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Non-splitting in Kirchberg's Ideal-related *KK*-Theory

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Abstract. A. Bonkat obtained a universal coefficient theorem in the setting of Kirchberg's ideal-related KK-theory in the fundamental case of a C^*-algebra with one specified ideal. The universal coefficient sequence was shown to split, unnaturally, under certain conditions. Employing certain K-theoretical information derivable from the given operator algebras using a method introduced here, we shall demonstrate that Bonkat's UCT does not split in general. Related methods lead to information on the complexity of the K-theory which must be used to classify $*$-isomorphisms for purely infinite C^*-algebras with one non-trivial ideal.

1 Introduction

The KK-theory introduced by Kasparov [8] is one of the most important tools in the theory of classification of C^*-algebras, of use especially for simple C^*-algebras. Recently, Kirchberg has developed the so-called ideal-related KK-theory—a generalisation of Kasparov's KK-theory that takes into account the ideal structure of the algebras considered—and obtained strong isomorphism theorems for stable, nuclear, separable, strongly purely infinite C^*-algebras [9]. The results obtained by Kirchberg establish ideal-related KK-theory as an essential tool in the classification theory of non-simple C^*-algebras.

KK-theory is a bivariant functor; to obtain a real classification result one needs a univariant classification functor instead. For ordinary KK-theory this is obtained (within the bootstrap category) by invoking the Universal Coefficient Theorem (UCT) of Rosenberg and Schochet.

Theorem 1.1 **(Rosenberg-Schochet's UCT, [16])** *Let A and B be separable C^*-algebras in the bootstrap category $\mathcal{N}$. Then there is a short exact sequence*

$$\operatorname{Ext}^1_{\mathbb{Z}}(K_*(A), K_*(SB)) \hookrightarrow KK(A,B) \overset{\gamma}{\twoheadrightarrow} \operatorname{Hom}_{\mathbb{Z}}(K_*(A), K_*(B)),$$

where $K_*(-)$ denotes the graded group $K_0(-) \oplus K_1(-)$. The sequence is natural in both A and B, and splits (unnaturally, in general). Moreover, an element x in $KK(A,B)$ is invertible if and only if $\gamma(x)$ is an isomorphism.

This UCT allows us to turn isomorphism results (such as Kirchberg–Phillips' theorem [10]) into strong classification theorems. Moreover, using the splitting, it allows us to determine completely the additive structure of the KK-groups.

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To transform Kirchberg's general result into a strong classification theorem, one would need a UCT for ideal-related KK-theory. This was achieved by A. Bonkat [1] in the special case where the specified ideal structure is just a single ideal. Progress into more general cases with finitely many ideals has recently been announced by Meyer–Nest [11] and by the second author [13], but in this paper we will only consider the case with one specified ideal.

Theorem 1.2 (Bonkat's UCT, [1, Satz 7.5.3, Satz 7.7.1, and Proposition 7.7.2]) *Let e_1 and e_2 be extensions of separable nuclear C^*-algebras in the bootstrap category $\mathcal{N}$. Then there is a short exact sequence*

$$\mathrm{Ext}^1_{\mathcal{Z}_6}(K_{\mathrm{six}}(e_1), K_{\mathrm{six}}(Se_2)) \hookrightarrow KK_{\mathcal{E}}(e_1, e_2) \xrightarrow{\Gamma} \mathrm{Hom}_{\mathcal{Z}_6}(K_{\mathrm{six}}(e_1), K_{\mathrm{six}}(e_2)),$$

where $K_{\mathrm{six}}(-)$ is the standard cyclic six term exact sequence, $\mathcal{Z}_6$ is the category of cyclic six term chain complexes, and Se denotes the extension obtained by tensoring all the C^*-algebras in the extension e with $C_0(0,1)$. The sequence is natural in both e_1 and e_2. Moreover, an element $x \in KK_{\mathcal{E}}(e_1, e_2)$ is invertible if and only if $\Gamma(x)$ is an isomorphism.

Bonkat leaves open the question of whether this UCT splits in general. We prove here that this is not always the case, even in the fundamental case considered by Bonkat (see Proposition 3.1(i)).

This observation tells us—in contrast to the ordinary KK-theory—that we cannot, in general, completely determine the additive structure of $KK_{\mathcal{E}}$ just by using the UCT. It is comforting to note, as may be inferred from the results in [6, 14, 15], that this has only marginal impact on the usefulness of Bonkat's result in the context of classification of the C^*-algebras considered by Kirchberg. But as we shall see, it has several repercussions concerning the classification of homomorphisms and automorphisms of such C^*-algebras, and opens an intriguing discussion—which it is our ambition to close elsewhere in the important special case of Cuntz–Krieger algebras satisfying condition (II) — on the nature of an invariant classifying such morphisms.

Indeed, examples abound in classification theory in which the invariant needed to classify automorphims up to approximate unitary equivalence on a certain class of C^*-algebras is more complicated than the classifying invariant for the algebras themselves. For instance, even though $K_*(-)$ is a classifying invariant for stable Kirchberg algebras (*i.e.,* nuclear, separable, simple, purely infinite C^*-algebras,) one needs to turn to *total* K-theory — the collection of $K_*(-)$ and all torsion coefficient groups $K_*(-;\mathbb{Z}_n)$—in order to obtain exactness of

$$\{1\} \to \overline{\mathrm{Inn}}(A) \to \mathrm{Aut}(A) \to \mathrm{Aut}_\Lambda(\underline{K}(A)) \to \{1\}, \tag{1.1}$$

where $\overline{\mathrm{Inn}}(A)$ is the group of automorphisms of A that are approximately unitarily equivalent to id_A and the subscript Λ indicates that the group isomorphism on $\underline{K}(A)$ must commute with all the natural *Bockstein* operations.

