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Linear stability of a ferrofluid centred around a current-carrying wire

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Investigated first is the linear stability of a Newtonian ferrofluid centred on a rigid wire, surrounded by another ferrofluid with a different magnetic susceptibility. An electric current runs through the wire, generating an azimuthal magnetic field that produces a magnetic stress at the interface of the fluids. Three-dimensional disturbances to the system are considered, and the linearised Navier–Stokes equations are solved analytically in terms of an implicit expression for the growth rate of the disturbance. The growth rate is found numerically for arbitrary Reynolds number, and given explicitly in the inviscid and Stokes regimes. Investigated next is a ferrofluid whose magnetic susceptibility varies radially, centred on a rigid wire, subject to a non-uniform azimuthal field. It is proven that if the gradient of the susceptibility is positive anywhere in the fluid, then the system is linearly unstable. Moreover, it is proven that applying an axial field can stabilise disturbances for both continuous and discontinuous susceptibilities.

Key words: magnetic fluids

1. Introduction

The instability of liquid jets and cylindrical columns of ideal fluids has been researched rigorously for over a century. The well-known Plateau–Rayleigh instability describes the break-up of a capillary jet into droplets, for linear axisymmetric perturbations of wavelengths longer than the radius of the jet (Rayleigh 1878). It has been shown in the ferro-hydrodynamic literature that the Plateau–Rayleigh instability for a ferrofluid jet can be stabilised by a sufficiently strong azimuthal magnetic field (Bashtovoi & Krakov 1978; Arkhipenko *et al.* 1980; Rannacher & Engel 2006). Ferrofluids are colloidal fluids, consisting of magnetic solids such as magnetite, suspended in a carrier solution, usually water, kerosene or oils. The liquid becomes magnetised in the presence of a magnetic field, and to prevent agglomeration, the nanoparticles are either electrically charged or coated in a surfactant. An investigation into the stability of jets or columns of ferrofluid

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could be useful for technological, industrial and biomedical applications. In industry, they are most commonly used for dynamic sealing and heat dissipation (Scherer & Figueiredo Neto 2005), and there has been investigation into their use in inkjet printing (Charles 1987; Abdel Fattah, Ghosh & Puri 2016) and three-dimensional printing (Löwa *et al.* 2019), where the jet disintegrates into drops that are then directed by a magnetic field. Moreover, there have been recent biomedical advances in their use in hyperthermia treatment (Zhang, Gu & Wang 2007) and magnetic drug targeting, experimentally (Asfer, Saroj & Panigrahi 2017) and theoretically (Voltairas, Fotiadis & Michalis 2002; Gonella *et al.* 2020).

The governing equations are discussed in § 2, and in § 3, the linear stability of a column of ferrofluid centred on a straight rigid wire, surrounded by another ferrofluid of different magnetic susceptibility, is investigated. A current runs through the wire, producing an azimuthal magnetic field, resulting in a magnetic stress at the interface of the fluids. The critical parameter is the magnetic Bond number, which measures the ratio between the capillary pressure and magnetic forcing from the current through the wire. The linear stability of a ferrofluid jet has been studied in the literature but with limiting assumptions. Arkhipenko *et al.* (1980) consider an irrotational inviscid ferrofluid jet, and show that increasing the current such that $B > 1$ produces a stable system. Of the previous works, Bashtovoi & Krakov (1978) and Rannacher & Engel (2006) consider non-axisymmetric disturbances, where they assume an inviscid irrotational system, but the experimental work performed by Arkhipenko *et al.* (1980) and Bourdin, Barci & Falcon (2010) suggests that viscous effects are important in the development of the instability. Cornish (2018) considers the highly viscous limit, and Canu & Renoult (2021) consider a Newtonian ferrofluid jet, surrounded by a Newtonian non-magnetic fluid. Both works consider axisymmetric disturbances, and the results obtained by Canu & Renoult (2021) show that accounting for viscosity agrees better with the experimental results of Arkhipenko *et al.* (1980) and Bourdin *et al.* (2010) than the inviscid system. Moreover, Canu & Renoult (2021) highlight the importance of having a surrounding liquid to mirror experimental conditions and for drug targeting applications. Blyth & Parau (2014) and Doak & Vanden-Broeck (2019) also consider the effect of a non-magnetic fluid surrounding the jet, but in the inviscid limit. Korovin (2004) considers the surrounding liquid being a ferrofluid, filling a cuvette, rather than an infinite domain. Performing axisymmetric perturbations localised to the interface between the two fluids, the dispersion relation is derived using a modified equation of motion. He finds the thickness of the inner fluid to be of importance, and that the drops produced from the perturbation are different to when a gas is the surrounding medium. Thus we allow both fluids to be ferrofluids, and consider both axisymmetric and non-axisymmetric disturbances to the system with arbitrary Reynolds number. We give an analytical solution to the perturbed linearised Navier–Stokes equation and an implicit expression for the growth rate of the disturbance. For a given Reynolds number, a numerical root solver is used to find the growth rate for given wavenumbers, magnetic susceptibilities, strength of current and the wire radius. In the inviscid and highly viscous limits, a dispersion relation is obtained analytically, giving a stability condition.

For a non-ferrofluid jet, Christiansen (1955) found axisymmetric modes to be the most unstable (for non-axisymmetry with inertial effects), and we prove that this is true when the inner ferrofluid has a larger susceptibility than the outer. In this case, the system is linearly unstable to axisymmetric disturbances only, and $B > 1$ results in stability of all wavelengths, supporting previous works. In contrast, when the outer fluid has a larger susceptibility, axisymmetric and non-axisymmetric modes are unstable, and as the wire radius shrinks, non-axisymmetric disturbances become the most unstable at low Reynolds number. Moreover, increasing the current in the wire will not suppress all unstable modes if the outer fluid has a higher susceptibility than the inner fluid. Bashtovoi & Krakov (1978) show that adding an axial field will stabilise an inviscid irrotational ferrofluid column, and we prove this for our system in both the inviscid and highly viscous regimes, irrespective of which fluid has a higher susceptibility. We show numerically that this holds for arbitrary Reynolds number too.

When a magnetic fluid is subject to a non-uniform magnetic field, the magnetic particles are attracted to the region of highest field intensity to obtain the minimum energy configuration (Scherer & Figueiredo Neto 2005). The results outlined in § 3 and the analysis performed by Zelazo & Melcher (1969) for a two-fluid layer in a planar domain agree that when the field decreases outwards (upwards), and the stronger ferrofluid is the inner (lower) fluid, magnetic forcing is stabilising. Yet if the field decreases outwards (upwards), and the stronger ferrofluid is the outer (upper) fluid, then magnetic forcing is destabilising due to the region with largest magnetic susceptibility not being located where the field is strongest. This motivates the investigation of the stability of one ferrofluid, whose susceptibility varies continuously with radius, centred on a current-carrying wire, with an associated field decreasing as the reciprocal of the radius. In § 4, we prove that the stability of the system is determined by the sign of the gradient of the susceptibility with respect to the field strength, and prove that adding an axial field can suppress the instability.

Some works have used nonlinear theory to analyse the behaviour of a ferrofluid jet. Blyth & Parau (2014) use a fully nonlinear numerical model to show that axisymmetric solitary waves propagate at the surface of an inviscid column of ferrofluid, and compare their results with the experimental work by Bourdin *et al.* (2010), who show the existence of axisymmetric periodic and solitary waves at the interface of a ferrofluid jet in a cylindrical domain. Doak & Vanden-Broeck (2019), as well as studying the linear stability, use a numerical model to find stable travelling wave solutions on a ferrofluid jet. Cornish (2018) uses weakly nonlinear stability theory and long wave theory for both a highly viscous and an inviscid axisymmetric jet, studying the resultant drop formation. In this paper, we focus solely on linear stability analysis.

There is a direct analogue between a ferrofluid subject to a magnetic field in ferro-hydrodynamics and a dielectric exposed to a gradient electric field in electro-hydrodynamics (Zelazo & Melcher 1969; Rosensweig 1985). Nayyar & Murty (1960) and Garcia *et al.* (1997) study, respectively, the stability of inviscid and viscous dielectric liquid columns, subject to a longitudinal electric field. Nayyar & Murty (1960) use an energy argument to show that the electric field has a stabilising effect. Garcia *et al.* (1997) consider axisymmetric perturbations to the system and produce a dispersion relation, showing that viscous dissipation and dielectric forces at the interface work to stabilise the system. A crucial difference between the dielectric work on jets and the problem in this paper is the presence of a wire and the associated azimuthal field.

2. Magnetic force and stress tensor

We assume that the magnetisation $\boldsymbol{M}$ is collinear with the magnetic field $\boldsymbol{H}$ such that $\boldsymbol{M} = \chi \boldsymbol{H}$, where χ is the magnetic susceptibility. The field satisfies

$$\nabla \times \boldsymbol{H} = 0, \tag{2.1}$$

and the induced field $\boldsymbol{B}$ satisfies

$$\nabla \cdot \boldsymbol{B} = 0. \tag{2.2}$$

Equation (2.1) allows us to define a magnetic potential ϕ such that

$$\boldsymbol{H} = \nabla \phi. \tag{2.3}$$

Due to collinearity, $\boldsymbol{B} = \mu_0(1+\chi)\boldsymbol{H}$, where μ_0 is the magnetic permeability of the fluid, and therefore

$$\nabla \cdot ((1+\chi)\,\nabla\phi) = 0. \tag{2.4}$$

At an interface, we require continuity of the normal component of $\boldsymbol{B}$,

$$[\mu_0(1+\chi)\,\nabla\phi \cdot \boldsymbol{n}] = 0, \tag{2.5}$$

and continuity of the tangential component of $\boldsymbol{H}$,

$$[\nabla\phi \cdot \boldsymbol{\tau}] = 0, \tag{2.6}$$

where $\boldsymbol{n}$ and $\boldsymbol{\tau}$ are respectively the unitary normal and tangential vectors to the interface, and the square brackets denote the jump across it.

Rosensweig (1985) gives the stress tensor for a Newtonian isothermal ferrofluid as

$$\boldsymbol{T} = -\mu_0\left(\int_0^H \chi H\,\mathrm{d}H + \tfrac{1}{2}H^2\right)\boldsymbol{I} - p\boldsymbol{I} + \boldsymbol{B}\boldsymbol{H}^{\mathrm{T}} + \eta(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^{\boldsymbol{T}}), \tag{2.7}$$

where $H = |\boldsymbol{H}|$, η is the viscosity, p is the pressure, $\boldsymbol{u}$ is the velocity of the fluid, and strictive effects have been neglected on account of there being no physiochemical phase change in the flow. The force density due to magnetic effects, for a ferrofluid subject to $\boldsymbol{H}$, is

$$\boldsymbol{f} = -\nabla\left(\mu_0\int_0^H (1+\chi)H\,\mathrm{d}H\right) + \mu_0(1+\chi)H\,\nabla H, \tag{2.8}$$

and if χ is independent of $\boldsymbol{H}$, then

$$\boldsymbol{f} = -\frac{\mu_0 H^2}{2}\,\nabla\chi. \tag{2.9}$$

Now, $\boldsymbol{f}$ appears in the Navier–Stokes equation as

$$\rho\,\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \nabla p = \eta\,\nabla^2\boldsymbol{u} + \boldsymbol{f}, \tag{2.10}$$

where ρ is the density of the fluid. Taking the curl of (2.10) gives

$$\rho\,\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} = \eta((\boldsymbol{\omega}\cdot\nabla)\boldsymbol{u} + \nabla^2\boldsymbol{\omega}) + \mu_0 H\,\nabla\chi \times \nabla H, \tag{2.11}$$

where $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$. Equation (2.11) holds for both (2.8) and (2.9), and it follows that if $H = H(\chi)$, then we have a stationary state.

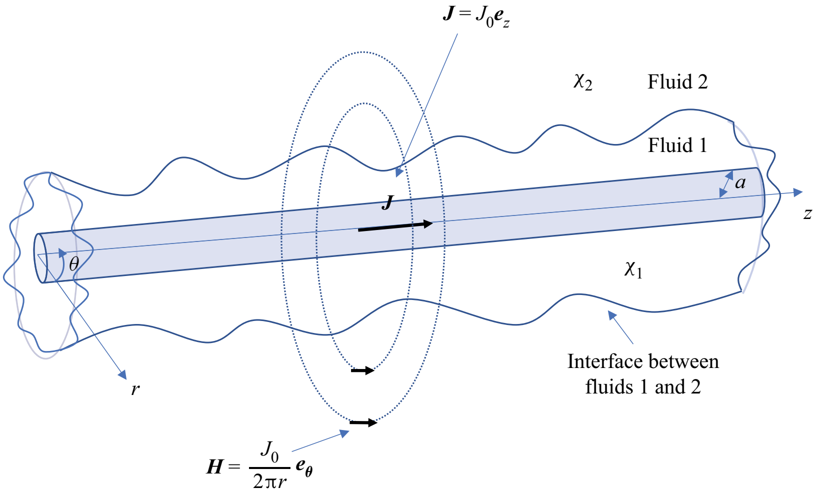

Figure 1. Schematic of the two-fluid system.

3. Two-fluid system

3.1. *Formulation of the problem*

We consider a column of ferrofluid, fluid 1, with magnetic susceptibility $\chi = \chi_1$, centred on a rigid wire with radius a. We choose the cylindrical system (r, θ, z) such that r and θ are the radial and azimuthal coordinates, and z points along the wire. In the stationary state, fluid 1 is in the region $a < r < R$ and is surrounded by another ferrofluid, fluid 2, whose domain is unbounded, with susceptibility $\chi = \chi_2$. Both fluids are incompressible and isothermal, with constant density ρ, viscosity η, and permeability μ_0. A steady electric current $\boldsymbol{J} = J_0 \boldsymbol{e}_z$ runs through the wire, producing an azimuthal magnetic field $\boldsymbol{H} = J_0/2\pi r \boldsymbol{e}_\theta$, where $\boldsymbol{e}_z$ is the unit vector in the z direction, and $\boldsymbol{e}_\theta$ is the unit vector in the anticlockwise, azimuthal direction. The set-up is shown in figure 1.

Take R as the length scale, and define $a = a_* R$. We non-dimensionalise pressure as $p = \sigma p_*/R$, where σ is the surface tension, and non-dimensionalise the field as $\boldsymbol{H} = J_0 \boldsymbol{H}_*/2\pi R$, and pick the time scale, T, to be $T = \eta R/\sigma$, such that the velocity is non-dimensionalised as $\boldsymbol{u} = \sigma/\eta \boldsymbol{u}_*$. The starred variables are dimensionless, but we drop the stars from here on. Equation (2.10) becomes

$$Re \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \nabla p = \nabla^2 \boldsymbol{u} + B\boldsymbol{f}, \tag{3.1}$$

where

$$Re = \frac{\rho \sigma R}{\eta^2} \tag{3.2}$$

is the Reynolds number, and

$$B = \frac{\mu_0 J_0^2}{4\pi^2 R \sigma} \tag{3.3}$$

is the magnetic Bond number. Since

$$\left.\begin{aligned} \chi &= 0, && \text{for } r \leq a,\\ \chi &= \chi_1, && \text{for } a < r < R,\\ \chi &= \chi_2, && \text{for } r \geq R, \end{aligned}\right\} \tag{3.4}$$

χ is always constant, resulting in $\boldsymbol{f} = \mathbf{0}$ in (3.1), and magnetic effects are felt only at the interface.

