Proceedings of the Royal Society of Edinburgh, **150**, 2682–2718, *2020*
DOI:10.1017/prm.2019.60

Nonlinear fractional Laplacian problems with nonlocal 'gradient terms'

Boumediene Abdellaoui
Laboratoire d'Analyse Nonlinéaire et Mathématiques Appliquées, Département de Mathématiques, Université Abou Bakr Belkaïd, Tlemcen 13000, Algeria (boumediene.abdellaoui@inv.uam.es)

Antonio J. Fernández
Univ. Polytechnique Hauts-de-France, EA 4015 - LAMAV - FR CNRS 2956, F-59313 Valenciennes, France
Laboratoire de Mathématiques (UMR 6623), Université de Bourgogne-Franche-Comté, 16 route de Gray, 25030 Besançon Cedex, France (antonio_jesus.fernandez_sanchez@univ-fcomte.fr)

(MS Received 8 March 2019; Accepted 11 September 2019)

Let $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$, be a smooth bounded domain. For $s \in (1/2, 1)$, we consider a problem of the form

$$\begin{cases} (-\Delta)^s u = \mu(x)\, \mathbb{D}_s^2(u) + \lambda f(x), & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $\lambda > 0$ is a real parameter, f belongs to a suitable Lebesgue space, $\mu \in L^\infty(\Omega)$ and $\mathbb{D}_s^2$ is a nonlocal 'gradient square' term given by

$$\mathbb{D}_s^2(u) = \frac{a_{N,s}}{2} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}}\, dy.$$

Depending on the real parameter $\lambda > 0$, we derive existence and non-existence results. The proof of our existence result relies on sharp Calderón–Zygmund type regularity results for the fractional Poisson equation with low integrability data. We also obtain existence results for related problems involving different nonlocal diffusion terms.

Keywords: Fractional Laplacian; Nonlocal gradient; Calderón–Zygmund; Regularity; Fractional Poisson equation

2010 *Mathematics subject classification:* 35B65; 35J62; 35R09; 47G20

1. Introduction and main results

In the last 15 years, there has been an increasing interest in the study of partial differential equations involving integro-differential operators. In particular, the case of the fractional Laplacian has been widely studied and is nowadays a very active field of research. This is due not only to its mathematical richness.

The fractional Laplacian has appeared in a great number of equations modelling real-world phenomena, especially those which take into account nonlocal effects. Among others, let us mention applications in quasi-geostrophic flows [**15**], quantum mechanic [**33**], mathematical finances [**6**, **18**], obstacle problems [**8**, **9**, **14**] and crystal dislocation [**23**, **24**, **42**].

The first aim of the present paper is to discuss, depending on the real parameter $\lambda > 0$, the existence and non-existence of solutions to the Dirichlet problem

$$\begin{cases}(-\Delta)^s u = \mu(x)\,\mathbb{D}_s^2(u) + \lambda f(x), & \text{in}\,\Omega,\\ u = 0, & \text{in}\,\mathbb{R}^N \setminus \Omega,\end{cases} \tag{P_λ}$$

under the assumption

$$\begin{cases}\Omega \subset \mathbb{R}^N,\ N \geqslant 2,\ \text{is a bounded domain with } \partial\Omega \text{ of class } \mathcal{C}^2,\\ s \in (1/2, 1),\\ f \in L^m(\Omega) \text{ for some } m > N/2s \text{ and } \mu \in L^\infty(\Omega).\end{cases} \tag{A_1}$$

Throughout the work, $(-\Delta)^s$ stands for the, by know classical, fractional Laplacian operator. For a smooth function u and $s \in (0,1)$, it can be defined as

$$(-\Delta)^s u(x) := a_{N,s}\ \text{p.v.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}}\,\mathrm{d}y,$$

where

$$a_{N,s} := \left(\int_{\mathbb{R}^N} \frac{1 - \cos(\xi_1)}{|\xi|^{N+2s}}\mathrm{d}\xi\right)^{-1} = -\frac{2^{2s}\Gamma\left(N/2 + s\right)}{\pi^{N/2}\Gamma(-s)},$$

is a normalization constant and 'p.v.' is an abbreviation for 'in the principal value sense'. In (P_λ), $\mathbb{D}_s^2$ is a nonlocal diffusion term. It plays the role of the 'gradient square' in the nonlocal case and is given by

$$\mathbb{D}_s^2(u) = \frac{a_{N,s}}{2} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}}\,\mathrm{d}y. \tag{1.1}$$

Since they will not play a role in this work, we normalize the constants appearing in the definitions of $(-\Delta)^s$ and $\mathbb{D}_s^2$ and we omit the p.v. sense. However, let us stress that these constants guarantee

$$\lim_{s \to 1^-} (-\Delta)^s u(x) = -\Delta u(x), \quad \forall\, u \in \mathcal{C}_0^\infty(\mathbb{R}^N), \tag{1.2}$$

and

$$\begin{aligned}\lim_{s \to 1^-} \mathbb{D}_s^2(u(x)) &= \lim_{s \to 1^-} \frac{c_N(1 - s)}{2} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}}\,\mathrm{d}y \\ &= |\nabla u(x)|^2, \quad \forall\, u \in \mathcal{C}_0^\infty(\mathbb{R}^N).\end{aligned} \tag{1.3}$$

We refer to [**22**] and [**13**] respectively for a proof of (1.2) and (1.3). Hence, at least formally, if $s \to 1^-$ in (P_λ), we recover the local problem

$$\begin{cases}-\Delta u = \mu(x)|\nabla u|^2 + \lambda f(x), & \text{in}\,\Omega,\\ u = 0, & \text{on}\,\partial\Omega.\end{cases} \tag{1.4}$$

https://doi.org/10.1017/prm.2019.60 Published online by Cambridge University Press

This equation corresponds to the stationary case of the Kardar–Parisi–Zhang model of growing interfaces introduced in [**30**]. The existence and multiplicity of solutions to problem (1.4) and of its different extensions have been extensively studied and it is still an active field of research. See for instance [**3**, **7**, **10**, **21**, **25**, **28**]. In most of these papers, the existence of solutions is proved using either a priori estimates or, when it is possible, a suitable change of variable to obtain an equivalent semilinear problem. However, neither of these techniques seem to be appropriate to deal with the nonlocal case (P_λ).

Let us also point out that in [**17**], using pointwise estimates on the Green function for the fractional Laplacian, the authors deal with the nonlocal–local problem

$$\begin{cases}(-\Delta)^s u = |\nabla u|^q + \lambda f(x), & \text{in } \Omega,\\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega.\end{cases} \tag{1.5}$$

For $s \in (1/2, 1)$, $1 < q < N/(N-(2s-1))$, $f \in L^1(\Omega)$ and $\lambda > 0$ small enough they obtained the existence of a solution to (1.5). This existence result was later completed in [**4**] where, under suitable assumptions on f, the authors showed the existence of a solution to (1.5) for all $1 < q < \infty$ and $\lambda > 0$ small enough.

Following [**16**, **17**] we introduce the following notion of weak solution to (P_λ).

Definition 1.1. We say that u is a *weak solution* to (P_λ) if u and $\mathbb{D}_s^2(u)$ belong to $L^1(\Omega)$, $u \equiv 0$ in $\mathcal{C}\Omega := \mathbb{R}^N \setminus \Omega$ and

$$\int_\Omega u(-\Delta)^s \phi \,\mathrm{d}x = \int_\Omega \big(\mu(x)\mathbb{D}_s^2(u) + \lambda f(x)\big)\,\phi \,\mathrm{d}x, \quad \forall\, \phi \in \mathbb{X}_s, \tag{1.6}$$

where

$$\mathbb{X}_s := \Big\{\xi \in \mathcal{C}(\mathbb{R}^N) : \operatorname{Supp} \xi \subset \overline{\Omega}, (-\Delta)^s \xi(x) \text{ exists } \forall\, x \in \Omega \text{ and } |(-\Delta)^s \xi(x)| \leqslant C \text{ for some } C > 0\Big\}. \tag{1.7}$$

In the spirit of the existing results for the local case, our first main result shows the existence of a weak solution to (P_λ) under a smallness condition on λf.

Theorem 1.1. *Assume that (A_1) holds. Then there exists $\lambda^* > 0$ such that, for all $0 < \lambda \leqslant \lambda^*$, ($P_\lambda$) has a weak solution $u \in W_0^{s,2}(\Omega) \cap \mathcal{C}^{0,\alpha}(\Omega)$ for some $\alpha > 0$.*

Remark 1.1.

(a) The definition of $W_0^{s,2}(\Omega)$ will be introduced in § 2.

(b) In 1983, L. Boccardo, F. Murat and J.P. Puel [**10**] already pointed out that the existence of solution to (1.4) is not guaranteed for every $\lambda f \in L^\infty(\Omega)$. Some extra conditions are needed. Hence, the smallness condition appearing in theorem 1.1 was somehow expected.

(c) For $\lambda f \equiv 0$, $u \equiv 0$ is a solution to (P_λ) that obviously belongs to $W_0^{s,2}(\Omega) \cap \mathcal{C}^{0,\alpha}(\Omega)$. Hence, there is no loss of generality to assume that $\lambda > 0$.

The counterpart of $|\nabla u|^2$ in (1.4) is played in (P_λ) by $\mathbb{D}_s^2(u)$. This term appears in several applications. For instance, let us mention [**13**, **35**, **38**] where it naturally appears as the equivalent of $|\nabla u|^2$ when considering fractional harmonic maps into the sphere.

Let us now give some ideas of the proof of theorem 1.1. As in the local case, see for instance [**36**], the existence of solutions to (P_λ) is related to the regularity of the solutions to a linear equation of the form

$$\begin{cases}(-\Delta)^s v = h(x), & \text{in } \Omega,\\ v = 0, & \text{in } \mathbb{R}^N \setminus \Omega.\end{cases} \tag{1.8}$$

In § 3, we obtain sharp Calderón–Zygmund type regularity results for the fractional Poisson equation (1.8) with low integrability data. We believe these results are of independent interest and will be useful in other settings. Actually, § 3 can be read as an independent part of the present work. In particular, we refer the interested reader to propositions 3.1, 3.3 and 3.4.

Having at hand suitable regularity results for (1.8) and inspired by [**36**, § 6], we develop a fixed point argument to obtain a solution to (P_λ). Note that, due to the nonlocality of the operator and of the 'gradient term', the approach of [**36**] has to be adapted significantly. In particular, the form of the set where we apply the fixed point argument seems to be new in the literature. We consider a subset of $W_0^{s,1}(\Omega)$ where, in some sense, we require more 'differentiability' and more integrability. This extra 'differentiability' is a purely nonlocal phenomena and it is related to our regularity results for (1.8). See § 4 for more details.

Let us also stress that the restriction $s \in (1/2, 1)$ comes from the regularity results of § 3. If suitable regularity results for (1.8) with $s \in (0, 1/2]$ were available, our fixed point argument would provide the desired existence results to (P_λ). See § 3 and 7 for more details.

Next, let us prove that the smallness condition imposed in theorem 1.1 is somehow necessary.

THEOREM 1.2. *Assume* (A_1) *and suppose that* $\mu(x) \geqslant \mu_1 > 0$ *and* $f^+ \not\equiv 0$. *Then there exists* $\lambda^{**} > 0$ *such that, for all* $\lambda > \lambda^{**}$, (P_λ) *has no weak solutions in* $W_0^{s,2}(\Omega)$.

REMARK 1.2.

(a) Observe that, if v is a solution to

$$\begin{cases}(-\Delta)^s v = -\mu(x)\, \mathbb{D}_s^2(v) - \lambda f(x), & \text{in } \Omega,\\ v = 0, & \text{in } \mathbb{R}^N \setminus \Omega,\end{cases}$$

then $u = -v$ is a solution to (P_λ). Hence, if $\mu(x) \leqslant -\mu_1 < 0$ and $f^- \not\equiv 0$ we recover the same kind of non-existence result and the smallness condition is also required.

(b) Since we do not use the regularity results of § 3, the restriction $s \in (1/2, 1)$ is not necessary in the proof of theorem 1.2. The result holds for all $s \in (0, 1)$.

Also, in order to show that the regularity imposed on the data f is almost optimal, we provide a counterexample to our existence result when the regularity condition on f is not satisfied. The proof makes use of the Hardy potential.

THEOREM 1.3. *Let $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$, be a bounded domain with $\partial\Omega$ of class $\mathcal{C}^2$, let $s \in (0,1)$ and let $\mu \in L^\infty(\Omega)$ such that $\mu(x) \geqslant \mu_1 > 0$. Then, for all $1 \leqslant p < N/2s$, there exists $f \in L^p(\Omega)$ such that (P_λ) has no weak solutions in $W_0^{s,2}(\Omega)$ for any $\lambda > 0$.*

REMARK 1.3.

(a) Our first main existence result, theorem 1.1, goes on the line of the existence results to (1.5) proved in [**4**, **17**]. In particular, it seems natural to compare theorem 1.1 with [**4**, theorem 5.6]. Both theorems give an existence result for $\lambda > 0$ small enough. On the other hand, let us stress that, when dealing with problem (1.5), it is not known if non-existence type results on the line of theorems 1.2 and 1.3 hold.

(b) Let us define

$$\overline{\lambda} := \sup\{\lambda \in \mathbb{R} : \ (P_\lambda) \text{ has a solution}\}.$$

By theorem 1.1 it is clear that, if (A_1) holds, then $0 < \lambda^* \leqslant \overline{\lambda}$. On the other hand, if $\mu(x) \geqslant \mu_1 > 0$ and $f^+ \not\equiv 0$, by theorem 1.2, it follows that $\overline{\lambda} \leqslant \lambda^{**}$. Nevertheless, we do not know if $\overline{\lambda} = \lambda^* = \lambda^{**}$.

Using the same kind of approach as in theorem 1.1, i.e., regularity results for (1.8) and a fixed point argument, one can obtain existence results for related problems involving different nonlocal diffusion terms and different nonlinearities.

First, we deal with the Dirichlet problem

$$\begin{cases} (-\Delta)^s u = \mu(x)\, u\, \mathbb{D}_s^2(u) + \lambda f(x), & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \qquad (\widetilde{P}_\lambda)$$

For $\mu(x) \equiv 1$, this problem can be seen as a particular case of the fractional harmonic maps problem considered in [**13**, **35**].

REMARK 1.4. The notion of weak solution to ($\widetilde{P}_\lambda$) is essentially the same as in definition 1.1. The only difference is that we now require that u and $u\,\mathbb{D}_s^2(u)$ belong to $L^1(\Omega)$.

We derive the following existence result for λf small enough.

THEOREM 1.4. *Assume that (A_1) holds. Then, there exists $\lambda^* > 0$ such that, for all $0 < \lambda \leqslant \lambda^*$, ($\widetilde{P}_\lambda$) has a weak solution $u \in W_0^{s,2}(\Omega) \cap \mathcal{C}^{0,\alpha}(\Omega)$ for some $\alpha > 0$.*

Next, motivated by some other results on fractional harmonic maps into the sphere [**19**, **20**] and some classical results of harmonic analysis [**41**, chapter V],

https://doi.org/10.1017/prm.2019.60 Published online by Cambridge University Press

we consider a different diffusion term. Depending on the real parameter $\lambda > 0$, we study the existence of solutions to the Dirichlet problem

$$\begin{cases}(-\Delta)^s u = \mu(x)|(-\Delta)^{s/2}u|^q + \lambda f(x), & \text{in } \Omega,\\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega,\end{cases} \tag{Q_λ}$$

under the assumption

$$\begin{cases}\Omega \subset \mathbb{R}^N,\ N \geqslant 2,\ \text{is a bounded domain with } \partial\Omega \text{ of class } \mathcal{C}^2,\\ f \in L^m(\Omega) \text{ for some } m \geqslant 1 \text{ and } \mu \in L^\infty(\Omega),\\ s \in (1/2, 1) \quad \text{and } 1 < q < N/(N - ms).\end{cases} \tag{B_1}$$

REMARK 1.5. If $m \geqslant N/s$, we just need to assume $1 < q < \infty$ in (B_1).

Since the diffusion term considered in (Q_λ) is different from the ones in (P_λ) and ($\widetilde{P}_\lambda$), we shall make precise the notion of weak solution to (Q_λ).

