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A coupled variational principle for 2D interactions between water waves and a rigid body containing fluid

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New variational principles are given for the two-dimensional interactions between gravity-driven water waves and a rotating and translating rectangular vessel dynamically coupled to its interior potential flow with uniform vorticity. The complete set of equations of motion for the exterior water waves, the exact nonlinear hydrodynamic equations of motion for the vessel in the roll/pitch, sway/surge and heave directions, and also the full set of equations of motion for the interior fluid of the vessel, relative to the body coordinate system attached to the rotating–translating vessel, are derived from two Lagrangian functionals.

Key words: variational methods, wave–structure interactions

1. Introduction

Luke (1967) presented a variational principle for the classical water-wave problem described by the equations

$$\left.\begin{aligned} \Delta\Phi := \Phi_{XX} + \Phi_{YY} = 0 \quad &\text{for } -H(X) < Y < \Gamma(X,t), \\ \Phi_t + \tfrac{1}{2}\nabla\Phi\cdot\nabla\Phi + gY = 0 \quad &\text{on } Y = \Gamma(X,t), \\ \Phi_Y = \Gamma_t + \Phi_X\Gamma_X \quad &\text{on } Y = \Gamma(X,t), \\ \Phi_Y + \Phi_X H_X = 0 \quad &\text{on } Y = -H(X), \end{aligned}\right\} \tag{1.1}$$

where (X, Y) is the spatial coordinate system and $\Phi(X, Y, t)$ is the velocity potential of an irrotational fluid lying between $Y = -H(X)$ and $Y = \Gamma(X, t)$, with the gravity acceleration g acting in the negative Y direction. In the horizontal direction X, the fluid

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domain is cut off by a vertical surface Σ which extends from the bottom to the free surface. Then Luke's variational principle reads

$$\delta\mathcal{L}(\Phi, \Gamma) = \delta \int_{t_1}^{t_2} \int_{X_1}^{X_2} \int_{-H(X)}^{\Gamma(X,t)} -\rho\left(\Phi_t + \tfrac{1}{2}\nabla\Phi \cdot \nabla\Phi + gY\right) \mathrm{d}Y\, \mathrm{d}X\, \mathrm{d}t = 0, \tag{1.2}$$

with variations in $\Phi(X, Y, t)$ and $\Gamma(X, t)$ subject to the restrictions $\delta\Phi = 0$ at the end points of the time interval, t_1 and t_2. In (1.2), the gradient vector field is denoted by ∇, and ρ is the water density.

Miloh (1984) presented an extension of Luke's variational principle for water waves interacting with several bodies on or below a free surface which oscillate at a common frequency. van Daalen (1993) and van Daalen, van Groesen & Zandbergen (1993), hereafter DGZ, extended the Hamiltonian formulation of surface waves due to Zakharov (1968), Broer (1974) and Miles (1977) to water waves in hydrodynamic interaction with freely floating bodies, starting from a variational principle of the form

$$\delta\mathcal{L} = \delta \int_{t_1}^{t_2} \int_{\Omega(t)} -\rho\left(\Phi_t + \tfrac{1}{2}\nabla\Phi \cdot \nabla\Phi + gY\right) \mathrm{d}\Omega\, \mathrm{d}t + \delta \int_{t_1}^{t_2} (\mathsf{KE}^{vessel} - \mathsf{PE}^{vessel})\, \mathrm{d}t = 0, \tag{1.3}$$

where KE^{vessel} is the kinetic energy of the vessel and PE^{vessel} is the potential energy of the vessel. In this action integral, the system under consideration, $\Omega(t)$, consists of a fluid bounded by the impermeable bottom $Y = -H(X)$, the free surface $Y = \Gamma(X, t)$ and the wetted surface S of a rigid body. In the horizontal direction X, the fluid domain is cut off by a vertical surface Σ at $X = X_1, X_2$ which extends from the bottom to the free surface. DGZ used this Lagrangian action to derive the complete set of equations of motion, i.e. equations (1.1) and the hydrodynamic equations of motion for the rigid body. However, they did not present the exact nonlinear equations for the rigid-body motion, due to the approximation in their definition for the body angular velocity in KE^{vessel}. This can be seen by comparing the second term in equation (3) of DGZ with the third term in the last line of (1.4). Moreover, the second term in the last line of (1.4) is absent in equation (3) of DGZ. van Groesen & Andonowati (2017) presented a Boussinesq-type Hamiltonian formulation for wave–ship interactions.

The variational principle presented by DGZ was for an empty rigid body in hydrodynamic interaction with exterior water waves. However, in the present article, in order to take into account the coupled dynamics between fluid sloshing in a vessel while in an ambient wave field, with coupling to the vessel motion, the second part of the variational principle (1.3) should be modified to include the kinetic and potential energies of the interior fluid. To do this, we first present the general form of a three-dimensional Lagrangian action for a rigid body with interior fluid motion, and then, in § 3, we show how a reduced two-dimensional version of this functional can be derived for the purposes of this paper. Alemi Ardakani (2010) derived the exact form of a three-dimensional Lagrangian action for a rigid body containing fluid that undergoes 3D rotational and translational motions. The action integral takes the form

$$\begin{aligned}\mathcal{L}(\omega, \boldsymbol{q}) &= \int_{t_1}^{t_2} \int_{\Omega'} (\mathsf{KE}^{fluid} - \mathsf{PE}^{fluid})\, \mathrm{d}\Omega'\, \mathrm{d}t + \int_{t_1}^{t_2} (\mathsf{KE}^{vessel} - \mathsf{PE}^{vessel})\, \mathrm{d}t \\ &= \int_{t_1}^{t_2} \int_{\Omega'} \left(\tfrac{1}{2}\|\dot{\boldsymbol{x}}\|^2 + \dot{\boldsymbol{x}} \cdot (\omega \times (\boldsymbol{x} + \boldsymbol{d}) + \boldsymbol{Q}^{\mathrm{T}}\dot{\boldsymbol{q}}) + \boldsymbol{Q}^{\mathrm{T}}\dot{\boldsymbol{q}} \cdot (\omega \times (\boldsymbol{x} + \boldsymbol{d})) + \tfrac{1}{2}\|\dot{\boldsymbol{q}}\|^2\right.\end{aligned}$$