Initially, both fluids are at rest and the interface between the two fluids is located at $r = 1$. We consider perturbations to the surface such that the surface is located at

$$r = 1 + \epsilon\, \mathrm{Re}(\hat{S}\zeta), \tag{3.5}$$

where $\zeta = \exp\,(\mathrm{i}(kz + m\theta) + st)$, $\epsilon \ll 1$, k, m are real and positive wavenumbers, $\hat{S}$ may be a complex constant (or it could be unity), and s is the growth rate of the disturbance and could be complex. In (3.5), the real part of the perturbation is taken, and this is done from here on for the other variables, but it is not written explicitly. The normal vector to the surface becomes

$$\boldsymbol{n} = \left(1, -\frac{\epsilon \mathrm{i} m \hat{S}\zeta}{r}, -\epsilon \mathrm{i} k \hat{S}\zeta\right)^{\mathrm{T}}, \tag{3.6}$$

and the tangential vectors are

$$\boldsymbol{\tau_1} = (\epsilon \mathrm{i} k \hat{S}\zeta, 0, 1)^{\mathrm{T}}, \quad \boldsymbol{\tau_2} = \left(\frac{\epsilon \mathrm{i} m \hat{S}\zeta}{r}, 1, 0\right)^{\mathrm{T}}. \tag{3.7a,b}$$

The perturbed pressure $p^{(i)}$ and velocity $\boldsymbol{u}^{(i)}$, where $i = 1, 2$ for fluids 1 and 2, are

$$p^{(i)} = p_0 + \epsilon\, \hat{p}^{(i)}(r)\, \zeta \quad \text{and} \quad \boldsymbol{u}^{(i)} = \epsilon\, \hat{\boldsymbol{u}}^{(i)}(r)\, \zeta, \tag{3.8a,b}$$

where p_0 is constant. The perturbations satisfy

$$\boldsymbol{\nabla} \cdot \hat{\boldsymbol{u}} = 0 \quad \text{and} \quad s\, Re\, \hat{\boldsymbol{u}} + \boldsymbol{\nabla} \hat{p} = \nabla^2 \hat{\boldsymbol{u}}. \tag{3.9a,b}$$

In component form we have

$$(r\hat{u})' + \mathrm{i} m \hat{v} + \mathrm{i} k r \hat{w} = 0, \tag{3.10}$$

$$r^2 \hat{p}' = -2\mathrm{i} m \hat{v} - \hat{u} + r^2 \hat{u}'' + r\hat{u}' - (m^2 + \bar{k}^2 r^2)\hat{u}, \tag{3.11}$$

$$\mathrm{i} m r \hat{p} = -\hat{v} + 2\mathrm{i} m \hat{u} + r^2 \hat{v}'' + r\hat{v}' - (m^2 + \bar{k}^2 r^2)\hat{v}, \tag{3.12}$$

$$\mathrm{i} k r^2 \hat{p} = r^2 \hat{w}'' + r w' - (m^2 + \bar{k}^2 r^2)\hat{w}, \tag{3.13}$$

where $'$ denotes the first derivative with respect to r, and $\bar{k} = \sqrt{k^2 + s\, Re}$. The general solution is in terms of the modified Bessel functions of the first and second kind, $\mathrm{I}_n(z)$ and $\mathrm{K}_n(z)$, respectively, where we write $\mathrm{I}_n, \mathrm{K}_n$ when $z = kr$, and $\bar{\mathrm{I}}_n, \bar{\mathrm{K}}_n$ when $z = \bar{k} r$, but give the argument otherwise. The general solution of (3.9a,b) is modified from Saville (1971) and Mestel (1996) to account for the inner wire at $r = a$, and is found to be

$$\hat{p}^{(i)} = c_1^{(i)}\mathrm{I}_m + c_2^{(i)}\mathrm{K}_m, \tag{3.14a}$$

$$\hat{u}^{(i)} = -\frac{1}{(s\,Re)^2 r}\left(c_1^{(i)}(kr\mathrm{I}_{m+1} + m\mathrm{I}_m) + c_2^{(i)}(m\mathrm{K}_m - kr\mathrm{K}_{m+1})\right) - \frac{\mathrm{i}k}{\bar{k}}\left(c_3^{(i)}\bar{\mathrm{I}}_{m+1} + c_4^{(i)}\bar{\mathrm{K}}_{m+1}\right) + \frac{2m}{\bar{k}r}\left(c_5^{(i)}\bar{\mathrm{I}}_m + c_6^{(i)}\bar{\mathrm{K}}_m\right), \tag{3.14b}$$

$$\hat{v}^{(i)} = \left(\frac{-c_3^{(i)}k}{\bar{k}} + 2\mathrm{i}c_5^{(i)}\right)\bar{\mathrm{I}}_{m+1} + \frac{2\mathrm{i}m}{\bar{k}r}\left(c_5^{(i)}\bar{\mathrm{I}}_m + c_6^{(i)}\bar{\mathrm{K}}_m\right) - \left(\frac{c_4^{(i)}k}{\bar{k}} + 2\mathrm{i}c_6^{(i)}\right)\bar{\mathrm{K}}_{m+1} - \frac{\mathrm{i}m}{(s\,Re)^2 r}\left(c_2^{(i)}\mathrm{K}_m + c_1^{(i)}\mathrm{I}_m\right), \tag{3.14c}$$

$$\hat{w}^{(i)} = \frac{-\mathrm{i}k}{(s\,Re)^2}\left(c_1^{(i)}\mathrm{I}_m + c_2^{(i)}\mathrm{K}_m\right) + c_3^{(i)}\bar{\mathrm{I}}_m - c_4^{(i)}\bar{\mathrm{K}}_m, \tag{3.14d}$$

for constants $c_1^{(i)}, \ldots, c_6^{(i)}$. To satisfy $u^{(2)} \to 0$ as $r \to \infty$, $\mathrm{Re}(\bar{k}) > 0$ and $c_1^{(2)} = c_3^{(2)} = c_5^{(2)} = 0$.

We perturb the magnetic potential such that

$$\phi^{(l)} = \theta + \epsilon\,\hat{\phi}^{(l)}(r)\,\zeta, \tag{3.15}$$

where $l = 0, 1, 2$ for the wire, inner fluid and outer fluid, respectively. Equation (2.4) gives

$$r^2\hat{\phi}''^{(l)} + \hat{\phi}'^{(l)} - (m^2 + k^2r^2)\hat{\phi}^{(l)} = 0, \tag{3.16}$$

with general solution

$$\hat{\phi}(r)^{(l)} = q_1^{(l)}\mathrm{I}_m + q_2^{(l)}\mathrm{K}_m, \tag{3.17}$$

for constants $q_1^{(l)}, q_2^{(l)}$. For $\hat{\phi}^{(0)}$ regular at $r = 0$,

$$\hat{\phi}^{(0)} = q_1^{(0)}\mathrm{I}_m, \tag{3.18}$$

and imposing $\phi^{(2)} \to 0$, as $r \to \infty$ gives

$$\hat{\phi}^{(2)} = q_2^{(2)}\mathrm{K}_m. \tag{3.19}$$

Equations (2.5) and (2.6) give

$$(1+\chi_2)\hat{\phi}'^{(2)} - (1+\chi_1)\hat{\phi}'^{(1)} + im(\chi_1 - \chi_2)\hat{S} = 0, \quad \hat{\phi}^{(1)} = \hat{\phi}^{(2)} \tag{3.20a,b}$$

at $r = 1$, and

$$(1+\chi_1)\hat{\phi}'^{(1)} - \hat{\phi}'^{(0)} = 0, \quad \hat{\phi}^{(1)} = \hat{\phi}^{(0)} \tag{3.21a,b}$$

at $r = a$, determining the constants $q_1^{(0)}, q_1^{(1)}, q_2^{(1)}, q_2^{(2)}$ given in (A1)–(A4) in Appendix A.

At $r = a$,

$$\hat{\boldsymbol{u}}^{(1)} = 0, \tag{3.22}$$

and at $r = 1$,

$$\hat{\boldsymbol{u}}^{(1)} = \hat{\boldsymbol{u}}^{(2)}. \tag{3.23}$$

At the interface of the fluids, there is a normal stress condition

$$[\boldsymbol{n} \cdot \boldsymbol{T} \cdot \boldsymbol{n}] = \sigma \, \nabla \cdot \boldsymbol{n}, \tag{3.24}$$

and two tangential stress conditions

$$[\boldsymbol{n} \cdot \boldsymbol{T} \cdot \boldsymbol{\tau_1}] = 0, \tag{3.25}$$

$$[\boldsymbol{n} \cdot \boldsymbol{T} \cdot \boldsymbol{\tau_2}] = 0. \tag{3.26}$$

Non-dimensionalising (3.24), substituting the perturbed variables, linearising, and invoking continuity of the normal component of $\boldsymbol{B}$, we obtain

$$(m^2 + k^2 - 1 + B(\chi_1 - \chi_2))\hat{S} = \mathrm{i}mB(\chi_1\hat{\phi}^{(1)} - \chi_2\hat{\phi}^{(2)}) + \hat{p}^{(1)} - \hat{p}^{(2)} - 2\hat{u}'^{(1)} + 2\hat{u}'^{(2)} \tag{3.27}$$

at $r = 1$. Similarly, (3.25) and (3.26) give

$$\hat{v}'^{(1)} - \hat{v}'^{(2)} = 0 \tag{3.28}$$

and

$$\hat{w}'^{(1)} - \hat{w}'^{(2)} = 0 \tag{3.29}$$

at $r = 1$. Equations (3.22), (3.23) (3.27)–(3.29) determine the constants $c_1^{(i)}, \ldots, c_6^{(i)}$ given in (A13)–(A21).

The growth rate appears in the kinematic condition at $r = 1$:

$$s\hat{S} - \hat{u}^{(i)} = 0. \tag{3.30}$$

Consequently, substituting $\hat{u}^{(i)}$ into (3.30), we obtain

$$s = -gf(f_1(k^2 + m^2 - 1 + B(\chi_1 - \chi_2)) + f_2B(\chi_1 - \chi_2)^2m^2), \tag{3.31}$$

where $g, f_1, f_2 > 0$. Here, g, f_1, f_2 are functions of m, k, a, χ_1, χ_2, and f is a function of $\bar{k}, m, k, a, \chi_1, \chi_2$, all given in (A10)–(A12). Since $\bar{k}$ is a function of s, f is a function of s, and therefore (3.31) is an implicit relation that must be solved numerically.

3.2. *Highly viscous and inviscid limits*

In the limit $Re \to 0$, $f \to f_v$, where $f_v > 0$ and f_v is no longer a function of s, given in (A22). The growth rate in the highly viscous regime, s_v, is expressed explicitly as

$$s_v = -gf_v(f_1(k^2 + m^2 - 1 + B(\chi_1 - \chi_2)) + f_2B(\chi_1 - \chi_2)^2m^2). \tag{3.32}$$

In the inviscid limit, $\eta \to 0$, and a more appropriate scaling for time, T_I, is $T_I = \sqrt{R^3\rho/\sigma}$. Since $T_I = \sqrt{Re}\,T$, we substitute $s = s_I/\sqrt{Re}$ into (3.31), where s_I is the inviscid

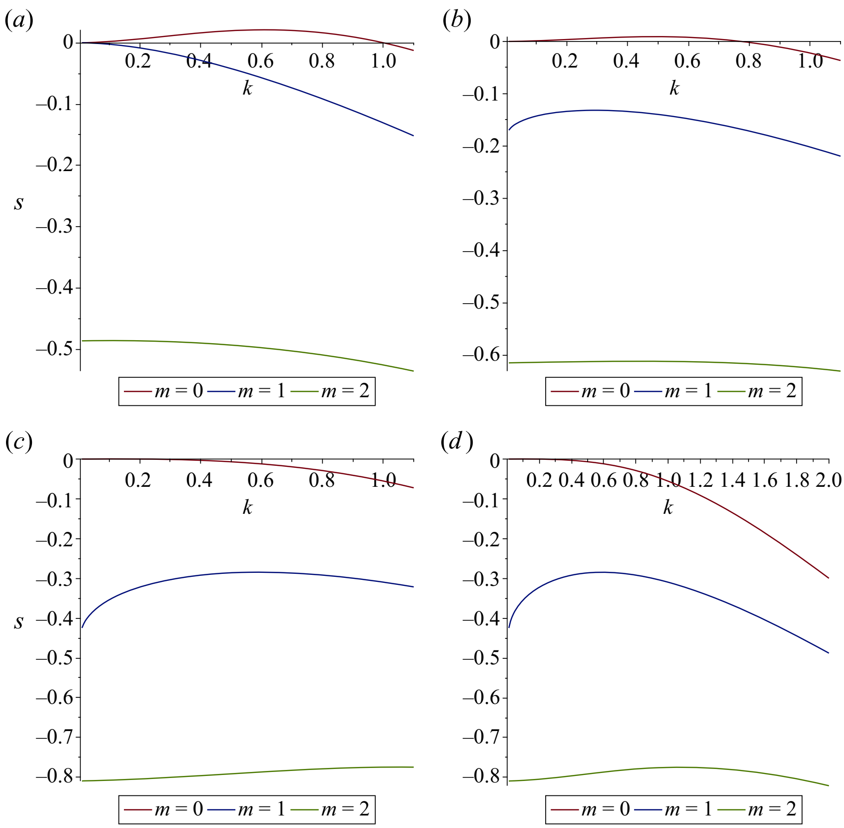

Figure 2. Viscous growth rate, $a = 0.1$, $\chi_1 = 5$, $\chi_2 = 1$: (*a*) $B = 0$, (*b*) $B = 0.1$, (*c*) $B = 0.25$, (*d*) $B = 4$.

growth rate. Taking the limit as $Re \to \infty$ gives $f \to f_I/s_I$, where $f_I > 0$ and f_I is no longer a function of s, given in (A23). Consequently,

$$s_I^2 = -gf_I(f_1(k^2 + m^2 - 1 + B(\chi_1 - \chi_2)) + f_2B(\chi_1 - \chi_2)^2m^2). \tag{3.33}$$

More simply, (3.33) can be obtained by taking the limit $\eta \to 0$ in the governing equations from the outset, and applying the boundary conditions for an inviscid system ($\eta = 0$), namely $\hat{u}^{(1)} = 0$ at the wire, and $\hat{u}^{(1)} = \hat{u}^{(2)}$, (3.27), (3.29) at the interface.

Since $f_I, f_v > 0$, the system is stable (or neutrally stable), in both the inviscid and highly viscous regimes, if and only if

$$f_1(k^2 + m^2 - 1 + B(\chi_1 - \chi_2)) + f_2B(\chi_1 - \chi_2)^2m^2 \geq 0. \tag{3.34}$$

Note that in the inviscid regime, if (3.34) holds, then the system is neutrally stable as the growth rate is imaginary. If the inner fluid has a higher susceptibility than the outer fluid, then the system can be unstable only as a result of capillary forces. Only axisymmetric modes can be unstable when $k < 1$, and increasing the current in the wire will stabilise the system provided that $B(\chi_1 - \chi_2) > 1$. Figures 2 and 3 show the growth rate of the modes being dampened as the current in the wire is increased for the viscous and inviscid regimes, respectively.

On the other hand, when the outer fluid has a higher susceptibility, capillary and magnetic forces may be destabilising. Increasing the current, thereby increasing the

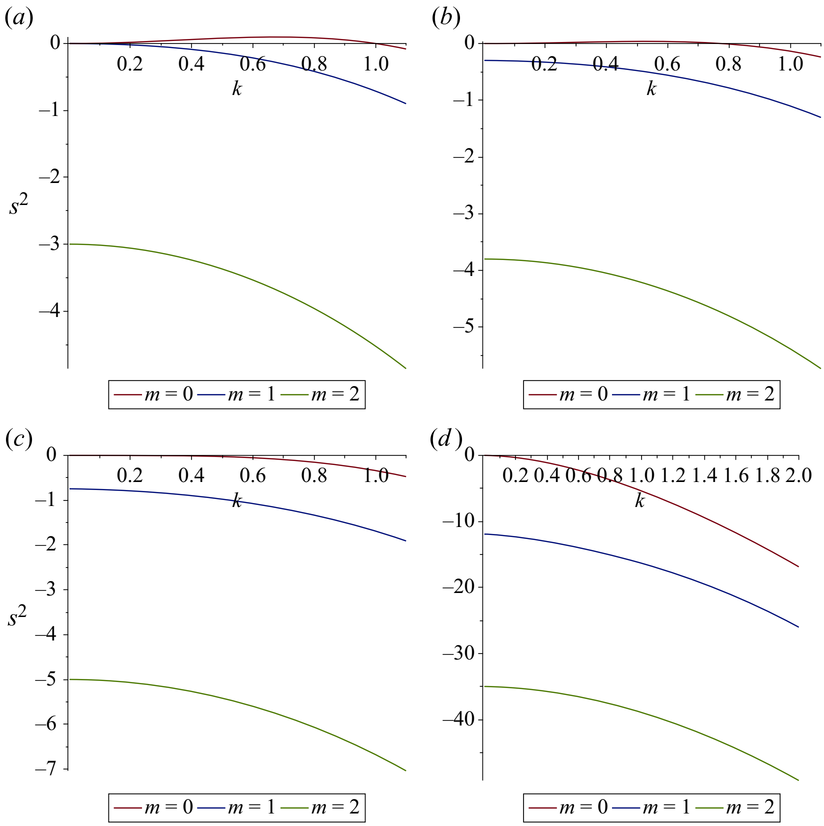

Figure 3. Inviscid growth rate, $a = 0.1$, $\chi_1 = 5$, $\chi_2 = 1$: (*a*) $B = 0$, (*b*) $B = 0.1$, (*c*) $B = 0.25$, (*d*) $B = 4$.

magnetic forcing at the interface, renders non-axisymmetric modes unstable, as well as axisymmetric modes, and for sufficiently large B, all modes m can be rendered unstable. Figures 4–6 show that increasing B results in an increase in unstable modes, and increases the magnitude of their growth rates. Moreover, in the highly viscous regime, figure 5 shows $m = 1$, $k \to 0$, is the most unstable mode for $a = 0.1$, and in fact $s \to \infty$ as $k \to 0$, whereas when $a = 0.5$ in figure 6, $m = 0$ remains the most unstable mode. Performing a series expansion on the viscous growth rate as $a, k \to 0$, we find $s \sim \ln(ka)$ when $m = 1$, thus $s \to \infty$ in the limit, but s converges to a constant when $m = 0$ or $m > 1$, a result seen in the context of electro-hydrodynamics too (Saville 1971; Mestel 1996). It is important to note that when comparing the inviscid regime with the viscous regime, s_I and s_v are on different time scales.