DEFINITION 1.2. We say that u is a *weak solution* to (Q_λ) if $u \in L^1(\Omega)$, $|(-\Delta)^{s/2}u| \in L^q(\Omega)$, $u \equiv 0$ in $\mathcal{C}\Omega$ and

$$\int_\Omega u(-\Delta)^s \phi \,\mathrm{d}x = \int_\Omega \Big(\mu(x)|(-\Delta)^{s/2}u|^q + \lambda f(x)\Big)\,\phi\,\mathrm{d}x, \quad \forall\, \phi \in \mathbb{X}_s, \tag{1.9}$$

where $\mathbb{X}_s$ is defined in (1.7).

THEOREM 1.5. *Assume that* (B_1) *holds. Then there exists* $\lambda^* > 0$ *such that, for all* $0 < \lambda \leqslant \lambda^*$, ($Q_\lambda$) *has a weak solution* $u \in W_0^{s,1}(\Omega)$.

REMARK 1.6. The regularity results for (1.8) that we need to prove theorem 1.5 are different from the ones used in theorems 1.1 and 1.4. Nevertheless, the restriction $s \in (1/2, 1)$ still arises out of these regularity results. See proposition 3.5 for more details.

Finally, for $s \in (0, 1)$ and $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$, following [**37**, **39**], we define the (distributional Riesz) *fractional gradient of order s* as the vector field $\nabla^s : \mathbb{R}^N \to \mathbb{R}^N$ given by

$$\nabla^s \phi(x) := \int_{\mathbb{R}^N} \frac{\phi(x) - \phi(y)}{|x - y|^s} \frac{x - y}{|x - y|} \frac{\mathrm{d}y}{|x - y|^N}, \quad \forall\, x \in \mathbb{R}^N. \tag{1.10}$$

Then we deal with the Dirichlet problem

$$\begin{cases}(-\Delta)^s u = \mu(x)|\nabla^s u|^q + \lambda f(x), & \text{in } \Omega,\\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega.\end{cases} \tag{$\widetilde{Q}_\lambda$}$$

REMARK 1.7. The notion of weak solution to ($\widetilde{Q}_\lambda$) has to be understood as in definition 1.2.

THEOREM 1.6. *Assume that* (B_1) *holds. Then there exists* $\lambda^* > 0$ *such that, for all* $0 < \lambda \leqslant \lambda^*$, ($\widetilde{Q}_\lambda$) *has a weak solution* $u \in W_0^{s,1}(\Omega)$.

https://doi.org/10.1017/prm.2019.60 Published online by Cambridge University Press

We end this section describing the organization of the paper. In § 2, we introduce the suitable functional setting to deal with our problems and we also recall some known results that will be useful. In § 3, which is independent of the rest of the work, we prove Calderón–Zygmund type regularity results for the fractional Poisson equation (1.8). Section 4 is devoted to the proofs of theorems 1.1 and 1.4. Section 5 contains the proofs of theorems 1.2 and 1.3. Section 6 deals with (Q_λ) and $(\widetilde{Q}_\lambda)$, i.e., it is devoted to the proofs of theorems 1.5 and 1.6. Finally, in § 7, we present some remarks and open problems.

Notation

(1) In $\mathbb{R}^N$, we use the notations $|x| = \sqrt{x_1^2 + \cdots + x_N^2}$ and $B_R(y) = \{x \in \mathbb{R}^N : |x - y| < R\}$.

(2) For a bounded open set $\Omega \subset \mathbb{R}^N$ we denote its complementary as $\mathcal{C}\Omega$, i.e., $\mathcal{C}\Omega = \mathbb{R}^N \setminus \Omega$.

(3) For $p \in (1, \infty)$, we denote by p' the conjugate exponent of p, namely $p' = p/(p-1)$ and by p_s^* the Sobolev critical exponent i.e., $p_s^* = Np/(N - sp)$ if $sp < N$ and $p_s^* = +\infty$ in case $sp \geqslant N$.

(4) For $u \in L^\infty(\Omega)$ we use the notation $\|u\|_\infty = \|u\|_{L^\infty(\Omega)} = \operatorname{esssup}_{x\in\Omega} |u(x)|$.

2. Functional setting and useful tools

In this section, we present the functional setting and some auxiliary results that will play an important role throughout the paper. We begin recalling the definition of the fractional Sobolev space.

Definition 2.1. Let Ω be an open set in $\mathbb{R}^N$ and $s \in (0, 1)$. For any $p \in [1, \infty)$, the fractional Sobolev space $W^{s,p}(\Omega)$ is defined as

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) : \iint_{\Omega\times\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \,\mathrm{d}x\,\mathrm{d}y < \infty \right\}.$$

It is a Banach space endowed with the usual norm

$$\|u\|_{W^{s,p}(\Omega)} := \left(\|u\|_{L^p(\Omega)}^p + \iint_{\Omega\times\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \,\mathrm{d}x\,\mathrm{d}y \right)^{1/p}.$$

Having at hand this definition we introduce the suitable space to deal with our problems and we refer to [**22**] for more details on fractional Sobolev spaces.

Definition 2.2. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with boundary $\partial\Omega$ of class $\mathcal{C}^{0,1}$ and $s \in (0, 1)$. For any $p \in [1, \infty)$. We define the space $W_0^{s,p}(\Omega)$ as

$$W_0^{s,p}(\Omega) := \left\{ u \in W^{s,p}(\mathbb{R}^N) : u = 0 \text{ in } \mathbb{R}^N \setminus \Omega \right\}.$$

It is a Banach space endowed with the norm

$$\|u\|_{W_0^{s,p}(\Omega)} := \left(\iint_{D_\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \,\mathrm{d}x\,\mathrm{d}y \right)^{1/p},$$

https://doi.org/10.1017/prm.2019.60 Published online by Cambridge University Press

where

$$D_\Omega := (\mathbb{R}^N \times \mathbb{R}^N) \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega) = (\Omega \times \mathbb{R}^N) \cup (\mathcal{C}\Omega \times \Omega).$$

In order to prove some of the Calderón–Zygmund type regularity results of § 3, we will use the relation between the fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ and the Bessel potential space defined below.

DEFINITION 2.3. Let $s \in (0,1)$. For any $p \in [1,\infty)$, the Bessel potential space $L^{s,p}(\mathbb{R}^N)$ is defined as

$$L^{s,p}(\mathbb{R}^N) := \left\{ u \in L^p(\mathbb{R}^N) \text{ such that } u = (I-\Delta)^{-s/2} f \text{ with } f \in L^p(\mathbb{R}^N) \right\}.$$

It is a Banach space endowed with the norm

$$|||u|||_{L^{s,p}(\mathbb{R}^N)} := \|u\|_{L^p(\mathbb{R}^N)} + \|f\|_{L^p(\mathbb{R}^N)}.$$

REMARK 2.1.

(a) Having in mind the *fractional gradient of order s* introduced in (1.10), let us point out that in [39, theorem 1.7] it is proved that

$$L^{s,p}(\mathbb{R}^N) = \left\{ u \in L^p(\mathbb{R}^N) \text{ such that } |\nabla^s u| \in L^p(\mathbb{R}^N) \right\},$$

with the equivalent norm

$$\|u\|_{L^{s,p}(\mathbb{R}^N)} := \|u\|_{L^p(\mathbb{R}^N)} + \|\nabla^s u\|_{L^p(\mathbb{R}^N)}.$$

(b) Notice also that in the case where s is an integer and $1 < p < \infty$, by [5, theorem 7.63, (f)] we know that $L^{s,p}(\mathbb{R}^N) = W^{s,p}(\mathbb{R}^N)$. Differently, in case $s \in (0,1)$, the two previous spaces does not coincide. However, for all $0 < \varepsilon < s$ and all $1 < p < \infty$, by [5, theorem 7.63, (g)], it follows that $L^{s+\varepsilon,p}(\mathbb{R}^N) \subset W^{s,p}(\mathbb{R}^N) \subset L^{s-\varepsilon,p}(\mathbb{R}^N)$ with continuous inclusions.

Finally, we recall here two known results of harmonic and functional analysis that will be key in the proofs of the results contained in § 3.

LEMMA 2.1. [41, theorem I, § 1.2, chapter V] *Let $0 < \lambda < N$ and $1 \leqslant p < \ell < \infty$ be such that $1/\ell + 1 = 1/p + \lambda/N$. For $g \in L^p(\mathbb{R}^N)$, we define*

$$J_\lambda(g)(x) = \int_{\mathbb{R}^N} \frac{g(y)}{|x-y|^\lambda}\, \mathrm{d}y.$$

Then, it follows that:

(a) *J_λ is well defined in the sense that the integral converges absolutely for almost all $x \in \mathbb{R}^N$.*

(b) *If $p > 1$, then $\|J_\lambda(g)\|_{L^\ell(\mathbb{R}^N)} \leqslant c_{p,q} \|g\|_{L^p(\mathbb{R}^N)}$.*

(c) *If $p = 1$, then $\left|\{x \in \mathbb{R}^N \,|J_\lambda(g)(x) > \sigma\}\right| \leqslant (A\|g\|_{L^1(\mathbb{R}^N)}/\sigma)^\ell$.*

https://doi.org/10.1017/prm.2019.60 Published online by Cambridge University Press

Let us introduce the real numbers $0 \leqslant s_1 \leqslant \eta \leqslant s_2 \leqslant 1$ and $1 \leqslant p_1, p_2, p \leqslant \infty$ and assume that they satisfy

$$\eta = \theta s_1 + (1-\theta) s_2 \quad \text{and} \quad \frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2} \quad \text{with } 0 < \theta < 1. \tag{2.1}$$

Moreover, let us introduce the condition

$$s_2 = p_2 = 1 \quad \text{and} \quad \frac{1}{p_1} \leqslant s_1. \tag{2.2}$$

Lemma 2.2. [**12**, theorem 1] *Assume that* (2.1) *holds and* (2.2) *fails. Then, for every* $\theta \in (0,1)$, *there exists a constant* $C = C(s_1, s_2, p_1, p_2, \theta) > 0$ *such that*

$$\|w\|_{W^{\eta,p}(\mathbb{R}^N)} \leqslant C \|w\|^{\theta}_{W^{s_1,p_1}(\mathbb{R}^N)} \|w\|^{1-\theta}_{W^{s_2,p_2}(\mathbb{R}^N)}, \quad \forall\, w \in W^{s_1,p_1}(\mathbb{R}^N) \cap W^{s_2,p_2}(\mathbb{R}^N).$$

3. Regularity results for the fractional Poisson equation

The main goal of this section, which is independent of the rest of the work, is to prove sharp Calderón–Zygmund type regularity results for the fractional Poisson equation

$$\begin{cases} (-\Delta)^s v = h(x), & \text{in } \Omega, \\ v = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{3.1}$$

under the assumption

$$\begin{cases} \Omega \subset \mathbb{R}^N,\ N \geqslant 2, \text{ is a bounded domain with } \partial\Omega \text{ of class } \mathcal{C}^2, \\ s \in (1/2, 1), \\ h \in L^m(\Omega) \text{ for some } m \geqslant 1. \end{cases} \tag{3.2}$$

First of all, let us precise the notion of weak solution to (3.1).

Definition 3.1. We say that v is a *weak solution* to (3.1) if $v \in L^1(\Omega)$, $v \equiv 0$ in $\mathcal{C}\Omega := \mathbb{R}^N \setminus \Omega$ and

$$\int_\Omega v(-\Delta)^s \phi \, \mathrm{d}x = \int_\Omega h(x) \phi \, \mathrm{d}x, \quad \forall\, \phi \in \mathbb{X}_s,$$

where $\mathbb{X}_s$ is defined in (1.7).

Under our assumption (3.2), the existence and uniqueness of solutions to (3.1) is a particular case of [**16**, proposition 2.4] (see also [**1**, **34**, § 1]). Having this in mind, we prove several regularity results for (3.1). Our first main result reads as follows:

Proposition 3.1. *Assume* (3.2) *and let* v *be the unique weak solution to* (3.1) *and* $t \in (0,1)$*:*

(1) *If* $m = 1$, *then* $v \in W_0^{t,p}(\Omega)$ *for all* $1 \leqslant p < N/(N-(2s-t))$ *and there exists* $C_1 = C_1(s,t,p,\Omega) > 0$ *such that*

$$\|v\|_{W_0^{t,p}(\Omega)} \leqslant \|v\|_{W^{t,p}(\mathbb{R}^N)} \leqslant C_1 \|h\|_{L^1(\Omega)}.$$

(2) *If* $1 < m < N/2s$, *then* $v \in W_0^{t,p}(\Omega)$ *for all* $1 \leqslant p \leqslant mN/(N - m(2s - t))$ *and there exists* $C_1 = C_1(m, s, t, p, \Omega) > 0$ *such that*

$$\|v\|_{W_0^{t,p}(\Omega)} \leqslant \|v\|_{W^{t,p}(\mathbb{R}^N)} \leqslant C_1 \|h\|_{L^m(\Omega)}.$$

(3) *If* $N/2s \leqslant m < N/(2s - 1)$, *then* $v \in W_0^{t,p}(\Omega)$ *for all* $1 \leqslant p < mN/(t(N - m(2s - 1)))$ *and there exists* $C_1 = C_1(m, s, t, p, \Omega) > 0$ *such that*

$$\|v\|_{W_0^{t,p}(\Omega)} \leqslant \|v\|_{W^{t,p}(\mathbb{R}^N)} \leqslant C_1 \|h\|_{L^m(\Omega)}.$$

(4) *If* $m \geqslant N/(2s - 1)$, *then* $v \in W_0^{t,p}(\Omega)$ *for all* $1 \leqslant p < \infty$.

Remark 3.1.

(a) The previous results are sharp in the sense that, if 'we take $t = s = 1$', we recover the classical sharp regularity results for the local case and those cannot be improved. See for instance [**37**, chapter 5].

(b) In the particular case of the fractional Laplacian of order $s \in (1/2, 1)$ and for $h \in L^1(\Omega)$, we improve the regularity results of [**1**, **31**, **34**]. Note however that in the three quoted papers the authors deal with more general operators and cover the full range $s \in (0, 1)$. Furthermore, in [**31**] the authors also deal with measures as data.

(c) Since $s \in (1/2, 1)$, observe that $t < 2s$ for all $t \in (0, 1)$.

As we believe it has its own interest, let us highlight a particular case of the previous result which follows directly from proposition 3.1 considering $t = s$.

Corollary 3.2. *Assume* (3.2) *and let* v *be the unique weak solution to* (3.1)*:*

(1) *If* $m = 1$, *then* $v \in W_0^{s,p}(\Omega)$ *for all* $1 \leqslant p < N/(N - s)$ *and there exists* $C_1 = C_1(s, p, \Omega) > 0$ *such that*

$$\|v\|_{W_0^{s,p}(\Omega)} \leqslant \|v\|_{W^{s,p}(\mathbb{R}^N)} \leqslant C_1 \|h\|_{L^1(\Omega)}.$$

(2) *If* $1 < m < N/2s$, *then* $v \in W_0^{s,p}(\Omega)$ *for all* $1 \leqslant p \leqslant mN/(N - ms)$ *and there exists* $C_1 = C_1(m, s, p, \Omega) > 0$ *such that*

$$\|v\|_{W_0^{s,p}(\Omega)} \leqslant \|v\|_{W^{s,p}(\mathbb{R}^N)} \leqslant C_1 \|h\|_{L^m(\Omega)}.$$

(3) *If* $N/2s \leqslant m < N/(2s - 1)$, *then* $v \in W_0^{s,p}(\Omega)$ *for all* $1 \leqslant p < mN/(s(N - m(2s - 1)))$ *and there exists* $C_1 = C_1(m, s, p, \Omega) > 0$ *such that*

$$\|v\|_{W_0^{s,p}(\Omega)} \leqslant \|v\|_{W^{s,p}(\mathbb{R}^N)} \leqslant C_1 \|h\|_{L^m(\Omega)}.$$

(4) *If* $m \geqslant N/(2s - 1)$, *then* $v \in W_0^{s,p}(\Omega)$ *for all* $1 \leqslant p < \infty$.