$$+\tfrac{1}{2}\omega\cdot\left(\|\boldsymbol{x}+\boldsymbol{d}\|^2\,\boldsymbol{I}-(\boldsymbol{x}+\boldsymbol{d})\otimes(\boldsymbol{x}+\boldsymbol{d})\right)\omega-g(\boldsymbol{Q}(\boldsymbol{x}+\boldsymbol{d})+\boldsymbol{q})\cdot\boldsymbol{E}_2\Big)\rho\,\mathrm{d}\Omega'\,\mathrm{d}t$$
$$+\int_{t_1}^{t_2}\left(\tfrac{1}{2}m_v\,\|\dot{\boldsymbol{q}}\|^2+(\omega\times m_v\overline{\boldsymbol{x}}_v)\cdot\boldsymbol{Q}^{\mathrm{T}}\dot{\boldsymbol{q}}+\tfrac{1}{2}\omega\cdot\boldsymbol{I}_v\omega-m_vg(\boldsymbol{Q}\overline{\boldsymbol{x}}_v+\boldsymbol{q})\cdot\boldsymbol{E}_2\right)\mathrm{d}t, \tag{1.4}$$

where the body frame, which is attached to the moving rigid body, has coordinates $\boldsymbol{x}=(x, y, z)$, the distance between the centre of rotation and the origin of the body frame is $\boldsymbol{d}=(d_1, d_2, d_3)$, the fluid-tank system has a uniform translation $\boldsymbol{q}(t)=(q_1, q_2, q_3)$ relative to the spatial frame $\boldsymbol{X}=(X, Y, Z)$, the integral is over the volume Ω' of the interior fluid, $\otimes$ denotes the tensor product, $\boldsymbol{I}$ is the 3×3 identity matrix, $\boldsymbol{I}_v$ is the dry-vessel mass moment of inertia relative to the point of rotation, m_v is the mass of the dry vessel, $\overline{\boldsymbol{x}}_v=(\overline{x}_v, \overline{y}_v, \overline{z}_v)$ is the centre of mass of the dry vessel relative to the body frame, $\boldsymbol{E}_2$ is the unit vector in the Y direction and $\omega(t)=(\omega_1(t), \omega_2(t), \omega_3(t))$ is the body angular velocity vector with entries determined from the rotation tensor $\boldsymbol{Q}(t)$ by

$$\boldsymbol{Q}^{\mathrm{T}}\dot{\boldsymbol{Q}}=\begin{bmatrix}0 & -\omega_3 & \omega_2\\ \omega_3 & 0 & -\omega_1\\ -\omega_2 & \omega_1 & 0\end{bmatrix}:=\widehat{\boldsymbol{\omega}}. \tag{1.5}$$

The convention for the entries of the skew-symmetric matrix $\widehat{\boldsymbol{\omega}}$ is such that $\widehat{\boldsymbol{\omega}}\boldsymbol{r}=\omega\times\boldsymbol{r}$ for any $\boldsymbol{r}\in\mathbb{R}^3$. The relation between the spatial displacement and the body displacement, and the relation between the body velocity and the space velocity are respectively

$$\boldsymbol{X}=\boldsymbol{Q}(\boldsymbol{x}+\boldsymbol{d})+\boldsymbol{q}\quad\text{and}\quad\dot{\boldsymbol{X}}=\boldsymbol{Q}(\dot{\boldsymbol{x}}+\omega\times(\boldsymbol{x}+\boldsymbol{d})+\boldsymbol{Q}^{\mathrm{T}}\dot{\boldsymbol{q}}). \tag{1.6a,b}$$

This formulation is consistent with the theory of rigid-body motion, where an arbitrary motion can be described by the pair $(\boldsymbol{Q}(t), \boldsymbol{q}(t))$. The exact equations of motion for the rigid body can be derived from the Lagrangian action (1.4).

The interest in this paper is to derive a coupled variational principle for two-dimensional interactions between water waves and a floating rectangular vessel with interior fluid motion which gives the exact nonlinear Euler–Lagrange equations for the coupled dynamics. The vessel is free to undergo roll/pitch motion (θ), sway/surge motion (q_1) and heave motion (q_2), which are rotation about the centre of rotation in the Z-direction relative to the rest keel, horizontal displacement along the X-axis and vertical displacement along the Y-axis respectively. It is shown in § 3 that the addition of Luke's Lagrangian action (1.2) to a two-dimensional variant of the Lagrangian action (1.4) gives

$$\delta\mathcal{L}(\Phi, \Gamma, \theta, q_1, q_2)=\delta\int_{t_1}^{t_2}\int_{\Omega(t)}-\rho\left(\Phi_t+\tfrac{1}{2}\nabla\Phi\cdot\nabla\Phi+gY\right)\mathrm{d}\Omega\,\mathrm{d}t$$
$$+\delta\int_{t_1}^{t_2}\Bigg(\int_0^L\int_0^{\eta(x,t)}\left[\tfrac{1}{2}(\phi_x^2+\phi_y^2)+\phi_x(\dot{q}_1\cos\theta+\dot{q}_2\sin\theta)+\tfrac{1}{2}(\dot{q}_1^2+\dot{q}_2^2)\right.$$
$$\left.+\phi_y(-\dot{q}_1\sin\theta+\dot{q}_2\cos\theta)-g(\sin\theta(x+d_1)+\cos\theta(y+d_2)+q_2)\right]\rho\,\mathrm{d}y\,\mathrm{d}x$$
$$+\tfrac{1}{2}m_v(\dot{q}_1^2+\dot{q}_2^2)-m_v\overline{y}_v\dot{\theta}(\dot{q}_1\cos\theta+\dot{q}_2\sin\theta)+m_v\overline{x}_v\dot{\theta}(-\dot{q}_1\sin\theta+\dot{q}_2\cos\theta)$$
$$+\tfrac{1}{2}m_v(\overline{x}_v^2+\overline{y}_v^2)\dot{\theta}^2-m_vg(\overline{x}_v\sin\theta+\overline{y}_v\cos\theta+q_2)\Bigg)\mathrm{d}t=0, \tag{1.7}$$

