Omega}^0_U)$.

Therefore $H^i_P(U, \mathcal{O}_U) \to \mathbb{H}^i_P(U, \underline{\Omega}^0_U)$ is surjective. By assumption it is also injective, hence an isomorphism.

The cohomology sheaves of $F^{\cdot}$ are supported on $P$, so $\mathbb{H}^i(U\backslash P, F^{\cdot}) = 0$ for all $i$. Hence $\mathbb{H}^i(U, F^{\cdot}) = \mathbb{H}^i_P(U, F^{\cdot}) = 0$. Using again that $\dim P \leqslant 0$, one finds that $\mathbb{H}^i(U, F^{\cdot}) = H^0(U, h^i(F^{\cdot}))$, so in fact $h^i(F^{\cdot}) = 0$ for all $i$, thus $\mathcal{O}_U \simeq \underline{\Omega}^0_U$. □

THEOREM 2.3. *Let $U$ be a complex scheme such that $\mathcal{O}_U \to \underline{\Omega}^0_U$ has a left inverse, then $U$ has Du Bois singularities.*

*Proof.* The statement is local, so one may assume that $U$ is affine. Since $\mathcal{O}_H \simeq \mathcal{O}_U \otimes_{\mathrm{L}} \mathcal{O}_H$ and $\underline{\Omega}^0_H \simeq_{\mathrm{qis}} \underline{\Omega}^0_U \otimes_{\mathrm{L}} \mathcal{O}_H$ for a general hyperplane $H$ (cf. [K95, 12.6.2]), one can use (2.1), and then (2.2) can be applied. □

COROLLARY 2.4 (cf. [K95, 12.8]). *Let $V$ be a complex scheme with Du Bois singularities and $f\colon V \to U$ a morphism to a complex scheme such that $\mathcal{O}_U \to Rf_*\mathcal{O}_V$ has a left inverse. Then $U$ has Du Bois singularities as well.*

*Proof.* $\mathcal{O}_U \to \underline{\Omega}^0_U \to Rf_*\mathcal{O}_V$ has a left inverse, so the statement follows by (2.3). □

COROLLARY 2.5 (cf. [K95, 12.8.2]). *Let $V$ be a complex scheme with Du Bois singularities and $f\colon V \to U$ a finite and dominant morphism to a normal variety. Then $U$ also has Du Bois singularities.*

*Proof.* $R^i f_*\mathcal{O}_V = 0$ for $i > 0$ and the normalized trace map splits $\mathcal{O}_U \to f_*\mathcal{O}_V$. □

COROLLARY 2.6 (cf. [S83, 3.7], [K95, 12.9]). *Let $U$ be a complex variety with rational singularities. Then $U$ has Du Bois singularities.*



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*Proof.* Let $f\colon V \to U$ be a resolution of singularities. Then $\mathcal{O}_U \simeq_{\text{qis}} Rf_*\mathcal{O}_V$, so the statement again follows by (2.4). □

COROLLARY 2.7. *Let $U$ be a complex variety with log terminal singularities. Then $U$ has Du Bois singularities.*

*Proof.* First assume that $U$ has canonical singularities of index 1. Let $f\colon V \to U$ be a resolution of singularities. Now $f^*\omega_U$ is a subsheaf of $\omega_V$, so by the Grauert–Riemenschneider vanishing theorem $Rf_*\omega_V \simeq_{\text{qis}} \omega_U$. Then $\omega_U \to Rf_*f^*\omega_U \to Rf_*\omega_V \simeq_{\text{qis}} \omega_U$ shows that $\omega_U \to Rf_*f^*\omega_U$ has a left inverse, so tensoring by $\omega_U^{-1}$ one obtains that $\mathcal{O}_U \to Rf_*\mathcal{O}_V$ has a left inverse. Then (2.4) implies the result.

The statement is local, so in the general case one can take the index 1 cover, $U'$, of $U$ which has canonical singularities of index 1 (cf. [R87, 3.6]). Then $U'$ has Du Bois singularities by the first part, so $U$ has Du Bois singularities by (2.5). □

*Remark* 2.7.1. The last result certainly follows also from (2.6), but then one would have to appeal to the nontrivial fact that log terminal singularities are rational (cf. [E81], [F81], [KMM87]), whereas the above proof is considerably short and simple.

## 3. Log Canonical Singularities

The following notation will be used through the rest of the article.