In the following two results we complete the information obtained in proposition 3.1 when $h \in L^m(\Omega)$ for some $m > N/2s$.

https://doi.org/10.1017/prm.2019.60 Published online by Cambridge University Press

PROPOSITION 3.3. *Assume* (3.2) *and let* v *be the unique weak solution to* (3.1) *and* $t \in (0, s)$:

(1) *If* $N/2s \leqslant m < N/(2s - t)$ *then* $v \in W_0^{t,p}(\Omega)$ *for all* $1 \leqslant p < mN/(N - m(2s - t))$ *and there exists* $C_2 = C_2(m, s, t, p, \Omega) > 0$ *such that*

$$\|v\|_{W_0^{t,p}(\Omega)} \leqslant \|v\|_{W^{t,p}(\mathbb{R}^N)} \leqslant C_2 \|h\|_{L^m(\Omega)}.$$

(2) *If* $m \geqslant N/(2s - t)$ *then* $v \in W_0^{t,p}(\Omega)$ *for all* $1 \leqslant p < \infty$ *and there exists* $C_2 = C_2(m, s, t, p, \Omega) > 0$ *such that*

$$\|v\|_{W_0^{t,p}(\Omega)} \leqslant \|v\|_{W^{t,p}(\mathbb{R}^N)} \leqslant C_2 \|h\|_{L^m(\Omega)}.$$

PROPOSITION 3.4. *Assume* (3.2) *and let* v *be the unique weak solution to* (3.1) *and* $t \in (s, 1)$. *If* $N/2s \leqslant m < N/s$ *then* $v \in W_0^{t,p}(\Omega)$ *for all* $1 \leqslant p < mN/(N - m(2s - t))$ *and there exists* $C_2 = C_2(m, s, t, p, \Omega) > 0$ *such that*

$$\|v\|_{W_0^{t,p}(\Omega)} \leqslant \|v\|_{W^{t,p}(\mathbb{R}^N)} \leqslant C_2 \|h\|_{L^m(\Omega)}.$$

REMARK 3.2. Notice that, in the case where $t \in (s, 1)$, propositions 3.1 and 3.4 complete and somehow give a more precise information than the result obtained in [**32**].

REMARK 3.3. The proofs of propositions 3.1, 3.3 and 3.4 are postponed to subsection 3.1

Due to the nonlocality of the fractional Laplacian, several notions of regularity can be studied. The following results, which generalize the fractional regularity proved in [**34**, theorem 24] with a different approach, can be seen as the counterpart of proposition 3.1 to deal with (Q_λ) and ($\widetilde{Q}_\lambda$).

PROPOSITION 3.5. *Assume* (3.2) *and let* v *be the unique weak solution to* (3.1) *and* $t \in (0, s]$:

(1) *If* $m = 1$, *then* $(-\Delta)^{t/2} v \in L^p(\Omega)$ *for all* $1 \leqslant p < N/(N - (2s - t))$ *and there exists* $C_3 = C_3(s, t, p, \Omega) > 0$ *such that*

$$\|(-\Delta)^{t/2} v\|_{L^p(\Omega)} \leqslant C_3 \|h\|_{L^1(\Omega)}.$$

(2) *If* $1 < m < N/(2s - t)$, *then* $(-\Delta)^{t/2} v \in L^p(\Omega)$ *for all* $1 \leqslant p \leqslant mN/(N - m(2s - t))$ *and there exists* $C_3 = C_3(s, t, m, p, \Omega) > 0$ *such that*

$$\|(-\Delta)^{t/2} v\|_{L^p(\Omega)} \leqslant C_3 \|h\|_{L^m(\Omega)}.$$

https://doi.org/10.1017/prm.2019.60 Published online by Cambridge University Press

(3) *If* $m \geqslant N/(2s-t)$ *then* $(-\Delta)^{t/2}v \in L^p(\Omega)$ *for all* $1 \leqslant p < \infty$ *and there exists* $C_3 = C_3(s,t,m,p,\Omega) > 0$ *such that*

$$\|(-\Delta)^{t/2}v\|_{L^p(\Omega)} \leqslant C_3\|h\|_{L^m(\Omega)}.$$

COROLLARY 3.6. *Assume* (3.2) *and let* v *be the unique weak solution to* (3.1)*:*

(1) *If* $m = 1$, *then* $|\nabla^s v| \in L^p(\Omega)$ *for all* $1 \leqslant p < N/(N-s)$ *and there exists* $C_4 = C_4(s,p,\Omega) > 0$ *such that*

$$\|\nabla^s v\|_{L^p(\Omega)} \leqslant C_4\|h\|_{L^1(\Omega)}.$$

(2) *If* $1 < m < N/s$, *then* $|\nabla^s v| \in L^p(\Omega)$ *for all* $1 \leqslant p \leqslant mN/(N-ms)$ *and there exists* $C_4 = C_4(s,m,p,\Omega) > 0$ *such that*

$$\|\nabla^s v\|_{L^p(\Omega)} \leqslant C_4\|h\|_{L^m(\Omega)}.$$

(3) *If* $m \geqslant N/s$ *then* $|\nabla^s v| \in L^p(\Omega)$ *for all* $1 \leqslant p < \infty$ *and there exists* $C_4 = C_4(s,m,p,\Omega) > 0$ *such that*

$$\|\nabla^s v\|_{L^p(\Omega)} \leqslant C_4\|h\|_{L^m(\Omega)}.$$

REMARK 3.4. The proofs of proposition 3.5 and corollary 3.6 will be given in subsection 3.2

REMARK 3.5.

(a) The Calderón–Zygmund type regularity results that we are going to prove rely on [**4**, lemma 2.15]. Since the fractional Laplacian is somehow a derivative of order $2s$, it is not clear if [**4**, lemma 2.15] holds true for $s \in (0, 1/2]$. Hence, with this approach we are limited to deal with $s \in (1/2, 1)$. One of the main difficulties in order to deal with $s \in (0, 1/2]$ is that we cannot use the regularity of the gradient of the solutions to (3.1). Hence, a direct purely non-local approach is missing. In trying to develop this purely non-local approach, we used the representation formula and were lead to obtain pointwise estimates on the 'non-local gradient' of the corresponding Green function. Nevertheless, we did not succeed.

(b) Let us also stress that [**4**, lemma 2.15] depends on the representation formula for the fractional Laplacian and pointwise estimates on the corresponding Green function and its gradient. If the corresponding pointwise estimates (see for instance [**4**, lemma 2.10] and [**17**, proposition 2.2] where they are gathered) were available for more general divergence-like operators, [**4**, lemma 2.15] would directly extend and so would do our results. It may be also interesting to obtain the estimates without the explicit representation formula as pointed out in [**4**, § 6].

https://doi.org/10.1017/prm.2019.60 Published online by Cambridge University Press

3.1. Proofs of propositions 3.1, 3.3 and 3.4

***Proof of proposition* 3.1.**

(1) Let v be the unique weak solution to (3.1). By [34, theorem 23], we know that, for all $1 \leqslant p_1 < N/(N-2s)$, there exists $C_5 = C_5(s, p_1, \Omega) > 0$ such that

$$\|v\|_{L^{p_1}(\mathbb{R}^N)} \leqslant C_5 \|h\|_{L^1(\Omega)}. \tag{3.3}$$

On the other hand, by [4, lemma 2.15], we know that, for all $1 \leqslant p_2 < N/(N-(2s-1))$, there exists $C_6 = C_6(s, p_2, \Omega) > 0$ such that

$$\|v\|_{W^{1,p_2}(\mathbb{R}^N)} \leqslant C_6 \|h\|_{L^1(\Omega)}. \tag{3.4}$$

Also, by lemma 2.2 applied with $\eta = t$, $s_1 = 0$ and $s_2 = 1$, we have that

$$\|v\|_{W^{t,p}(\mathbb{R}^N)} \leqslant C \|v\|^{1-t}_{L^{p_1}(\mathbb{R}^N)} \|v\|^{t}_{W^{1,p_2}(\mathbb{R}^N)}. \tag{3.5}$$

The result follows from (3.3)–(3.5) using that

$$1 \geqslant \frac{1}{p} = \frac{1-t}{p_1} + \frac{t}{p_2} > \frac{(1-t)(N-2s) + t(N-(2s-1))}{N} = \frac{N-(2s-t)}{N}.$$

(2) Let v be the unique weak solution to (3.1). By [34, theorems 16 and 24], we know that, for all $1 \leqslant p_1 \leqslant mN/(N-2ms)$, there exists $C_5 = C_5(s, m, p_1, \Omega) > 0$ such that

$$\|v\|_{L^{p_1}(\mathbb{R}^N)} \leqslant C_5 \|h\|_{L^m(\Omega)}. \tag{3.6}$$

On the other hand, by [4, lemma 2.15], we know that, for all $1 \leqslant p_2 \leqslant mN/(N-m(2s-1))$, there exists $C_6 = C_6(s, m, p_2, \Omega) > 0$ such that

$$\|v\|_{W^{1,p_2}(\mathbb{R}^N)} \leqslant C_6 \|h\|_{L^m(\Omega)}. \tag{3.7}$$

Finally, by lemma 2.2 applied with $\eta = t$, $s_1 = 0$ and $s_2 = 1$, we know that (3.5) holds. The result follows from (3.5), (3.6) and (3.7) using that

$$\begin{aligned} 1 \geqslant \frac{1}{p} = \frac{1-t}{p_1} + \frac{t}{p_2} &\geqslant \frac{(1-t)(N-2ms) + t(N-m(2s-1))}{mN} \\ &= \frac{N-m(2s-t)}{mN}. \end{aligned}$$

(3) Let v be the unique weak solution to (3.1). By [34, theorem 15], we know that, for all $1 \leqslant p_1 < \infty$, there exists $C_5 = C_5(s, m, p_1, \Omega) > 0$ such that

$$\|v\|_{L^{p_1}(\mathbb{R}^N)} \leqslant C_5 \|h\|_{L^m(\Omega)}. \tag{3.8}$$

By [4, lemma 2.15] we know that (3.7) holds. Moreover, by lemma 2.2 applied with $\eta = t$, $s_1 = 0$ and $s_2 = 1$, it follows that (3.5) holds. The result follows from (3.5), (3.7) and (3.8) using that

$$1 \geqslant \frac{1}{p} = \frac{1-t}{p_1} + \frac{t}{p_2} > \frac{t(N-m(2s-1))}{mN}.$$

https://doi.org/10.1017/prm.2019.60 Published online by Cambridge University Press

(4) Let v be the unique weak solution to (3.1). By [**34**, theorem 15] we know that (3.8) holds. On the other hand, by [**4**, lemma 2.15], we know that, for all $1 \leqslant p_2 < \infty$, there exists $C_6 = C_6(s, m, p_2, \Omega) > 0$ such that

$$\|v\|_{W^{1,p_2}(\mathbb{R}^N)} \leqslant C_6 \|h\|_{L^m(\Omega)}. \tag{3.9}$$

The result follows from lemma 2.2 applied with $\eta = t, s_1 = 0$ and $s_2 = 1$.

□

Now, using the Bessel potential space $L^{s,p}(\mathbb{R}^N)$ and [**5**, theorems 7.58 and 7.63, (g)], we prove propositions 3.3 and 3.4. We begin proving a regularity result in the Bessel potential space and some useful consequences.

PROPOSITION 3.7. *Assume* (3.2) *and let v be the unique weak solution to* (3.1)*:*

(1) *If $m = 1$, then $v \in L^{s,p}(\mathbb{R}^N)$ for all $1 \leqslant p < N/(N - s)$ and there exists $C_7 = C_7(s, p, \Omega) > 0$ such that*

$$\|v\|_{L^{s,p}(\mathbb{R}^N)} \leqslant C_7 \|h\|_{L^1(\Omega)}.$$

(2) *If $1 < m < N/s$, then $v \in L^{s,p}(\mathbb{R}^N)$ for all $1 \leqslant p \leqslant mN/(N - ms)$ and there exists $C_7 = C_7(m, s, p, \Omega) > 0$ such that*

$$\|v\|_{L^{s,p}(\mathbb{R}^N)} \leqslant C_7 \|h\|_{L^m(\Omega)}.$$

(3) *If $m \geqslant N/s$, then $v \in L^{s,p}(\mathbb{R}^N)$ for all $1 \leqslant p < \infty$ and there exists $C_7 = C_7(m, s, p, \Omega) > 0$ such that*

$$\|v\|_{L^{s,p}(\mathbb{R}^N)} \leqslant C_7 \|h\|_{L^m(\Omega)}.$$

Proof. Assume (3.2) and let v be the unique weak solution to (3.1). Taking into account remark 2.1, (a), we have just to show the regularity of $|\nabla^s v|$ where ∇^s is defined in (1.10). By a density argument and [**37**, lemma 15.9] we have that

$$|\nabla^s v(x)| \leqslant \frac{1}{N - (1 - s)} \int_{\mathbb{R}^N} \frac{|\nabla v(y)|}{|x - y|^{N-(1-s)}} \, \mathrm{d}y \quad \text{a.e. in } \mathbb{R}^N. \tag{3.10}$$

Let us then split into three cases:

(1) $m = 1$.
By [**4**, lemma 2.15] we get $v \in W^{1,q}(\mathbb{R}^N)$ for all $1 \leqslant q < N/(N - (2s - 1))$. Thus, by lemma 2.1, we conclude that $|\nabla^s v(x)| \in L^p(\mathbb{R}^N)$ for all $1 \leqslant p < N/(N - s)$.

(2) $1 < m < N/s$.
The result follows arguing on the same way.

(3) $m \geqslant N/s$.
In this case, since $m \geqslant N/s$ and Ω is a bounded domain then $f \in L^{\bar{m}}(\Omega)$ for all $\bar{m} < N/s$. In particular, using the second point it follows that $v \in L^{s,\bar{p}}(\mathbb{R}^N)$

https://doi.org/10.1017/prm.2019.60 Published online by Cambridge University Press

for all $1 \leqslant \bar{p} \leqslant \bar{m}N/(N-s\bar{m})$. Letting $\bar{m} \uparrow N/s$, we reach that $\bar{p} \uparrow \infty$. Hence $v \in L^{s,p}(\mathbb{R}^N)$ for all $1 \leqslant p < \infty$.

□

Corollary 3.8. *Assume* (3.2) *and let* v *be the unique weak solution to* (3.1)*:*

(1) If $N/2s \leqslant m < N/s$ *then* $v \in W_0^{s',p}(\Omega)$ *for all* $0 < s' < s$ *and all* $1 \leqslant p \leqslant mN/(N-ms)$*. Moreover, there exists* $C_8 = C_8(m,s,s',p,\Omega) > 0$ *such that*

$$\|v\|_{W_0^{s',p}(\Omega)} \leqslant \|v\|_{W^{s',p}(\mathbb{R}^N)} \leqslant C_8\|h\|_{L^m(\Omega)}.$$

(2) If $m \geqslant N/s$*, then* $v \in W_0^{s',p}(\Omega)$ *for all* $0 < s' < s$ *and all* $1 \leqslant p < \infty$*. Moreover, there exists* $C_8 = C_8(m,s,s',p,\Omega) > 0$ *such that*

$$\|v\|_{W_0^{s',p}(\Omega)} \leqslant \|v\|_{W^{s',p}(\mathbb{R}^N)} \leqslant C_8\|h\|_{L^m(\Omega)}.$$

Remark 3.6. Observe that without loss of generality we can assume in the proof that $p > 1$. For $p = 1$ the result follows from proposition 3.1 and the continuous embedding $W^{s,p}(\mathbb{R}^N) \subset W^{s',p}(\mathbb{R}^N)$.

Proof.

(1) Let v be the unique weak solution to (3.1). By proposition 3.7, (2) we know that $v \in L^{s,p}(\mathbb{R}^N)$ for all $1 \leqslant p \leqslant mN/(N-ms)$. Thus, by [**5**, theorem 7.63, (g)] (see also remark 2.1) we conclude that $v \in W_0^{s',p}(\Omega)$ for all $0 < s' < s$ and

$$\|v\|_{W_0^{s',p}(\Omega)} \leqslant C\|v\|_{L^{s,p}(\mathbb{R}^N)} \leqslant C_8\|h\|_{L^m(\Omega)}.$$

(2) Let v be the unique weak solution to (3.1). In this case, by proposition 3.7, we know that $v \in L^{s,p}(\mathbb{R}^N)$ for all $1 \leqslant p < \infty$. Hence, by [**5**, theorem 7.63, (g)], we conclude.