The appearance of total K-theory in (1.1) is explained by the Universal Multicoefficient Theorem (UMCT) obtained by Dadarlat and Loring in [4].

Theorem 1.3 (Dadarlat–Loring's UMCT, [4]) *Let A and B be separable C^*-algebras in the bootstrap category $\mathcal{N}$. Then there is a short exact sequence*

$$\mathrm{Pext}^1_{\mathbb{Z}}(K_*(A), K_*(SB)) \hookrightarrow KK(A,B) \twoheadrightarrow \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B)),$$

where $\mathrm{Pext}^1_{\mathbb{Z}}$ denotes the subgroup of $\mathrm{Ext}^1_{\mathbb{Z}}$ consisting of pure extensions, and Hom_Λ denotes the group of homomorphisms respecting the Bockstein operations. The sequence is natural in both A and B, and an element x in $KK(A,B)$ is invertible if and only if the induced element is an isomorphism. Moreover, $\mathrm{Pext}^1_{\mathbb{Z}}(K_*(A), K_*(SB))$ is zero whenever the K-theory of A is finitely generated.

Dadarlat has pointed out to us that although [4] states that the UMCT splits in general, this is not true. The problem can be traced to [17–19].

In the stably finite case, as exemplified by stable real rank zero *AD* algebras, the UMCT leads to exactness of

$$\{1\} \longrightarrow \overline{\mathrm{Inn}}(A) \longrightarrow \mathrm{Aut}(A) \longrightarrow \mathrm{Aut}_{\Lambda,+}(\underline{K}(A)) \longrightarrow \{1\},$$

in which the subscript "+" indicates the presence of positivity conditions (see [4] for details). Noting the way the usage of a six term exact sequence in [15] parallels the usage of positivity in the stably finite case (cf. [3]) it is natural to speculate (as indeed the first author did at The First Abel Symposium, cf. [6]) that by combining all coefficient six term exact sequences into an invariant $\underline{K}_{\mathrm{six}}(-)$ one obtains an exact sequence of the form

$$\{1\} \longrightarrow \overline{\mathrm{Inn}}(e) \longrightarrow \mathrm{Aut}(e) \longrightarrow \mathrm{Aut}_\Lambda(\underline{K}_{\mathrm{six}}(e)) \longrightarrow \{1\}, \tag{1.2}$$

and to search for a corresponding UMCT along the lines of Theorem 1.3.

This sequence is clearly a chain complex, but as we will see, the natural map from $KK_{\mathcal{E}}(e_1, e_2)$ to $\mathrm{Hom}_\Lambda(\underline{K}_{\mathrm{six}}(e_1), \underline{K}_{\mathrm{six}}(e_2))$ is not injective nor is it surjective in general for extensions e_1 and e_2 with finitely generated K-theory (see Proposition 3.1(ii),(iii)), and we will give an example of an extension of stable Kirchberg algebras in the bootstrap category $\mathcal{N}$ with finitely generated K-theory, such that (1.2) is only exact at $\overline{\mathrm{Inn}}(e)$, telling us in unmistakable terms that this is the wrong invariant to use.

Our methods are based on computations related to a class of extensions that, we believe, should be thought of as a substitute for the total K-theory of relevance in the classification of, *e.g.,* non-simple, stably finite C^*-algebras with real rank zero. We shall undertake a more systematic study of these objects elsewhere and show how they may be employed to the task of computing Kirchberg's groups $KK_{\mathcal{E}}(-,-)$.

2 Preliminaries

We first set up some notation that will be used throughout.

Definition 2.1 Let $n \geq 2$ be an integer and denote the non-unital dimension drop algebra by $\mathbb{I}_n^0 = \{f \in C_0((0,1], \mathsf{M}_n) : f(1) \in \mathbb{C}1_{\mathsf{M}_n}\}$. Then $\mathbb{I}_n^0$ fits into the short exact sequence

$$\mathfrak{e}_{n,0} \colon S\mathsf{M}_n \hookrightarrow \mathbb{I}_n^0 \twoheadrightarrow \mathbb{C}.$$

It is well known that $K_0(\mathbb{I}_n^0) = 0$ and $K_1(\mathbb{I}_n^0) = \mathbb{Z}_n$, where $\mathbb{Z}_n$ denotes the cyclic abelian group with n elements.

Let $\mathfrak{e}_{n,1} \colon S\mathbb{C} \hookrightarrow \mathbb{I}_n^1 \twoheadrightarrow \mathbb{I}_n^0$ be the extension obtained from the mapping cone of the map $\mathbb{I}_n^0 \twoheadrightarrow \mathbb{C}$. The diagram

$$\begin{array}{ccccc}
0 & \longrightarrow & S\mathbb{C} & = & S\mathbb{C} \\
\downarrow & & \downarrow & & \downarrow \\
S\mathsf{M}_n & \hookrightarrow & \mathbb{I}_n^1 & \twoheadrightarrow & C\mathbb{C} \\
\| & & \downarrow & & \downarrow \\
S\mathsf{M}_n & \hookrightarrow & \mathbb{I}_n^0 & \twoheadrightarrow & \mathbb{C}
\end{array}$$

is commutative and the columns and rows are short exact sequences. Note that the $*$-homomorphism from $S\mathsf{M}_n$ to $\mathbb{I}_n^1$ induces a KK-equivalence.