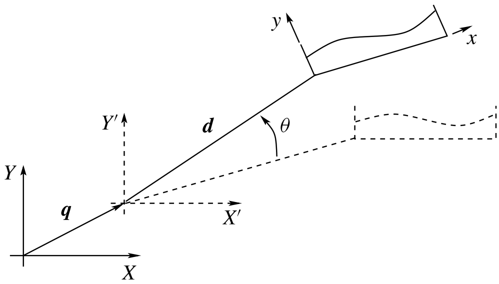

FIGURE 1. Diagram showing the coordinate systems for a rotating–translating vessel. The coordinate system (X', Y') is the translation by $\boldsymbol{q} = (q_1, q_2)$ of the fixed coordinate system (X, Y). The distance from the centre of rotation to the origin of the body coordinate system (x, y) is $\boldsymbol{d} = (d_1, d_2)$.

where $\varOmega(t)$ is defined as in (1.3), L is the length of the vessel, $\eta(x, t)$ is the fluid height relative to the body coordinate system (x, y), which is attached to the moving vessel, $(\overline{x}_v, \overline{y}_v)$ is the centre of mass of the dry vessel relative to the body frame, m_v is the mass of the dry vessel, (d_1, d_2) is the distance between the centre of rotation and the origin of the body frame and $\phi(x, y, t)$ is the velocity potential for the interior fluid motion, yet to be determined. See figure 1 for a sketch of the coordinate systems. The second part of (1.7) is the kinetic and potential energy of the whole domain, for the fluid in the vessel in moving coordinates such that the extra fictitious forces emerge. The Lagrangian action (1.7) can be used for derivation of the set of equations of motion for the classical water-wave problem (1.1) and also the hydrodynamic equations of motion for the rigid body in the roll/pitch, sway/surge and heave directions.

The second variational principle is a variant of Luke's variational principle (1.2) for the interior rotating–translating fluid motion of the vessel. It is shown in § 2 that the complete set of equations for the interior fluid motion relative to the body coordinate system can be obtained from the Lagrangian action

$$\begin{aligned}\delta\mathcal{L}(\phi, \eta) = \delta \int_{t_1}^{t_2}\int_0^L\int_0^{\eta(x,t)} \big[&-\big(\phi_t + \tfrac{1}{2}\nabla\phi\boldsymbol{\cdot}\nabla\phi + \dot{\theta}(y+d_2)\phi_x - \dot{\theta}(x+d_1)\phi_y\big)\\ &+ \tfrac{1}{2}(x+d_1)(-g\sin\theta - \ddot{q}_1\cos\theta - \ddot{q}_2\sin\theta)\\ &+ \tfrac{1}{2}(y+d_2)(-g\cos\theta + \ddot{q}_1\sin\theta - \ddot{q}_2\cos\theta)\big]\rho\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t = 0. \qquad (1.8)\end{aligned}$$

This paper starts with the derivation of a variational principle for the fluid motion in a rotating–translating rectangular vessel in § 2. In § 3, a variational principle is presented for interactions between the exterior water waves and the rigid body containing fluid. The exact nonlinear hydrodynamic equations for the rigid-body motion are derived. The paper ends with concluding remarks in § 4.

2. A variational principle for the interior fluid motion

The configuration of the fluid in a rotating–translating vessel is shown in figure 1. The fluid occupies the region $0 \leqslant y \leqslant \eta(x, t)$, with $0 \leqslant x \leqslant L$. The two-dimensional Euler equations relative to a rotating–translating coordinate system (x, y) given by Alemi Ardakani & Bridges (2012) are

$$\left.\begin{aligned}\frac{\mathrm{D}u}{\mathrm{D}t}+\frac{1}{\rho}\frac{\partial p}{\partial x}&=-g\sin\theta+2\dot{\theta}v+\ddot{\theta}(y+d_2)+\dot{\theta}^2(x+d_1)-\ddot{q}_1\cos\theta-\ddot{q}_2\sin\theta,\\ \frac{\mathrm{D}v}{\mathrm{D}t}+\frac{1}{\rho}\frac{\partial p}{\partial y}&=-g\cos\theta-2\dot{\theta}u-\ddot{\theta}(x+d_1)+\dot{\theta}^2(y+d_2)+\ddot{q}_1\sin\theta-\ddot{q}_2\cos\theta,\\ \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}&=0,\end{aligned}\right\} \tag{2.1}$$

where $\mathrm{D}u/\mathrm{D}t=u_t+uu_x+vu_y$. The velocity field (u, v) is relative to the body frame and p is the pressure field. Relative to the body frame, the boundary conditions are

$$u=0 \quad \text{at } x=0 \quad \text{and} \quad x=L, \qquad v=0 \quad \text{at } y=0 \tag{2.2a–c}$$

and

$$p=0 \quad \text{and} \quad \eta_t+u\eta_x=v \quad \text{at } y=\eta(x, t), \tag{2.3a,b}$$

where the surface tension is neglected in the boundary condition for the pressure. The vorticity, $\mathcal{V}=v_x-u_y$, satisfies the equation $\mathrm{D}\mathcal{V}/\mathrm{D}t=-2\ddot{\theta}$.

Now, we introduce a velocity potential $\phi(x, y, t)$ such that

$$u(x, y, t)=\phi_x+\dot{\theta}(y+d_2) \quad \text{and} \quad v(x, y, t)=\phi_y-\dot{\theta}(x+d_1). \tag{2.4a,b}$$