□

***Proof of proposition* 3.3**. First observe that, since $t \in (0,s)$ it follows that $N/(2s-t) < N/s$. We then consider separately the two cases:

(1) Let v be the unique weak solution to (3.1). By corollary 3.8, (1), we have that $v \in W^{s',p_1}(\mathbb{R}^N)$ for all $0 < s' < s$ and all $1 \leqslant p_1 \leqslant mN/(N-ms)$. Moreover, there exists $C_8 = C_8(m,s,s',p_1,\Omega) > 0$ such that

$$\|v\|_{W_0^{s',p_1}(\Omega)} \leqslant \|v\|_{W^{s',p_1}(\mathbb{R}^N)} \leqslant C_8\|h\|_{L^m(\Omega)}. \tag{3.11}$$

On the other hand, by [**5**, theorem 7.58], we have that $v \in W^{\eta,q_1}(\mathbb{R}^N)$ for all $0 < \eta < s' < s$ and all $1 \leqslant q_1 \leqslant Np_1/(N-p_1(s'-\eta)) \leqslant$

https://doi.org/10.1017/prm.2019.60 Published online by Cambridge University Press

$mN/(N-m(s+s'-\eta))$. Moreover, there exists $C>0$ such that

$$\|v\|_{W^{\eta,q_1}(\mathbb{R}^N)} \leqslant C\|v\|_{W^{s',p_1}(\mathbb{R}^N)} \leqslant C\,C_8\|h\|_{L^m(\Omega)}. \tag{3.12}$$

We fix then $p\in[1,mN/(N-m(2s-t)))$ and observe that we can find $\overline{\eta}\in(t,s')$ such that

$$1\leqslant p\leqslant \frac{mN}{N-m(s+s'-\overline{\eta})}.$$

The result follows from (3.12) using the continuous embedding $W^{\overline{\eta},p}(\mathbb{R}^N)\subset W^{t,p}(\mathbb{R}^N)$.

(2) The result follows arguing as in the proof of proposition 3.7, (3) using proposition 3.3, (1).

□

***Proof of proposition* 3.4.** Let v be the unique weak solution to (3.1). By corollary 3.8, (1), we know that, for all $0<s'<s$ and all $1\leqslant p_1\leqslant mN/(N-ms)$, there exists $C_8=C_8(m,s,s',p_1,\Omega)>0$ such that

$$\|v\|_{W_0^{s',p_1}(\Omega)} \leqslant \|v\|_{W^{s',p_1}(\mathbb{R}^N)} \leqslant C_8\|h\|_{L^m(\Omega)}. \tag{3.13}$$

On the other hand, by [4, lemma 2.15], we know that, for all $1\leqslant p_2\leqslant mN/(N-m(2s-1))$, there exists $C_6=C_6(s,m,p_2,\Omega)>0$ such that

$$\|v\|_{W^{1,p_2}(\mathbb{R}^N)} \leqslant C_6\|h\|_{L^m(\Omega)}. \tag{3.14}$$

Also, by lemma 2.2 applied with $\eta=t$, $s_1=s'$ and $s_2=1$, we know that

$$\|v\|_{W^{t,p'}(\mathbb{R}^N)} \leqslant C\|v\|_{W^{s',p_1}(\mathbb{R}^N)}^{(1-t)/(1-s')}\|v\|_{W^{1,p_2}(\mathbb{R}^N)}^{(t-s')/(1-s')}, \tag{3.15}$$

with

$$\frac{1}{p'}=\frac{1}{1-s'}\left(\frac{1-t}{p_1}+\frac{t-s'}{p_2}\right).$$

We fix then an arbitrary $1\leqslant p<mN/(N-m(2s-t))$ and observe that we can choose $s'<s$ such that $p'=p$. Hence, the result follows from (3.13)–(3.15). □

3.2. Proofs of proposition 3.5 and corollary 3.6

Next, using again [4, lemma 2.15] but with a different approach, we prove proposition 3.5. As a consequence we will obtain corollary 3.6.

https://doi.org/10.1017/prm.2019.60 Published online by Cambridge University Press

***Proof of proposition* 3.5.** Let v be the unique weak solution to (3.1) and define, for $x \in \mathbb{R}^N$ arbitrary,

$$S_1 := \{y \in \mathbb{R}^N : \operatorname{dist}(y,\Omega) > 2\} \quad \text{and}$$
$$S_2 := \{y \in \mathbb{R}^N : \operatorname{dist}(y,\Omega) \leqslant 2 \text{ and } |x-y| \geqslant 1\}.$$

Then, observe that, for all $x \in \Omega$,

$$\begin{aligned} |(-\Delta)^{t/2} v(x)| &\leqslant \int_{\mathbb{R}^N} \frac{|v(x)-v(y)|}{|x-y|^{N+t}}\,\mathrm{d}y \\ &\leqslant \int_{S_1} \frac{|v(x)-v(y)|}{|x-y|^{N+t}}\,\mathrm{d}y + \int_{S_2} \frac{|v(x)-v(y)|}{|x-y|^{N+t}}\,\mathrm{d}y \\ &\quad + \int_{B_1(x)} \frac{|v(x)-v(y)|}{|x-y|^{N+t}}\,\mathrm{d}y \\ &=: I_1(x) + I_2(x) + I_3(x). \end{aligned} \tag{3.16}$$

Now, let us estimate each one of the three terms. First observe that

$$\begin{aligned} I_1(x) &= \int_{S_1} \frac{|v(x)|}{|x-y|^{N+t}}\,\mathrm{d}y \leqslant \int_{S_1} \frac{|v(x)|}{\operatorname{dist}(y,\Omega)^{N+t}}\,\mathrm{d}y \\ &= |v(x)| \int_{S_1} \frac{\mathrm{d}y}{\operatorname{dist}(y,\Omega)^{N+t}}\,\mathrm{d}y = c_1(N,t,\Omega)|v(x)|, \quad \forall\, x \in \Omega. \end{aligned} \tag{3.17}$$

Next, using that Ω is a bounded domain and the triangular inequality, we deduce that

$$I_2(x) \leqslant \int_{S_2} |v(x)-v(y)|\,\mathrm{d}y \leqslant c_2(\Omega)|v(x)| + \|v\|_{L^1(\Omega)}, \quad \forall\, x \in \Omega. \tag{3.18}$$

Finally, following the arguments of [**22**, proposition 2.2], we deduce that

$$\begin{aligned} I_3(x) &= \int_{B_1(0)} \frac{|v(x)-v(x+z)|}{|z|} \frac{1}{|z|^{N+t-1}}\mathrm{d}z = \int_{B_1(0)} \int_0^1 \frac{|\nabla v(x+\tau z)|}{|z|^{N+t-1}}\,\mathrm{d}\tau\,\mathrm{d}z \\ &\leqslant \int_0^1 \int_{\mathbb{R}^N} \frac{|\nabla v(w)|}{|w-x|^{N+t-1}} \tau^{t-1}\,\mathrm{d}w\,\mathrm{d}\tau \\ &= \left(\int_{\mathbb{R}^N} \frac{|\nabla v(w)|}{|w-x|^{N+t-1}}\,\mathrm{d}w \right) \left(\int_0^1 \tau^{t-1}\,\mathrm{d}\tau \right) \\ &= \frac{1}{t} \int_{\mathbb{R}^N} \frac{|\nabla v(w)|}{|w-x|^{N+t-1}}\,\mathrm{d}w, \quad \forall\, x \in \Omega. \end{aligned} \tag{3.19}$$

From (3.16)–(3.19), we deduce that

$$|(-\Delta)^{t/2} v(x)| \leqslant c(s,t,\Omega) \left(|v(x)| + \int_{\mathbb{R}^N} \frac{|\nabla v(w)|}{|w-x|^{N+t-1}}\,\mathrm{d}w + \|v\|_{L^1(\Omega)} \right), \quad \forall\, x \in \Omega, \tag{3.20}$$

and so, exploiting again the fact that Ω is a bounded domain and using the Hölder and triangular inequalities, for all $1 \leqslant p < \infty$, we obtain that

$$\|(-\Delta)^{t/2}v\|_{L^p(\Omega)} \leqslant c_2(s,t,\Omega)\left(\|v\|_{L^p(\Omega)} + \left\|\int_{\mathbb{R}^N} \frac{|\nabla v(w)|}{|w-x|^{N+t-1}}\,\mathrm{d}w\right\|_{L^p(\mathbb{R}^N)}\right). \tag{3.21}$$

Now, let us split the rest of the proof into three parts:

(1) $m = 1$.
By [4, lemma 2.15] we know that $v \in W_0^{1,\sigma}(\Omega)$ for all $1 \leqslant \sigma < N/(N-(2s-1))$ and that there exists $C_6 = C_6(s,\sigma,\Omega) > 0$ such that

$$\|\nabla v\|_{L^\sigma(\mathbb{R}^N)} \leqslant C_6\|h\|_{L^1(\Omega)}.$$

Thus, applying lemma 2.1, we deduce that

$$\left\|\int_{\mathbb{R}^N} \frac{|\nabla v(w)|}{|w-x|^{N+t-1}}\,\mathrm{d}w\right\|_{L^\ell(\mathbb{R}^N)} \leqslant C\,C_6\|h\|_{L^1(\Omega)}, \quad \forall\, 1 \leqslant \ell < \frac{N}{N-(2s-t)}. \tag{3.22}$$

Also, by [34, theorem 23], we know that, for all $1 \leqslant \gamma < N/(N-2s)$, there exists $C_5 = C_5(s,\gamma,\Omega) > 0$ such that

$$\|v\|_{L^\gamma(\Omega)} \leqslant C_5\|h\|_{L^1(\Omega)}. \tag{3.23}$$

Taking into account (3.22)–(3.23), the result follows from (3.21).

(2) $1 < m < N/(2s-t)$.
The result follows arguing as in (1) using [34, theorems 15, 16 and 24] instead of [34, theorem 23].

(3) $m \geqslant N/(2s-t)$.
It follows from proposition 3.5, (2) arguing as in the proof of proposition 3.7, (3).

□

Proof of corollary 3.6. By [37, lemma 15.9] we know that

$$\nabla^s u(x) = \frac{1}{N-(1-s)}\int_{\mathbb{R}^N} \frac{\nabla u(y)}{|x-y|^{N+s-1)}}\,\mathrm{d}y. \tag{3.24}$$

Hence, we have that

$$|\nabla^s u(x)| \leqslant C\int_{\mathbb{R}^N} \frac{|\nabla u(y)|}{|x-y|^{N+s-1}}\,\mathrm{d}y. \tag{3.25}$$

The result then follows arguing as in the proof of proposition 3.5. □

https://doi.org/10.1017/prm.2019.60 Published online by Cambridge University Press

3.3. Convergence and compactness

We end this section presenting a result of convergence and one of compactness for the fractional Poisson equation (3.1). They will be used in the proofs of theorems 1.1, 1.4, 1.5 and 1.6.

PROPOSITION 3.9. *Let $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$, be a bounded domain with $\partial\Omega$ of class $\mathcal{C}^2$, let $s \in (1/2, 1)$, let $\{h_n\} \subset L^1(\Omega)$ be a sequence such that $h_n \to h$ in $L^1(\Omega)$ and let v_n be the unique weak solution to*

$$\begin{cases} (-\Delta)^s v_n = h_n(x), & \text{in}\,\Omega, \\ v_n = 0, & \text{in}\,\mathbb{R}^N \setminus \Omega, \end{cases}$$

for all $n \in \mathbb{N}$, and v be the unique weak solution to

$$\begin{cases} (-\Delta)^s v = h(x), & \text{in } \Omega, \\ v = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Then $v_n \to v$ in $W_0^{s,p}(\Omega)$ for all $1 \leqslant p < N/(N-s)$.

Proof. First of all observe that the existence of v_n and v are insured by [**16**, proposition 2.4]. Now, let us define $w_n = v_n - v$ and observe that w_n satisfies

$$\begin{cases} (-\Delta)^s w_n = h_n(x) - h(x), & \text{in } \Omega, \\ w_n = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

By proposition 3.1, (1), we know that, for all $1 \leqslant p < N/(N-s)$, there exists $C_3 = C_3(s,p,\Omega) > 0$ such that

$$\|w_n\|_{W_0^{s,p}(\Omega)} \leqslant C_3 \|h_n - h\|_{L^1(\Omega)}.$$

Hence, since $h_n \to h$ in $L^1(\Omega)$, it follows that $w_n \to 0$ in $W_0^{s,p}(\Omega)$ for all $1 \leqslant p < N/(N-s)$ and so, that $v_n \to v$ in $W_0^{s,p}(\Omega)$ for all $1 \leqslant p < N/(N-s)$, as desired. □

PROPOSITION 3.10. *Let $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$, be a bounded domain with $\partial\Omega$ of class $\mathcal{C}^2$, let $s \in (1/2, 1)$ and let $h \in L^1(\Omega)$. Then the operator $\mathcal{S} : L^1(\Omega) \to W_0^{s,p}(\Omega)$ given by $\mathcal{S}(h) = v$ with v the unique weak solution to (3.1) is compact for all $1 \leqslant p < N/(N-s)$.*

Proof. Let $\{f_n\} \subset L^1(\Omega)$ be a bounded sequence. By [**17**, proposition 2.4] we know that $\mathcal{S}$ is a compact operator from $L^1(\Omega)$ to $W_0^{1,p_1}(\Omega)$ for all $1 \leqslant \theta < N/(N-(2s-1))$. Hence, for all $1 \leqslant \theta < N/(N-(2s-1))$, up to a subsequence we have that $\mathcal{S}(f_n) \to v$ for some $v \in W_0^{1,\theta}(\Omega)$. By Sobolev inequality, this implies, for all $1 \leqslant \sigma < N/(N-2s)$, that $\mathcal{S}(f_n) \to v$ in $L^\sigma(\Omega)$ and $v \in L^\sigma(\Omega)$.

https://doi.org/10.1017/prm.2019.60 Published online by Cambridge University Press

Now, applying lemma 2.2 with $\eta = s$, $s_1 = 0$ and $s_2 = 1$, we obtain that

$$\begin{aligned}\|\mathcal{S}(f_n) - v\|_{W_0^{s,p}(\Omega)} &\leqslant C\|\mathcal{S}(f_n) - v\|_{L^\sigma(\mathbb{R}^N)}^{1-s}\|\mathcal{S}(f_n) - v\|_{W^{1,\theta}(\mathbb{R}^N)}^{s} \\ &= C\|\mathcal{S}(f_n) - v\|_{L^\sigma(\Omega)}^{1-s}\|\mathcal{S}(f_n) - v\|_{W_0^{1,\theta}(\Omega)}^{s}, \qquad (3.26)\end{aligned}$$

for p satisfying $1/p = (1-s)/\sigma + s/\theta$. Hence, the result follows from (3.26) using that

$$1 \geqslant \frac{1}{p} > \frac{N-s}{N}.$$

□

4. Proofs of theorems 1.1 and 1.4

This section is devoted to prove theorems 1.1 and 1.4. As indicated in § 1, once we have the regularity results of § 3, we follow the approach first develop in [**36**, § 6]. Let us begin with two elementary technical lemmas that will be useful in the proofs of both theorems.

LEMMA 4.1. *Let $a, b > 0$, $p > 1$ and $c^* := ((p-1)/p)(1/pa^pb)^{1/(p-1)}$. Then, the function $g : [0, \infty) \to \mathbb{R}$ given by*

$$g(t) = a^p(bt + c^*)^p - t,$$

has exactly one root $t^* \in (0, \infty)$.