3.3. *Arbitrary Reynolds number*

A numerical root solver in the program Maple is used on (3.31) for specific values of $k, a, m, \chi_1, \chi_2, B, Re$, to find the associated growth rate of the mode. The stability condition (3.34) appears to hold for all Reynolds numbers. Figure 7 is the growth rate plotted when $a = 0.1, k = 0.5, \chi_1 = 1, \chi_2 = 5, B = 0.1$, for a range of Reynolds numbers, showing two stable branches when $m = 1$ and an unstable branch for $m = 0$. Given a stable mode, as $Re \to 0$, there are two branches, both real: one branch tends to s_v, and the other tends to $-\infty$, the latter a result of $Re \to 0$ for the chosen time scale. As Re is increased,

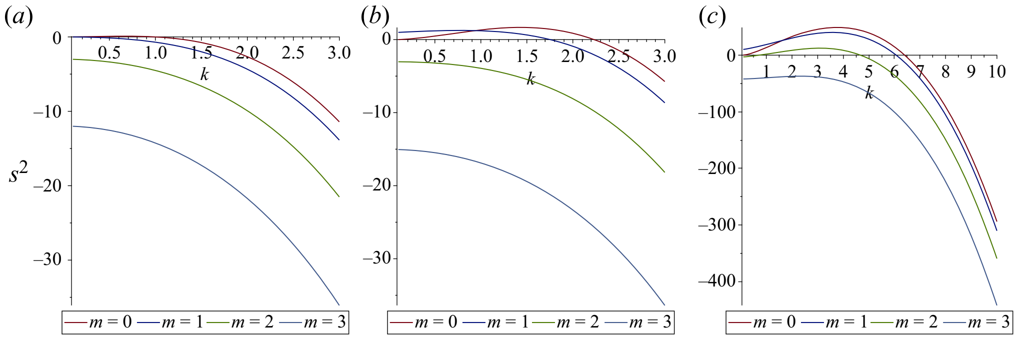

Figure 4. Inviscid growth rate, $a = 0.1$, $\chi_1 = 1$, $\chi_2 = 5$: (*a*) $B = 0$, (*b*) $B = 1$, (*c*) $B = 10$.

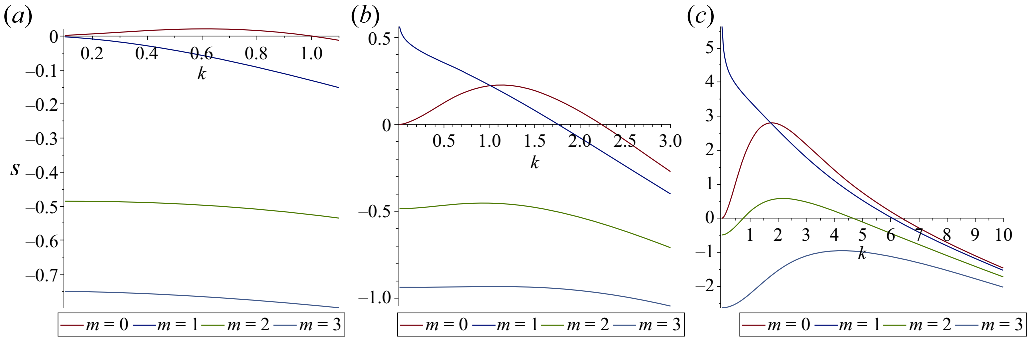

Figure 5. Viscous growth rate, $a = 0.1$, $\chi_1 = 1$, $\chi_2 = 5$: (*a*) $B = 0$, (*b*) $B = 1$, (*c*) $B = 10$.

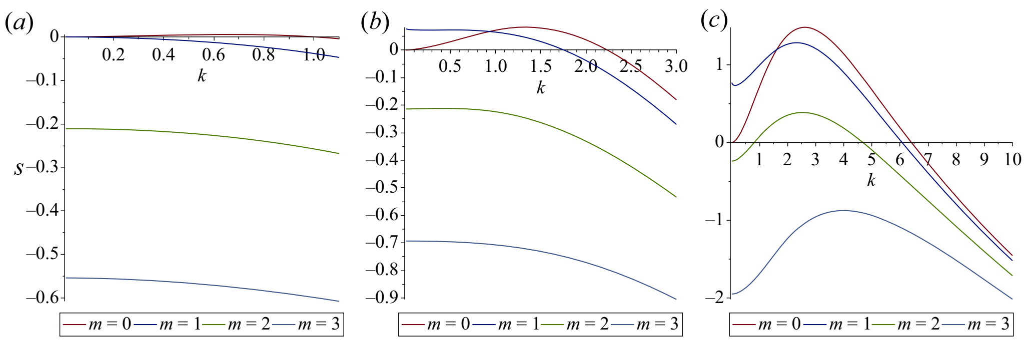

Figure 6. Viscous growth rate, $a = 0.5$, $\chi_1 = 1$, $\chi_2 = 5$: (*a*) $B = 0$, (*b*) $B = 1$, (*c*) $B = 10$.

the two branches meet and then split, becoming complex conjugates of each other, tending towards $\pm s_I$, where s_I is purely imaginary for a stable mode. Given an unstable mode, we get one branch, starting at s_v and tending to $|s_I|$. There exists a branch that tends to the negative inviscid root, $-|s_I|$, but this is invalid for finite *Re*, since the boundary conditions require $\mathrm{Re}(\bar{k}) \geq 0$. When $\chi_2 > \chi_1$, and a, k are sufficiently small, $m = 1$ is more unstable than $m = 0$. Yet for all k, $m = 1$ is the most unstable mode only for sufficiently small *Re*. This is shown in figure 8, where the growth rate is plotted against k for different Reynolds numbers, showing that when $Re = 0.001, 0.1$, $m = 1$ is the most unstable mode, but for the other Reynolds numbers shown, it is not. We find that increasing the current in the wire

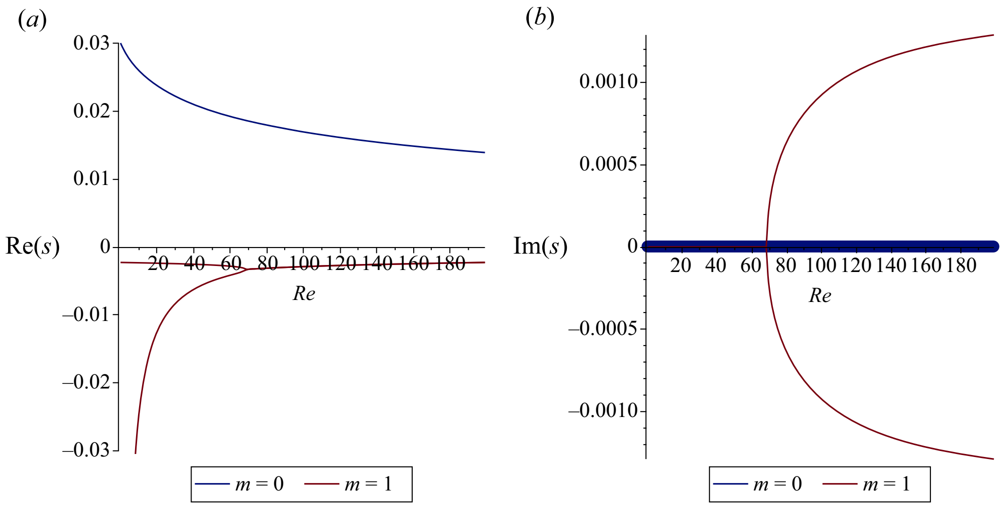

Figure 7. Growth rate plotted when $a = 0.1$, $k = 0.5$, $\chi_1 = 1$, $\chi_2 = 5$ and $B = 0.1$ for arbitrary *Re*. Panels (*a*) and (*b*) plot, respectively, the real and imaginary parts of s. The $m = 0$ branch is purely real, but the $m = 1$ branch starts as real for low *Re* and then becomes a complex conjugate pair.

stabilises the system if $\chi_1 > \chi_2$ for all Reynolds numbers. Figures 7 and 9, where $\chi_2 > \chi_1$, show that increasing B from $B = 0.1$ to $B = 0.5$ renders the mode $m = 1$ unstable, and we find that for all Reynolds numbers, increasing the current does not stabilise the system if $\chi_2 > \chi_1$, but renders more modes unstable.

3.4. *Stabilisation with an axial field*

To stabilise the system, irrespective of whether $\chi_2 > \chi_1$ or $\chi_1 > \chi_2$, we consider $\boldsymbol{H}_0 = (0, 1/r, Z)^{\mathrm{T}}$, Z constant, thereby adding an axial field. It follows that

$$\phi_0 = \theta + Zz, \tag{3.35}$$

and we perform analysis analogous to that in § 3.1. The general solutions (3.14), (3.17) still hold, and $\hat{\phi}^{(0)}$, $\hat{\phi}^{(2)}$ are still given by (3.18) and (3.19), respectively. At $r = a$, we apply (3.21) and (3.22). At $r = 1$, we apply (3.23), (3.28), (3.29), but (2.5), (2.6), (3.24) now give

$$(1 + \chi_2)\hat{\phi}'^{(2)} - (1 + \chi_1)\hat{\phi}'^{(1)} + \mathrm{i}(m + kZ)(\chi_1 - \chi_2)\hat{S} = 0, \quad \hat{\phi}^{(1)} = \hat{\phi}^{(2)}, \tag{3.36a,b}$$

and

$$(m^2 + k^2 - 1 + B(\chi_1 - \chi_2))\hat{S} = \mathrm{i}(m + kZ)B(\chi_1\hat{\phi}^{(1)} - \chi_2\hat{\phi}^{(2)}) + \hat{p}^{(1)} - \hat{p}^{(2)} - 2\hat{u}'^{(1)} + 2\hat{u}'^{(2)}. \tag{3.37}$$

We now obtain

$$s = -gf(f_1(k^2 + m^2 - 1 + B(\chi_1 - \chi_2)) + f_2B(\chi_1 - \chi_2)^2(Zk + m)^2), \tag{3.38}$$

$$s_v = -gf_v(f_1(k^2 + m^2 - 1 + B(\chi_1 - \chi_2)) + f_2B(\chi_1 - \chi_2)^2(Zk + m)^2), \tag{3.39}$$

$$s_I^2 = -gf_I(f_1(k^2 + m^2 - 1 + B(\chi_1 - \chi_2)) + f_2B(\chi_1 - \chi_2)^2(Zk + m)^2), \tag{3.40}$$

analogously to (3.31), (3.32) and (3.33). Equations (3.39) and (3.40) show that a sufficiently large kZ will stabilise all modes in the inviscid and highly viscous regimes,

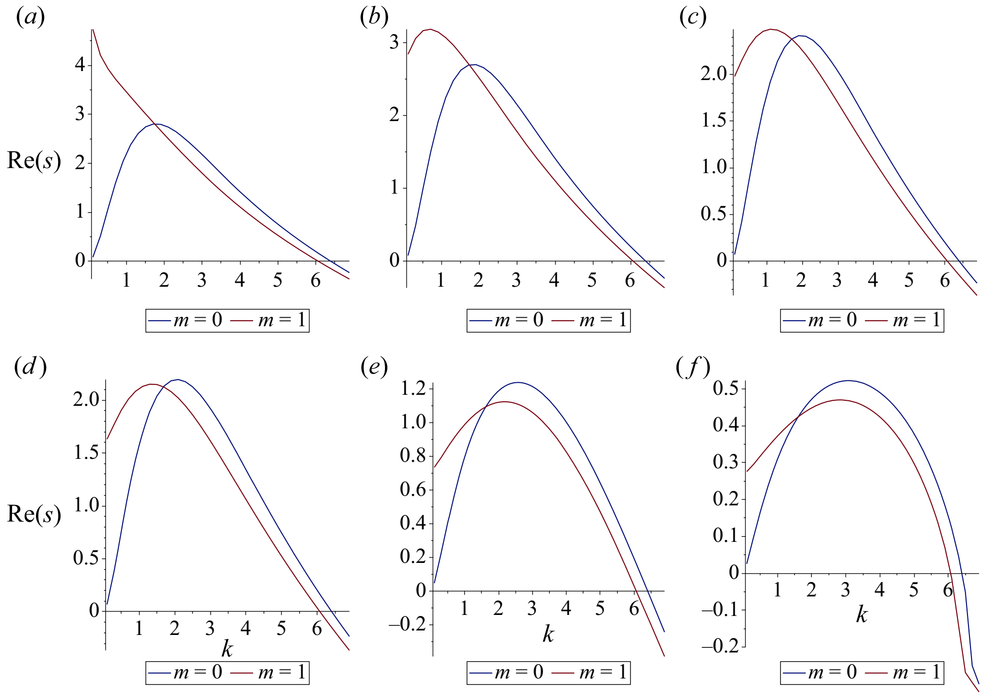

Figure 8. Growth rate plotted against k, for arbitrary Re, when $a = 0.1$, $\chi_1 = 1$, $\chi_2 = 5$, $B = 10$: (a) $Re = 0.0001$, (b) $Re = 0.1$, (c) $Re = 0.5$, (d) $Re = 1$, (e) $Re = 10$, (f) $Re = 100$.

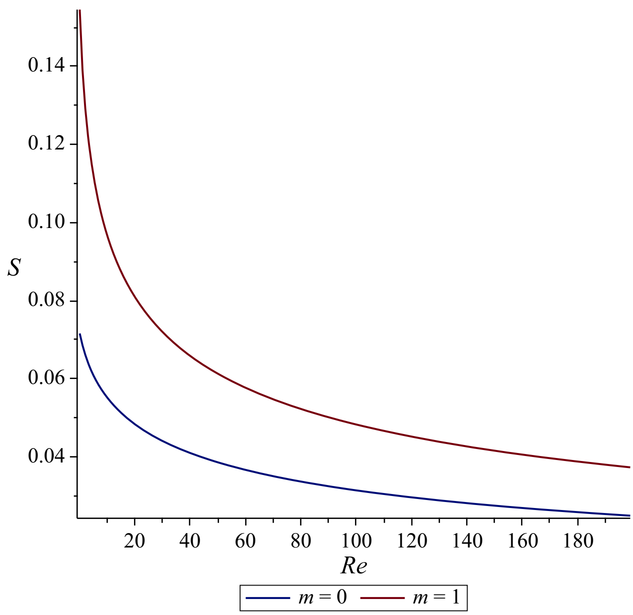

Figure 9. Growth rate plotted when $a = 0.1$, $k = 0.5$, $\chi_1 = 1$, $\chi_2 = 5$ and $B = 0.5$ for arbitrary Re.