The velocity field in (2.4) satisfies the vorticity equation. The vorticity is constant in space and satisfies $\mathcal{V}=-2\dot{\theta}$. Substitution of (2.4) into the continuity equation in (2.1) leads to Laplace's equation for $\phi(x, y, t)$,

$$\phi_{xx}+\phi_{yy}=0 \quad \text{in } 0\leqslant y\leqslant\eta(x, t), 0\leqslant x\leqslant L, \tag{2.5}$$

and substitution of (2.4) into the momentum equations in (2.1) leads to Bernoulli's equation for the pressure field,

$$\begin{aligned}\frac{p}{\rho}&+\phi_t+\frac{1}{2}(\phi_x^2+\phi_y^2)-\frac{1}{2}(x+d_1)(-g\sin\theta-\ddot{q}_1\cos\theta-\ddot{q}_2\sin\theta)\\ &-\frac{1}{2}(y+d_2)(-g\cos\theta+\ddot{q}_1\sin\theta-\ddot{q}_2\cos\theta)\\ &+\dot{\theta}(y+d_2)\phi_x-\dot{\theta}(x+d_1)\phi_y=Be(t),\end{aligned} \tag{2.6}$$

where $Be(t)$ is the Bernoulli function which can be absorbed into $\phi(x, y, t)$. Therefore, the dynamic free-surface boundary condition in (2.3) at $y=\eta(x, t)$ becomes

$$\begin{aligned}&\phi_t+\tfrac{1}{2}(\phi_x^2+\phi_y^2)+\dot{\theta}(\eta+d_2)\phi_x-\tfrac{1}{2}(x+d_1)(-g\sin\theta-\ddot{q}_1\cos\theta-\ddot{q}_2\sin\theta)\\ &\quad-\tfrac{1}{2}(\eta+d_2)(-g\cos\theta+\ddot{q}_1\sin\theta-\ddot{q}_2\cos\theta)-\dot{\theta}(x+d_1)\phi_y=0.\end{aligned} \tag{2.7}$$

In terms of the velocity potential $\phi(x, y, t)$, the kinematic free-surface boundary condition in (2.3) becomes

$$\eta_t+(\phi_x+\dot{\theta}(\eta+d_2))\eta_x=\phi_y-\dot{\theta}(x+d_1) \quad \text{at } y=\eta(x, t) \tag{2.8}$$

and the rigid-wall boundary conditions in (2.2) become

$$\phi_x=-\dot{\theta}(y+d_2) \quad \text{at } x=0, L, \qquad \phi_y=\dot{\theta}(x+d_1) \quad \text{at } y=0. \tag{2.9a,b}$$

Following Luke, the variational principle for the interior rotating–translating fluid is

$$\delta\mathcal{L}(\phi,\eta)=\delta\int_{t_1}^{t_2}\int_0^L\int_0^{\eta(x,t)} p(x,y,t)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t=0. \tag{2.10}$$

Now, substituting for $p(x, y, t)$ from (2.6) into (2.10), we obtain (1.8). Taking the variations in (1.8), we obtain

$$\begin{aligned}\delta\mathcal{L}(\phi,\eta)=&\int_{t_1}^{t_2}\int_0^L(\eta_t+\eta_x\phi_x-\phi_y+\eta_x\dot{\theta}(\eta+d_2)+\dot{\theta}(x+d_1))\delta\phi\big|^{y=\eta}\rho\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_{t_1}^{t_2}\int_0^L\int_0^{\eta(x,t)}(\phi_{xx}+\phi_{yy})\delta\phi\rho\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t-\int_{t_1}^{t_2}\int_0^L(\boldsymbol{\nabla}\phi\boldsymbol{\cdot}\boldsymbol{n}+\dot{\theta}(x+d_1))\delta\phi\big|_{y=0}\rho\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_{t_1}^{t_2}\int_0^{\eta}(-\boldsymbol{\nabla}\phi\boldsymbol{\cdot}\boldsymbol{n}+\dot{\theta}(y+d_2))\delta\phi\big|_{x=0}\rho\,\mathrm{d}y\,\mathrm{d}t\\
&+\int_{t_1}^{t_2}\int_0^{\eta}(-\boldsymbol{\nabla}\phi\boldsymbol{\cdot}\boldsymbol{n}-\dot{\theta}(y+d_2))\delta\phi\big|_{x=L}\rho\,\mathrm{d}y\,\mathrm{d}t+\int_{t_1}^{t_2}\int_0^L p(x,\eta,t)\delta\eta\,\mathrm{d}x\,\mathrm{d}t=0.\end{aligned} \tag{2.11}$$

A detailed derivation of (2.11) is given in appendix A. From (2.11), it is obvious that invariance of $\mathcal{L}$ with respect to a variation in the free-surface elevation η yields the dynamic free-surface boundary condition (2.7). Similarly, the invariance of $\mathcal{L}$ with respect to a variation in the velocity potential ϕ yields the field equation (2.5). Moreover, the invariance of $\mathcal{L}$ with respect to a variation in the velocity potential ϕ at $y=0$, $x=0$ and $x=L$ recovers the rigid-wall boundary conditions in (2.9), and the invariance of $\mathcal{L}$ with respect to a variation in the velocity potential ϕ at $y=\eta$ recovers the kinematic free-surface boundary condition (2.8).

3. A variational principle for the exterior water waves and the motion of the rigid body containing fluid

The complete set of differential equations for the exterior water waves interacting with the rigid body in the plane can be obtained from a variant of the Lagrangian action (1.3) which takes the form

$$\begin{aligned}\delta\mathcal{L}\;=\;&\delta\int_{t_1}^{t_2}\int_{\varOmega(t)}-\rho\left(\varPhi_t+\tfrac{1}{2}\boldsymbol{\nabla}\varPhi\boldsymbol{\cdot}\boldsymbol{\nabla}\varPhi+gY\right)\mathrm{d}\varOmega\,\mathrm{d}t\\
&+\delta\int_{t_1}^{t_2}\int_{\varOmega'}(\mathsf{KE}^{fluid}-\mathsf{PE}^{fluid})\,\mathrm{d}\varOmega'\,\mathrm{d}t+\delta\int_{t_1}^{t_2}(\mathsf{KE}^{vessel}-\mathsf{PE}^{vessel})\,\mathrm{d}t=0,\end{aligned} \tag{3.1}$$

where the second line in (3.1) is a 2D variant of the Lagrangian action (1.4), which can be obtained by substituting $\dot{\boldsymbol{x}}=(u,v,0)=(\phi_x+\dot{\theta}(y+d_2),\phi_y-\dot{\theta}(x+d_1),0)$, $\boldsymbol{q}=(q_1,q_2,0)$, $\omega=(0,0,\dot{\theta})$, $\boldsymbol{d}=(d_1,d_2,0)$, $\bar{\boldsymbol{x}}_v=(\bar{x}_v,\bar{y}_v,0)$ and

$$\boldsymbol{Q}=\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix} \tag{3.2}$$

into (1.4). Then, the variational principle (3.1) reduces to (1.7).