Let $\mathfrak{e}_{n,2} \colon S\mathbb{I}_n^0 \hookrightarrow \mathbb{I}_n^2 \twoheadrightarrow \mathbb{I}_n^1$ be the extension obtained from the mapping cone of the canonical map $\mathbb{I}_n^1 \twoheadrightarrow \mathbb{I}_n^0$. Then the diagram

$$\begin{array}{ccccc}
0 & \longrightarrow & S\mathbb{I}_n^0 & = & S\mathbb{I}_n^0 \\
\downarrow & & \downarrow & & \downarrow \\
S\mathbb{C} & \hookrightarrow & \mathbb{I}_n^2 & \twoheadrightarrow & C\mathbb{I}_n^0 \\
\| & & \downarrow & & \downarrow \\
S\mathbb{C} & \hookrightarrow & \mathbb{I}_n^1 & \twoheadrightarrow & \mathbb{I}_n^0
\end{array}$$

is commutative and the columns and rows are short exact sequences. Note that the $*$-homomorphism from $S\mathbb{C}$ to $\mathbb{I}_n^2$ induces a KK-equivalence. This implies, with a little more work, that we get no new K-theoretical information from considering objects $\mathbb{I}_n^k$ or $\mathfrak{e}_{n,k}$ for $k > 2$. Note also that the C^*-algebras $\mathbb{I}_n^0, \mathbb{I}_n^1$, and $\mathbb{I}_n^2$ are NCCW complexes of dimension 1, 1, and 2, respectively, in the sense of [5]. See Figure 2.1.

Let $e \colon A_0 \hookrightarrow A_1 \twoheadrightarrow A_2$ be an extension of C^*-algebras. We have an "ideal-related K-theory with $\mathbb{Z}_n$-coefficients" denoted by $K_{\mathrm{six}}(e; \mathbb{Z}_n)$. More precisely, $K_{\mathrm{six}}(e; \mathbb{Z}_n)$

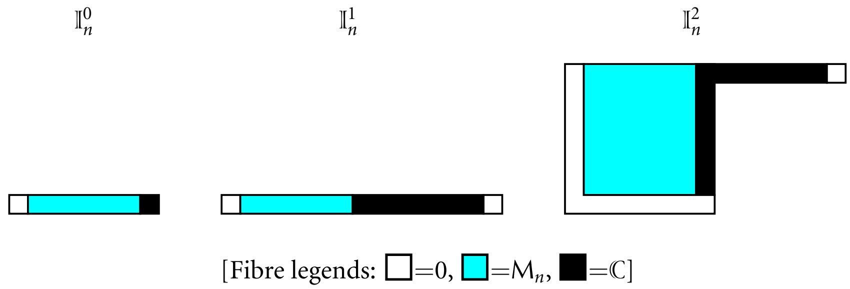

Figure 2.1: NCCW structure of $\mathbb{I}_n^i$

denotes the six term exact sequence

$$\begin{array}{ccccc} K_0(A_0;\mathbb{Z}_n) & \longrightarrow & K_0(A_1;\mathbb{Z}_n) & \longrightarrow & K_0(A_2;\mathbb{Z}_n) \\ \uparrow & & & & \downarrow \\ K_1(A_2;\mathbb{Z}_n) & \longleftarrow & K_1(A_1;\mathbb{Z}_n) & \longleftarrow & K_1(A_0;\mathbb{Z}_n) \end{array}$$

obtained by applying the covariant functor $KK^*(\mathbb{I}_n^0,-)$ to the extension e.

Let us denote the standard six term exact sequence in K-theory by $K_{\text{six}}(e)$. The collection consisting of $K_{\text{six}}(e)$ and $K_{\text{six}}(e;\mathbb{Z}_n)$ for all $n \geq 2$ will be denoted by $\underline{K}_{\text{six}}(e)$. A homomorphism from $\underline{K}_{\text{six}}(e_1)$ to $\underline{K}_{\text{six}}(e_2)$ consists of a morphism from $K_{\text{six}}(e_1)$ to $K_{\text{six}}(e_2)$ along with an infinite family of morphisms from $K_{\text{six}}(e_1;\mathbb{Z}_n)$ to $K_{\text{six}}(e_2;\mathbb{Z}_n)$ respecting the Bockstein operations in Λ. We will denote the group of homomorphisms from $\underline{K}_{\text{six}}(e_1)$ to $\underline{K}_{\text{six}}(e_2)$ by $\text{Hom}_\Lambda(\underline{K}_{\text{six}}(e_1),\underline{K}_{\text{six}}(e_2))$. We turn $\underline{K}_{\text{six}}$ into a functor in the obvious way.