NOTATION 3.1. Let $U$ be a complex scheme and $\Sigma = \operatorname{Sing} U$ the set of singular points of $U$. Further let $\Sigma_r$ denote the smallest closed subset of $U$, such that $U \setminus \Sigma_r$ has rational singularities. $\Sigma$ and $\Sigma_r$ will be considered with the reduced induced subscheme structure. Let $f\colon V \to U$ be a resolution of singularities such that it is an isomorphism outside $\Sigma$ and $E = f^{-1}(\Sigma)$ is a divisor with normal crossings. Finally let $\mathcal{Q} = \omega_U / f_*\omega_V$. Note that $\operatorname{Supp} \mathcal{Q} \subseteq \Sigma_r$.

Grothendieck duality will be used in the following form (cf. [H66, III.11.1]):

For $f\colon V \to U$ as above and for all $F^{\cdot}$ bounded complexes of $\mathcal{O}_V$-modules $Rf_*R\mathcal{H}om_V(F^{\cdot}, \omega_V^{\cdot}) \simeq R\mathcal{H}om_U(Rf_*F^{\cdot}, \omega_U^{\cdot})$.

LEMMA 3.2. *Let $U$ be a complex variety of dimension $n$ with log canonical Gorenstein singularities. Then the natural maps*

$$R^i f_*\mathcal{O}_V \to R^i f_*\mathcal{O}_E, \qquad h^i(\underline{\Omega}^0_U) \to h^i(\underline{\Omega}^0_\Sigma)$$

*are injective for all $i > 0$.*

*Proof.* $U$ has log canonical singularities, so $f^*\omega_U \subseteq \omega_V(E)$, and thus $\omega_U \simeq f_*\omega_V(E)$ by [KMM87, 1-3-2]. Consider the following commutative diagram

$$\begin{array}{ccc} f_*\omega_V & \longrightarrow & f_*\omega_V(E) \simeq \omega_U \\ \downarrow & & \downarrow \\ Rf_*\omega_V & \longrightarrow & Rf_*\omega_V(E). \end{array}$$

By the Grauert–Riemenschneider vanishing theorem, the first vertical arrow is a quasi-isomorphism, so the natural morphism on the bottom factors through $\omega_U$

$$Rf_*\omega_V \to \omega_U \to Rf_*\omega_V(E).$$

Hence the same holds for the dualizing complexes

$$Rf_*\omega^{\cdot}_V \to \omega^{\cdot}_U \to Rf_*\omega_V(E)[n]. \tag{3.2.1}$$

Next apply $R\mathcal{H}om_U(–, \omega^{\cdot}_U)$ to (3.2.1). By Grothendieck duality

$$Rf_*\mathcal{O}_V(-E) \simeq_{\text{qis}} R\mathcal{H}om_U(Rf_*\omega_V(E)[n], \omega^{\cdot}_U),$$

$$\mathcal{O}_U \simeq_{\text{qis}} R\mathcal{H}om_U(\omega^{\cdot}_U, \omega^{\cdot}_U),$$

$$Rf_*\mathcal{O}_V \simeq_{\text{qis}} R\mathcal{H}om_U(Rf_*\omega^{\cdot}_V, \omega^{\cdot}_U),$$

so (3.2.1) implies that the natural morphism $Rf_*\mathcal{O}_V(-E) \to Rf_*\mathcal{O}_V$ factors through $\mathcal{O}_U$

$$Rf_*\mathcal{O}_V(-E) \to \mathcal{O}_U \to Rf_*\mathcal{O}_V. \tag{3.2.2}$$

Observe that the natural morphism $\mathcal{O}_U \to Rf_*\mathcal{O}_V$ factors through $\underline{\Omega}^0_U$, so (3.2.2) gives a natural morphism $Rf_*\mathcal{O}_V(-E) \to \underline{\Omega}^0_U$ that factors through $\mathcal{O}_U$

$$Rf_*\mathcal{O}_V(-E) \to \mathcal{O}_U \to \underline{\Omega}^0_U. \tag{3.2.3}$$

By (1.1.6) there exists a distinguished triangle $\underline{\Omega}^0_U \to \underline{\Omega}^0_\Sigma \oplus Rf_*\mathcal{O}_V \to Rf_*\mathcal{O}_E \xrightarrow{+1}$.