In order to take the variations in (1.7), the variational Reynold's transport theorem should be used, since the domain of integration $\varOmega$ is time-dependent. See Flanders (1973), Daniliuk (1976) and Gagarina, van der Vegt & Bokhove (2013) for the background mathematics on the variational analogue of Reynold's transport theorem. Then, according to the usual procedure in the calculus of variations, equation (1.7) for all variations in the free-surface elevation $\varGamma$, the velocity potential $\varPhi$, the vessel position $\boldsymbol{q}=(q_1,q_2)$ and the vessel orientation θ becomes

$$
\begin{aligned}
\delta\mathcal{L}(\varPhi,\varGamma,\theta,q_1,q_2) &= \int_{t_1}^{t_2}\int_{X_1}^{X_2} -\left(\varPhi_t+\tfrac{1}{2}\nabla\varPhi\boldsymbol{\cdot}\nabla\varPhi+gY\right)\Big|^{Y=\varGamma}\rho\,\delta\varGamma\,\mathrm{d}X\,\mathrm{d}t \\
&+\int_{t_1}^{t_2}\int_S P(X,Y,t)(\delta\boldsymbol{X}_s\boldsymbol{\cdot}\boldsymbol{n})\,\mathrm{d}s\,\mathrm{d}t-\int_{t_1}^{t_2}\int_{\varOmega(t)}(\delta\varPhi_t+\nabla\varPhi\boldsymbol{\cdot}\nabla\delta\varPhi)\rho\,\mathrm{d}\varOmega\,\mathrm{d}t \\
&+\int_{t_1}^{t_2}\int_0^L\int_0^{\eta(x,t)}\big[\phi_x(\delta\dot{q}_1\cos\theta-\dot{q}_1\sin\theta\delta\theta+\delta\dot{q}_2\sin\theta+\dot{q}_2\cos\theta\delta\theta) \\
&+\phi_y(-\delta\dot{q}_1\sin\theta-\dot{q}_1\cos\theta\delta\theta+\delta\dot{q}_2\cos\theta-\dot{q}_2\sin\theta\delta\theta) \\
&+(\dot{q}_1\delta\dot{q}_1+\dot{q}_2\delta\dot{q}_2)-g(\cos\theta(x+d_1)\delta\theta-\sin\theta(y+d_2)\delta\theta+\delta q_2)\big]\,\rho\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \\
&+\int_{t_1}^{t_2}\big(m_v(\dot{q}_1\delta\dot{q}_1+\dot{q}_2\delta\dot{q}_2)-m_v\overline{y}_v\delta\dot{\theta}(\dot{q}_1\cos\theta+\dot{q}_2\sin\theta) \\
&-m_v\overline{y}_v\dot{\theta}\,(\delta\dot{q}_1\cos\theta-\dot{q}_1\sin\theta\delta\theta+\delta\dot{q}_2\sin\theta+\dot{q}_2\cos\theta\delta\theta) \\
&+m_v\overline{x}_v\delta\dot{\theta}(-\dot{q}_1\sin\theta+\dot{q}_2\cos\theta)+m_v\overline{x}_v\dot{\theta} \\
&\times(-\delta\dot{q}_1\sin\theta-\dot{q}_1\cos\theta\delta\theta+\delta\dot{q}_2\cos\theta-\dot{q}_2\sin\theta\delta\theta) \\
&+m_v(\overline{x}_v^2+\overline{y}_v^2)\dot{\theta}\delta\dot{\theta}-m_vg(\overline{x}_v\cos\theta\delta\theta-\overline{y}_v\sin\theta\delta\theta+\delta q_2)\big)\,\mathrm{d}t=0,
\end{aligned}
\tag{3.3}
$$

where

$$
P(X,Y,t)=-\rho(\varPhi_t+\tfrac{1}{2}\nabla\varPhi\boldsymbol{\cdot}\nabla\varPhi+gY)\quad\text{on } S \tag{3.4}
$$

and it should be noted that these variations are subject to the restrictions that they vanish at the end points of the time interval and on the vertical boundary at infinity $\varSigma$. In (3.3), $\boldsymbol{X}_s$ denotes the position of a point on the wetted vessel surface S relative to the spatial frame (X,Y) and $\boldsymbol{n}$ is the unit normal vector along $\partial\varOmega\supset S$. The change in $\boldsymbol{X}_s$ due to variations in $\boldsymbol{q}$ and θ is given by

$$
\delta\boldsymbol{X}_s=\boldsymbol{Q}'\boldsymbol{x}_s\delta\theta+\delta\boldsymbol{q}\quad\text{with } \boldsymbol{Q}'=\begin{bmatrix}-\sin\theta & -\cos\theta & 0\\ \cos\theta & -\sin\theta & 0\\ 0 & 0 & 0\end{bmatrix}, \tag{3.5}
$$

where $\boldsymbol{x}_s$ is the position of a point on the wetted vessel surface relative to the body frame (x,y). Taking into account the motion of $\varOmega(t)$, we may write

$$
\begin{aligned}
-\frac{\mathrm{d}}{\mathrm{d}t}\int_{t_1}^{t_2}\int_{\varOmega}\delta\varPhi\rho\mathrm{d}\varOmega\,\mathrm{d}t &= -\int_{t_1}^{t_2}\int_{X_1}^{X_2}\varGamma_t\delta\varPhi\Big|^{Y=\varGamma}\rho\,\mathrm{d}X\,\mathrm{d}t-\int_{t_1}^{t_2}\int_S(\dot{\boldsymbol{X}}_s\boldsymbol{\cdot}\boldsymbol{n})\delta\varPhi\rho\,\mathrm{d}s\,\mathrm{d}t \\
&\quad-\int_{t_1}^{t_2}\int_{\varOmega}\delta\varPhi_t\rho\,\mathrm{d}\varOmega\,\mathrm{d}t.
\end{aligned}
\tag{3.6}
$$