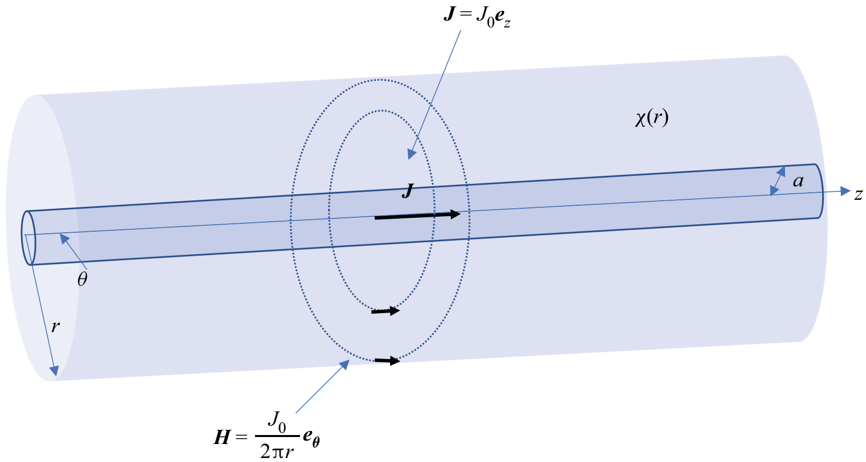

Figure 10. Schematic of the system.

irrespective of the sign of $(\chi_1 - \chi_2)$, provided that $B \neq 0$, and this result appears to hold for all Reynolds numbers. Although extremely long waves in the z direction, $k \to 0$, would remain unstable, by physical restrictions of the system, k is bounded from zero.

4. Continuous magnetic susceptibility

Now we allow χ to depend on position and the field. We consider one incompressible, isothermal, ferrofluid whose susceptibility varies radially. The ferrofluid is centred on the wire as shown in figure 10. Since χ varies with position, the magnetic forcing acts throughout the fluid. In § 3, magnetic forcing gave rise to an instability if $\chi_1 < \chi_2$; we thus expect that $\mathrm{d}\chi/\mathrm{d}r > 0$ will lead to an instability, as a result of the regions of fluid with largest χ being drawn to the regions of strongest field. Yet since there is no longer forcing due to surface tension at an interface, we expect $\mathrm{d}\chi/\mathrm{d}r < 0$ to be stable, as both χ and the strength of the field decrease with the radius.

Zelazo & Melcher (1969) require the physical properties of the ferrofluid, α_i, to obey

$$\frac{\mathrm{D}\alpha_i}{\mathrm{D}t} = 0, \tag{4.1}$$

but not the field dependent parts of χ. Here we assume

$$\frac{\mathrm{D}\chi}{\mathrm{D}t} = 0, \tag{4.2}$$

so that a displaced fluid parcel retains its dependence on H and its physical properties.

Since χ is no longer constant, $\boldsymbol{f} \neq 0$ in (3.1). We choose the same scaling as in § 3 to non-dimensionalise the equations. Non-dimensionalising (2.11) gives

$$Re\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} = (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{u} + \nabla^2\boldsymbol{\omega} + BH\,\nabla\chi \times \nabla H. \tag{4.3}$$

Equation (4.3) is satisfied by $\boldsymbol{u} = 0$, $\chi = \chi_0(r)$, $H = H_0(r)$, and it follows from (3.1) that p satisfies

$$\nabla p = -\frac{B(H_0(r))^2}{2}\nabla\chi_0(r), \tag{4.4}$$

giving $p = p_0(r)$, for a stationary state. Consider a perturbation to this stationary state such that

$$\chi = \chi_0(r) + \epsilon\chi_1 + O(\epsilon^2), \quad \boldsymbol{H} = \boldsymbol{H}_0 + \epsilon\boldsymbol{H}_1 + O(\epsilon^2), \quad \boldsymbol{u} = \epsilon\boldsymbol{u_1} + O(\epsilon^2), \tag{4.5a–c}$$

and

$$|\boldsymbol{H}| = H = H_0 + \epsilon H_1 + O(\epsilon^2), \quad \text{where } H_0 = \sqrt{\boldsymbol{H}_0\cdot\boldsymbol{H}_0}, H_1 = \frac{\boldsymbol{H}_0\cdot\boldsymbol{H}_1}{H_0}. \tag{4.6}$$

Substituting into (4.3) and linearising gives

$$Re\,\frac{\partial\boldsymbol{\omega}_1}{\partial t} = \nabla^2\boldsymbol{\omega_1} + BH_0\left(\nabla\chi_0\times\nabla H_1 + \nabla\chi_1\times\nabla H_0\right), \tag{4.7}$$

where $\boldsymbol{\omega}_1 = \nabla\times\boldsymbol{u}_1$. To satisfy (2.4)–(2.6), $\boldsymbol{H}_0 = (1/r)\boldsymbol{e_\theta}$ and $\chi_0(r)$ is any function of r. We consider perturbations such that

$$\boldsymbol{\omega}_1 = \nabla\times(\hat{\boldsymbol{u}}(r)\,\zeta), \quad \chi_1 = \hat{\chi}(r)\,\zeta \tag{4.8}$$

and

$$\boldsymbol{H}_1 = \nabla(\hat{\phi}(r)\,\zeta), \tag{4.9}$$

to give

$$H_1 = \frac{\mathrm{i}m}{r}\,\hat{\phi}(r)\,\zeta; \tag{4.10}$$

note that H_1 is defined by (4.6), and $H_1 \neq |\boldsymbol{H}_1|$.

Consequently, (4.7) in component form is

$$(s\,Re - \mathcal{L})\hat{\omega}_r + \frac{2\mathrm{i}m}{r^2}\,\hat{\omega}_\theta = 0, \tag{4.11a}$$

$$(s\,Re - \mathcal{L})\hat{\omega}_\theta - \frac{2\mathrm{i}m}{r^2}\,\hat{\omega}_r = B\left(\frac{mk\chi_0'\hat{\phi}}{r^2} - \frac{\mathrm{i}k\hat{\chi}}{r^3}\right), \tag{4.11b}$$

$$(s\,Re - \mathcal{D})\hat{\omega}_z = B\left(\frac{-m^2\chi_0'\hat{\phi}}{r^3} + \frac{\mathrm{i}m\hat{\chi}}{r^4}\right), \tag{4.11c}$$

where

$$\hat{\omega}_r = \frac{\mathrm{i}m\hat{w}}{r} - \mathrm{i}k\hat{v}, \quad \hat{\omega}_\theta = \mathrm{i}k\hat{u} - \hat{w}', \quad \hat{\omega}_z = \frac{1}{r}\left((r\hat{v})' - \mathrm{i}m\hat{u}\right), \tag{4.12}$$

$$\mathcal{L} = \frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{m^2}{r^2} - k^2 - \frac{1}{r^2}, \tag{4.13}$$

$$\mathcal{D} = \frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{m^2}{r^2} - k^2. \tag{4.14}$$

Also, $\hat{\boldsymbol{u}}$ satisfies (3.10), (3.22) and $\boldsymbol{u}\to 0$ as $r\to\infty$. Substituting the perturbed variables and linearising (4.2) gives

$$s\hat{\chi} = -\chi_0'\hat{u}, \tag{4.15}$$

and therefore $\chi = 0$ at $r = a$, and $\chi\to 0$ as $r\to\infty$. Equation (2.4) gives

$$(1+\chi_0)r^2\mathcal{D}\hat{\phi} + r^2\chi_0'\hat{\phi}' = -\mathrm{i}m\hat{\chi}. \tag{4.16}$$

Equations (2.5), (2.6) and (3.18) give

$$\hat{\phi}' = \frac{(\mathrm{I}_m)'\hat{\phi}}{(1+\chi_0)\mathrm{I}_m} \tag{4.17}$$

at $r = a$, and $\hat{\phi}, \hat{\phi}' \to 0$ as $r \to \infty$.

4.1. *Axisymmetric disturbances*

For axisymmetric disturbances, (4.11) becomes

$$(s\,Re - \mathcal{L}_0)\hat{\omega}_r = 0, \tag{4.18a}$$

$$(s\,Re - \mathcal{L}_0)\hat{\omega}_\theta = -\frac{\mathrm{i}kB\hat{\chi}}{r^3}, \tag{4.18b}$$

$$(s\,Re - \mathcal{D}_0)\hat{\omega}_z = 0, \tag{4.18c}$$

where

$$\mathcal{L}_0 = \frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} - k^2 - \frac{1}{r^2}, \tag{4.19}$$

$$\mathcal{D}_0 = \frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} - k^2. \tag{4.20}$$

Define a stream function Ψ such that $\hat{\boldsymbol{u}} = \nabla \times (0, \Psi/r, 0)$, and use the change of variables $\Psi = r\psi$ to give

$$\hat{\boldsymbol{\omega}} = -\hat{\boldsymbol{\theta}}\mathcal{L}_0\hat{\psi}, \quad \nabla^2\hat{\boldsymbol{\omega}} = -\hat{\boldsymbol{\theta}}\mathcal{L}_0^2\hat{\psi}, \tag{4.21}$$

where $\psi = \hat{\psi}(r)\,\mathrm{e}^{\mathrm{i}kz+st}$. It follows from the boundary conditions for $\boldsymbol{u}$ that $\hat{\psi}, \hat{\psi}' = 0$ at $r = a$ and as $r \to \infty$. Equations (4.15) and (4.18) give the eigenvalue equation

$$(s^2\,Re\,\mathcal{L}_0 - s\mathcal{L}_0^2)\hat{\psi} = -\frac{k^2B\chi_0'\hat{\psi}}{r^3}. \tag{4.22}$$

Rather than find the eigenvalues of (4.22) numerically, we prove a stability condition. Multiply (4.22) by $r\hat{\psi}^*$, where $\hat{\psi}^*$ is the complex conjugate of $\hat{\psi}$. Integrate over the domain, use integration by parts, and invoke the boundary conditions, to obtain

$$s^2\,Re\int_a^\infty \left(|\hat{\psi}'|^2 + \left(\frac{1}{r^2}+k^2\right)|\hat{\psi}|^2\right) r\,\mathrm{d}r + s\int_a^\infty |\mathcal{L}_0\hat{\psi}|^2 r\,\mathrm{d}r - k^2B\int_a^\infty \frac{\chi_0'|\hat{\psi}|^2}{r^2}\,\mathrm{d}r = 0, \tag{4.23}$$

an equation for s of the form $as^2 + bs + c = 0$, where a, b, c all depend on $\hat{\psi}$ and therefore s too, but a, b, c are real, as well as $a, b > 0$, bounded away from zero. Thus if $\chi_0' > 0$, then $c < 0$ and there exists a root with $s > 0$. Note that in this case, s and $\hat{\psi}$ are real, whereas when $\chi_0' < 0$, $c > 0$, $\mathrm{Re}(s) < 0$ and $s, \hat{\psi}$ could be complex. We conclude that if $\chi_0' > 0$ everywhere, then there exists an unstable mode, while if $\chi_0' \leq 0$ everywhere, then the axisymmetric modes are stable.

We prove a stronger stability condition using variational methods. Crucially, we have shown that if the flow is unstable, then s must be real and therefore $\hat{\psi}$ is real, thus it suffices to consider only y real in the following argument. Consider the functional

$$I(y) = \frac{-\int_a^\infty (\mathcal{L}_0 y)^2 r \,\mathrm{d}r + \sqrt{\left(\int_a^\infty (\mathcal{L}_0 y)^2 r \,\mathrm{d}r\right)^2 + W_0}}{2\,Re \int_a^\infty \left((y')^2 + \left(\frac{1}{r^2} + k^2\right) y^2\right) r \,\mathrm{d}r}, \tag{4.24}$$

where

$$W_0 = 4k^2\, Re\, B \int_a^\infty \left((y')^2 + \left(\frac{1}{r^2} + k^2\right) y^2\right) r \,\mathrm{d}r \int_a^\infty \frac{\chi_0' y^2}{r^2}\,\mathrm{d}r, \tag{4.25}$$

for all real functions $y(r)$ satisfying $y, y' = 0$ at $r = a$, $r \to \infty$, and proceed in a manner similar to the Rayleigh–Ritz argument. It can be shown that $I(y)$ is bounded above and has a maximum. Suppose that $y = y_0$ is a stationary point of I, and $I(y_0) = I_0$. Consider $y = y_0 + \epsilon y_1$, where $\epsilon \ll 1$, and y_1 satisfies the boundary conditions for y. Since $y = y_0$ is a stationary point of $I(y)$, the first variation is zero, and thus after Taylor expanding $I(y_0 + \epsilon y_1)$, it follows that

$$-Re\, I_0^2 \int_a^\infty \left(y_0' y_1' + \left(\frac{1}{r^2} + k^2\right) y_0 y_1\right) r \,\mathrm{d}r - I_0 \int_a^\infty \mathcal{L}_0 y_0 \mathcal{L}_0 y_1 r \,\mathrm{d}r = k^2 B \int_a^\infty \frac{\chi_0' y_0 y_1}{r^2}\,\mathrm{d}r. \tag{4.26}$$

Invoking the self-adjoint property of $\mathcal{L}_0$, using integration by parts, and the boundary conditions for y_0, y_1, we can write (4.26) as

$$Re\, I_0^2 \int_a^\infty y_1 r \mathcal{L}_0 y_0 \,\mathrm{d}r - I_0 \int_a^\infty r y_1 \mathcal{L}_0^2 y_0 \,\mathrm{d}r = k^2 B \int_a^\infty \frac{\chi_0' y_0 y_1}{r^2}\,\mathrm{d}r. \tag{4.27}$$

Equation (4.27) is valid for any y_1, and therefore

$$Re\, I_0^2 \mathcal{L}_0 y_0 - I_0 \mathcal{L}_0^2 y_0 = \frac{k^2 B \chi_0' y_0}{r^3}. \tag{4.28}$$

It follows that the stationary points of $I(y)$ satisfy (4.22) with real eigenvalues $s = I_0$, and therefore the stationary points of $I(y)$ correspond to the real eigenvalues of (4.22).

Suppose that

$$\chi_0' > 0, \quad \text{for } r_1 \le r \le r_2, \tag{4.29}$$

and pick an arbitrary real function $\hat{y}(r)$ that satisfies the boundary conditions of y such that

$$\left.\begin{aligned} \hat{y}(r) &\neq 0, \quad \text{for } r_1 \le r \le r_2, \\ \hat{y}(r) &= 0, \quad \text{for } r \notin [r_1, r_2]. \end{aligned}\right\} \tag{4.30}$$

Substitute $y = \hat{y}(r)$ into (4.24) to give $I(\hat{y}) = \eta_M$. It follows that $\eta_M > 0$, and either η_M is a stationary point of $I(y)$, and therefore a positive, real eigenvalue of (4.22), or there exists a positive stationary point of $I(y)$ greater than η_M, since I is bounded above. Thus there exists a positive eigenvalue of (4.22), resulting in an unstable mode. We conclude that if and only if $\chi_0' > 0$ anywhere in the domain, every axisymmetric mode is unstable.

4.2. *Two-dimensional modes*

By considering two-dimensional modes $k = 0$ in (3.10), and (4.11)–(4.16), we obtain an eigenvalue equation,

$$(s^2\, Re\, \mathcal{L}_m - s\mathcal{L}_m^2)\mathcal{M}_m\hat{\phi} = -m^2 B\left(\frac{\chi_0'\mathcal{M}_m\hat{\phi}}{r^5} + \frac{m^2\chi_0'\hat{\phi}}{r^3}\right), \tag{4.31}$$

where

$$\mathcal{L}_m = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}}{\mathrm{d}r}\right) - \frac{m^2}{r^2}, \quad \mathcal{M}_m = \frac{r^3}{\chi_0'}\left[(1+\chi_0)\mathcal{L}_m + \chi_0'\frac{\mathrm{d}}{\mathrm{d}r}\right], \tag{4.32}$$

and $\phi, \phi' \to 0$ as $r \to \infty$. Taking the limit $k \to 0$ in (4.17) gives

$$\hat{\phi}' = \frac{m}{r(1+\chi_0)}\hat{\phi}(r) \tag{4.33}$$

at $r = a$.