Then it is easy to see, that one has the following commutative diagram of distinguished triangles (cf. [DB90, 7.7])

$$\begin{array}{ccccccc} Rf_*\mathcal{O}_V(-E) & \longrightarrow & \underline{\Omega}^0_U & \longrightarrow & \underline{\Omega}^0_\Sigma & \xrightarrow{+1} & \\ \downarrow & & \downarrow & & \downarrow & & \\ Rf_*\mathcal{O}_V(-E) & \longrightarrow & Rf_*\mathcal{O}_V & \longrightarrow & Rf_*\mathcal{O}_E & \xrightarrow{+1} & . \end{array} \tag{3.2.4}$$



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Now (3.2.2) and (3.2.3) implies that the natural morphisms on the higher cohomology sheaves must be zero

$$R^i f_*\mathcal{O}_V(-E) \to 0 \to R^i f_*\mathcal{O}_V, \qquad R^i f_*\mathcal{O}_V(-E) \to 0 \to h^i(\underline{\Omega}^0_U).$$

Hence the statement follows by (3.2.4). □

The following is probably known to experts, but I could not find a reference, so a proof is included here

LEMMA 3.3. *Let $U$ be a complex Cohen–Macaulay scheme of dimension $n$. Then $R^i f_*\mathcal{O}_V = 0$ for $0 < i < n - \dim \Sigma_r - 1$.*

*Proof.* Apply $R\mathcal{H}om_U(-, \omega_U)$ to the short exact sequence $0 \to f_*\omega_V \to \omega_U \to \mathcal{Q} \to 0$.

By the Grauert–Riemenschneider vanishing theorem, $f_*\omega_V \simeq_{\text{qis}} Rf_*\omega_V$, so by Grothendieck duality $R\mathcal{H}om_U(\mathcal{Q}, \omega_U) \to \mathcal{O}_U \to Rf_*\mathcal{O}_V \overset{+1}{\to}$ forms a distinguished triangle, hence for $i > 0$, $R^i f_*\mathcal{O}_V \simeq \mathcal{E}xt^{i+1}_U(\mathcal{Q}, \omega_U)$.

Let $x \in U$ be a closed point. The local ring of $x$, $\mathcal{O}_x$, is a Cohen–Macaulay ring of dimension $n$, so $(R^i f_*\mathcal{O}_V)_x \simeq \mathcal{E}xt^{i+1}_U(\mathcal{Q}, \omega_U)_x \simeq \operatorname{Ext}^{i+1}_{\mathcal{O}_x}(\mathcal{Q}_x, \omega_{\mathcal{O}_x}) = 0$ for $1 < i + 1 < \dim \mathcal{O}_x - \dim \mathcal{Q}_x$ (cf. BH93, 3.5.11). Now the statement follows since $\dim \mathcal{Q}_x \leqslant \dim \Sigma_r$. □

*Remark* 3.3.1. In this lemma $f$ may be an arbitrary resolution of singularities (as opposed to the assumption in (3.1)).

THEOREM 3.4. *Let $U$ be a complex variety of dimension $n$ with log canonical Gorenstein singularities. Assume that $h^i(\underline{\Omega}^0_\Sigma) = 0$ for $i \geqslant n - \dim \Sigma_r - 1$ Then $U$ has Du Bois singularities.*

*Proof.* Consider the following commutative diagram where the rows are distinguished triangles

$$\begin{array}{ccccccc}
\mathcal{O}_U & \longrightarrow & \underline{\Omega}^0_U & \longrightarrow & F^{\cdot} & \overset{+1}{\longrightarrow} & \\
\downarrow \simeq & & \downarrow & & \downarrow & & \\
\mathcal{O}_U & \longrightarrow & Rf_*\mathcal{O}_V & \longrightarrow & R^{\cdot} & \overset{+1}{\longrightarrow} &
\end{array}$$

For $i > 0$, $h^i(F^{\cdot}) \simeq h^i(\underline{\Omega}^0_U) \subset h^i(\underline{\Omega}^0_\Sigma)$ by (3.2), so $h^i(F^{\cdot}) = 0$ for $i \geqslant n - \dim \Sigma_r - 1$.

This remains true after taking general hyperplane sections, so as in (2.1), one may assume that $\dim \Sigma_{DB} \leqslant 0$, i.e., for a finite set of points, $P$, $U \backslash P$ has Du Bois singularities.

The cohomology sheaves of $F^{\cdot}$ are supported on $P$, so $\mathbb{H}^i(U \backslash P, F^{\cdot}) = 0$ for all $i$ and since $\dim P \leqslant 0$, one finds that $\mathbb{H}^i_P(U, F^{\cdot}) = H^0_P(U, h^i(F^{\cdot}))$.