This is the same as the variational Reynold's transport theorem but with variational derivatives replaced by time derivatives. Noting that the left-hand side vanishes due to the restriction $\delta\varPhi=0$ at times $t=t_1$ and $t=t_2$, this expression simplifies to

$$
-\int_{t_1}^{t_2}\int_{\varOmega}\delta\varPhi_t\rho\,\mathrm{d}\varOmega\,\mathrm{d}t=\int_{t_1}^{t_2}\int_{X_1}^{X_2}\varGamma_t\delta\varPhi\Big|^{Y=\varGamma}\rho\,\mathrm{d}X\,\mathrm{d}t+\int_{t_1}^{t_2}\int_S(\dot{\boldsymbol{X}}_s\boldsymbol{\cdot}\boldsymbol{n})\delta\varPhi\rho\,\mathrm{d}s\,\mathrm{d}t. \tag{3.7}
$$

With Green's first identity, we may write

$$\begin{aligned}\int_{t_1}^{t_2}\int_{\Omega}\nabla\Phi\cdot\nabla\delta\Phi\rho\,\mathrm{d}\Omega\,\mathrm{d}t &= -\int_{t_1}^{t_2}\int_{\Omega}\Delta\Phi\delta\Phi\rho\,\mathrm{d}\Omega\,\mathrm{d}t+\int_{t_1}^{t_2}\int_{\partial\Omega}(\nabla\Phi\cdot\boldsymbol{n})\delta\Phi\rho\,\mathrm{d}s\,\mathrm{d}t\\ &= -\int_{t_1}^{t_2}\int_{\Omega}\Delta\Phi\delta\Phi\rho\,\mathrm{d}\Omega\,\mathrm{d}t+\int_{t_1}^{t_2}\int_{X_1}^{X_2}(-\Gamma_X\Phi_X+\Phi_Y)\delta\Phi\big|^{Y=\Gamma}\rho\,\mathrm{d}X\,\mathrm{d}t\\ &\quad+\int_{t_1}^{t_2}\int_{X_1}^{X_2}(\Phi_X H_X+\Phi_Y)\delta\Phi\big|_{Y=-H}\rho\,\mathrm{d}X\,\mathrm{d}t+\int_{t_1}^{t_2}\int_S\frac{\partial\Phi}{\partial n}\delta\Phi\rho\,\mathrm{d}s\,\mathrm{d}t.\end{aligned}\tag{3.8}$$

Now, using the expressions (3.5), (3.7) and (3.8), integrating by parts and noting that $\delta\boldsymbol{q}$ and $\delta\theta$ vanish at the end points of the time interval, the variational principle (3.3) simplifies to

$$\begin{aligned}\delta\mathcal{L}(\Phi,\Gamma,\theta,q_1,q_2) &= \int_{t_1}^{t_2}\int_{X_1}^{X_2}-\left(\Phi_t+\frac{1}{2}\nabla\Phi\cdot\nabla\Phi+gY\right)\bigg|^{Y=\Gamma}\rho\delta\Gamma\,\mathrm{d}X\,\mathrm{d}t\\ &+\int_{t_1}^{t_2}\int_{X_1}^{X_2}(\Gamma_t+\Gamma_X\Phi_X-\Phi_Y)\delta\Phi\big|^{Y=\Gamma}\rho\,\mathrm{d}X\,\mathrm{d}t+\int_{t_1}^{t_2}\int_S\left(\dot{\boldsymbol{X}}_s\cdot\boldsymbol{n}-\frac{\partial\Phi}{\partial n}\right)\delta\Phi\rho\,\mathrm{d}s\,\mathrm{d}t\\ &-\int_{t_1}^{t_2}\int_{X_1}^{X_2}(\Phi_X H_X+\Phi_Y)\delta\Phi\big|_{Y=-H}\rho\,\mathrm{d}X\,\mathrm{d}t+\int_{t_1}^{t_2}\int_{\Omega}\Delta\Phi\delta\Phi\rho\,\mathrm{d}\Omega\,\mathrm{d}t\\ &+\int_{t_1}^{t_2}\int_S P(X,Y,t)(\boldsymbol{Q}'\boldsymbol{x}_s\cdot\boldsymbol{n})\delta\theta\rho\,\mathrm{d}s\,\mathrm{d}t+\int_{t_1}^{t_2}\int_S P(X,Y,t)\boldsymbol{n}\cdot\delta\boldsymbol{q}\rho\,\mathrm{d}s\,\mathrm{d}t\\ &+\int_{t_1}^{t_2}\left(\int_0^L\int_0^{\eta}(-\phi_{xt}\cos\theta+\phi_x\dot{\theta}\sin\theta+\phi_{yt}\sin\theta+\phi_y\dot{\theta}\cos\theta-\ddot{q}_1\right.\\ &+(\phi_x+\dot{\theta}(y+d_2))(\phi_{yx}\sin\theta-\phi_{xx}\cos\theta)\\ &+(\phi_y-\dot{\theta}(x+d_1))(\phi_{yy}\sin\theta-\phi_{xy}\cos\theta))\,\rho\,\mathrm{d}y\,\mathrm{d}x\\ &\left.-m_v\ddot{q}_1+m_v\overline{y}_v(\ddot{\theta}\cos\theta-\dot{\theta}^2\sin\theta)+m_v\overline{x}_v(\ddot{\theta}\sin\theta+\dot{\theta}^2\cos\theta)\right)\delta q_1\,\mathrm{d}t\\ &+\int_{t_1}^{t_2}\left(\int_0^L\int_0^{\eta}(-\phi_{xt}\sin\theta-\phi_x\dot{\theta}\cos\theta-\phi_{yt}\cos\theta+\phi_y\dot{\theta}\sin\theta-\ddot{q}_2-g\right.\\ &-(\phi_{xx}\sin\theta+\phi_{yx}\cos\theta)(\phi_x+\dot{\theta}(y+d_2))\\ &-(\phi_{xy}\sin\theta+\phi_{yy}\cos\theta)(\phi_y-\dot{\theta}(x+d_1)))\,\rho\,\mathrm{d}y\,\mathrm{d}x\\ &\left.-m_v\ddot{q}_2+m_v\overline{y}_v(\ddot{\theta}\sin\theta+\dot{\theta}^2\cos\theta)-m_v\overline{x}_v(\ddot{\theta}\cos\theta-\dot{\theta}^2\sin\theta)-m_vg\right)\delta q_2\,\mathrm{d}t\\ &+\int_{t_1}^{t_2}\left(\int_0^L\int_0^{\eta}\left[\phi_x(-\dot{q}_1\sin\theta+\dot{q}_2\cos\theta)+\phi_y(-\dot{q}_1\cos\theta-\dot{q}_2\sin\theta)\right.\right.\\ &\left.-g(\cos\theta(x+d_1)-\sin\theta(y+d_2))\right]\rho\,\mathrm{d}y\,\mathrm{d}x+m_v\overline{y}_v(\ddot{q}_1\cos\theta+\ddot{q}_2\sin\theta)\\ &\left.-m_v\overline{x}_v(-\ddot{q}_1\sin\theta+\ddot{q}_2\cos\theta)-m_v(\overline{x}_v^2+\overline{y}_v^2)\ddot{\theta}-m_vg(\overline{x}_v\cos\theta-\overline{y}_v\sin\theta)\right)\delta\theta\,\mathrm{d}t=0.\end{aligned}\tag{3.9}$$