We multiply (4.31) by $r\mathcal{M}_m\hat{\phi}^*$, where $\hat{\phi}^*$ is the complex conjugate of $\hat{\phi}$, and integrate over the domain to obtain

$$\int_a^\infty (\mathcal{M}_m\hat{\phi}^*(s^2\, Re\, \mathcal{L}_m - s\mathcal{L}_m^2)\mathcal{M}_m\hat{\phi})r\,\mathrm{d}r = -Bm^2\int_a^\infty \left(\frac{\chi_0'\,|\mathcal{M}_m\hat{\phi}|^2}{r^4} + \frac{m^2\chi_0'}{r^2}\hat{\phi}\mathcal{M}_m\hat{\phi}^*\right)\mathrm{d}r. \tag{4.34}$$

Now,

$$\int_a^\infty \frac{1}{r^2}\chi_0'\hat{\phi}\mathcal{M}_m\hat{\phi}^*\,\mathrm{d}r = \int_a^\infty \left((1+\chi_0)\hat{\phi}(r(\hat{\phi}^*)')' + r\chi_0'\hat{\phi}(\hat{\phi}^*)' - \frac{m^2(1+\chi_0)|\hat{\phi}|^2}{r}\right)\mathrm{d}r, \tag{4.35}$$

and integration by parts gives

$$\int_a^\infty \frac{1}{r^2}\chi_0'\hat{\phi}\mathcal{M}_m\,\hat{\phi}^*\,\mathrm{d}r = -m|\hat{\phi}(a)|^2 - \int_a^\infty r(1+\chi_0)\left(|\hat{\phi}'|^2 + \frac{m^2|\hat{\phi}|^2}{r^2}\right)\mathrm{d}r. \tag{4.36}$$

Equation (4.36) and the self-adjoint property of $\mathcal{L}_m$ allow (4.34) to be written as

$$s^2\, Re\int_a^\infty \left(|(\mathcal{M}_m\hat{\phi})'|^2 + \frac{m^2|\mathcal{M}_m\hat{\phi}|^2}{r^2}\right)r\,\mathrm{d}r + s\int_a^\infty \left|\mathcal{L}_m(\mathcal{M}_m\hat{\phi})\right|^2 r\,\mathrm{d}r \\ + Bm^2\left(m^3|\hat{\phi}(a)|^2 + \int_a^\infty \left[\frac{-\chi_0'\,|\mathcal{M}_m\hat{\phi}|^2}{r^4} + m^2 r(1+\chi_0)\left(|\hat{\phi}'|^2 + \frac{m^2|\hat{\phi}|^2}{r^2}\right)\right]\mathrm{d}r\right) = 0. \tag{4.37}$$

It follows that if $\chi_0' < 0$ everywhere, then two-dimensional modes are stable. Yet if

$$\int_a^\infty \frac{\chi_0'\,|\mathcal{M}_m\hat{\phi}|^2}{r^4}r\,\mathrm{d}r > m^3|\hat{\phi}(a)|^2 + m^2\int_a^\infty r(1+\chi_0)\left(|\hat{\phi}'|^2 + \frac{m^2|\hat{\phi}|^2}{r^2}\right)\mathrm{d}r \tag{4.38}$$

holds for an eigenfunction $\hat{\phi}$, then two-dimensional modes are unstable.

Furthermore, by considering the functional

$$I(y)=\frac{-\int_a^\infty(\mathcal{L}_m(\mathcal{M}_m y))^2\,r\,\mathrm{d}r+\sqrt{(\int_a^\infty(\mathcal{L}_m(\mathcal{M}_m y))^2\,r\,\mathrm{d}r)^2-W_1}}{2\,Re\int_a^\infty\left((\mathcal{M}_m y')^2+\dfrac{m^2}{r^2}(\mathcal{M}_m y)^2\right)r\,\mathrm{d}r}, \tag{4.39}$$

where

$$W_1=4m^2B\,Re\left(\int_a^\infty\left((\mathcal{M}_m y')^2+\frac{m^2}{r^2}(\mathcal{M}_m y)^2\right)r\,\mathrm{d}r\right)\left(m^3|\hat{\phi}(a)|^2\right.$$
$$\left.+\int_a^\infty\left[\frac{-\chi_0'\,|\mathcal{M}_m\hat{\phi}|^2}{r^4}+m^2r(1+\chi_0)\left(|\hat{\phi}'|^2+\frac{m^2|\hat{\phi}|^2}{r^2}\right)\right]\mathrm{d}r\right), \tag{4.40}$$

for all real functions y satisfying the boundary conditions of $\hat{\phi}$, we prove that the stationary points of $I(y)$ correspond to the real eigenvalues of (4.31) by an argument analogous to that in § 4.1. Again, if (4.29) is true, then by picking a $y=\hat{y}(r)$, where $\hat{y}$ satisfies (4.30), with oscillatory behaviour for r in the interval $[r_1, r_2]$, then $I(\hat{y})>0$ and (4.38) holds as $(\mathcal{M}_m\hat{y})^2\gg\hat{y}'\gg\hat{y}$. Consequently, if and only if $\chi_0'>0$ anywhere in the fluid, every mode where $k=0$ is unstable.

4.3. *Highly viscous and inviscid limits*

Although we have proven that if $\chi_0'>0$ anywhere, all axisymmetric or two-dimensional disturbances are unstable, we are unable to prove, for arbitrary Reynolds number, that if $\chi_0'<0$ everywhere, then all modes are stable; we have proved only that axisymmetric or two-dimensional disturbances are stable. A global energy argument could be applied here by posing an argument analogous to the Rayleigh's stability argument for centrifugal instability. One considers the change in energy when two parcels of ferrofluid at different radii are interchanged while conserving $\chi(r)$, where the resulting condition for stability is that $\boldsymbol{\nabla}\chi$ must decrease continuously radially. Yet this argument would not account for viscous forces or three-dimensional disturbances.

By considering the inviscid limit of (4.11)–(4.17) and following an argument analogous to that in § 4.1, we prove that for all m, k, if and only if $\chi_0'>0$ anywhere in the fluid, the system is unstable, and it can be proved that s^2 and $\hat{\phi}$ are real. In the highly viscous limit of (4.11)–(4.17), the eigenvalues are also proven to be real for axisymmetric disturbances and modes $k=0$, and these modes are shown to be unstable if and only if $\chi_0'>0$ anywhere in the fluid.

4.4. *Stabilisation with an axial field*

We now show that by adding an axial field

$$\boldsymbol{H}_0=\left(0,\frac{1}{r},Z\right), \tag{4.41}$$

we can suppress unstable disturbances. It follows that

$$H_0=\left(\frac{1}{r^2}+Z^2\right)^{1/2},\quad H_1=\frac{\mathrm{i}m+\mathrm{i}kZr^2}{r^2H_0}\,\hat{\phi}(r)\,\zeta, \tag{4.42}$$

and (4.7) in component form is

$$(s\,Re-\mathcal{L})\hat{\omega}_r+\frac{2\mathrm{i}m}{r^2}\,\hat{\omega}_\theta=0, \tag{4.43a}$$

$$(s\,Re-\mathcal{L})\hat{\omega}_\theta-\frac{2\mathrm{i}m}{r^2}\,\hat{\omega}_r=B\left(\frac{k(m+kZr^2)\chi_0'\hat{\phi}}{r^2}+\mathrm{i}kH_0H_0'\hat{\chi}\right), \tag{4.43b}$$

$$(s\,Re-\mathcal{D})\hat{\omega}_z=B\left(-\frac{m(m+kZr^2)\chi_0'\hat{\phi}}{r^3}-\frac{\mathrm{i}mH_0H_0'\hat{\chi}}{r}\right). \tag{4.43c}$$

4.4.1. *Axisymmetric disturbances*

For solely axisymmetric disturbances, we have

$$H_1=\frac{\mathrm{i}kZ}{H_0}\,\hat{\phi}(r)\exp(\mathrm{i}(m\theta+kz)+st), \tag{4.44}$$

and (4.43) becomes

$$(s\,Re-\mathcal{L}_0)\hat{\omega}_r=0, \tag{4.45a}$$

$$(s\,Re-\mathcal{L}_0)\hat{\omega}_\theta=B(k^2Z\chi_0'\hat{\phi}+\mathrm{i}kH_0H_0'\hat{\chi}), \tag{4.45b}$$

$$(s\,Re-\mathcal{D}_0)\hat{\omega}_z=0. \tag{4.45c}$$

Equation (2.4) gives

$$\hat{\chi}=\frac{\mathrm{i}((1+\chi_0)\nabla^2\hat{\phi}+\chi_0'\hat{\phi}')}{kZ}, \tag{4.46}$$

and it follows from (4.15), for axisymmetric disturbances, that

$$\hat{\psi}=\frac{-\mathrm{i}s\hat{\chi}}{k\chi_0'}, \tag{4.47}$$

where $\hat{\psi}$ was defined in § 4.1.

Equations (4.45)–(4.47) give an eigenvalue equation for $\hat{\phi}$:

$$(s^2\,Re\,\mathcal{L}_0-s\mathcal{L}_0^2)\mathcal{P}_0\hat{\phi}=k^2B\chi_0'(H_0H_0'\mathcal{P}_0\hat{\phi}-k^2Z^2\hat{\phi}), \tag{4.48}$$

where

$$\mathcal{P}_0\hat{\phi}=\frac{(1+\chi_0)}{\chi_0'}\left(\frac{1}{r}\,(r\hat{\phi}')'-k^2\hat{\phi}\right)+\hat{\phi}'. \tag{4.49}$$

When $m=0$,

$$\hat{\phi}'=\frac{kI_1\hat{\phi}}{(1+\chi_0)I_0} \tag{4.50}$$

at $r=a$, and $\nabla\hat{\phi}\to 0$ as $r\to\infty$. Due to the boundary conditions for $\hat{\chi}$, $\mathcal{P}_0\hat{\phi}=0$ at $r=a$ and as $r\to\infty$.

Multiply (4.48) by $r\mathcal{P}_0\hat{\phi}^*$, and integrate over the domain to give

$$\int_a^\infty (s^2\, Re\, \mathcal{P}_0\hat{\phi}^*\mathcal{L}_0\mathcal{P}_0\hat{\phi} - s\mathcal{P}_0\hat{\phi}^*\mathcal{L}_0^2\mathcal{P}_0\hat{\phi})r\,\mathrm{d}r = k^2B\int_a^\infty r\chi_0'(H_0H_0'|\mathcal{P}_0\hat{\phi}|^2 - k^2Z^2\hat{\phi}\mathcal{P}_0\hat{\phi}^*)\mathrm{d}r. \tag{4.51}$$

Due to the self-adjoint property of $\mathcal{L}_0$ and the boundary conditions for $\hat{\phi}$, we can write (4.51) as

$$\int_a^\infty \left(s^2\, Re\left(|(\mathcal{P}_0\hat{\phi})'|^2 + \left(k^2 + \frac{1}{r^2}\right)|\mathcal{P}_0\hat{\phi}|^2\right) + s\,|\mathcal{L}_0\mathcal{P}_0\hat{\phi}|^2\right) r\,\mathrm{d}r$$
$$+ k^2B\int_a^\infty (H_0H_0'\chi_0'|\mathcal{P}_0\hat{\phi}|^2 + k^2Z^2(1+\chi_0)(|\hat{\phi}'|^2 + k^2|\hat{\phi}|^2))r\,\mathrm{d}r = 0. \tag{4.52}$$

Thus for sufficiently large kZ, $\mathrm{Re}(s) < 0$ provided that $B \neq 0$, therefore axisymmetric modes are stable. Moreover, in the inviscid regime, this result holds for all disturbances m, k.

5. Concluding remarks

This paper first looks at a ferrofluid column surrounded by another ferrofluid of different susceptibility centred on a current-carrying wire. The greatest growth rate is found when the ratio between the radius of the wire and the radius of the inner fluid, a/R, is at its smallest. When the inner fluid is more magnetic, only axisymmetric modes with $k < 1$ and $B(\chi_1 - \chi_2) < 1$ are unstable. When the outer fluid is more magnetic, both non-axisymmetric and axisymmetric modes can be unstable. Interestingly, for sufficiently small Reynolds numbers, the non-axisymmetric mode $m = 1$ is the most unstable; otherwise, $m = 0$ is the most unstable. Sufficient current in the wire suppresses instabilities due to surface tension only if the inner fluid is more magnetic than the outer fluid. When the outer fluid is more magnetic, instabilities are due to magnetic forcing at the interface, produced from the current in the wire, as well as capillary forcing from the surface tension. Thus when $\chi_2 > \chi_1$, increasing the current in the wire will only increase the strength of the forcing at the interface, thereby increasing the growth rate of the perturbation. However, adding a large enough axial field will suppress all disturbances, irrespective of which fluid has a higher susceptibility, provided that there is some current in the wire.

Considering a ferrofluid whose susceptibility varies radially, centred on a wire, we proved if $\mathrm{d}\chi_0/\mathrm{d}r > 0$ anywhere in the fluid, then the system is unstable, but adding a large enough axial field will suppress the unstable axisymmetric modes for arbitrary Reynolds number, and all disturbances in the inviscid regime. When $\mathrm{d}\chi_0/\mathrm{d}r < 0$ everywhere, axisymmetric disturbances and two-dimensional disturbances are proven to be stable for arbitrary Reynolds number, and every three-dimensional disturbance is stable in the inviscid regime. In the inviscid limit, we proved that if and only if $\mathrm{d}\chi_0/\mathrm{d}r > 0$ somewhere in the fluid, the system is unstable, and we conjecture that this holds for arbitrary Reynolds number. Moreover, these results hold for χ depending explicitly on H, so long as $\mathrm{D}\chi/\mathrm{D}t = 0$ holds, and thus can be applied to a ferrofluid with nonlinear magnetisation characteristics. It should be noted that by assuming $\mathrm{D}\chi/\mathrm{D}t = 0$, we are neglecting any effects due to relative motion of the magnetic particles and the fluid.

Physically, for an instability to occur, a source of energy is needed, enabling a perturbation to grow. Since $H_0 = 1/r$, when $\mathrm{d}\chi_0/\mathrm{d}r > 0$, $\mathrm{d}\chi_0/\,\mathrm{d}H_0 < 0$, implying that when the regions of fluid with highest susceptibility do not coincide with the regions of strongest field, an instability may occur to achieve a minimum energy configuration. Although $\mathrm{d}\chi_0/\mathrm{d}r > 0$ is a local condition, a global instability occurs, suggesting that when $\mathrm{d}\chi_0/\,\mathrm{d}H_0 < 0$ somewhere in the fluid, a release of energy locally suffices to drive a global instability. We surmise that given a more general geometry where the equilibrium satisfies $\chi_0 = \chi_0(H_0)$, if $\mathrm{d}\chi_0/\,\mathrm{d}H_0 > 0$ everywhere, then the system would be stable, whereas there may be an instability if $\mathrm{d}\chi_0/\mathrm{d}H_0 < 0$ somewhere, a result that could be used to determine the stability of a stationary state in a more complicated geometry.