Proof. First observe that, $g'(t) = 0$ if and only if

$$t = t^* := \frac{1}{b}\left(\frac{1}{pa^pb}\right)^{1/(p-1)} - \frac{c^*}{b} = \frac{1}{pb}\left(\frac{1}{pa^pb}\right)^{1/(p-1)} \in (0, \infty).$$

Moreover, observe that

$$g''(t^*) = (p-1)pa^pb^2\left(\frac{1}{pa^pb}\right)^{(p-2)/(p-1)} > 0.$$

Thus, we deduce that g has an strict global minimum on $t = t^*$. Finally, observe that

$$\begin{aligned}g(t^*) = a^p\left(\frac{1}{pa^pb}\right)^{p/(p-1)} &- \frac{1}{b}\left(\frac{1}{pa^pb}\right)^{1/(p-1)} \\ &+ \frac{p-1}{pb}\left(\frac{1}{pa^pb}\right)^{1/(p-1)} = 0, \quad g(0) > 0 \quad \text{and} \quad \lim_{t\to\infty} g(t) = \infty.\end{aligned}$$

Hence, we conclude that g has exactly one root $t^* \in (0, \infty)$. □

https://doi.org/10.1017/prm.2019.60 Published online by Cambridge University Press

LEMMA 4.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with boundary $\partial\Omega$ of class $\mathcal{C}^{0,1}$ and let $s \in (0,1)$. For all $\varepsilon > 0$ satisfying $0 < s-\varepsilon < s+\varepsilon < 1$ and all $1 \leqslant \sigma < r$ there exists $C_9 = C_9(s,\varepsilon,\sigma,r,\Omega) > 0$ such that*

$$\left\| \left(\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^\sigma}{|x-y|^{N+s\sigma}} \,\mathrm{d}y \right)^{1/\sigma} \right\|_{L^r(\Omega)} \leqslant C_9 \|u\|_{W_0^{s+\varepsilon,r}(\Omega)}, \quad \forall\, u \in W_0^{s+\varepsilon,r}(\Omega). \tag{4.1}$$

Proof. First of all, observe that

$$\begin{aligned} &\int_\Omega \left(\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^\sigma}{|x-y|^{N+s\sigma}} \,\mathrm{d}y \right)^{r/\sigma} \mathrm{d}x \\ &= \int_\Omega \left(\int_{\mathbb{R}^N \cap \{|x-y|<1\}} \frac{|u(x)-u(y)|^\sigma}{|x-y|^{N+s\sigma}} \,\mathrm{d}y \right. \\ &\quad \left. + \int_{\mathbb{R}^N \cap \{|x-y|\geqslant 1\}} \frac{|u(x)-u(y)|^\sigma}{|x-y|^{N+s\sigma}} \,\mathrm{d}y \right)^{r/\sigma} \mathrm{d}x \\ &\leqslant c_{r,\sigma} \left[\int_\Omega \left(\int_{\mathbb{R}^N \cap \{|x-y|<1\}} \frac{|u(x)-u(y)|^\sigma}{|x-y|^{N+s\sigma}} \,\mathrm{d}y \right)^{r/\sigma} \mathrm{d}x \right. \\ &\quad \left. + \int_\Omega \left(\int_{\mathbb{R}^N \cap \{|x-y|\geqslant 1\}} \frac{|u(x)-u(y)|^\sigma}{|x-y|^{N+s\sigma}} \,\mathrm{d}y \right)^{r/\sigma} \mathrm{d}x \right] \\ &=: c_{r,\sigma}(J_1 + J_2). \end{aligned} \tag{4.2}$$

Let us then estimate J_1. Applying Hölder inequality, we have that

$$\begin{aligned} J_1 &= \int_\Omega \left(\int_{\mathbb{R}^N \cap \{|x-y|<1\}} \frac{|u(x)-u(y)|^\sigma}{|x-y|^{N\sigma/r+(s+\varepsilon)\sigma}} \frac{|x-y|^{\varepsilon\sigma}}{|x-y|^{N-N\sigma/r}} \,\mathrm{d}y \right)^{r/\sigma} \mathrm{d}x \\ &\leqslant \int_\Omega \left[\left(\int_{\mathbb{R}^N \cap \{|x-y|<1\}} \frac{|u(x)-u(y)|^r}{|x-y|^{N+(s+\varepsilon)r}} \,\mathrm{d}y \right) \right. \\ &\quad \left. \times \left(\int_{\mathbb{R}^N \cap \{|x-y|<1\}} \frac{\mathrm{d}y}{|x-y|^{N-(\varepsilon\sigma r/(r-\sigma))}} \right)^{(r-\sigma)/\sigma} \right] \mathrm{d}x. \end{aligned}$$

Furthermore, since

$$\int_{\mathbb{R}^N \cap \{|x-y|<1\}} \frac{\mathrm{d}y}{|x-y|^{N-(\varepsilon\sigma r/(r-\sigma))}} = \int_{B_1(0)} \frac{\mathrm{d}z}{|z|^{N-(\varepsilon\sigma r/(r-\sigma))}} = C_{J_1}(\varepsilon,\sigma,r) < \infty,$$

we deduce that

$$J_1 \leqslant \widetilde{C}_{J_1} \int_\Omega \left(\int_{\mathbb{R}^N \cap \{|x-y|<1\}} \frac{|u(x)-u(y)|^r}{|x-y|^{N+(s+\varepsilon)r}} \,\mathrm{d}y \right) \mathrm{d}x \leqslant \widetilde{C}_{J_1} \|u\|^r_{W_0^{s+\varepsilon,r}(\Omega)}. \tag{4.3}$$

https://doi.org/10.1017/prm.2019.60 Published online by Cambridge University Press

Now, arguing as with J_1, we obtain that

$$J_2 = \int_\Omega \left(\int_{\mathbb{R}^N \cap \{|x-y| \geqslant 1\}} \frac{|u(x)-u(y)|^\sigma}{|x-y|^{N\sigma/r+(s-\varepsilon)\sigma}} \frac{\mathrm{d}y}{|x-y|^{N-N\sigma/r+\varepsilon\sigma}} \right)^{r/\sigma} \mathrm{d}x$$
$$\leqslant \int_\Omega \left(\int_{\mathbb{R}^N \cap \{|x-y| \geqslant 1\}} \frac{|u(x)-u(y)|^r}{|x-y|^{N+(s-\varepsilon)r}} \mathrm{d}y \right)$$
$$\times \left(\int_{\mathbb{R}^N \cap \{|x-y| \geqslant 1\}} \frac{\mathrm{d}y}{|x-y|^{N+(\varepsilon\sigma r/(r-\sigma))}} \right)^{(r-\sigma)/\sigma} \mathrm{d}x.$$

Hence, since

$$\int_{\mathbb{R}^N \cap \{|x-y| \geqslant 1\}} \frac{\mathrm{d}y}{|x-y|^{N+(\varepsilon\sigma r/(r-\sigma))}}$$
$$= \int_{\mathbb{R}^N \setminus B_1(0)} \frac{\mathrm{d}z}{|z|^{N+(\varepsilon\sigma r/(r-\sigma))}} = C_{J_2}(\varepsilon, \sigma, r) < \infty,$$

and $W_0^{s+\varepsilon,r}(\Omega) \subset W_0^{s-\varepsilon,r}(\Omega)$, it follows that

$$J_2 \leqslant \widetilde{C}_{J_2} \int_\Omega \left(\int_{\mathbb{R}^N \cap \{|x-y| \geqslant 1\}} \frac{|u(x)-u(y)|^r}{|x-y|^{N+(s-\varepsilon)r}} \mathrm{d}y \right) \mathrm{d}x \leqslant \widetilde{C}_{J_2} \|u\|^r_{W_0^{s-\varepsilon,r}(\Omega)}$$
$$\leqslant \overline{C}_{J_2} \|u\|^r_{W_0^{s+\varepsilon,r}(\Omega)}. \tag{4.4}$$

The result follows from (4.2), (4.3) and (4.4). □

REMARK 4.1. Observe that the constant $C_9 = C_9(s, \varepsilon, \sigma, r, \Omega) > 0$ obtained in the previous lemma is not stable when $s \to 1^-$. More precisely, if $s \to 1^-$ then $\varepsilon \to 0$ and this implies that $C_9 \to +\infty$.

4.1. Proof of theorem 1.1

Let us begin recalling that, under the assumption (A_1), $f \in L^m(\Omega)$ for some $m > N/2s$. Hence, since we are working in a bounded domain, without loss of generality, we can assume that $m \in (N/2s, N/(2s-1))$. Moreover, observe that, for $\lambda f \equiv 0$, $u \equiv 0$ is a solution to (P_λ) and, for $\mu \equiv 0$, (P_λ) reduces to (3.1). Hence, we may assume that $\|\mu\|_\infty \neq 0$ and $\|f\|_{L^m(\Omega)} \neq 0$.

Next, we fix some notation that we use throughout this subsection. First, we fix $r = r(m, s) > 0$ such that

$$1 < 2m < r < \frac{mN}{s(N-m(2s-1))},$$

and $\varepsilon = \varepsilon(r, m, s) > 0$ such that

$$1 < r < \frac{mN}{(s+\varepsilon)(N-m(2s-1))} < \frac{mN}{s(N-m(2s-1))}, \quad s+\varepsilon < 1,$$
$$\text{and} \quad s - \varepsilon > \frac{1}{2}.$$

https://doi.org/10.1017/prm.2019.60 Published online by Cambridge University Press

Also, we introduce and fix the constants C_1, given by proposition 3.1, (3) applied with $t = s + \varepsilon$ and $p = r$, $C_{10} := C_9^2 |\Omega|^{(r-2m)/rm}$, where C_9 is the constant given by lemma 4.2, and

$$\lambda^* := \frac{1}{4\|f\|_{L^m(\Omega)} C_1^2 C_{10} \|\mu\|_\infty}.$$

By the definition of λ^* and lemma 4.1, we know there exists and unique $l \in (0, \infty)$ such that

$$C_1(C_{10}\|\mu\|_\infty l + \lambda^* \|f\|_{L^m(\Omega)}) = l^{1/2}. \tag{4.5}$$

Having fixed the above constants, we introduce

$$E := \left\{ v \in W_0^{s,1}(\Omega) : \iint_{D_\Omega} \frac{|u(x)-u(y)|^r}{|x-y|^{N+(s+\varepsilon)r}} \, \mathrm{d}x \, \mathrm{d}y \leqslant l^{r/2} \right\},$$

which is a closed convex set of $W_0^{s,1}(\Omega)$. Then, we define $T : E \to W_0^{s,1}(\Omega)$ by $T(\varphi) = u$, where u is a weak solution to

$$\begin{cases} (-\Delta)^s u = \mu(x)\, \mathbb{D}_s^2(\varphi) + \lambda f(x), & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{4.6}$$

and observe that problem (P_λ) is equivalent to the fixed point problem $u = T(u)$. Hence, to prove theorem 1.1, we shall show that T has fixed point belonging to $W_0^{s,2}(\Omega) \cap \mathcal{C}^{0,\alpha}(\Omega)$ for some $\alpha > 0$.

Lemma 4.3. *Assume that* (A_1) *holds. Then T is well defined.*

Proof. First of all, by Hölder inequality and lemma 4.2, observe that for all $\varphi \in E$,

$$\int_\Omega \mathbb{D}_s^2(\varphi) \, \mathrm{d}x \leqslant c(r, \Omega) \left(\int_\Omega (\mathbb{D}_s^2(\varphi))^{r/2} \, \mathrm{d}x \right)^{2/r} \leqslant c\, C_9^2 \|\varphi\|^2_{W_0^{s+\varepsilon,r}(\Omega)} = c\, C_9^2 l. \tag{4.7}$$

Hence, for all $\varphi \in E$, it follows that

$$\|\mu(x)\mathbb{D}_s^2(\varphi) + \lambda f(x)\|_{L^1(\Omega)} \leqslant c\, C_9^2 \|\mu\|_\infty l + |\lambda| \, \|f\|_{L^1(\Omega)} = C < \infty. \tag{4.8}$$

Thanks to [**34**, theorem 23] and proposition 3.1, if the right hand side in (4.6) belongs to $L^1(\Omega)$, problem (4.6) has an unique weak solution and it belongs to $W_0^{s,1}(\Omega)$. Thus, the result follows from (4.8). □

Lemma 4.4. *Assume* (A_1) *and let* $0 < \lambda \leqslant \lambda^*$. *Then* $T(E) \subset E$.

https://doi.org/10.1017/prm.2019.60 Published online by Cambridge University Press

Proof. For an arbitrary $\varphi \in E$, we define $u = T(\varphi)$. Now, by proposition 3.1 and since $0 < \lambda \leqslant \lambda^*$, it follows that

$$\left(\iint_{D_\Omega} \frac{|u(x)-u(y)|^r}{|x-y|^{N+(s+\varepsilon)r}}\,\mathrm{d}x\,\mathrm{d}y\right)^{1/r} \leqslant C_1 \left\|\mu(x)\mathbb{D}_s^2(\varphi)+\lambda f(x)\right\|_{L^m(\Omega)} \leqslant C_1\|\mu\|_\infty \left\|\mathbb{D}_s^2(\varphi)\right\|_{L^m(\Omega)} + C_1\lambda^*\|f\|_{L^m(\Omega)}. \tag{4.9}$$

Also, by lemma 4.2, Hölder inequality and the definition of C_{10}, we obtain that

$$\|\mathbb{D}_s^2(\varphi)\|_{L^m(\Omega)} \leqslant |\Omega|^{(r-2m)/rm}\|(\mathbb{D}_s^2(\varphi))^{1/2}\|^2_{L^r(\Omega)} \leqslant C_9^2|\Omega|^{(r-2m)/rm}\|\varphi\|^2_{W_0^{s+\varepsilon,r}(\Omega)} = C_{10}\|\varphi\|^2_{W_0^{s+\varepsilon,r}(\Omega)}.$$

Thus, since $\varphi \in E$, we have that

$$\|\mathbb{D}_s^2(\varphi)\|_{L^m(\Omega)} \leqslant C_{10}l. \tag{4.10}$$

From (4.5), (4.9) and (4.10), it follows that

$$\left(\iint_{D_\Omega} \frac{|u(x)-u(y)|^r}{|x-y|^{N+(s+\varepsilon)r}}\,\mathrm{d}x\,\mathrm{d}y\right)^{1/r} \leqslant C_1(C_{10}\|\mu\|_\infty\, l + \lambda^*\|f\|_{L^m(\Omega)}) = l^{1/2}.$$

Hence, since by proposition 3.1 we also know that $u \in W_0^{s,1}(\Omega)$, we conclude that $u \in E$ and so, that $T(E) \subset E$. □

Lemma 4.5. *Assume that* (A_1) *holds. Then* T *is continuous.*

Proof. Let $\{\varphi_n\} \subset E$ be a sequence such that $\varphi_n \to \varphi$ in $W_0^{s,1}(\Omega)$ and define $u_n = T(\varphi_n)$, for all $n \in \mathbb{N}$, and $u = T(\varphi)$. To show that $u_n \to u$ in $W_0^{s,1}(\Omega)$, and so, that T is continuous, we prove that

$$g_n(x) := \mathbb{D}_s^2(\varphi_n) + \lambda f(x) \to g(x) := \mathbb{D}_s^2(\varphi) + \lambda f(x), \quad \text{in } L^1(\Omega). \tag{4.11}$$

Indeed, if (4.11) holds, the result follows from proposition 3.9.

First of all, using the notation $\psi_n = \varphi_n - \varphi$ and the reverse triangle inequality, we obtain that

$$\begin{aligned}
&\|\mathbb{D}_s^2(\varphi_n) - \mathbb{D}_s^2(\varphi)\|_{L^1(\Omega)} \\
&\quad = \int_\Omega \left|\int_{\mathbb{R}^N} \frac{|\varphi_n(x)-\varphi_n(y)|^2 - |\varphi(x)-\varphi(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}y\right|\,\mathrm{d}x \\
&\quad \leqslant \int_\Omega \left|\int_{\mathbb{R}^N} \frac{(|\varphi_n(x)-\varphi_n(y)| + |\varphi(x)-\varphi(y)|)\,|\psi_n(x)-\psi_n(y)|}{|x-y|^{N+2s}}\,\mathrm{d}y\right|\,\mathrm{d}x \\
&\quad = \int_\Omega \left(\int_{\mathbb{R}^N} \frac{|\varphi_n(x)-\varphi_n(y)| + |\varphi(x)-\varphi(y)|}{|x-y|^{N/2+s}} \cdot \frac{|\psi_n(x)-\psi_n(y)|}{|x-y|^{N/2+s}}\,\mathrm{d}y\right)\,\mathrm{d}x.
\end{aligned}$$

https://doi.org/10.1017/prm.2019.60 Published online by Cambridge University Press

Applying then Hölder inequality, we deduce that

$$\begin{aligned}
&\|\mathbb{D}_s^2(\varphi_n) - \mathbb{D}_s^2(\varphi)\|_{L^1(\Omega)} \\
&\quad \leqslant \int_\Omega \left[\left(\int_{\mathbb{R}^N} \frac{(|\varphi_n(x) - \varphi_n(y)| + |\varphi(x) - \varphi(y)|)^2}{|x-y|^{N+2s}} \, \mathrm{d}y \right)^{1/2} \right. \\
&\qquad \left. \times \left(\int_{\mathbb{R}^N} \frac{|\psi_n(x) - \psi_n(y)|^2}{|x-y|^{N+2s}} \, \mathrm{d}y \right)^{1/2} \right] \mathrm{d}x \\
&\quad \leqslant \left(\int_\Omega \left(\int_{\mathbb{R}^N} \frac{(|\varphi_n(x) - \varphi_n(y)| + |\varphi(x) - \varphi(y)|)^2}{|x-y|^{N+2s}} \, \mathrm{d}y \right) \mathrm{d}x \right)^{1/2} \\
&\qquad \times \left(\int_\Omega \left(\int_{\mathbb{R}^N} \frac{|\psi_n(x) - \psi_n(y)|^2}{|x-y|^{N+2s}} \, \mathrm{d}y \right) \mathrm{d}x \right)^{1/2} \\
&\quad = \left(\int_\Omega \left(\int_{\mathbb{R}^N} \frac{(|\varphi_n(x) - \varphi_n(y)| + |\varphi(x) - \varphi(y)|)^2}{|x-y|^{N+2s}} \, \mathrm{d}y \right) \mathrm{d}x \right)^{1/2} \|\mathbb{D}_s^2(\varphi_n - \varphi)\|_{L^1(\Omega)}^{1/2} \\
&\quad =: I_1 \cdot I_2 \, .
\end{aligned}$$

Taking into account the above inequality, if we show that I_1 is bounded and I_2 goes to zero, we deduce that $\|\mathbb{D}_s^2(\varphi_n) - \mathbb{D}_s^2(\varphi)\|_{L^1(\Omega)} \to 0$.