Lemma 2.2 *There is a natural homomorphism*

$$\Gamma_{e_1,e_2}\colon KK_{\mathcal{E}}(e_1,e_2) \longrightarrow \text{Hom}_\Lambda(\underline{K}_{\text{six}}(e_1),\underline{K}_{\text{six}}(e_2)).$$

Proof A computation shows that $K_{\text{six}}(-;\mathbb{Z}_n)$ is a stable, homotopy invariant, split exact functor, since KK satisfies these properties. Therefore, $\underline{K}_{\text{six}}(-)$ is a stable, homotopy invariant, split exact functor. Hence, for every fixed extension e_1 of C^*-algebras, $\text{Hom}_\Lambda(\underline{K}_{\text{six}}(e_1),\underline{K}_{\text{six}}(-))$ is a stable, homotopy invariant, split exact functor. By [1, Satz 3.5.9], we have a natural transformation $\Gamma_{e_1,-}$ from $KK_{\mathcal{E}}(e_1,-)$ to $\text{Hom}_\Lambda(\underline{K}_{\text{six}}(e_1),\underline{K}_{\text{six}}(-))$ such that $\Gamma_{e_1,-}$ sends $[\text{id}_{e_1}]$ to $\underline{K}_{\text{six}}(\text{id}_{e_1})$. Arguing as in the proof of [7, Lemma 3.2], we have that

$$\Gamma_{e_1,e_2}: KK_{\mathcal{E}}(e_1,e_2) \longrightarrow \text{Hom}_\Lambda(\underline{K}_{\text{six}}(e_1),\underline{K}_{\text{six}}(e_2))$$

is a group homomorphism. ∎

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Another collection of groups that we will use in this paper is the following: for each $n \geq 2$, set

$$K_{\mathcal{E}}(e;\mathbb{Z}_n) = \bigoplus_{i=0}^{2}\left(KK^*_{\mathcal{E}}(\mathfrak{e}_{n,i}, e) \oplus KK^*(\mathbb{I}_n^0, A_i) \oplus KK^*(\mathbb{C}, A_i)\right).$$

3 Examples

Accompanying the groups $KK^*_{\mathcal{E}}(\mathfrak{e}_{n,i}, e)$ are naturally defined diagrams, which will be systematically described in the second author's doctoral thesis [13]. For now, we will use these groups to show the following.

Proposition 3.1 (i) *The UCT of Bonkat (Theorem* 1.2*) does not split in general.*
(ii) *There exist e_1 and e_2 extensions of separable nuclear C^*-algebras in the bootstrap category $\mathcal{N}$ of Rosenberg and Schochet [16] such that the six term exact sequence of K-groups associated with e_1 is finitely generated and*

$$\Gamma_{e_1,e_2} \colon KK_{\mathcal{E}}(e_1, e_2) \longrightarrow \operatorname{Hom}_{\Lambda}(\underline{K}_{\mathrm{six}}(e_1), \underline{K}_{\mathrm{six}}(e_2))$$

is not injective.
(iii) *There exist e_1 and e_2 extensions of separable nuclear C^*-algebras in the bootstrap category $\mathcal{N}$ of Rosenberg and Schochet [16] such that the six term exact sequence of K-groups associated to e_1 is finitely generated and*

$$\Gamma_{e_1,e_2} \colon KK_{\mathcal{E}}(e_1, e_2) \longrightarrow \operatorname{Hom}_{\Lambda}(\underline{K}_{\mathrm{six}}(e_1), \underline{K}_{\mathrm{six}}(e_2))$$

is not surjective.

The proposition will be proved through a series of examples. The following example shows that the UCT of Bonkat does not split in general. Also it shows that there exist extensions e_1 and e_2 of separable nuclear C^*-algebras in $\mathcal{N}$ with finitely generated K-theory, such that Γ_{e_1,e_2} is not injective.

Example 3.2 Let n be a prime number. By [1, Korollar 7.1.6], we have that

$$\mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow KK^1_{\mathcal{E}}(\mathfrak{e}_{n,0}, \mathfrak{e}_{n,1}) \longrightarrow 0$$

is an exact sequence. Therefore, $KK^1_{\mathcal{E}}(\mathfrak{e}_{n,0}, \mathfrak{e}_{n,1})$ is a cyclic group. By [1, Korollar 7.1.6], $KK^1_{\mathcal{E}}(\mathfrak{e}_{n,0}, \mathfrak{e}_{n,1})$ fits into the following exact sequence

$$0 \longrightarrow \mathbb{Z}_n \longrightarrow KK^1_{\mathcal{E}}(\mathfrak{e}_{n,0}, \mathfrak{e}_{n,1}) \longrightarrow \mathbb{Z}_n \longrightarrow 0.$$

So, $KK^1_{\mathcal{E}}(\mathfrak{e}_{n,0}, \mathfrak{e}_{n,1})$ is isomorphic to $\mathbb{Z}_{n^2}$.

An easy computation shows that $\operatorname{Hom}(K_{\mathrm{six}}(\mathfrak{e}_{n,0}), K_{\mathrm{six}}(S\mathfrak{e}_{n,1}))$ is isomorphic to $\mathbb{Z}_n$. Using this fact and the fact that $KK_{\mathcal{E}}(\mathfrak{e}_{n,0}, S\mathfrak{e}_{n,1}) \cong KK^1_{\mathcal{E}}(\mathfrak{e}_{n,0}, \mathfrak{e}_{n,1})$ is $\mathbb{Z}_{n^2}$, we immediately see that the UCT of Bonkat does not split in this case.