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Appendix A

$$
\begin{aligned}
q_2^{(2)} = {} & (\mathrm{i}m\hat{S}(\chi_1-\chi_2)(ak\,\mathrm{I}_m(k)\,\mathrm{I}_m(ka)\,(1+\chi_1)\,\mathrm{K}_{m+1}(ka)+ka(\chi_1\,\mathrm{K}_m(k)\,\mathrm{I}_m(ka)\\
& +\mathrm{K}_m(ka)\,\mathrm{I}_m(k))\,\mathrm{I}_{m+1}(ka)-m\chi_1\,\mathrm{I}_m(ka)\,(\mathrm{K}_m(ka)\,\mathrm{I}_m(k)-\mathrm{K}_m(k)\,\mathrm{I}_m(ka))))\\
& \times(\mathrm{I}_m(ka)\,a(1+\chi_1)k(k\,\mathrm{I}_m(k)\,(1+\chi_2)\,\mathrm{K}_{m+1}(k)+(k(1+\chi_1)\,\mathrm{I}_{m+1}(k)\\
& +m\,\mathrm{I}_m(k)\,(\chi_1-\chi_2))\,\mathrm{K}_m(k))\,\mathrm{K}_{m+1}(ka)+a(-\chi_1\,\mathrm{K}_m(k)\,(\chi_1-\chi_2)\\
& \times(\mathrm{K}_{m+1}(k)\,k-m\,\mathrm{K}_m(k))\,\mathrm{I}_m(ka)+\mathrm{K}_m(ka)\,(k\,\mathrm{I}_m(k)\,(1+\chi_2)\,\mathrm{K}_{m+1}(k)\\
& +(k(1+\chi_1)\,\mathrm{I}_{m+1}(k)+m\,\mathrm{I}_m(k)\,(\chi_1-\chi_2))\,\mathrm{K}_m(k)))k\,\mathrm{I}_{m+1}(ka)\\
& -\mathrm{I}_m(ka)\,(\mathrm{K}_m(k)\,(\chi_1-\chi_2)(\mathrm{K}_{m+1}(k)\,k-m\,\mathrm{K}_m(k))\,\mathrm{I}_m(ka)\\
& +\mathrm{K}_m(ka)\,(k\,\mathrm{I}_m(k)\,(1+\chi_2)\,\mathrm{K}_{m+1}(k)\\
& +(k(1+\chi_1)\,\mathrm{I}_{m+1}(k)+m\,\mathrm{I}_m(k)\,(\chi_1-\chi_2))\,\mathrm{K}_m(k)))m\chi_1)^{-1},
\end{aligned}
\tag{A1}
$$

$$
\begin{aligned}
q_1^{(0)} = {} & \left(\frac{\mathrm{i}}{a}\hat{S}\,\mathrm{K}_m(k)(1+\chi_1)ma(\chi_1-\chi_2)\right)\\
& \times(k(1+\chi_1)a\,\mathrm{I}_m(ka)\,(k\,\mathrm{I}_m(k)\,(1+\chi_2)\,\mathrm{K}_{m+1}(k)+\mathrm{K}_m(k)\,(k(1+\chi_1)\,\mathrm{I}_{m+1}(k)\\
& +m\,\mathrm{I}_m(k)\,(\chi_1-\chi_2)))\,\mathrm{K}_{m+1}(ka)+k(-\chi_1\,\mathrm{K}_m(k)\,(\chi_1-\chi_2)\\
& \times(\mathrm{K}_{m+1}(k)\,k-m\,\mathrm{K}_m(k))\,\mathrm{I}_m(ka)+\mathrm{K}_m(ka)\,(k\,\mathrm{I}_m(k)(1+\chi_2)\,\mathrm{K}_{m+1}(k)+\mathrm{K}_m(k)\\
& \times(k(1+\chi_1)\,\mathrm{I}_{m+1}(k)+m\,\mathrm{I}_m(k)\,(\chi_1-\chi_2))))a\,\mathrm{I}_{m+1}(ka)-m(\mathrm{K}_m(k)\,(\chi_1-\chi_2)\\
& \times(\mathrm{K}_{m+1}(k)\,k-m\,\mathrm{K}_m(k))\,\mathrm{I}_m(ka)+\mathrm{K}_m(ka)\,(k\,\mathrm{I}_m(k)\,(1+\chi_2)\,\mathrm{K}_{m+1}(k)\\
& +\mathrm{K}_m(k)\,(k(1+\chi_1)\,\mathrm{I}_{m+1}(k)+m\,\mathrm{I}_m(k)\,(\chi_1-\chi_2))))\,\mathrm{I}_m(ka)\,\chi_1)^{-1},
\end{aligned}
\tag{A2}
$$

$$
\begin{aligned}
q_1^{(1)} = {} & (\mathrm{i}\hat{S}\,\mathrm{K}_m(k)\,(ak\,\mathrm{I}_m(ka)\,(1+\chi_1)\,\mathrm{K}_{m+1}(ka)+\mathrm{K}_m(ka)\,(ak\,\mathrm{I}_{m+1}(ka)\\
& -m\chi_1\,\mathrm{I}_m(ka)))m(\chi_1-\chi_2))(k(1+\chi_1)a\,\mathrm{I}_m(ka)\,(k\,\mathrm{I}_m(k)\,(1+\chi_2)\,\mathrm{K}_{m+1}(k)\\
& +\mathrm{K}_m(k)\,(k(1+\chi_1)\,\mathrm{I}_{m+1}(k)+m\,\mathrm{I}_m(k)\,(\chi_1-\chi_2)))\,\mathrm{K}_{m+1}(ka)\\
& +k(-\chi_1\,\mathrm{K}_m(k)\,(\chi_1-\chi_2)(\mathrm{K}_{m+1}(k)\,k-m\,\mathrm{K}_m(k))\,\mathrm{I}_m(ka)\\
& +\mathrm{K}_m(ka)\,(k\,\mathrm{I}_m(k)\,(1+\chi_2)\,\mathrm{K}_{m+1}(k)+\mathrm{K}_m(k)\,(k(1+\chi_1)\,\mathrm{I}_{m+1}(k)
\end{aligned}
$$

$$
\begin{aligned}
&+ m\,\mathrm{I}_m(k)\,(\chi_1-\chi_2))))a\,\mathrm{I}_{m+1}(ka) - m(\mathrm{K}_m(k)\,(\chi_1-\chi_2)(\mathrm{K}_{m+1}(k)\,k\\
&- m\,\mathrm{K}_m\,(k))\,\mathrm{I}_m(ka) + \mathrm{K}_m(ka)\,(k\,\mathrm{I}_m(k)(1+\chi_2)\,\mathrm{K}_{m+1}(k)\\
&+ \mathrm{K}_m(k)\,(k(1+\chi_1)\,\mathrm{I}_{m+1}(k) + m\,\mathrm{I}_m(k)\,(\chi_1-\chi_2))))\,\mathrm{I}_m(ka)\,\chi_1)^{-1}
\end{aligned}
\tag{A3}
$$

and

$$
\begin{aligned}
q_2^{(1)} = {}& (\mathrm{i}\hat{S}\,\mathrm{I}_m(ka)\,\chi_1\,\mathrm{K}_m(k)\,m(\chi_1-\chi_2)\big(ak\,\mathrm{I}_{m+1}(ka) + m\,\mathrm{I}_m(ka)\big))\\
&\times (k(1+\chi_1)a\,\mathrm{I}_m(ka)\,(k\,\mathrm{I}_m(k)\,(1+\chi_2)\,\mathrm{K}_{m+1}(k) + \mathrm{K}_m(k)\,(k(1+\chi_1)\,\mathrm{I}_{m+1}(k)\\
&+ m\,\mathrm{I}_m(k)\,(\chi_1-\chi_2)))\,\mathrm{K}_{m+1}(ka) + k(-\chi_1\,\mathrm{K}_m(k)\,(\chi_1-\chi_2)(\mathrm{K}_{m+1}(k)\,k\\
&- m\,\mathrm{K}_m(k))\,\mathrm{I}_m(ka) + \mathrm{K}_m(ka)\,(k\,\mathrm{I}_m(k)(1+\chi_2)\,\mathrm{K}_{m+1}(k)\\
&+ \mathrm{K}_m(k)\,(k(1+\chi_1)\,\mathrm{I}_{m+1}(k) + m\,\mathrm{I}_m(k)\,(\chi_1-\chi_2))))a\,\mathrm{I}_{m+1}(ka)\\
&- m(\mathrm{K}_m(k)\,(\chi_1-\chi_2)(\mathrm{K}_{m+1}(k)\,k - m\,\mathrm{K}_m(k))\,\mathrm{I}_m(ka)\\
&+ \mathrm{K}_m(ka)\,(k\,\mathrm{I}_m(k)\,(1+\chi_2)\,\mathrm{K}_{m+1}(k) + \mathrm{K}_m(k)\,(k(1+\chi_1)\,\mathrm{I}_{m+1}(k) + m\,\mathrm{I}_m(k)\\
&\times (\chi_1-\chi_2))))\,\mathrm{I}_m(ka)\,\chi_1)^{-1}.
\end{aligned}
\tag{A4}
$$

Now, f in (3.31) is given by

$$
f = \frac{F_1}{F_2},
\tag{A5}
$$

where

$$
\begin{aligned}
F_1 = {}& k^2a^2\bar{k}((-\bar{k}^2k(\mathrm{I}_m(k)\,m + \mathrm{I}_{m+1}(k)\,k)\,\mathrm{K}_{m+1}(k) + \bar{k}k^2(\bar{k}\,\mathrm{I}_{m+1}(\bar{k}) + m\,\mathrm{I}_m(\bar{k}))\,\mathrm{K}_{m+1}(\bar{k})\\
&+ m(\bar{k}^2\,\mathrm{I}_{m+1}(k)\,k\,K_m(k) - \bar{k}k^2\,\mathrm{K}_m(\bar{k})\,\mathrm{I}_{m+1}(\bar{k}) + m(\mathrm{I}_m(\bar{k})\,(\bar{k}^2-2k^2)\,\mathrm{K}_m(\bar{k})\\
&+ \bar{k}^2\,\mathrm{I}_m(k)\,\mathrm{K}_m(k))))\,\mathrm{K}_m(ka) + \bar{k}^2(-\mathrm{I}_m(ka)\,(\mathrm{K}_{m+1}(k))^2k^2 + k(\bar{k}\,\mathrm{K}_{m+1}(\bar{k})\,\mathrm{I}_m(\bar{k}a)\\
&+ 2(\tfrac{1}{2}\,\mathrm{K}_m(\bar{k})\,(\mathrm{I}_m(k)\,m\,\mathrm{K}_m(k)-1)\,\mathrm{I}_m(\bar{k}a) + \mathrm{I}_m(ka)\,\mathrm{K}_m(k))m)\mathrm{K}_{m+1}(k)\\
&- (\bar{k}\,\mathrm{K}_{m+1}(\bar{k})\,\mathrm{I}_m(\bar{k}a) + m\,\mathrm{K}_m(k)(-\mathrm{I}_{m+1}(k)\,k\,\mathrm{K}_m(\bar{k})\,\mathrm{I}_m(\bar{k}a) + \mathrm{I}_m(ka)))m\,\mathrm{K}_m(k)))\\
&\times (\mathrm{K}_{m+1}(\bar{k}a))^2 - a(k((-\bar{k}^2k(\mathrm{I}_m(k)\,m + \mathrm{I}_{m+1}(k)k)\,\mathrm{K}_{m+1}(k)\\
&+ \bar{k}k^2(\bar{k}\,\mathrm{I}_{m+1}(\bar{k}) + m\,\mathrm{I}_m(\bar{k}))\,\mathrm{K}_{m+1}(\bar{k}) + m(\bar{k}^2\,\mathrm{I}_{m+1}(k)\,k\,\mathrm{K}_m(k) - \bar{k}k^2\,\mathrm{K}_m(\bar{k})\,\mathrm{I}_{m+1}(\bar{k})\\
&+ m(\mathrm{I}_m(\bar{k})\,(\bar{k}^2-2k^2)\,\mathrm{K}_m(\bar{k}) + \bar{k}^2\,\mathrm{I}_m(k)\,\mathrm{K}_m(k))))\,\mathrm{K}_m(\bar{k}a) + k^2\,\mathrm{I}_m(\bar{k}a)\,(\bar{k}\,\mathrm{K}_{m+1}(\bar{k})\\
&- \mathrm{K}_m(\bar{k})\,m)^2)a\bar{k}^2\,\mathrm{K}_{m+1}(ka) - \bar{k}k^2a(\bar{k}^2(k\,\mathrm{K}_{m+1}(k) - m\,\mathrm{K}_m(k))(\bar{k}\,\mathrm{K}_{m+1}(\bar{k})\\
&- \mathrm{K}_m(\bar{k})\,m)\,\mathrm{K}_m(\bar{k}a) - (\bar{k}^2\,(\mathrm{K}_{m+1}(\bar{k}))^2k^2 - 2\bar{k}\,\mathrm{K}_m(\bar{k})\,\mathrm{K}_{m+1}(\bar{k})\,k^2m\\
&- m^2(\mathrm{K}_m(\bar{k}))^2(\bar{k}^2-2k^2))\,\mathrm{K}_m(ka))\mathrm{I}_{m+1}(\bar{k}a) + \bar{k}^4\,\mathrm{K}_m(\bar{k}a)\,ak(k\,\mathrm{K}_{m+1}(k)\\
&- m\,\mathrm{K}_m(k))^2\,\mathrm{I}_{m+1}(ka) + ((k(-2\bar{k}k^2m\,\mathrm{I}_m(\bar{k})\,\mathrm{I}_m(k)\mathrm{K}_{m+1}(\bar{k}) + (\bar{k}^2k - 2k^3)\,\mathrm{I}_{m+1}(k)\\
&+ \mathrm{I}_m(k)\,m(\bar{k}-k)(\bar{k}+k))\bar{k}^2\,\mathrm{K}_{m+1}(k) + ((-\bar{k}^4k^2 + 2\bar{k}^2k^4)\,\mathrm{I}_{m+1}(\bar{k})\\
&+ 2\bar{k}k^4m\,\mathrm{I}_m(\bar{k}))\,\mathrm{K}_{m+1}(\bar{k}) - (\bar{k}^2k\,\mathrm{K}_m(k)\,(-2\bar{k}k^2\,\mathrm{K}_m(\bar{k})\,\mathrm{I}_{m+1}(\bar{k}) + \bar{k}^2 - k^2)\,I_{m+1}(k)
\end{aligned}
$$

$$
\begin{aligned}
&+ 2\bar{k}k^4\,\mathrm{K}_m(\bar{k})\,\mathrm{I}_{m+1}(\bar{k}) + m(\mathrm{I}_m(\bar{k})(\bar{k}^2 - 2k^2)\mathrm{K}_m(\bar{k}) + \bar{k}^2\,\mathrm{I}_m(k)\,\mathrm{K}_m(k))\\
&\times (\bar{k}^2 - 2k^2))m)\,\mathrm{K}_m(ka) + (\mathrm{I}_m(ka)\,k^2(\bar{k}^2 - 2k^2)(\mathrm{K}_{m+1}(k))^2\\
&- 2k(-\tfrac{1}{2}\bar{k}\,\mathrm{K}_{m+1}(\bar{k})\,k^2\,\mathrm{I}_m(\bar{k}a) + m(-\tfrac{1}{2}\bar{k}^2\,\mathrm{K}_m(\bar{k})\,(m\,\mathrm{I}_m(k)\,\mathrm{K}_m(k)\\
&- 1)\,\mathrm{I}_m(\bar{k}a) + \mathrm{I}_m(ka)\,\mathrm{K}_m(k)\,(\bar{k}^2 - 2k^2)))\,\mathrm{K}_{m+1}(k) + (-\bar{k}\,\mathrm{K}_{m+1}(\bar{k})\,k^2\,\mathrm{I}_m(\bar{k}a)\\
&+ (\bar{k}^2\,\mathrm{I}_{m+1}(k)\,k\,\mathrm{K}_m(\bar{k})\,\mathrm{I}_m(\bar{k}a) + \mathrm{I}_m(ka)\,(\bar{k}^2 - 2k^2))m\,\mathrm{K}_m(k))\,m\,\mathrm{K}_m(k))\,\bar{k}^2)m\,\mathrm{K}_m(\bar{k}a)\\
&- k^2(m\,\mathrm{I}_m(\bar{k}a)\,(\bar{k} - k)(\bar{k} + k)(\bar{k}\,\mathrm{K}_{m+1}(\bar{k}) - 2\,\mathrm{K}_m(\bar{k})\,m)^2\mathrm{K}_m(ka) + \bar{k}^2(k\,\mathrm{K}_{m+1}(k)\\
&- m\,\mathrm{K}_m(k))\,(\bar{k}\,\mathrm{K}_{m+1}(\bar{k}) - \mathrm{K}_m(\bar{k})\,m)))\mathrm{K}_{m+1}(\bar{k}a)\\
&+ \mathrm{K}_m(\bar{k}a)\,m(-k(\bar{k}(\mathrm{K}_m(\bar{k}))^2am(\bar{k} - k)(\bar{k} + k)\,\mathrm{I}_{m+1}(\bar{k}a)\\
&+ (\bar{k}^2k(\mathrm{I}_m(k)\,m + \mathrm{I}_{m+1}(k)k)\mathrm{K}_{m+1}(k) - \bar{k}k^2(\bar{k}\,\mathrm{I}_{m+1}(\bar{k}) + m\,\mathrm{I}_m(\bar{k}))\,\mathrm{K}_{m+1}(\bar{k})\\
&- m(\bar{k}^2\,\mathrm{I}_{m+1}(k)\,k\,\mathrm{K}_m(k) - \bar{k}k^2\,\mathrm{K}_m(\bar{k})\,\mathrm{I}_{m+1}(\bar{k}) + m(\mathrm{I}_m(\bar{k})\,(\bar{k}^2 - 2k^2)\,\mathrm{K}_m(\bar{k})\\
&+ \bar{k}^2\,\mathrm{I}_m(k)\,\mathrm{K}_m(k))))\,\mathrm{K}_m(\bar{k}a) - \bar{k}^2\,\mathrm{I}_m(ka)\,\mathrm{K}_{m+1}(k)\,km\,\mathrm{K}_m(\bar{k}) - \mathrm{I}_m(\bar{k}a)\\
&\times (\bar{k}^2(\mathrm{K}_{m+1}(\bar{k}))^2k^2 - 2\bar{k}\,\mathrm{K}_m(\bar{k})\,\mathrm{K}_{m+1}(\bar{k})\,k^2m - m^2(\mathrm{K}_m(\bar{k}))^2(\bar{k}^2 - 2k^2)))a\,\mathrm{K}_{m+1}(ka)\\
&+ \bar{k}((\mathrm{K}_m(\bar{k})\,m\bar{k} - \mathrm{K}_{m+1}(\bar{k})\,k^2)(\bar{k}(k\,\mathrm{K}_{m+1}(k) - m\,\mathrm{K}_m(k))\,\mathrm{K}_m(\bar{k}a)\\
&+ (\mathrm{K}_m(\bar{k})\,m\bar{k} - \mathrm{K}_{m+1}(\bar{k})\,k^2)\,\mathrm{K}_m(ka))a\,\mathrm{I}_{m+1}(\bar{k}a)\\
&+ k((k\,\mathrm{K}_{m+1}(k) - m\,\mathrm{K}_m(k))^2\,\mathrm{K}_m(\bar{k}a) + \mathrm{K}_m(ka)\,\mathrm{K}_{m+1}(k)\,km\mathrm{K}_m(\bar{k}))a\bar{k}\,\mathrm{I}_{m+1}(ka)\\
&- \mathrm{K}_{m+1}(\bar{k})\,\mathrm{K}_{m+1}(k)\,k^3 - m\,\mathrm{K}_m(k)(\mathrm{K}_m(\bar{k})\,m\bar{k} - \mathrm{K}_{m+1}(\bar{k})\,k^2)))\bar{k}
\end{aligned}
\tag{A6}
$$