Claim 1: I_1 *is bounded.* Directly observe that

$$\begin{aligned}
I_1 &\leqslant 2 \left[\int_\Omega \left(\int_{\mathbb{R}^N} \frac{|\varphi_n(x) - \varphi_n(y)|^2}{|x-y|^{N+2s}} \, \mathrm{d}y \right) \mathrm{d}x + \int_\Omega \left(\int_{\mathbb{R}^N} \frac{|\varphi(x) - \varphi(y)|^2}{|x-y|^{N+2s}} \, \mathrm{d}y \right) \mathrm{d}x \right] \\
&= 2 \left[\|\mathbb{D}_s^2(\varphi_n)\|_{L^1(\Omega)} + \|\mathbb{D}_s^2(\varphi)\|_{L^1(\Omega)} \right] . \qquad (4.12)
\end{aligned}$$

Since $\varphi_n, \ \varphi \in E$ for all $n \in \mathbb{N}$, by (4.7), we have that

$$\left[\|\mathbb{D}_s^2(\varphi_n)\|_{L^1(\Omega)} + \|\mathbb{D}_s^2(\varphi)\|_{L^1(\Omega)} \right] \leqslant 2c\, C_9^2\, l < \infty,$$

and so, that I_1 is bounded.

Claim 2: I_2 *goes to zero.* Let $\theta \in (0,1)$ be small enough to ensure that $(2-\theta)/(1-\theta) < r$. By Hölder inequality, it follows that

$$\begin{aligned}
&\|\mathbb{D}_s^2(\varphi_n - \varphi)\|_{L^1(\Omega)} \\
&\quad = \int_\Omega \left(\int_{\mathbb{R}^N} \frac{|\psi_n(x) - \psi_n(y)|^2}{|x-y|^{N+2s}} \, \mathrm{d}y \right) \mathrm{d}x \\
&\quad = \int_\Omega \left(\int_{\mathbb{R}^N} \frac{|\psi_n(x) - \psi_n(y)|^\theta}{|x-y|^{(N+s)\theta}} \frac{|\psi_n(x) - \psi_n(y)|^{2-\theta}}{|x-y|^{N(1-\theta)+s(2-\theta)}} \, \mathrm{d}y \right) \mathrm{d}x \\
&\quad \leqslant \int_\Omega \left[\left(\int_{\mathbb{R}^N} \frac{|\psi_n(x) - \psi_n(y)|}{|x-y|^{N+s}} \, \mathrm{d}y \right)^\theta \left(\int_{\mathbb{R}^N} \frac{|\psi_n(x) - \psi_n(y)|^{(2-\theta)/(1-\theta)}}{|x-y|^{N+s((2-\theta)/(1-\theta))}} \, \mathrm{d}y \right)^{1-\theta} \right] \mathrm{d}x
\end{aligned}$$

https://doi.org/10.1017/prm.2019.60 Published online by Cambridge University Press

$$\leqslant \left(\int_\Omega \left(\int_{\mathbb{R}^N} \frac{|\psi_n(x)-\psi_n(y)|}{|x-y|^{N+s}}\,\mathrm{d}y\right)\,\mathrm{d}x\right)^{\theta}$$
$$\times \left(\int_\Omega \left(\int_{\mathbb{R}^N} \frac{|\psi_n(x)-\psi_n(y)|^{(2-\theta)/(1-\theta)}}{|x-y|^{N+s((2-\theta)/(1-\theta))}}\,\mathrm{d}y\right)\,\mathrm{d}x\right)^{1-\theta}. \tag{4.13}$$

Hence, since $\varphi_n \to \varphi$ in $W_0^{s,1}(\Omega)$ implies that

$$\int_\Omega \left(\int_{\mathbb{R}^N} \frac{|\psi_n(x)-\psi_n(y)|}{|x-y|^{N+s}}\,\mathrm{d}y\right)\,\mathrm{d}x \to 0, \tag{4.14}$$

if we prove that

$$\int_\Omega \left(\int_{\mathbb{R}^N} \frac{|\psi_n(x)-\psi_n(y)|^{(2-\theta)/(1-\theta)}}{|x-y|^{N+s((2-\theta)/(1-\theta))}}\,\mathrm{d}y\right)\,\mathrm{d}x \tag{4.15}$$

is bounded, we can conclude that I_2 goes to zero, as desired. Since we have chosen $\theta \in (0,1)$ small enough in order to ensure that $(2-\theta)/(1-\theta) < r$ and Ω is a bounded domain, it follows that

$$\int_\Omega \left(\int_{\mathbb{R}^N} \frac{|\psi_n(x)-\psi_n(y)|^{(2-\theta)/(1-\theta)}}{|x-y|^{N+s((2-\theta)/(1-\theta))}}\,\mathrm{d}y\right)\,\mathrm{d}x$$
$$\leqslant C(r,\Omega)\left(\int_\Omega \left(\int_{\mathbb{R}^N} \frac{|\psi_n(x)-\psi_n(y)|^{(2-\theta)/(1-\theta)}}{|x-y|^{N+s((2-\theta)/(1-\theta))}}\,\mathrm{d}y\right)^{r/((2-\theta)/(1-\theta))}\,\mathrm{d}x\right)^{((2-\theta)(1-\theta))/r}. \tag{4.16}$$

Applying then lemma 4.2 and the triangular inequality we have that

$$\begin{aligned}\int_\Omega \left(\int_{\mathbb{R}^N} \frac{|\psi_n(x)-\psi_n(y)|^{(2-\theta)/(1-\theta)}}{|x-y|^{N+s((2-\theta)/(1-\theta)}}\,\mathrm{d}y\right)\,\mathrm{d}x &\leqslant \overline{C}\|\psi_n\|_{W_0^{s+\varepsilon,r}(\Omega)}\\ &\leqslant \widetilde{C}\left[\|\varphi_n\|_{W_0^{s+\varepsilon,r}(\Omega)} + \|\varphi\|_{W_0^{s+\varepsilon,r}(\Omega)}\right]\\ &\leqslant 2\,\widetilde{C}\,l^{1/2} = \widehat{C},\end{aligned} \tag{4.17}$$

where $\overline{C}, \widetilde{C}$ and $\widehat{C}$ are positive constants independent of n. Thus, we conclude that (4.15) is indeed bounded.

From claims 1 and 2 we deduce that $\|\mathbb{D}_s^2(\varphi_n) - \mathbb{D}_s^2(\varphi)\|_{L^1(\Omega)} \to 0$. This implies that $g_n \to g$ in $L^1(\Omega)$, as desired, and the result follows. □

LEMMA 4.6. *Assume that* (A_1) *holds. Then* T *is compact.*

Proof. Let $\{\varphi_n\} \subset E$ be a bounded sequence in $W_0^{s,1}(\Omega)$ and define $u_n = T(\varphi_n)$ for all $n \in \mathbb{N}$. We have to show that $u_n \to u$ in $W_0^{s,1}(\Omega)$ for some $u \in W_0^{s,1}(\Omega)$.

Since $\{\varphi_n\} \subset E$ for all $n \in \mathbb{N}$, arguing as in lemma 4.3, we deduce that $\{\mathbb{D}_s^2(\varphi_n)\}$ is a bounded sequence in $L^1(\Omega)$. Hence, if we define

$$g_n(x) := \mathbb{D}_s^2(\varphi_n) + \lambda f(x), \quad \forall\, n \in \mathbb{N},$$

we have that $\{g_n\}$ is a bounded sequence in $L^1(\Omega)$. The result then follows from proposition 3.10. □

https://doi.org/10.1017/prm.2019.60 Published online by Cambridge University Press

***Proof of theorem* 1.1**. Since E is a closed convex set of $W_0^{s,1}(\Omega)$ and, by lemmas 4.3, 4.4, 4.5 and 4.6, we know that T is continuous, compact and satisfies $T(E) \subset E$, we can apply the Schauder fixed point Theorem to obtain $u \in E$ such that $T(u) = u$. Thus, we conclude that (P_λ) has a weak solution for all $0 < \lambda \leqslant \lambda^*$. Finally, since $u \in W_0^{s,1}(\Omega) \cap W_0^{s,r}(\Omega)$ for some $1 < 2 < r$, by lemma 2.2 applied with $s_1 = s_2 = s$, we deduce that $u \in W_0^{s,2}(\Omega)$. Moreover, since $r > N/s$, by [**22**, theorem 8.2], we know that every $\varphi \in E$ belongs to $\mathcal{C}^{0,\alpha}(\Omega)$ for some $\alpha > 0$. □

4.2. Proof of theorem 1.4

First observe that, as before, without loss of generality we can assume $m \in (N/2s, N/s)$, $\|\mu\|_\infty \neq 0$ and $\|f\|_{L^m(\Omega)} \neq 0$. Next, let us fix some notation. We fix

$$r = \frac{3mN}{N+ms} \tag{4.18}$$

and $\varepsilon := \varepsilon(r, m, s) > 0$ such that

$$1 < r < \frac{mN}{N - m(s-\varepsilon)} < \frac{mN}{N-ms}, \quad s + \varepsilon < 1 \quad \text{and} \quad s - \varepsilon > \frac{1}{2}.$$

Also, we introduce and fix the constants C_2, given by corollary 3.4 applied with $t = s + \varepsilon$ and $p = r$, $C_{11} := S_{N,r} C C_9^{2m}$, where $S_{N,r}$ is the optimal constant in the Sobolev inequality (see for instance [**37**, proposition 15.5] for a very beautiful proof), C is the smallest constant guaranteeing the continuous embedding $W_0^{s+\varepsilon,r}(\Omega) \subset W_0^{s,r}(\Omega)$ and C_9 is the constant given by lemma 4.2, and

$$\lambda^* := \frac{2}{3\|f\|_{L^m(\Omega),}} \left(\frac{1}{3C_2^3 C_{11} \|\mu\|_\infty} \right)^{1/2}.$$

Then, by lemma 4.1 we know that there exists and unique $l \in (0, \infty)$ such that

$$C_2(C_{11}\|\mu\|_\infty l + \lambda^* \|f\|_{L^m(\Omega)}) = l^{1/3}. \tag{4.19}$$

Having fixed all these constants, we define

$$E_1 := \left\{ v \in W_0^{s,1}(\Omega) : \iint_{D_\Omega} \frac{|u(x) - u(y)|^r}{|x-y|^{N+(s+\varepsilon)r}} \, \mathrm{d}x \, \mathrm{d}y \leqslant l^{r/3} \right\},$$

which is a closed convex set of $W_0^{s,1}(\Omega)$, and $T_1 : E_1 \to W_0^{s,1}(\Omega)$ by $T_1(\varphi) = u$, with u a weak solution to

$$\begin{cases} (-\Delta)^s u = \mu(x)\, \varphi\, \mathbb{D}_s^2(\varphi) + \lambda f(x), & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{4.20}$$

Observe that $(\widetilde{P}_\lambda)$ is equivalent to the fixed point problem $u = T_1(u)$. Hence, we shall prove that T_1 has a fixed point belonging to $W_0^{s,2}(\Omega) \cap \mathcal{C}^{0,\alpha}(\Omega)$ for some $\alpha > 0$.

Lemma 4.7. *For all $\varphi \in W_0^{s+\varepsilon,r}(\Omega)$, it follows that*

$$\|\varphi \mathbb{D}_s^2(\varphi)\|_{L^m(\Omega)} \leqslant C_{11} \|\varphi\|^3_{W_0^{s+\varepsilon,r}(\Omega)}. \tag{4.21}$$

https://doi.org/10.1017/prm.2019.60 Published online by Cambridge University Press

Proof. First observe that, with the above notation, we have that

$$2 < \frac{2mr_s^*}{r_s^* - m} = r.$$

Hence, by Hölder and Sobolev inequalities and using that $W_0^{s+\varepsilon,r}(\Omega) \subset W_0^{s,r}(\Omega)$ with continuous inclusion, we obtain that

$$\|\varphi \mathbb{D}_s^2(\varphi)\|_{L^m(\Omega)}^m \leqslant S_{N,r} C \, \|\varphi\|_{W_0^{s+\varepsilon,r}(\Omega)}^m \left\| \left(\int_{\mathbb{R}^N} \frac{|\varphi(x) - \varphi(y)|^2}{|x-y|^{N+2s}} \, \mathrm{d}y \right)^{1/2} \right\|_{L^r(\Omega)}^{2m}.$$

Since $r > 2$, the result follows from lemma 4.2. □

Corollary 4.8. *Assume that* (A_1) *holds. Then* T_1 *is well defined.*

Proof. Since Ω is a bounded domain and $m > N/2s > 1$ the result follows from lemma 4.7 arguing as in the proof of lemma 4.3. □

Lemma 4.9. *Assume* (A_1) *and let* $0 < \lambda \leqslant \lambda^*$. *Then* $T_1(E_1) \subset E_1$.

Proof. Let us consider an arbitrary $\varphi \in E$ and define $u = T_1(\varphi)$. By corollary 3.4, since that $0 < \lambda \leqslant \lambda^*$, we have that

$$\left(\iint_{D_\Omega} \frac{|u(x) - u(y)|^r}{|x-y|^{N+(s+\varepsilon)r}} \, \mathrm{d}x \, \mathrm{d}y \right)^{1/r} \leqslant C_2 \|\mu\|_\infty \left\| \varphi \, \mathbb{D}_s^2(\varphi) \right\|_{L^m(\Omega)} + C_2 \lambda^* \|f\|_{L^m(\Omega)}. \tag{4.22}$$

Hence, since $\varphi \in E$, by lemma 4.7 and (4.19), it follows that

$$\left(\iint_{D_\Omega} \frac{|u(x) - u(y)|^r}{|x-y|^{N+(s+\varepsilon)r}} \, \mathrm{d}x \, \mathrm{d}y \right)^{1/r} \leqslant C_2 (C_{11} \|\mu\|_\infty \, l + \lambda^* \|f\|_{L^m(\Omega)}) = l^{1/3}.$$

Thus, as by proposition 3.1 we also know that $u \in W_0^{s,1}(\Omega)$, we conclude that $u \in E_1$ and so, that $T_1(E_1) \subset E_1$. □

Lemma 4.10. *Assume* (A_1). *Then* T_1 *is continuous.*

Proof. Let $\{\varphi_n\} \subset E$ be a sequence such that $\varphi_n \to \varphi$ in $W_0^{s,1}(\Omega)$ and define $u_n = T_1(\varphi_n)$, for all $n \in \mathbb{N}$, and $u = T_1(\varphi)$. Arguing as in the proof of lemma 4.5, we just have to prove that

$$\varphi_n \mathbb{D}_s^2(\varphi_n) \to \varphi \mathbb{D}_s^2(\varphi), \quad \text{in } L^1(\Omega). \tag{4.23}$$

First observe that, since $r > N/s > N/(s+\varepsilon)$, for all $\varphi \in E_1$, it follows that

$$\|\varphi\|_{L^\infty(\Omega)} \leqslant C \iint_{D_\Omega} \frac{|\varphi(x) - \varphi(y)|^r}{|x-y|^{N+(s+\varepsilon)r}} \, \mathrm{d}x \, \mathrm{d}y \leqslant l^{r/3}. \tag{4.24}$$

Hence, since $\varphi_n \to \varphi$ in $W_0^{s,1}(\Omega)$, by Vitali's Convergence Theorem we deduce that $\varphi_n \to \varphi$ in $L^\alpha(\Omega)$ for all $1 \leqslant \alpha < \infty$.