We would also like to point out another consequence of this example. Since n is prime and $\operatorname{Ext}^1_{\mathcal{Z}_6}(K_{\mathrm{six}}(\mathfrak{e}_{n,0}), K_{\mathrm{six}}(\mathfrak{e}_{n,1}))$ injects into a proper subgroup of $KK^1_{\mathcal{E}}(\mathfrak{e}_{n,0}, \mathfrak{e}_{n,1})$, we have that $\operatorname{Ext}^1_{\mathcal{Z}_6}(K_{\mathrm{six}}(\mathfrak{e}_{n,0}), K_{\mathrm{six}}(\mathfrak{e}_{n,1}))$ is isomorphic to $\mathbb{Z}_n$. Therefore, n annihilates all torsional K-theory information, but n does not annihilate the torsion group $KK^1_{\mathcal{E}}(\mathfrak{e}_{n,0}, \mathfrak{e}_{n,1})$.

We will now show that the natural map $\Gamma_{\mathfrak{e}_{n,0}, S\mathfrak{e}_{n,1}}$ from $KK_{\mathcal{E}}(\mathfrak{e}_{n,0}, S\mathfrak{e}_{n,1})$ to $\operatorname{Hom}_{\Lambda}(\underline{K}_{\mathrm{six}}(\mathfrak{e}_{n,0}), \underline{K}_{\mathrm{six}}(S\mathfrak{e}_{n,1}))$ is not injective. Let $A_0 \hookrightarrow A_1 \twoheadrightarrow A_2$ and $B_0 \hookrightarrow B_1 \twoheadrightarrow B_2$ denote the extensions $\mathfrak{e}_{n,0}$ and $S\mathfrak{e}_{n,1}$, respectively. Note that the corresponding six term exact sequences are (isomorphic to)

$$\begin{array}{ccccc} 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} \\ \uparrow & & & & \downarrow \\ 0 & \longleftarrow & \mathbb{Z}_n & \longleftarrow & \mathbb{Z} \end{array} \quad \text{and} \quad \begin{array}{ccccc} \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}_n \\ \uparrow & & & & \downarrow \\ 0 & \longleftarrow & 0 & \longleftarrow & 0 \end{array},$$

respectively. Using the UCT of Rosenberg and Schochet, a short computation shows that $n\prod_{i=0}^{2} KK(A_i, B_i) = 0$. Since all the K-theory is finitely generated, we have by Dadarlat and Loring's UMCT that $\operatorname{Hom}_{\Lambda}(\underline{K}_{\mathrm{six}}(\mathfrak{e}_{n,0}), \underline{K}_{\mathrm{six}}(S\mathfrak{e}_{n,1}))$ is isomorphic to a subgroup of $\prod_{i=0}^{2} KK(A_i, B_i)$. Since the latter group has no element of order n^2 and $KK_{\mathcal{E}}(\mathfrak{e}_{n,0}, S\mathfrak{e}_{n,1})$ is isomorphic to $\mathbb{Z}_{n^2}$, we have that $\Gamma_{\mathfrak{e}_{n,0}, S\mathfrak{e}_{n,1}}$ is not injective.

The above example also provides a counterexample to [1, Satz 7.7.6]. The arguments in the proof of Satz 7.7.6 are correct, but it appears that Bonkat overlooked the case where the six term exact sequences are of the form

$$\begin{array}{ccccc} 0 & \longrightarrow & 0 & \longrightarrow & * \\ \uparrow & & & & \downarrow \\ 0 & \longleftarrow & * & \longleftarrow & * \end{array} \quad \text{and} \quad \begin{array}{ccccc} 0 & \longrightarrow & 0 & \longrightarrow & 0 \\ \uparrow & & & & \downarrow \\ * & \longleftarrow & * & \longleftarrow & * \end{array}.$$

Our next example shows that there exist extensions e_1 and e_2 of separable nuclear C^*-algebras in $\mathcal{N}$ with finitely generated K-groups, such that Γ_{e_1,e_2} is not surjective.

Example 3.3 Let n be a prime number. Consider the following short exact sequences of extensions:

$$\begin{array}{ccccc} S\mathbb{C} & = & S\mathbb{C} & \longrightarrow & 0 \\ \| & & \downarrow & & \downarrow \\ S\mathbb{C} & \hookrightarrow & \mathbb{I}_n^1 & \twoheadrightarrow & \mathbb{I}_n^0 \\ \downarrow & & \downarrow & & \| \\ 0 & \longrightarrow & \mathbb{I}_n^0 & = & \mathbb{I}_n^0 \end{array} \tag{3.1}$$

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$$
\begin{array}{ccccc}
SM_n & = & SM_n & \longrightarrow & 0 \\
\| & & \downarrow & & \downarrow \\
SM_n & \hookrightarrow & \mathbb{I}_n^0 & \twoheadrightarrow & \mathbb{C} \\
\downarrow & & \downarrow & & \| \\
0 & \longrightarrow & \mathbb{C} & = & \mathbb{C}
\end{array}
\tag{3.2}
$$

By applying the bivariant functor $KK^*_{\mathcal{E}}(-,-)$ to the above exact sequences of extensions with (3.1) in the first variable and (3.2) in the second variable and by Lemma 7.1.5 of [1], we get that the diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}_n & \longrightarrow & \mathbb{Z}_n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & KK_{\mathcal{E}}(\mathfrak{e}_{n,1}, \mathfrak{e}_{n,0}) & \longrightarrow & \mathbb{Z}_n & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle n} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z} & = & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z}_n & \longrightarrow & \mathbb{Z}_n & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

is commutative. By [1, Korollar 3.4.6], the columns and rows of the above diagram are exact sequences. Therefore, we have that $KK_{\mathcal{E}}(\mathfrak{e}_{n,1}, \mathfrak{e}_{n,0})$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_n$.