From (3.9), we conclude that invariance of $\mathcal{L}$ with respect to a variation in the free-surface elevation Γ yields the dynamic free-surface boundary condition in (1.1), invariance of $\mathcal{L}$ with respect to a variation in the velocity potential Φ yields the field equation in $\Omega(t)$, invariance of $\mathcal{L}$ with respect to a variation in the velocity potential Φ at $Y=-H(X)$ gives the bottom boundary condition in (1.1), invariance of $\mathcal{L}$ with respect to a variation in the velocity potential Φ at $Y=\Gamma(X,t)$ gives the kinematic free-surface boundary condition in (1.1) and invariance of $\mathcal{L}$ with respect to a variation in the velocity potential Φ on S gives the contact condition on the wetted surface of the vessel,

$$\frac{\partial \Phi}{\partial n} = \dot{\boldsymbol{X}}_s \cdot \boldsymbol{n} \quad \text{on } S. \tag{3.10}$$

Finally, invariance of $\mathcal{L}$ with respect to variations in q_1, q_2 and θ gives the hydrodynamic equations of motion for the rigid body in the sway/surge (q_1), heave (q_2) and roll/pitch (θ) directions respectively, which are

(i)

$$\begin{aligned}(m_v+m_f)\ddot{q}_1 - \int_0^L\int_0^\eta & (-\phi_{xt}\cos\theta + \phi_{yt}\sin\theta + \dot{\theta}(\phi_x\sin\theta+\phi_y\cos\theta) \\ & + (\phi_x + \dot{\theta}(y+d_2))(\phi_{yx}\sin\theta - \phi_{xx}\cos\theta) \\ & + (\phi_y - \dot{\theta}(x+d_1))(\phi_{yy}\sin\theta - \phi_{xy}\cos\theta))\,\rho\,\mathrm{d}y\,\mathrm{d}x \\ & - m_v\bar{y}_v(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) - m_v\bar{x}_v(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta) \\ & - \int_S P(X,Y,t)n_1\,\mathrm{d}s = 0,\end{aligned} \tag{3.11}$$

(ii)

$$\begin{aligned}(m_v+m_f)\,(g+\ddot{q}_2) + \int_0^L\int_0^\eta & (\,\phi_{xt}\sin\theta + \phi_{yt}\cos\theta + \dot{\theta}(\phi_x\cos\theta - \phi_y\sin\theta) \\ & + (\phi_{xx}\sin\theta + \phi_{yx}\cos\theta)(\phi_x + \dot{\theta}(y+d_2)) \\ & + (\phi_{xy}\sin\theta + \phi_{yy}\cos\theta)(\phi_y - \dot{\theta}(x+d_1)))\,\rho\,\mathrm{d}y\,\mathrm{d}x \\ & - m_v\bar{y}_v(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta) + m_v\bar{x}_v(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) \\ & - \int_S P(X,Y,t)n_2\,\mathrm{d}s = 0,\end{aligned} \tag{3.12}$$

(iii)

$$\begin{aligned}m_v(\bar{x}_v^2+\bar{y}_v^2)\ddot{\theta} - \int_0^L\int_0^\eta & [\,\phi_x(-\dot{q}_1\sin\theta + \dot{q}_2\cos\theta) + \phi_y(-\dot{q}_1\cos\theta - \dot{q}_2\sin\theta) \\ & - g(\cos\theta(x+d_1) - \sin\theta(y+d_2))\,]\,\rho\,\mathrm{d}y\,\mathrm{d}x - m_v\bar{y}_v(\ddot{q}_1\cos\theta + \ddot{q}_2\sin\theta) \\ & + m_v\bar{x}_v(-\ddot{q}_1\sin\theta + \ddot{q}_2\cos\theta) + m_v g(\bar{x}_v\cos\theta - \bar{y}_v\sin\theta) \\ & - \int_S P(X,Y,t)(\boldsymbol{Q}'\boldsymbol{x}_s\cdot\boldsymbol{n})\,\mathrm{d}s = 0,\end{aligned} \tag{3.13}$$

where $P(X,Y,t)$ is defined in (3.4) and $m_f = \int_0^L\int_0^\eta \rho\,\mathrm{d}y\,\mathrm{d}x$ is independent of time.

In summary, the equations of motion for the exterior water waves in $\Omega(t)$ are (1.1) with the contact boundary condition (3.10). The equations of motion for the interior fluid motion are the field equation (2.5) with the boundary conditions (2.7)–(2.9) which are dynamically coupled to the hydrodynamic equations of motion for the rigid

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body (3.11)–(3.13). The terms including the pressure field $P(X, Y, t)$ in the q_1, q_2 equations and also in the θ equation in (3.11)–(3.13) are the forces and moment respectively acting on the rigid body due to the exterior water waves. Similarly, the integral terms including the derivatives of $\phi(x, y, t)$ are the forces and moment acting on the rigid body due to the interior fluid motion.