https://doi.org/10.1017/prm.2019.60 Published online by Cambridge University Press

Next, observe that

$$
\begin{aligned}
&\|\varphi_n \mathbb{D}_s^2(\varphi_n) - \varphi \mathbb{D}_s^2(\varphi)\|_{L^1(\Omega)} \\
&\quad = \|\varphi_n(\mathbb{D}_s^2(\varphi_n) - \mathbb{D}_s^2(\varphi)) + \mathbb{D}_s^2(\varphi)(\varphi_n - \varphi)\|_{L^1(\Omega)} \\
&\quad \leqslant \|\varphi_n(\mathbb{D}_s^2(\varphi_n) - \mathbb{D}_s^2(\varphi))\|_{L^1(\Omega)} + \|\mathbb{D}_s^2(\varphi)(\varphi_n - \varphi)\|_{L^1(\Omega)} \\
&\quad \leqslant \|\varphi_n\|_\infty \|\mathbb{D}_s^2(\varphi_n) - \mathbb{D}_s^2(\varphi)\|_{L^1(\Omega)} + \|\mathbb{D}_s^2(\varphi)\|_{L^m(\Omega)} \|\varphi_n - \varphi\|_{L^{m'}(\Omega)} \\
&\quad =: I_1 + I_2
\end{aligned}
\tag{4.25}
$$

Then, arguing exactly as in lemma 4.5 and using that $\|\varphi_n\|_\infty \leqslant C$ (independent of n) we deduce that $I_1 \to 0$. On the other hand, we know that $\|\mathbb{D}_s^2(\varphi)\|_{L^m(\Omega)} < \infty$. Hence, since $\varphi_n \to \varphi$ in $L^\alpha(\Omega)$ for all $1 \leqslant \alpha < \infty$, we also obtain that $I_2 \to 0$. We then conclude that (4.23) holds, as desired. □

***Proof of theorem* 1.4.** Observe that the compactness of T_1 follows arguing exactly as in lemma 4.6. Hence, since E_1 is a closed convex set of $W_0^{s,1}(\Omega)$ and, by lemmas 4.9, 4.8 and 4.10 we know that T_1 is well defined, continuous and satisfies $T_1(E_1) \subset E_1$, we can apply the Schauder fixed point Theorem to obtain $u \in E_1$ such that $T_1(u) = u$. Thus, we conclude that $(\widetilde{P}_\lambda)$ has a weak solution for all $0 < \lambda \leqslant \lambda^*$. Finally, since $u \in W_0^{s,1}(\Omega) \cap W_0^{s,r}(\Omega)$ for some $1 < 2 < r$, by lemma 2.2 we deduce that $u \in W_0^{s,2}(\Omega)$. Moreover, since $r > N/s$, by [**22**, theorem 8.2], we know that every $\varphi \in E_1$ belongs to $\mathcal{C}^{0,\alpha}(\Omega)$ for some $\alpha > 0$. □

5. Proofs of theorems 1.2 and 1.3

In this section, we prove theorems 1.2 and 1.3. The aim of these theorems is to justify the hypotheses considered in theorem 1.1. First we prove that (P_λ) has no solutions for λ large and so, that the smallness condition is somehow necessary to have existence of solution.

***Proof of theorem* 1.2.** Assume that (P_λ) has a solution $u \in W_0^{s,2}(\Omega)$ and let $\phi \in \mathcal{C}_0^\infty(\Omega)$ be an arbitrary function such that

$$\int_\Omega f(x)\phi^2(x)\,\mathrm{d}x > 0,$$

Considering ϕ^2 as test function in (P_λ) we observe that

$$\int_\Omega (-\Delta)^s u\,\phi^2(x)\,\mathrm{d}x = \int_\Omega \mu(x) \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \phi^2(x)\,\mathrm{d}y\,\mathrm{d}x + \lambda \int_\Omega f(x)\phi^2(x)\mathrm{d}x. \tag{5.1}$$

Now, on one hand, since $\mu(x) \geqslant \mu_1 > 0$ and $\mathbb{D}_s^2$ is symmetric in x, y, it follows that

$$
\begin{aligned}
&\int_\Omega \mu(x) \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \phi^2(x)\,\mathrm{d}y\,\mathrm{d}x \\
&\quad = \iint_{D_\Omega} \mu(x) \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \phi^2(x)\,\mathrm{d}y\,\mathrm{d}x
\end{aligned}
$$

https://doi.org/10.1017/prm.2019.60 Published online by Cambridge University Press

$$\geqslant \mu_1 \iint_{D_\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\phi^2(x)\,\mathrm{d}y\,\mathrm{d}x$$
$$= \frac{\mu_1}{2} \iint_{D_\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\phi^2(x)\,\mathrm{d}y\,\mathrm{d}x + \frac{\mu_1}{2} \iint_{D_\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\phi^2(y)\,\mathrm{d}y\,\mathrm{d}x$$
$$\geqslant \frac{\mu_1}{4} \iint_{D_\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}(\phi(x)+\phi(y))^2\,\mathrm{d}y\,\mathrm{d}x. \tag{5.2}$$

On the other hand, by Young's inequality, it follows that

$$\begin{aligned}
&\int_\Omega (-\Delta)^s u\phi^2(x)\mathrm{d}x \\
&= \iint_{D_\Omega} \frac{(u(x)-u(y))(\phi^2(x)-\phi^2(y))}{|x-y|^{N+2s}}\,\mathrm{d}y\,\mathrm{d}x \\
&= \iint_{D_\Omega} \frac{(u(x)-u(y))(\phi(x)-\phi(y))(\phi(x)+\phi(y))}{|x-y|^{N+2s}}\,\mathrm{d}y\mathrm{d}x \\
&\leqslant \iint_{D_\Omega} \frac{|u(x)-u(y)||\phi(x)+\phi(y)|}{|x-y|^{\frac{N}{2}+s}} \cdot \frac{|\phi(x)-\phi(y)|}{|x-y|^{\frac{N}{2}+s}}\,\mathrm{d}y\mathrm{d}x \\
&\leqslant \frac{\mu_1}{4} \iint_{D_\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}(\phi(x)+\phi(y))^2\,\mathrm{d}y\,\mathrm{d}x + \frac{1}{\mu_1}\iint_{D_\Omega} \frac{|\phi(x)-\phi(y)|^2}{|x-y|^{N+2s'}}\,\mathrm{d}y\,\mathrm{d}x
\end{aligned} \tag{5.3}$$

Hence, substituting (5.2) and (5.3) into (5.1), we deduce that, if (P_λ) has a solution, then

$$\frac{1}{\mu_1}\iint_{D_\Omega} \frac{|\phi(x)-\phi(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}y\,\mathrm{d}x \geqslant \lambda \int_\Omega f(x)\phi^2(x)\,\mathrm{d}x, \tag{5.4}$$

which gives a contradiction for λ large enough. □

Now, we prove theorem 1.3. This theorem shows that the regularity considered on f is almost optimal. Just the limit case $f \in L^{N/2s}(\Omega)$ remains open. In order to prove this result, we make use of the following proposition which is a consequence of the fractional Hardy inequality [**27**, theorem 1.1].

PROPOSITION 5.1. *Let $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$, be a bounded domain with boundary $\partial\Omega$ of class $\mathcal{C}^2$ such that $0 \in \Omega$, $0 < s < 1$ and $p > 1$. Then:*

(1) [**2**, lemma 3.4] *If we set*

$$\Lambda(\Omega) := \inf\left\{ \frac{\displaystyle\iint_{D_\Omega} \frac{|\phi(x)-\phi(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y}{\displaystyle\int_\Omega \frac{|\phi(x)|^p}{|x|^{ps}}\mathrm{d}x} : \phi \in \mathcal{C}_0^\infty(\Omega)\setminus\{0\} \right\},$$

it follows that $\Lambda(\Omega) = \Lambda_{N,s,p}$ where $\Lambda_{N,s,p} > 0$ is the optimal constant in the fractional Hardy inequality given in [**27**, theorem 1.1].

https://doi.org/10.1017/prm.2019.60 Published online by Cambridge University Press

(2) *The weight* $|x|^{-ps}$ *is optimal in the sense that, for all* $\varepsilon > 0$, *if follows that*

$$\inf\left\{\frac{\displaystyle\iint_{D_\Omega}\frac{|\phi(x)-\phi(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y}{\displaystyle\int_\Omega\frac{|\phi(x)|^p}{|x|^{ps+\varepsilon}}\mathrm{d}x} : \phi\in\mathcal{C}_0^\infty(\Omega)\setminus\{0\}\right\}=0.$$

Proof. Since the proof of (1) can be found in [**2**, lemma 3.4], we just provide the proof of (2). Let $\varepsilon > 0$ be fixed but arbitrarily small. We assume by contradiction that there exists a smooth bounded domain $\Omega \subset \mathbb{R}^N$ such that $0 \in \Omega$ and

$$\Lambda_\varepsilon(\Omega) := \inf\left\{\frac{\displaystyle\iint_{D_\Omega}\frac{|\phi(x)-\phi(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y}{\displaystyle\int_\Omega\frac{|\phi(x)|^p}{|x|^{ps+\varepsilon}}\mathrm{d}x} : \phi\in\mathcal{C}_0^\infty(\Omega)\setminus\{0\}\right\}>0. \tag{5.5}$$

Let us then observe that for any $B_r(0) \subset \Omega$, it follows that

$$0 < \Lambda_\varepsilon(\Omega) \leqslant \Lambda_\varepsilon(B_r(0)). \tag{5.6}$$

Moreover, observe that for $\phi \in \mathcal{C}_0^\infty(B_r(0))$ we have that

$$\int_{B_r(0)}\frac{|\phi(x)|^p}{|x|^{ps+\varepsilon}}\mathrm{d}x \geqslant \frac{1}{r^\varepsilon}\int_{B_r(0)}\frac{|\phi(x)|^p}{|x|^{ps}}\,\mathrm{d}x. \tag{5.7}$$

Hence, gathering (5.6)–(5.7), it follows that, for all $\phi \in \mathcal{C}_0^\infty(B_r(0))$,

$$\begin{aligned} 0 < \Lambda_\varepsilon(\Omega) \leqslant \Lambda_\varepsilon(B_r(0)) &\leqslant \frac{\displaystyle\iint_{D_{B_r(0)}}\frac{|\phi(x)-\phi(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y}{\displaystyle\int_{B_r(0)}\frac{|\phi(x)|^p}{|x|^{ps+\varepsilon}}\mathrm{d}x} \\ &\leqslant r^\varepsilon\,\frac{\displaystyle\iint_{D_{B_r(0)}}\frac{|\phi(x)-\phi(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y}{\displaystyle\int_{B_r(0)}\frac{|\phi(x)|^p}{|x|^{ps}}\mathrm{d}x}. \end{aligned}$$

Thus, by the definition of $\Lambda(B_r(0))$ and (1), we deduce that $0 < \Lambda_\varepsilon(\Omega)/r^\varepsilon \leqslant \Lambda_{B_r(0)} = \Lambda_{N,s,p}$. Since (by assumption) $\Lambda_\varepsilon(\Omega) > 0$ and $\Lambda_{N,s,p}$ is independent of Ω, letting $r \to 0$, we obtain a contradiction and the result follows. ☐

Proof of theorem 1.3. Without loss of generality we choose a bounded domain Ω with boundary $\partial\Omega$ of class $\mathcal{C}^2$ such that $0 \in \Omega$. Consider then

$$f(x) = \frac{1}{|x|^{(N-\varepsilon)/m}}, \tag{5.8}$$

for some $\varepsilon \in (0, 1)$ to be chosen later and observe that, since Ω is bounded, $f \in L^m(\Omega)$. We assume by contradiction that, for all $\varepsilon > 0$, there exists $\lambda_\varepsilon > 0$ such

https://doi.org/10.1017/prm.2019.60 Published online by Cambridge University Press

that (P_λ) has a solution $u \in W_0^{s,2}(\Omega)$. Arguing as in the proof of theorem 1.2, we conclude that, for all $\phi \in \mathcal{C}_0^\infty(\Omega) \setminus \{0\}$,

$$\frac{1}{\mu_1} \iint_{D_\Omega} \frac{|\phi(x) - \phi(y)|^2}{|x-y|^{N+2s}} \, dy \, dx \geqslant \lambda_\varepsilon \int_\Omega f(x)\phi^2(x) dx = \lambda_\varepsilon \int_\Omega \frac{\phi^2(x)}{|x|^{(N-\varepsilon)/m}} \, dx. \tag{5.9}$$

Thus, we deduce that

$$0 < \mu_1 \lambda_\varepsilon \inf \left\{ \frac{\displaystyle\iint_{D_\Omega} \frac{|\phi(x) - \phi(y)|^2}{|x-y|^{N+2s}} \, dy \, dx}{\displaystyle\int_\Omega \frac{\phi^2(x)}{|x|^{(N-\varepsilon)/m}} dx} : \phi \in \mathcal{C}_0^\infty(\Omega) \setminus \{0\} \right\}. \tag{5.10}$$

Nevertheless, since $m < N/2s$, we can choose $\varepsilon > 0$ small enough to ensure that $(N-\varepsilon)/m > 2s$. In that case, by proposition 5.1, (2), we have that

$$\inf \left\{ \frac{\displaystyle\iint_{D_\Omega} \frac{|\phi(x) - \phi(y)|^2}{|x-y|^{N+2s}} \, dy \, dx}{\displaystyle\int_\Omega \frac{|\phi(x)|^2}{|x|^{(N-\varepsilon)/m}} dx} : \phi \in \mathcal{C}_0^\infty(\Omega) \setminus \{0\} \right\} = 0,$$

which contradicts (5.10). Hence, the result follows. □

6. Proofs of theorems 1.5 and 1.6

This section is devoted to the proofs of theorems 1.5 and 1.6. First, having at hand proposition 3.5, we prove theorem 1.5 using again a fixed point argument. The proof is similar to the ones performed in § 4. Hence, we skip some details.

Since Ω is bounded, without loss of generality, we assume that $1 \leqslant m < N/s$. Also, if $\lambda f \equiv 0$, it follows that $u \equiv 0$ is a solution to (Q_λ) and, if $\mu \equiv 0$, (Q_λ) reduces to (3.1). Hence, we may also assume that $\|\mu\|_\infty \neq 0$ and $\|f\|_{L^m(\Omega)} \neq 0$.

Next, we fix some notation that will be used throughout the section. First, we fix $r = r(m, s, q) > 0$ such that

$$1 < qm < r < \frac{mN}{N - ms},$$

C_3 the constant given by proposition 3.5 with $p = r$ and

$$\lambda^* = \frac{q-1}{q\|f\|_{L^m(\Omega)}} \left(\frac{1}{qC_3^q |\Omega|^{(r-qm)/r} \|\mu\|_\infty} \right)^{1/(q-1)}.$$

Then, by the definition of λ^* and lemma 4.1, we know that there exists an unique $l \in (0, \infty)$ such that

$$C_3(\|\mu\|_{L^\infty(\Omega)} |\Omega|^{(r-qm)/mr} l + \lambda^* \|f\|_{L^m(\Omega)}) = l^{1/q}. \tag{6.1}$$

With the above constants fixed, we introduce

$$E_2 := \left\{ v \in W_0^{s,1}(\Omega) : \|(-\Delta)^{s/2} v\|_{L^r(\Omega)} \leqslant l^{1/q} \right\},$$

https://doi.org/10.1017/prm.2019.60 Published online by Cambridge University Press

and observe that E_2 is a closed convex set of $W_0^{s,1}(\Omega)$. Then, we define $T_2 : E_2 \to W_0^{s,1}(\Omega)$ by $T_2(\varphi) = u$ with u the unique weak solution to

$$\begin{cases} (-\Delta)^s u = \mu(x)|(-\Delta)^{s/2}\varphi|^q + \lambda f(x), & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{6.2}$$

and observe that (Q_λ) is equivalent to the fixed point problem $u = T_2(u)$. Hence, we shall show that T_2 has a fixed point.

Lemma 6.1. *Assume* (B_1) *and let* $0 < \lambda \leqslant \lambda^*$. *Then* T_2 *is well defined,* $T_2(E_2) \subset E_2$ *and* T_2 *is compact.*

Proof. The proof of this lemma follows as in lemmas 4.3, 4.4 and 4.6 using proposition 3.5 instead of proposition 3.1. □

Remark 6.1. The only point in the proof of the previous lemma where we use $0 < \lambda \leqslant \lambda^*$ is to show that $T_2(E_2) \subset E_2$. The rest holds for every $\lambda \in \mathbb{R}$.