A straightforward computation gives that $K_{\text{six}}(\mathfrak{e}_{n,0})$ and $K_{\text{six}}(\mathfrak{e}_{n,0}; \mathbb{Z}_m)$ are given by

$$
\begin{array}{ccccc}
0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} \\
\uparrow & & & & \downarrow \\
0 & \longleftarrow & \mathbb{Z}_n & \longleftarrow & \mathbb{Z}
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccccc}
0 & \longrightarrow & \mathbb{Z}_{(m,n)} & \longrightarrow & \mathbb{Z}_n \\
\uparrow & & & & \downarrow \\
0 & \longleftarrow & \mathbb{Z}_{n/(m,n)} & \longleftarrow & \mathbb{Z}_n
\end{array}
,
$$

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and similarly, for $\mathfrak{e}_{n,1}$, by

$$\begin{array}{ccccc} 0 & \longrightarrow & 0 & \longrightarrow & 0 \\ \uparrow & & & & \downarrow \\ \mathbb{Z}_n & \longleftarrow & \mathbb{Z} & \longleftarrow & \mathbb{Z} \end{array} \qquad \text{and} \qquad \begin{array}{ccccc} 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}_{(n,m)} \\ \uparrow & & & & \downarrow \\ \mathbb{Z}_{n/(n,m)} & \longleftarrow & \mathbb{Z}_n & \longleftarrow & \mathbb{Z}_n \end{array} .$$

A map $K_{\mathrm{six}}(\mathfrak{e}_{n,1}) \oplus K_{\mathrm{six}}(\mathfrak{e}_{n,1}; \mathbb{Z}_n) \to K_{\mathrm{six}}(\mathfrak{e}_{n,0}) \oplus K_{\mathrm{six}}(\mathfrak{e}_{n,0}; \mathbb{Z}_n)$ is given by a 12-tuple

$$((0,0,0,x,a,0),(0,0,b,c,d,0))$$

where $x \in \mathbb{Z}$ and $a, b, c, d \in \mathbb{Z}_n$. To commute with the maps in the diagrams as well as the Bockstein maps of type ρ and β, we must have $d = a$ and $c = \bar{x}$, and straightforward computations show that this tuple extends uniquely to an element of $\mathrm{Hom}_\Lambda(\underline{K}_{\mathrm{six}}(\mathfrak{e}_{n,1}), \underline{K}_{\mathrm{six}}(\mathfrak{e}_{n,0}))$. Hence this group is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_n \oplus \mathbb{Z}_n$. Finally, note that no surjection $\mathbb{Z} \oplus \mathbb{Z}_n \to \mathbb{Z} \oplus \mathbb{Z}_n \oplus \mathbb{Z}_n$ exists.

Remark 3.4 The matrices

$$A = \begin{pmatrix} 1&1&0&0&0&0 \\ 1&1&1&0&0&0 \\ 0&1&1&0&0&0 \\ 0&0&0&1&1&1 \\ 0&0&0&1&1&1 \\ 1&0&0&1&1&1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1&1&1&0&0&0 \\ 1&1&1&0&0&0 \\ 1&1&1&0&0&0 \\ 0&0&0&1&1&0 \\ 0&0&0&1&1&1 \\ 1&0&0&0&1&1 \end{pmatrix}$$

satisfy condition (II) of Cuntz [2]. Hence, the Cuntz–Krieger algebras $\mathcal{O}_A$ and $\mathcal{O}_B$ are purely infinite C^*-algebras and have exactly one non-trivial ideal. Using the Smith normal form and [12, Proposition 3.4], we see that the six term exact sequence corresponding to $\mathcal{O}_A$ and $\mathcal{O}_B$ are (isomorphic to) the sequences

$$\begin{array}{ccccc} \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}_2 \\ \uparrow & & & & \downarrow{\scriptstyle 0} \\ 0 & \longleftarrow & \mathbb{Z} & \underset{\cong}{\longleftarrow} & \mathbb{Z} \end{array} \qquad \text{and} \qquad \begin{array}{ccccc} \mathbb{Z}_2 & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{\cong} & \mathbb{Z} \\ \uparrow & & & & \downarrow \\ \mathbb{Z} & \longleftarrow & \mathbb{Z} & \longleftarrow & 0 \end{array} ,$$

respectively. Using $KK_{\mathcal{E}}$-equivalent extensions, the fact that $KK_{\mathcal{E}}$ is split exact, and arguments similar to Example 3.2, one easily shows that the natural map Γ_{e_1,e_2} is not injective for the extensions e_1 and e_2 corresponding to the Cuntz–Krieger algebras $\mathcal{O}_A$ and $\mathcal{O}_B$, respectively. Similar considerations on

$$C = \begin{pmatrix} 1&1&0&0&0&0 \\ 1&1&1&0&0&0 \\ 0&1&1&0&0&0 \\ 0&0&0&1&1&1 \\ 0&0&0&1&1&1 \\ 1&0&0&1&1&1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 1&1&0&0&0&0 \\ 1&1&1&0&0&0 \\ 0&1&1&0&0&0 \\ 0&0&0&1&1&0 \\ 0&0&0&1&1&1 \\ 1&0&0&0&1&1 \end{pmatrix}$$

yield a version of Example 3.3 in the realm of Cuntz-Krieger algebras.