and

$$
\begin{aligned}
F_2 = -\bar{k}^2((\mathrm{K}_{m+1}(\bar{k}a))^2\,\mathrm{K}_m(ka)\,\bar{k}ak^2 + \mathrm{K}_m(\bar{k}a)\,(-a\bar{k}^2k\,\mathrm{K}_{m+1}(ka) + \mathrm{K}_m(ka)\,m(\bar{k}^2\\
- 2k^2))\,\mathrm{K}_{m+1}(\bar{k}a) + (\mathrm{K}_m(\bar{k}a))^2\,\mathrm{K}_{m+1}(ka)\,\bar{k}km)(\bar{k} + k)(\bar{k} - k)a.
\end{aligned}
\tag{A7}
$$

Let

$$
F = f_1(k^2 + m^2 - 1 + B(\chi_1 - \chi_2)) + f_2B(\chi_1 - \chi_2)^2m^2, \tag{A8}
$$

such that (3.31) is

$$
s = -gfF, \tag{A9}
$$

where

$$
\begin{aligned}
f_1 = (a\mathrm{I}_m(ka)\chi_1k(\mathrm{I}_m(k)\chi_2\mathrm{K}_{m+1}(k)k + \chi_1\mathrm{K}_m(k)\mathrm{I}_{m+1}(k)k + 1\\
+ m\mathrm{K}_m(k)\mathrm{I}_m(k)(\chi_1 - \chi_2))\mathrm{K}_{m+1}(ka) - ak\chi_1\mathrm{K}_m(k)\mathrm{I}_m(ka)(\chi_1 - \chi_2)\\
\times (k\mathrm{K}_{m+1}(k) - m\mathrm{K}_m(k))\mathrm{I}_{m+1}(ka) - \chi_1\mathrm{K}_m(k)m(\chi_1 - \chi_2)(k\mathrm{K}_{m+1}(k) - m\mathrm{K}_m(k))\\
\times (\mathrm{I}_m(ka))^2 - \mathrm{K}_m(ka)\chi_1m(\mathrm{I}_m(k)\chi_2\mathrm{K}_{m+1}(k)k\\
+ \chi_1\mathrm{K}_m(k)\mathrm{I}_{m+1}(k)k + 1 + m\mathrm{K}_m(k)\mathrm{I}_m(k)(\chi_1 - \chi_2))\mathrm{I}_m(ka) + \mathrm{I}_m(k)\chi_2\mathrm{K}_{m+1}(k)k\\
+ \chi_1\mathrm{K}_m(k)\mathrm{I}_{m+1}(k)k + 1 + m\mathrm{K}_m(k)\mathrm{I}_m(k)(\chi_1 - \chi_2)),
\end{aligned}
\tag{A10}
$$

$$f_2 = \mathrm{K}_m(k)(\mathrm{I}_m(ka)\mathrm{K}_m(k)\mathrm{I}_{m+1}(ka)a\chi_1 k + \mathrm{I}_m(ka)\mathrm{I}_m(k)\mathrm{K}_{m+1}(ka)a\chi_1 k$$
$$+ (\mathrm{I}_m(ka))^2\mathrm{K}_m(k)\chi_1 m - \mathrm{I}_m(ka)\mathrm{I}_m(k)\mathrm{K}_m(ka)\chi_1 m + \mathrm{I}_m(k)), \tag{A11}$$

$$g = -(k(-a\mathrm{I}_m(ka)(\mathrm{I}_m(k)\chi_2\mathrm{K}_{m+1}(k)k + k\mathrm{K}_m(k)(1+\chi_1)\mathrm{I}_{m+1}(k) + 1 + \mathrm{I}_m(k)m(\chi_1$$
$$- \chi_2 + 1)\mathrm{K}_m(k))\chi_1 k\mathrm{K}_{m+1}(ka) - a(-(\chi_1 - \chi_2)(k\mathrm{K}_{m+1}(k) - m\mathrm{K}_m(k))\mathrm{I}_m(ka)$$
$$+ \mathrm{K}_m(ka)(\mathrm{I}_m(k)m + \mathrm{I}_{m+1}(k)k))\chi_1 k\mathrm{K}_m(k)\mathrm{I}_{m+1}(ka) + \chi_1\mathrm{K}_m(k)m(\chi_1 - \chi_2)(k\mathrm{K}_{m+1}(k)$$
$$- m\mathrm{K}_m(k))(\mathrm{I}_m(ka))^2 + \mathrm{K}_m(ka)\chi_1 m(\mathrm{I}_m(k)\chi_2\mathrm{K}_{m+1}(k)k + \chi_1\mathrm{K}_m(k)\mathrm{I}_{m+1}(k)k + 1$$
$$+ m\mathrm{K}_m(k)\mathrm{I}_m(k)(\chi_1 - \chi_2))\mathrm{I}_m(ka) - \mathrm{I}_m(k)\chi_2\mathrm{K}_{m+1}(k)k + m\mathrm{K}_m(k)\mathrm{I}_m(k)\chi_2 - 1))^{-1}. \tag{A12}$$

The constants $c_1^{(i)}, \ldots, c_6^{(i)}$ in (3.14) are as follows: $c_{1,3,5}^{(2)} = 0$, and

$$c_1^{(1)} = gF\hat{S}(k\mathrm{K}_{(m+1)}(k) - m\mathrm{K}_m(k)), \tag{A13}$$

$$c_2^{(1)} = -gF\hat{S}(-a\bar{k}k^2((-\mathrm{K}_{m+1}(k)k + m\mathrm{K}_m(k))\mathrm{I}_m(ka) + \mathrm{I}_m(\bar{k}a)(\bar{k}\mathrm{K}_{m+1}(\bar{k}) - m\mathrm{K}_m(\bar{k})))$$
$$\times (\mathrm{K}_{m+1}(\bar{k}a))^2 + (-\bar{k}ak^2(\bar{k}\mathrm{K}_{m+1}(\bar{k}) - m\mathrm{K}_m(\bar{k}))\mathrm{I}_{m+1}(\bar{k}a) + \bar{k}^2 ak(\mathrm{K}_{m+1}(k)k$$
$$- m\mathrm{K}_m(k))\mathrm{I}_{m+1}(ka) + m((\bar{k}^2 - 2k^2)(K_{m+1}(k)k - m\mathrm{K}_m(k))\mathrm{I}_m(ka)$$
$$- \bar{k}\mathrm{I}_m(\bar{k}a)(\bar{k}\mathrm{K}_m(\bar{k})m - k^2\mathrm{K}_{m+1}(\bar{k}))))\mathrm{K}_m(\bar{k}a)\mathrm{K}_{m+1}(\bar{k}a)$$
$$- m(\mathrm{K}_m(\bar{k}a))^2\bar{k}((\bar{k}\mathrm{K}_m(\bar{k})m - k^2\mathrm{K}_{m+1}(\bar{k}))\mathrm{I}_{m+1}(\bar{k}a)$$
$$+ \mathrm{I}_{m+1}(ka)k(\mathrm{K}_{m+1}(k)k - m\mathrm{K}_m(k)))) \times ((\mathrm{K}_{m+1}(\bar{k}a))^2\mathrm{K}_m(ka)\bar{k}ak^2$$
$$+ (-\bar{k}^2 ak\mathrm{K}_{m+1}(ka) + \mathrm{K}_m(ka)m(\bar{k}^2 - 2k^2))\mathrm{K}_m(\bar{k}a)\mathrm{K}_{m+1}(\bar{k}a)$$
$$+ (\mathrm{K}_m(\bar{k}a))^2\mathrm{K}_{m+1}(ka)\bar{k}km)^{-1}, \tag{A14}$$

$$c_2^{(2)} = -gF\hat{S}(-a\bar{k}((-\mathrm{K}_{m+1}(k)k + m\mathrm{K}_m(k))\mathrm{I}_m(ka) + (-k\mathrm{I}_{m+1}(k) - m\mathrm{I}_m(k))\mathrm{K}_m(ka)$$
$$+ \mathrm{I}_m(\bar{k}a)(\bar{k}\mathrm{K}_{m+1}(\bar{k}) - m\mathrm{K}_m(\bar{k})))k^2(\mathrm{K}_{m+1}(\bar{k}a))^2 + \mathrm{K}_m(\bar{k}a)(-\bar{k}^2 ak(k\mathrm{I}_{m+1}(k)$$
$$+ m\mathrm{I}_m(k))\mathrm{K}_{m+1}(ka) - \bar{k}ak^2(\bar{k}\mathrm{K}_{m+1}(\bar{k}) - m\mathrm{K}_m(\bar{k}))\mathrm{I}_{m+1}(\bar{k}a) + \bar{k}^2 ak(\mathrm{K}_{m+1}(k)k$$
$$- m\mathrm{K}_m(k))\mathrm{I}_{m+1}(ka) + m((\bar{k}^2 - 2k^2)(k\mathrm{I}_{m+1}(k) + mI_m(k))\mathrm{K}_m(ka) + (\bar{k}^2 - 2k^2)$$
$$\times (\mathrm{K}_{m+1}(k)k - m\mathrm{K}_m(k))\mathrm{I}_m(ka) - \bar{k}\mathrm{I}_m(\bar{k}a)(\bar{k}\mathrm{K}_m(\bar{k})m - k^2 K_{m+1}(\bar{k}))))\mathrm{K}_{m+1}(\bar{k}a)$$
$$- m(\mathrm{K}_m(\bar{k}a))^2\bar{k}((-\mathrm{I}_{m+1}(k)k^2 - km\mathrm{I}_m(k))\mathrm{K}_{m+1}(ka)$$
$$+ (\bar{k}\mathrm{K}_m(\bar{k})m - k^2\mathrm{K}_{m+1}(\bar{k}))I_{m+1}(\bar{k}a) + \mathrm{I}_{m+1}(ka)k(K_{m+1}(k)k - m\mathrm{K}_m(k))))$$
$$\times ((\mathrm{K}_{m+1}(\bar{k}a))^2\mathrm{K}_m(ka)\bar{k}ak^2 + (-\bar{k}^2 ak\mathrm{K}_{m+1}(ka)$$
$$+ \mathrm{K}_m(ka)m(\bar{k}^2 - 2k^2))\mathrm{K}_m(\bar{k}a)\mathrm{K}_{m+1}(\bar{k}a) + (\mathrm{K}_m(\bar{k}a))^2\mathrm{K}_{m+1}(ka)\bar{k}km), \tag{A15}$$

$$c_3^{(1)} = \frac{\bar{k}gF\hat{S}(\mathrm{i}\bar{k}\mathrm{K}_{m+1}(\bar{k}) + m\mathrm{K}_m)}{(\bar{k}^2 - k^2)}, \tag{A16}$$

$$c_5^{(1)} = gF\hat{S}\frac{\mathrm{K}_{m+1}(\bar{k})\bar{k}k^2 + \mathrm{K}_m(\bar{k})m(\bar{k}^2 - 2k^2)}{2\bar{k}^3 - 2\bar{k}k^2}, \tag{A17}$$

$$\begin{aligned} c_4^{(1)} = gF\hat{S}(\mathrm{i}((-a(\mathrm{I}_m(\bar{k}a)(\bar{k}\mathrm{K}_{m+1}(\bar{k}) - m\mathrm{K}_m(\bar{k}))\\ - \mathrm{I}_m(ka)(\mathrm{K}_{m+1}(k)k - m\mathrm{K}_m(k)))\bar{k}^2 k\mathrm{K}_{m+1}(ka)\\ + (-\bar{k}ak^2(\bar{k}\mathrm{K}_{m+1}(\bar{k}) - m\mathrm{K}_m(\bar{k}))\mathrm{I}_{m+1}(\bar{k}a) + \bar{k}^2 ak(\mathrm{K}_{m+1}(k)k - m\mathrm{K}_m(k))\mathrm{I}_{m+1}(ka)\\ + \mathrm{I}_m(\bar{k}a)m(\bar{k} - k)(\bar{k} + k)(\bar{k}\mathrm{K}_{m+1}(\bar{k}) - 2m\mathrm{K}_m(\bar{k})))\mathrm{K}_m(ka))\mathrm{K}_{m+1}(\bar{k}a)\\ - m\mathrm{K}_m(\bar{k}a)\bar{k}(-(\mathrm{I}_m(\bar{k}a)(\bar{k}\mathrm{K}_{m+1}(\bar{k}) - m\mathrm{K}_m(\bar{k}))\\ - \mathrm{I}_m(ka)(\mathrm{K}_{m+1}(k)k - m\mathrm{K}_m(k)))k\mathrm{K}_{m+1}(ka)\\ + \mathrm{K}_m(ka)((\bar{k}\mathrm{K}_m(\bar{k})m - k^2 K_{m+1}(\bar{k}))\mathrm{I}_{m+1}(\bar{k}a)\\ + I_{m+1}(ka)k(\mathrm{K}_{m+1}(k)k - m\mathrm{K}_m(k)))))k)\\ \times ((\bar{k} + k)((\mathrm{K}_{m+1}(\bar{k}a))^2\mathrm{K}_m(ka)\bar{k}ak^2 + (-\bar{k}^2 ak\mathrm{K}_{m+1}(ka)\\ + \mathrm{K}_m(ka)m(\bar{k}^2 - 2k^2))\mathrm{K}_m(\bar{k}a)K_{m+1}(\bar{k}a) + (\mathrm{K}_m(\bar{k}a))^2\mathrm{K}_{m+1}(ka)\bar{k}km)(\bar{k} - k))^{-1}, \end{aligned} \tag{A18}$$