Lemma 6.2. *Assume that* (B_1) *holds. Then* T_2 *is continuous.*

Proof. Let $\{\varphi_n\} \subset E_2$ be a sequence such that $\varphi_n \to \varphi$ in $W_0^{s,1}(\Omega)$ and define $u_n = T_2(\varphi_n)$, for all $n \in \mathbb{N}$, and $u = T_2(\varphi)$. We shall show that $u_n \to u$ in $W_0^{s,1}(\Omega)$. Observe that $w_n = u_n - u$ satisfies

$$\begin{cases} (-\Delta)^s w_n = \mu(x)\left(|(-\Delta)^{s/2}\varphi_n|^q - |(-\Delta)^{s/2}\varphi|^q\right), & \text{in } \Omega, \\ w_n = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{6.3}$$

Hence, if we show that

$$\mu(x)\left(|(-\Delta)^{s/2}\varphi_n|^q - |(-\Delta)^{s/2}\varphi|^q\right) \to 0, \quad \text{in } L^1(\Omega), \tag{6.4}$$

the result follows from proposition 3.9. Directly, since $\varphi_n, \varphi \in E_2$ and $\mu \in L^\infty(\Omega)$, applying the Mean Value Theorem and Hölder inequality, we deduce that

$$\left\|\mu(x)\left(|(-\Delta)^{s/2}\varphi_n|^q - |(-\Delta)^{s/2}\varphi|^q\right)\right\|_{L^1(\Omega)} \leqslant C\left(\int_\Omega |(-\Delta)^{s/2}(\varphi_n - \varphi)|^q \,\mathrm{d}x\right)^{1/q}, \tag{6.5}$$

where C is a positive constant depending only on $\|\mu\|_{L^\infty(\Omega)}, l, q$ and Ω. By (6.5), if we show that

$$\int_\Omega |(-\Delta)^{s/2}(\varphi_n - \varphi)|^q \,\mathrm{d}x \to 0, \tag{6.6}$$

the continuity of the operator follows from proposition 3.9. Since $\varphi_n \to \varphi$ in $W_0^{s,1}(\Omega)$, it follows that $\varphi_n - \varphi \to 0$ almost everywhere in Ω. Furthermore, observe that, for all measurable subset $\omega \subset \Omega$, we have that

$$\int_\omega |(-\Delta)^{s/2}(\varphi_n - \varphi)|^q \,\mathrm{d}x \leqslant 2l|\omega|^{(r-q)/q}.$$

Hence, by Vitali's convergence Theorem, (6.6) holds and the result follows. □

https://doi.org/10.1017/prm.2019.60 Published online by Cambridge University Press

***Proof of theorem* 1.5.** Since E_2 is a closed convex set of $W_0^{s,1}(\Omega)$ and, by lemmas 6.1 and 6.2, we know that T_2 is continuous, compact and satisfies $T_2(E_2) \subset E_2$, we can apply the Schauder fixed point Theorem to obtain $u \in E_2$ such that $T_2(u) = u$. Thus, we conclude that (Q_λ) has a weak solution for all $0 < \lambda \leqslant \lambda^*$. □

***Proof of theorem* 1.6.** Having at hand corollary 3.6, the result follows arguing as in theorem 1.5. □

7. Further results and open problems

We end the paper pointing out some possible extensions of our results and formulating some open problems.

7.1. Further results

(1) In the spirit of the existence results of § 4, we can deal with more general nonlocal 'gradient terms'. Actually, we can consider a problem of the form

$$\begin{cases} (-\Delta)^s u = \mu(x)(\mathbb{B}_s^q(u))^\alpha + \lambda f(x), & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $1 < \alpha \leqslant q$, f belongs to a suitable Lebesgue space, $\mu \in L^\infty(\Omega)$ and $\mathbb{B}_s^q$ is given by

$$\mathbb{B}_s^q(u) = \left(\frac{a_{N,s}}{q} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^q}{|x-y|^{N+sq}} \, \mathrm{d}y \right)^{1/q}.$$

The existence of a solution for λf small enough can be obtained.

(2) On the line of § 6, we can consider a problem of the form

$$\begin{cases} (-\Delta)^s u = \mu(x)|(-\Delta)^{t/2} u|^q + \lambda f(x), & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

under the assumptions

$$\begin{cases} \Omega \subset \mathbb{R}^N,\ N \geqslant 2,\ \text{is a bounded domain with } \partial\Omega \text{ of class } \mathcal{C}^2, \\ f \in L^m(\Omega) \text{ for some } m \geqslant 1 \text{ and } \mu \in L^\infty(\Omega), \\ s \in (1/2, 1),\ t \in (0, s], \quad \text{and } 1 < q < N/(N - m(2s - t)). \end{cases}$$

A similar result to theorem 1.5 can be obtained. Concerning the multiplicity of solutions for this problem, let us consider $\Omega = B_1(0)$ and define

$$u(x) = A\left(\frac{1}{|x|^\theta} - 1 \right),$$

for some $\theta < N - 2s$. By direct computations, see for instance [26], we obtain that

$$(-\Delta)^s u = \frac{C}{|x|^{\theta+2s}} \quad \text{and } (-\Delta)^{t/2} u = \frac{\widehat{C}}{|x|^{\theta+t}},$$

https://doi.org/10.1017/prm.2019.60 Published online by Cambridge University Press

for some positive constants $C = C(N,s,\theta) > 0$ and $\widehat{C} = C(N,s,t,\theta) > 0$. Hence, choosing a suitable constant A, we have that u satisfies to

$$(-\Delta)^s u = |(-\Delta)^{t/2} u|^q \quad \text{in}\, \Omega,$$

with $q = (\theta + 2s)/(\theta + t)$. On the other hand, it is clear that $v \equiv 0$ is a solution to this problem. Hence, for this particular case, we may expect have at least two solutions. We cannot then expect uniqueness of weak solution for $q > N/(N - (2s - t))$.

7.2. Open problems

(1) The Calderón–Zygmund type regularity results proved in § 3 deeply rely on [**4**, lemma 2.15]. The restriction $s \in (1/2, 1)$ comes from this result. It is an open question if the regularity results of § 3 hold for $s \in (0, 1/2]$. Let us also stress that, if the corresponding regularity results with $s \in (0, 1/2]$ were available, our approaches to prove theorems 1.1, 1.4, 1.5 and 1.6 would directly provide the corresponding results.

(2) In the last few years there has been a renewed interest in classical problems of the form

$$-\Delta u = c(x)u + \mu(x)|\nabla u|^2 + h(x), \quad u \in H_0^1(\Omega) \cap L^\infty(\Omega).$$

Following [**29**, **40**], several works have appeared proving existence and multiplicity results. Does this kind of results hold in the nonlocal case? To be more precise, let us introduce the Dirichlet problem

$$\begin{cases} (-\Delta)^s u = c(x)u + \mu(x)\mathbb{D}_s^2(u) + \lambda f(x), & \text{in}\, \Omega, \\ u = 0, & \text{in}\, \mathbb{R}^N \setminus \Omega, \end{cases}$$

under the assumption (A_1) and $c \in L^\infty(\Omega)$. It seems interesting to address the following questions:

(a) Does the uniqueness of (smooth) solutions holds for $c(x) \leqslant 0$?

(b) Under the assumption $c(x) \leqslant \alpha_0 < 0$ a.e. in Ω. It is possible to remove the smallness condition imposed on λ?

(c) It is possible to prove the existence of more than one solution for $c(x) \gneqq 0$, $\mu(x) \geqslant \mu_1 > 0$ and $\lambda > 0$ small enough?

(3) Let us denote by $u_s \in W_0^{s,2}(\Omega) \cap \mathcal{C}^\alpha(\Omega)$ a solution to (P_λ). An interesting (in our opinion) open question is to analyse the behaviour of u_s when $s \to 1^-$. By the remark 4.1 it is clear that our solution u_s does not directly converge to a solution to (1.4). Nevertheless, we hope that an argument in the spirit of the seminal paper [**11**] would provide a suitable sequence of modified solutions that converge to a solution to (1.4). To that end, one needs to obtain uniform estimates on the constants involved in the proof of theorem 1.1.

(4) Similar questions to (2) and (3) can be formulated concerning the problems (Q_λ) and ($\widetilde{Q}_\lambda$).

Acknowledgements

Part of this work was done while the fist author was visiting the mathematics department of the University Bourgogne Franche-Comté. He would like to thank the LMB for the warm hospitality and financial support. The first author is partially supported by Project MTM-2016-80474-P, MINECO, Spain and by "Direction Générale de la Recherche Scientifique et du Développement Technologique" of Argelia. The second author thanks Prof. Tommaso Leonori for stimulating discussion concerning the subject of the present work. The authors would also like to thank anonymous referee for several comments and questions that helped to improve the paper.

References

1 B. Abdellaoui, A. Attar and R. Bentifour. On the Fractional p-Laplacian equations with weight and general datum. *Adv. Nonlinear Anal.* **8** (2019), 144–174.

2 B. Abdellaoui and R. Bentifour. Caffarelli–Kohn–Nirenberg type inequalities of fractional order with applications. *J. Funct. Anal.* **272** (2017), 3998–4029.

3 B. Abdellaoui, A. Dall'Aglio and I. Peral. Some remarks on elliptic problems with critical growth in the gradient. *J. Differ. Equ.* **222** (2006), 21–62.

4 B. Abdellaoui and I. Peral. Towards a deterministic KPZ equation with fractional diffusion: The stationary problem. *Nonlinearity* **31** (2018), 1260.

5 R. A. Adams. *Sobolev spaces* (New York: Academic Press, 1975).

6 D. Applebaum, *Lévy processes and stochastic calculus*. Cambridge Studies in Advanced Mathematics, vol. 93 (Cambridge: Cambridge University Press, 2004).

7 D. Arcoya, C. De Coster, L. Jeanjean and K. Tanaka. Continuum of solutions for an elliptic problem with critical growth in the gradient. *J. Funct. Anal.* **268** (2015), 2298–2335.

8 B. Barrios, A. Figalli and X. Ros-Oton. Free boundary regularity in the parabolic fractional obstacle problem. *Commun. Pure Appl. Math.* **71** (2018), 2129–2159.

9 B. Barrios, A. Figalli and X. Ros-Oton. Global regularity for the free boundary in the obstacle problem for the fractional Laplacian. *Am. J. Math.* **140** (2018), 415–447.

10 L. Boccardo, F. Murat and J.-P. Puel, Existence de solutions faibles pour des équations elliptiques quasi-linéaires à croissance quadratique. In *Nonlinear partial differential equations and their applications. Collège de France Seminar, Vol. IV (Paris, 1981/1982).* Res. Notes in Math., vol. 84, pp. 19–73 (Boston, MA, London: Pitman, 1983).

11 J. Bourgain, H. Brezis and P. Mironescu, Another look at Sobolev spaces. In *Optimal control and partial differential equations*. pp. 439–455. (Amsterdam: IOS, 2001).

12 H. Brezis and P. Mironescu. Gagliardo–Nirenberg inequalities and non-inequalities: The full story. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **35** (2018), 1355–1376.

13 L. Caffarelli and G. Dávila. Interior regularity for fractional systems. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **36** (2018), 165–180.

14 L. Caffarelli and A. Figalli. Regularity of solutions to the parabolic fractional obstacle problem. *J. Reine Angew. Math.* **680** (2013), 191–233.

15 L. Caffarelli and A. Vasseur. Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation. *Ann. Math.* **171** (2010), 1903–1930.

16 H. Chen and L. Véron. Semilinear fractional elliptic equations involving measures. *J. Differ. Equ.* **257** (2014), 1457–1486.

17 H. Chen and L. Véron. Semilinear fractional elliptic equations with gradient nonlinearity involving measures. *J. Funct. Anal.* **266** (2014), 5467–5492.

18 R. Cont and P. Tankov, *Financial modelling with jump processes*. Chapman & Hall/CRC Financial Mathematics Series (Boca Raton, FL: Chapman & Hall/CRC, 2004).

19 F. Da Lio and T. Rivière. Sub-criticality of non-local Schrödinger systems with antisymmetric potentials and applications to half-harmonic maps. *Adv. Math.* **227** (2011), 1300–1348.

20 F. Da Lio and A. Schikorra. n/p-Harmonic maps: Regularity for the sphere case. *Adv. Calc. Var.* **7** (2014), 1–26.

21 C. De Coster and A. J. Fernández. Existence and multiplicity for elliptic p-Laplacian problems with critical growth in the gradient. *Calc. Var. Partial Differ. Equ.* **57** (2018), 89.

22 E. Di Nezza, G. Palatucci and E. Valdinoci. Hitchhiker's guide to the fractional Sobolev spaces. *Bull. Sci. Math.* **136** (2012), 521–573.

23 S. Dipierro, A. Figalli and E. Valdinoci. Strongly nonlocal dislocation dynamics in crystals. *Commun. Partial Differ. Equ.* **39** (2014), 2351–2387.

24 S. Dipierro, G. Palatucci and E. Valdinoci. Dislocation dynamics in crystals: A macroscopic theory in a fractional Laplace setting. *Commun. Math. Phys.* **333** (2015), 1061–1105.

25 V. Ferone and F. Murat. Nonlinear problems having natural growth in the gradient: An existence result when the source terms are small. *Nonlinear Anal. Ser. A: Theory Methods)* **42** (2000), 1309–1326.

26 F. Ferrari and I. E. Verbitsky. Radial fractional Laplace operators and Hessian inequalities. *J. Differ. Equ.* **253** (2012), 244–272.

27 R. L. Frank and R. Seiringer. Non-linear ground state representations and sharp Hardy inequalities. *J. Funct. Anal.* **255** (2008), 3407–3430.

28 N. Grenon, F. Murat and A. Porretta. Existence and a priori estimate for elliptic problems with subquadratic gradient dependent terms. *C. R. Math. Acad. Sci. Paris* **342** (2006), 23–28.

29 L. Jeanjean and B. Sirakov. Existence and multiplicity for elliptic problems with quadratic growth in the gradient. *Commun. Partial Differ. Equ.* **38** (2013), 244–264.

30 M. Kardar, G. Parisi and Y.-C. Zhang. Dynamic scaling of growing interfaces. *Phys. Rev. Lett.* **56** (1986), 889–892.

31 T. Kuusi, G. Mingione and Y. Sire. Nonlocal equations with measure data. *Commun. Math. Phys.* **337** (2015), 1317–1368.

32 T. Kuusi, G. Mingione and Y. Sire. Nonlocal self-improving properties. *Anal. PDE* **8** (2015), 57–114.

33 N. Laskin. Fractional quantum mechanics and lévy path integrals. *Phys. Lett. A* **268** (2000), 298–305.

34 T. Leonori, I. Peral, A. Primo and F. Soria. Basic estimates for solutions of a class of nonlocal elliptic and parabolic equations. *Discrete Contin. Dyn. Syst.* **35** (2015), 6031–6068.

35 V. Millot and Y. Sire. On a fractional Ginzburg–Landau equation and 1/2-harmonic maps into spheres. *Arch. Ration. Mech. Anal.* **215** (2015), 125–210.

36 N. C. Phuc. Morrey global bounds and quasilinear Riccati type equations below the natural exponent. *J. Math. Pure Appl.* **102** (2014), 99–123.

37 A. C. Ponce, *Elliptic PDEs, measures and capacities. From the Poisson equations to nonlinear Thomas-Fermi problems.* EMS Tracts in Mathematics, vol. 23 (Zürich: European Mathematical Society (EMS), 2016).

38 A. Schikorra. Integro-differential harmonic maps into spheres. *Commun. Partial Differ. Equ.* **40** (2015), 506–539.

39 T.-T. Shieh and D. Spector. On a new class of fractional partial differential equations. *Adv. Calc. Var.* **8** (2015), 321–336.

40 B. Sirakov. Solvability of uniformly elliptic fully nonlinear PDE. *Arch. Ration. Mech. Anal.* **195** (2010), 579–607.

41 E. M. Stein, *Singular integrals and differentiability properties of functions.* Princeton Mathematical Series, No. 30 (Princeton, NJ: Princeton University Press, 1970).

42 J. F. Toland. The Peierls–Nabarro and Benjamin–Ono equations. *J. Funct. Anal.* **145** (1997), 136–150.