One may ask if Γ_{e_1,e_2} is ever surjective, and the answer is yes. If e_1 is an extension of a separable nuclear C^*-algebra in $\mathcal{N}$ such that the K-groups of $K_{\mathrm{six}}(e_1)$ are torsion free, then $\mathrm{Hom}_\Lambda(\underline{K}_{\mathrm{six}}(e_1), \underline{K}_{\mathrm{six}}(e_2))$ is naturally isomorphic to $\mathrm{Hom}_{\mathbb{Z}_6}(K_{\mathrm{six}}(e_1), K_{\mathrm{six}}(e_2))$ such that the composition of Γ_{e_1,e_2} with this natural isomorphism is the natural map from $KK_{\mathcal{E}}(e_1, e_2)$ to $\mathrm{Hom}_{\mathbb{Z}_6}(K_{\mathrm{six}}(e_1), K_{\mathrm{six}}(e_2))$. Hence, by the UCT of Bonkat, we have that Γ_{e_1,e_2} is surjective.

4 Automorphisms of Extensions of Kirchberg Algebras

The class $\mathcal{R}$ of C^*-algebras considered by Rørdam in [15] consists of all C^*-algebras A_1 fitting in an essential extension $e\colon A_0 \hookrightarrow A_1 \twoheadrightarrow A_2$, where A_0 and A_2 are Kirchberg algebras in $\mathcal{N}$ (with A_0 necessarily being stable). For convenience we shall often identify e and A_1 in this setting, as indeed we can without risk of confusion. As explained in [15], one needs to consider three distinct cases: (1) A_1 is stable; (2) A_1 is unital; and (3) A_1 is neither stable nor unital.

A functor F is called a *classification functor*, if $A \cong B \Leftrightarrow F(A) \cong F(B)$ (for all algebras A and B in the class considered). Such a functor F is called a *strong classification functor* if every isomorphism from $F(A)$ to $F(B)$ is induced by an isomorphism from A to B (for all algebras A and B in the class considered).

In [15], Rørdam showed K_{six} to be a classification functor for stable algebras in $\mathcal{R}$. More recently, the authors in [6, 14] showed that K_{six} (respectively K_{six} together with the class of the unit) is a strong classification functor for stable (respectively unital) algebras in $\mathcal{R}$. Moreover, they also showed that K_{six} is a classification functor for non-stable non-unital algebras in $\mathcal{R}$.

In this section we will address some questions regarding the automorphism group of e, where e is in $\mathcal{R}$. If $e\colon A_0 \hookrightarrow A_1 \twoheadrightarrow A_2$ is an essential extension of separable C^*-algebas, then an automorphism of e is a triple (ϕ_0, ϕ_1, ϕ_2) such that ϕ_i is an automorphism of A_i and the diagram

$$\begin{array}{ccccc} A_0 & \hookrightarrow & A_1 & \twoheadrightarrow & A_2 \\ \downarrow{\scriptstyle \phi_0} & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_2} \\ A_0 & \hookrightarrow & A_1 & \twoheadrightarrow & A_2 \end{array}$$

is commutative. We denote the group of automorphisms of e by $\mathrm{Aut}(e)$. If A_0 and A_2 are simple C^*-algebras, then $\mathrm{Aut}(e)$ and $\mathrm{Aut}(A_1)$ are canonically isomorphic. Two automorphisms (ϕ_0, ϕ_1, ϕ_2) and (ψ_0, ψ_1, ψ_2) of e are said to be approximately unitarily equivalent if ϕ_1 and ψ_1 are approximately unitarily equivalent. A consequence of Kirchberg's results in [9] is that $KK_{\mathcal{E}}(e, e)$ classifies automorphisms of stable algebras in $\mathcal{R}$.

In [6] the first and second authors asked whether the canonical map from $\mathrm{Aut}(e)$ to $\mathrm{Aut}_\Lambda(\underline{K}_{\mathrm{six}}(e))$ was surjective, cf. (1.2). We answer this in the negative as follows.

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Proposition 4.1 *There is a C^*-algebra $e \in \mathcal{R}$ with finitely generated K-theory such that* (1.2) *is exact only at*

$$\{1\} \longrightarrow \overline{\mathrm{Inn}}(e) \longrightarrow \mathrm{Aut}(e)$$

Before proving the above proposition, we first need to set up some notation. For ϕ in $\mathrm{Aut}(e)$, the element in $KK_{\mathcal{E}}(e, e)$ induced by ϕ will be denoted by $KK_{\mathcal{E}}(\phi)$ and the element in $\mathrm{Hom}_{\Lambda}(\underline{K}_{\mathrm{six}}(e), \underline{K}_{\mathrm{six}}(e))$ induced by ϕ will be denoted by $\underline{K}_{\mathrm{six}}(\phi)$. We will also need the following result.