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$H^0_P(U, h^i(F^\cdot)) = 0$ for $i \geqslant n - \dim \Sigma_r - 1$, so $\mathbb{H}^q_P(U, F^\cdot) = 0$ for $q \geqslant n - \dim \Sigma_r - 1$.

On the other hand, $h^i(R^\cdot) = 0$ for $i \leqslant 0$, since $\mathcal{O}_U \simeq f_*\mathcal{O}_V$, and $h^i(R^\cdot) \simeq R^i f_*\mathcal{O}_V = 0$ for $0 < i < n - \dim \Sigma_r - 1$ by (3.3). Then $H^j_P(U, h^i(R^\cdot)) = 0$ for $i < n - \dim \Sigma_r - 1$ and all $j$ trivially, and then $\mathbb{H}^q_P(U, R^\cdot) = 0$ for $q < n - \dim \Sigma_r - 1$.

Therefore $\mathbb{H}^q_P(U, F^\cdot) \to \mathbb{H}^q_P(U, R^\cdot)$ is zero for all $q$. Now the following commutative diagram

$$\begin{array}{ccccc}
\mathbb{H}^{i-1}_P(U, F^\cdot) & \longrightarrow & H^i_P(U, \mathcal{O}_U) & \longrightarrow & \mathbb{H}^i_P(U, \underline{\Omega}^0_U) \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \simeq} & & \downarrow \\
\mathbb{H}^{i-1}_P(U, R^\cdot) & \longrightarrow & H^i_P(U, \mathcal{O}_U) & \longrightarrow & \mathbb{H}^i_P(U, Rf_*\mathcal{O}_V)
\end{array}$$

implies, that $\mathbb{H}^{i-1}_P(U, F^\cdot) \to H^i_P(U, \mathcal{O}_U)$ is zero, thus $H^i_P(U, \mathcal{O}_U) \to \mathbb{H}^i_P(U, \underline{\Omega}^0_U)$ is injective. The result now follows by (2.2). □

COROLLARY 3.5. *Let $U$ be a complex variety with log canonical Cohen–Macaulay singularities. Assume that either* $\dim \Sigma + \dim \Sigma_r + 1 < \dim U$ *or* $\Sigma$ *has Du Bois singularities. Then $U$ has Du Bois singularities.*

*Proof*. Passing to the index 1 cover, as in Corollary 2.7, one may assume that $U$ has Gorenstein singularities. By (1.1.5) $\dim \operatorname{Supp} h^i(\underline{\Omega}^0_\Sigma) \leqslant \dim \Sigma - i$, so the assumptions imply that $\dim \operatorname{Supp} h^i(\underline{\Omega}^0_\Sigma) < \dim U - \dim \Sigma_r - 1 - i$ for all $i > 0$. Then the statement follows by (3.4). □

Finally let us regard the opposite direction, namely that Du Bois singularities are not far from being log canonical.

THEOREM 3.6. *Let $U$ be a normal variety and assume that $K_U$ is Cartier and $U$ has Du Bois singularities. Then $U$ is log canonical.*

*Proof*. The distinguished triangle of (1.1.6) $\underline{\Omega}^0_U \to \underline{\Omega}^0_\Sigma \oplus Rf_*\mathcal{O}_V \to Rf_*\mathcal{O}_E \xrightarrow{+1}$ implies that the natural morphism $Rf_*\mathcal{O}_V(-E) \to Rf_*\mathcal{O}_V$ factors through $\underline{\Omega}^0_U \simeq_{\text{qis}} \mathcal{O}_U$. Hence there exists a morphism $Rf_*\mathcal{O}_V(-E) \to \mathcal{O}_U$ that is a quasi-isomorphism on $U \setminus \Sigma$. Applying $R\mathcal{H}om_U(-, \omega^\cdot_U)$ to this morphism one obtains $\omega^\cdot_U \to Rf_*\omega_V(E)[n]$ and taking the $-n$th cohomology gives a morphism $\omega_U \to f_*\omega_V(E)$ that is an isomorphism on $U \setminus \Sigma$ (in particular it is not the zero morphism). By adjointness this gives a nonzero morphism $f^*\omega_U \to \omega_V(E)$. $f^*\omega_U$ is a line bundle, so this implies that $f^*\omega_U \subseteq \omega_V(E)$, and therefore $U$ is log canonical. □

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