$$\begin{aligned} c_4^{(2)} = gF\hat{S}(\mathrm{i}k(\bar{k}\mathrm{K}_m(ka)ak^2(\bar{k}\mathrm{I}_{m+1}(\bar{k}) + m\mathrm{I}_m(\bar{k}))(\mathrm{K}_{m+1}(\bar{k}a))^2 + (-a\bar{k}^2((\bar{k}\mathrm{I}_{m+1}(\bar{k})\\ + m\mathrm{I}_m(\bar{k}))\mathrm{K}_m(\bar{k}a) + \mathrm{I}_m(\bar{k}a)(\bar{k}\mathrm{K}_{m+1}(\bar{k}) - m\mathrm{K}_m(\bar{k})) - \mathrm{I}_m(ka)(\mathrm{K}_{m+1}(k)k\\ - m\mathrm{K}_m(k)))k\mathrm{K}_{m+1}(ka) + \mathrm{K}_m(ka)(-\bar{k}ak^2(\bar{k}\mathrm{K}_{m+1}(\bar{k}) - m\mathrm{K}_m(\bar{k}))\mathrm{I}_{m+1}(\bar{k}a)\\ + \bar{k}^2 ak(\mathrm{K}_{m+1}(k)k - m\mathrm{K}_m(k))\mathrm{I}_{m+1}(ka) + ((\bar{k}^2 - 2k^2)(\bar{k}\mathrm{I}_{m+1}(\bar{k}) + m\mathrm{I}_m(\bar{k}))\mathrm{K}_m(\bar{k}a)\\ + \mathrm{I}_m(\bar{k}a)(\bar{k} - k)(\bar{k} + k)(\bar{k}\mathrm{K}_{m+1}(\bar{k}) - 2m\mathrm{K}_m(\bar{k})))m))\mathrm{K}_{m+1}(\bar{k}a)\\ + m\mathrm{K}_m(\bar{k}a)\bar{k}(((\bar{k}\mathrm{I}_{m+1}(\bar{k})\\ + m\mathrm{I}_m(\bar{k}))\mathrm{K}_m(\bar{k}a) + \mathrm{I}_m(\bar{k}a)(\bar{k}\mathrm{K}_{m+1}(\bar{k}) - m\mathrm{K}_m(\bar{k})) - \mathrm{I}_m(ka)(\mathrm{K}_{m+1}(k)k\\ - m\mathrm{K}_m(k)))k\mathrm{K}_{m+1}(ka) - \mathrm{K}_m(ka)((\bar{k}\mathrm{K}_m(\bar{k})m - k^2 K_{m+1}(\bar{k}))\mathrm{I}_{m+1}(\bar{k}a)\\ + I_{m+1}(ka)k(\mathrm{K}_{m+1}(k)k - m\mathrm{K}_m(k)))))) \times ((\bar{k} + k)((\mathrm{K}_{m+1}(\bar{k}a))^2\mathrm{K}_m(ka)\bar{k}ak^2\\ + (-\bar{k}^2 ak\mathrm{K}_{m+1}(ka) + \mathrm{K}_m(ka)m(\bar{k}^2 - 2k^2))\mathrm{K}_m(\bar{k}a)K_{m+1}(\bar{k}a)\\ + (\mathrm{K}_m(\bar{k}a))^2\mathrm{K}_{m+1}(ka)\bar{k}km)(\bar{k} - k))^{-1}, \end{aligned} \tag{A19}$$

$$\begin{aligned} c_6^{(1)} = gF\hat{S}\Bigg(-2\Bigg(-\frac{a}{2}(\mathrm{I}_m(\bar{k}a)(\bar{k}\mathrm{K}_{m+1}(\bar{k}) - m\mathrm{K}_m(\bar{k})) - \mathrm{I}_m(ka)(\mathrm{K}_{m+1}(k)k - m\mathrm{K}_m(k)))\\ \times \bar{k}^2 k\mathrm{K}_{m+1}(ka) + \Bigg(-\frac{a\bar{k}}{2}(\mathrm{K}_{m+1}(\bar{k})\bar{k}k^2 + \mathrm{K}_m(\bar{k})m(\bar{k}^2 - 2k^2))\mathrm{I}_{m+1}(\bar{k}a)\\ + \frac{\bar{k}^2 ak}{2}(\mathrm{K}_{m+1}(k)k - m\mathrm{K}_m(k))\mathrm{I}_{m+1}(ka) \end{aligned}$$

$$+ \mathrm{I}_m(\bar{k}a)m(\bar{k}-k)(\bar{k}+k)(\bar{k}\mathrm{K}_{m+1}(\bar{k}) - 2m\mathrm{K}_m(\bar{k}))\Big)\mathrm{K}_m(ka)\Big)k^2K_{m+1}(\bar{k}a)$$
$$+ m\mathrm{K}_m(\bar{k}a)(-(\bar{k}\mathrm{K}_m(\bar{k})a(\bar{k}-k)(\bar{k}+k)\mathrm{I}_{m+1}(\bar{k}a) + (\mathrm{K}_{m+1}(\bar{k})\bar{k}k^2$$
$$+ \mathrm{K}_m(\bar{k})m(\bar{k}^2 - 2k^2))\mathrm{I}_m(\bar{k}a) - \bar{k}^2(\mathrm{K}_{m+1}(k)k - m\mathrm{K}_m(k))\mathrm{I}_m(ka))k\mathrm{K}_{m+1}(ka)$$
$$+ \bar{k}^2\mathrm{K}_m(ka)((\bar{k}\mathrm{K}_m(\bar{k})m - k^2\mathrm{K}_{m+1}(\bar{k}))\mathrm{I}_{m+1}(\bar{k}a)$$
$$+ \mathrm{I}_{m+1}(ka)k(\mathrm{K}_{m+1}(k)k - m\mathrm{K}_m(k))))\bar{k}\Big)$$
$$\times ((2\bar{k}+2k)\bar{k}((\mathrm{K}_{m+1}(\bar{k}a))^2\mathrm{K}_m(ka)\bar{k}ak^2 + (-\bar{k}^2ak\mathrm{K}_{m+1}(ka) + \mathrm{K}_m(ka)m(\bar{k}^2$$
$$- 2k^2))\mathrm{K}_m(\bar{k}a)\mathrm{K}_{m+1}(\bar{k}a) + (\mathrm{K}_m(\bar{k}a))^2\mathrm{K}_{m+1}(ka)\bar{k}km)(\bar{k}-k))^{-1}, \qquad \text{(A20)}$$

$$c_6^{(2)} = gF\hat{S}(a\bar{k}(-\bar{k}\mathrm{I}_{m+1}(\bar{k})k^2 + m\mathrm{I}_m(\bar{k})(\bar{k}^2 - 2k^2))\mathrm{K}_m(ka)k^2(\mathrm{K}_{m+1}(\bar{k}a))^2$$
$$+ (-a\bar{k}^2k((-\bar{k}\mathrm{I}_{m+1}(\bar{k})k^2 + m\mathrm{I}_m(\bar{k})(\bar{k}^2 - 2k^2))\mathrm{K}_m(\bar{k}a)$$
$$- (\mathrm{I}_m(\bar{k}a)(\bar{k}\mathrm{K}_{m+1}(\bar{k}) - m\mathrm{K}_m(\bar{k}))$$
$$- \mathrm{I}_m(ka)(K_{m+1}(k)k - m\mathrm{K}_m(k)))k^2)\mathrm{K}_{m+1}(ka) + \mathrm{K}_m(ka)(a\bar{k}(\mathrm{K}_{m+1}(\bar{k})\bar{k}k^2$$
$$+ \mathrm{K}_m(\bar{k})m(\bar{k}^2 - 2k^2))k^2\mathrm{I}_{m+1}(\bar{k}a) - \bar{k}^2ak^3(\mathrm{K}_{m+1}(k)k - m\mathrm{K}_m(k))\mathrm{I}_{m+1}(ka)$$
$$+ m((\bar{k}^2 - 2k^2)(-\bar{k}\mathrm{I}_{m+1}(\bar{k})k^2 + m\mathrm{I}_m(\bar{k})(\bar{k}^2 - 2k^2))\mathrm{K}_m(\bar{k}a)$$
$$- 2\mathrm{I}_m(\bar{k}a)k^2(\bar{k}-k)(\bar{k}+k)(\bar{k}\mathrm{K}_{m+1}(\bar{k}) - 2m\mathrm{K}_m(\bar{k})))))\mathrm{K}_{m+1}(\bar{k}a)$$
$$+ m\mathrm{K}_m(\bar{k}a)\bar{k}(-(\bar{k}\mathrm{K}_m(\bar{k})a(\bar{k}-k)(\bar{k}+k)\mathrm{I}_{m+1}(\bar{k}a)$$
$$+ (\bar{k}\mathrm{I}_{m+1}(\bar{k})k^2 - m\mathrm{I}_m(\bar{k})(\bar{k}^2 - 2k^2))\mathrm{K}_m(\bar{k}a)$$
$$+ (\mathrm{K}_{m+1}(\bar{k})\bar{k}k^2 + \mathrm{K}_m(\bar{k})m(\bar{k}^2 - 2k^2))\mathrm{I}_m(\bar{k}a)$$
$$- \bar{k}^2(\mathrm{K}_{m+1}(k)k - m\mathrm{K}_m(k))\mathrm{I}_m(ka))k\mathrm{K}_{m+1}(ka)$$
$$+ \bar{k}^2\mathrm{K}_m(ka)((\bar{k}K_m(\bar{k})m - k^2\mathrm{K}_{m+1}(\bar{k}))\mathrm{I}_{m+1}(\bar{k}a)$$
$$+ \mathrm{I}_{m+1}(ka)k(\mathrm{K}_{m+1}(k)k - m\mathrm{K}_m(k)))))$$
$$\times ((2\bar{k}+2k)\bar{k}((\mathrm{K}_{m+1}(\bar{k}a))^2\mathrm{K}_m(ka)\bar{k}ak^2 + (-\bar{k}^2ak\mathrm{K}_{m+1}(ka)$$
$$+ \mathrm{K}_m(ka)m(\bar{k}^2 - 2k^2))\mathrm{K}_m(\bar{k}a)\mathrm{K}_{m+1}(\bar{k}a) + (\mathrm{K}_m(\bar{k}a))^2\mathrm{K}_{m+1}(ka)\bar{k}km)(\bar{k}-k))^{-1}. \qquad \text{(A21)}$$

In the highly viscous limit, s_v is given by (3.32), where

$$f_v = -\frac{1}{2k^2}\Bigg(2\Bigg(\Bigg(k\mathrm{I}_{m+1}(k) - \frac{1}{2}(k^2 + m^2 - 2\,m)\mathrm{I}_m(k)\Bigg)k\mathrm{K}_{m+1}(k)$$
$$+ \frac{1}{2}\mathrm{K}_m(k)(k^2 + m^2 - 2\,m)(k\mathrm{I}_{m+1}(k) + 2\mathrm{I}_m(k)m)\Bigg)a^2k^2(\mathrm{K}_{m+1}(ka))^3$$
$$+ a(-2ka((\mathrm{K}_{m+1}(k))^2k^2 + k\mathrm{K}_m(k)(k^2 + m^2 - 2\,m)\mathrm{K}_{m+1}(k) - m(\mathrm{K}_m(k))^2$$

$$
\begin{aligned}
&\times (k^2+m^2-2\,m))\mathrm{I}_{m+1}(ka) + (k^2(a^2k^2+m^2-2\,m)(\mathrm{K}_{m+1}(k))^2 - 2m\mathrm{K}_m(k)\\
&\times ((a^2+1)k^2+2m^2-4\,m)k\mathrm{K}_{m+1}(k) + (\mathrm{K}_m(k))^2(k^4+m^2(a^2+4)k^2+4m^4\\
&-8m^3))\mathrm{I}_m(ka) - 6\left((k\mathrm{I}_{m+1}(k) - \frac{1}{2}\mathrm{I}_m(k)(k^2+m^2-2\,m))k\mathrm{K}_{m+1}(k)\right.\\
&\left.+\frac{1}{2}\mathrm{K}_m(k)(k^2+m^2-2\,m)(k\mathrm{I}_{m+1}(k)+2\mathrm{I}_m(k)m)\right)(m+2/3)\mathrm{K}_m(ka))k(\mathrm{K}_{m+1}(ka))^2\\
&+\mathrm{K}_m(ka)((((a^2k^2+(m+2)^2)k^2(\mathrm{K}_{m+1}(k))^2 - 2\mathrm{K}_m(k)((a^2m-2m-2)k^2-m^3\\
&+4\,m)k\mathrm{K}_{m+1}(k) + (\mathrm{K}_m(k))^2(k^4+((a^2-2)m^2-4\,m)k^2-2m^4+8m^2))ak\mathrm{I}_{m+1}(ka)\\
&+(-k^2(m+2)(a^2k^2+m^2-2\,m)(\mathrm{K}_{m+1}(k))^2 - 2\mathrm{K}_m(k)(a^2k^4\\
&+(-2a^2m-m^2-2\,m)k^2-2m^4+8m^2)k\mathrm{K}_{m+1}(k) + 2\,m\left(\left(a^2-\frac{1}{2}\right)k^4\right.\\
&\left.+\left(\frac{1}{2}a^2m^2-a^2m-2m^2-4\,m\right)k^2-2m^4+8m^2\right)(\mathrm{K}_m(k))^2)\mathrm{I}_m(ka)\\
&-2\left(\left(k\mathrm{I}_{m+1}(k)-\frac{1}{2}\mathrm{I}_m(k)(k^2+m^2-2\,m)\right)k\mathrm{K}_{m+1}(k)\right.\\
&\left.+\frac{1}{2}K_m(k)(k^2+m^2-2\,m)(k\mathrm{I}_{m+1}(k)+2\mathrm{I}_m(k)m)\right)\\
&\times (a^2k^2-2m^2-4\,m)\mathrm{K}_m(ka))\mathrm{K}_{m+1}(ka) - m(((a^2k^2+(m+2)^2)(\mathrm{K}_{m+1}(k))^2\\
&-2k\mathrm{K}_m(k)(a^2m-m-2)\mathrm{K}_{m+1}(k)\\
&+(k^2+a^2m(m-2))(\mathrm{K}_m(k))^2)k\mathrm{I}_{m+1}(ka) - 2a(((\mathrm{K}_{m+1}(k))^2k^2+k\mathrm{K}_m(k)(k^2+m^2\\
&-2\,m)\mathrm{K}_{m+1}(k) - m(\mathrm{K}_m(k))^2(k^2+m^2-2\,m))\mathrm{I}_m(ka)\\
&+\left(\left(k\mathrm{I}_{m+1}(k)-\frac{1}{2}\mathrm{I}_m(k)(k^2+m^2-2\,m)\right)k\mathrm{K}_{m+1}(k)\right.\\
&\left.+\frac{1}{2}\mathrm{K}_m(k)(k^2+m^2-2\,m)(k\mathrm{I}_{m+1}(k)+2\mathrm{I}_m(k)m)\right)\mathrm{K}_m(ka)))(\mathrm{K}_m(ka))^2k\Big)\\
&\times(-k^2a^2(\mathrm{K}_{m+1}(ka))^3 + 3(m+2/3)\mathrm{K}_m(ka)ak(\mathrm{K}_{m+1}(ka))^2\\
&+(\mathrm{K}_m(ka))^2(a^2k^2-2m^2-4\,m)\mathrm{K}_{m+1}(ka) - ka(\mathrm{K}_m(ka))^3m)^{-1}.
\end{aligned}
\tag{A22}
$$

In this inviscid limit, s_I is given by (3.33), where

$$
\begin{aligned}
f_I = {}&((ka(k\mathrm{I}_{m+1}(k)\\
&+\mathrm{I}_m(k)m)\mathrm{K}_{m+1}(ka) - ka(k\mathrm{K}_{m+1}(k)-m\mathrm{K}_m(k))\mathrm{I}_{m+1}(ka) - m((k\mathrm{K}_{m+1}(k)\\
&-m\mathrm{K}_m(k))\mathrm{I}_m(ka) + \mathrm{K}_m(ka)(k\mathrm{I}_{m+1}(k)+\mathrm{I}_m(k)m)))(k\mathrm{K}_{m+1}(k)-m\mathrm{K}_m(k)))\\
&\times\frac{1}{(\mathrm{K}_{m+1}(ka)ak-\mathrm{K}_m(ka)m)}.
\end{aligned}
\tag{A23}
$$

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