4. Concluding remarks

This paper is devoted to the derivation of new variational principles for 2D interactions between water waves and a rigid body with interior fluid sloshing. The complete set of equations of motion and boundary conditions for the exterior water waves and for the interior fluid motion of the vessel relative to the rotating–translating coordinate system attached to the moving vessel, and also the exact nonlinear Euler–Lagrange equations for the rigid-body motion in the sway/surge, heave and roll/pitch directions, are derived from the presented variational principles. The proposed variational principles are applicable to ocean engineering problems. The exact differential equations can be used for the coupled dynamical analysis of a freely floating ship with water on deck or with interior fluid sloshing in the tanks which interacts with exterior water waves.

Another interesting application of the presented coupled variational principles is for the dynamical response analysis of floating ocean wave energy converters (WECs) such as the OWEL wave energy converter proposed by Offshore Wave Energy Ltd, a schematic of which can be found on the website http://www.owel.co.uk/. OWEL is a floating vessel with variable topography and cross-section, open at one end to capture ocean waves. Once they are trapped, the waves undergo interior fluid sloshing while the vessel is interacting with exterior waves. A rise in the wave height is induced within the duct, mainly due to the internal geometry of the WEC. The wave then creates a seal with the rigid lid, resulting in a moving trapped pocket of air ahead of the wave front which drives the power take-off.

The proposed variational principles can be used for mathematical modelling of the pendulum–slosh problem. The rigid-body equation for the pendular motion of a rectangular vessel suspended from a single rigid pivoting rod, partially filled with an inviscid fluid, can be derived from a simplified version of the variational principle (1.7). One can consider the second part of (1.7) with $q_1 = q_2 = 0$ and take the variation with respect to θ to obtain the rigid-body equation. The complete set of equations of motion for the interior fluid of the pendulum can be obtained by setting $q_1 = q_2 = 0$ in (1.8) and taking the variations with respect to ϕ and η.

A direction of great interest is to use the variational symplectic methods of Gagarina *et al.* (2014, 2016), Bokhove & Kalogirou (2016) and Kalogirou & Bokhove (2016) to develop energy preserving numerical solvers for the proposed variational principles (1.7) and (1.8) for interactions between ocean waves and floating structures dynamically coupled to interior fluid sloshing, and also for the variational principle (1.8) for the problem of fluid sloshing in vessels undergoing prescribed rigid-body motion in two dimensions.

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Appendix A. Proof of the variational principle (2.11)

According to the usual procedure in the calculus of variations, equation (1.8) becomes

$$\delta\mathcal{L}(\phi,\eta) = \int_{t_1}^{t_2}\int_0^L\int_0^{\eta(x,t)} -\left(\delta\phi_t + \nabla\phi\cdot\nabla\delta\phi + \dot{\theta}(y+d_2)\delta\phi_x - \dot{\theta}(x+d_1)\delta\phi_y\right)\rho\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t + \int_{t_1}^{t_2}\int_0^L p(x,\eta,t)\delta\eta\,\mathrm{d}x\,\mathrm{d}t, \tag{A 1}$$

but

$$\int_{t_1}^{t_2}\int_0^L\int_0^{\eta(x,t)} -\delta\phi_t\rho\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t = \int_{t_1}^{t_2}\int_0^L \eta_t\delta\phi\big|^{y=\eta}\rho\,\mathrm{d}x\,\mathrm{d}t, \tag{A 2}$$

noting that $\delta\phi=0$ at $t=t_1$ and $t=t_2$. Moreover,

$$-\int_{t_1}^{t_2}\int_0^L\int_0^{\eta(x,t)} \dot{\theta}(y+d_2)\delta\phi_x\rho\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t = -\int_{t_1}^{t_2}\int_0^{\eta}\dot{\theta}(y+d_2)\delta\phi\big|_{x=L}\rho\,\mathrm{d}y\,\mathrm{d}t + \int_{t_1}^{t_2}\int_0^{\eta}\dot{\theta}(y+d_2)\delta\phi\big|_{x=0}\rho\,\mathrm{d}y\,\mathrm{d}t + \int_{t_1}^{t_2}\int_0^L \eta_x\dot{\theta}(\eta+d_2)\delta\phi\big|^{y=\eta}\rho\,\mathrm{d}x\,\mathrm{d}t \tag{A 3}$$

and

$$\int_{t_1}^{t_2}\int_0^L\int_0^{\eta(x,t)} \dot{\theta}(x+d_1)\delta\phi_y\rho\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t = \int_{t_1}^{t_2}\int_0^L \dot{\theta}(x+d_1)\delta\phi\big|^{y=\eta}\rho\,\mathrm{d}x\,\mathrm{d}t - \int_{t_1}^{t_2}\int_0^L \dot{\theta}(x+d_1)\delta\phi\big|_{y=0}\rho\,\mathrm{d}x\,\mathrm{d}t, \tag{A 4}$$

and using Green's first identity,

$$-\int_{t_1}^{t_2}\int_0^L\int_0^{\eta(x,t)} \nabla\phi\cdot\nabla\delta\phi\rho\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t = \int_{t_1}^{t_2}\int_0^L\int_0^{\eta(x,t)} (\phi_{xx}+\phi_{yy})\delta\phi\rho\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t - \int_{t_1}^{t_2}\int_0^L (-\eta_x\phi_x+\phi_y)\delta\phi\big|^{y=\eta}\rho\,\mathrm{d}x\,\mathrm{d}t - \int_{t_1}^{t_2}\int_0^L \nabla\phi\cdot\boldsymbol{n}\delta\phi\big|_{y=0}\rho\,\mathrm{d}x\,\mathrm{d}t - \int_{t_1}^{t_2}\int_0^{\eta} \nabla\phi\cdot\boldsymbol{n}\delta\phi\big|_{x=0}\rho\,\mathrm{d}y\,\mathrm{d}t - \int_{t_1}^{t_2}\int_0^{\eta} \nabla\phi\cdot\boldsymbol{n}\delta\phi\big|_{x=L}\rho\,\mathrm{d}y\,\mathrm{d}t, \tag{A 5}$$

where $\boldsymbol{n}$ is the unit outward normal vector along the rigid walls. Hence, (A 1) is converted to (2.11).

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