Proposition 4.2 *Let e be any extension of separable C^*-algebras. Define*

$$\Lambda_{\mathfrak{e}_{n,i},e} \colon KK_{\mathcal{E}}(\mathfrak{e}_{n,i}, e) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(KK_{\mathcal{E}}(\mathfrak{e}_{n,i}, \mathfrak{e}_{n,i}), KK_{\mathcal{E}}(\mathfrak{e}_{n,i}, e))$$

by $\Lambda_{\mathfrak{e}_{n,i},e}(x)(y) = y \times x$, where $y \times x$ is the generalized Kasparov product (see [1]). Then $\Lambda_{\mathfrak{e}_{n,i},e}$ is an isomorphism for $i = 0, 1, 2$.

Proof We will only prove the case when $i = 0$; the other cases are similar. By the UCT of Bonkat one shows that $KK_{\mathcal{E}}(\mathfrak{e}_{n,0}, \mathfrak{e}_{n,0})$ is isomorphic to $\mathbb{Z}$ and is generated by $KK_{\mathcal{E}}(\mathrm{id}_{\mathfrak{e}_{n,0}})$. Therefore, if $\Lambda_{\mathfrak{e}_{n,0},e}(x) = 0$, then

$$x = KK_{\mathcal{E}}(\mathrm{id}_{\mathfrak{e}_{n,0}}) \times x = \Lambda_{\mathfrak{e}_{n,0},e}(x)(KK_{\mathcal{E}}(\mathrm{id}_{\mathfrak{e}_{n,0}})) = 0.$$

Hence, $\Lambda_{\mathfrak{e}_{n,0},e}$ is injective. Suppose α is a homomorphism from $KK_{\mathcal{E}}(\mathfrak{e}_{n,0}, \mathfrak{e}_{n,0})$ to $KK_{\mathcal{E}}(\mathfrak{e}_{n,0}, e)$. Set $x = \alpha(KK_{\mathcal{E}}(\mathrm{id}_{\mathfrak{e}_{n,0}}))$. Then

$$\Lambda_{\mathfrak{e}_{n,0},e}(x)(KK_{\mathcal{E}}(\mathrm{id}_{\mathfrak{e}_{n,0}})) = x = \alpha(KK_{\mathcal{E}}(\mathrm{id}_{\mathfrak{e}_{n,0}})).$$

Therefore, $\Lambda_{\mathfrak{e}_{n,0},e}$ is surjective. ∎

Proof of Proposition 4.1 Set $e_1 = S\mathfrak{e}_{p,1} \oplus \mathfrak{e}_{p,1} \oplus \mathfrak{e}_{p,0}$ where p is a prime number. Let ι_1 be the embedding of $S\mathfrak{e}_{p,1}$ to e_1 and π_1 be the projection from e_1 to $\mathfrak{e}_{p,0}$. Note that

$$KK_{\mathcal{E}}(\iota_1) \times (-) \colon KK_{\mathcal{E}}(\mathfrak{e}_{p,0}, S\mathfrak{e}_{p,1}) \longrightarrow KK_{\mathcal{E}}(\mathfrak{e}_{p,0}, e_1)$$

and

$$(-) \times KK_{\mathcal{E}}(\pi_1) \colon KK_{\mathcal{E}}(\mathfrak{e}_{p,0}, e_1) \longrightarrow KK_{\mathcal{E}}(e_1, e_1)$$

are injective homomorphisms. Hence

$$\eta_1 = ((-) \times KK_{\mathcal{E}}(\pi_1)) \circ (KK_{\mathcal{E}}(\iota_1) \times (-))$$

is injective. Since $\Gamma_{-,-}$ is natural,

$$\begin{array}{ccc}
KK_{\mathcal{E}}(\mathfrak{e}_{p,0}, S\mathfrak{e}_{p,1}) & \xrightarrow{\quad\eta_1\quad} & KK_{\mathcal{E}}(e_1, e_1) \\
\Big\downarrow{\scriptstyle \Gamma_{\mathfrak{e}_{p,0},S\mathfrak{e}_{p,1}}} & & \Big\downarrow{\scriptstyle \Gamma_{e_1,e_1}} \\
\mathrm{Hom}_{\Lambda}(\underline{K}_{\mathrm{six}}(\mathfrak{e}_{p,0}), \underline{K}_{\mathrm{six}}(S\mathfrak{e}_{p,1})) & \xrightarrow[\theta_1]{} & \mathrm{Hom}_{\Lambda}(\underline{K}_{\mathrm{six}}(e_1), \underline{K}_{\mathrm{six}}(e_1))
\end{array}$$

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is commutative. By Example 3.2, $\Gamma_{\mathfrak{e}_{p,0},S\mathfrak{e}_{p,1}}$ is not injective. Therefore, Γ_{e_1,e_1} is not injective.

Let π_2 be the projection of e_1 to $\mathfrak{e}_{p,0}$ and let ι_2 be the embedding of $\mathfrak{e}_{p,1}$ to $\mathfrak{e}_1$. Note that

$$KK_{\mathcal{E}}(\pi_2) \times (-)\colon KK_{\mathcal{E}}(e_1, e_1) \longrightarrow KK_{\mathcal{E}}(e_1, \mathfrak{e}_{p,0})$$

and

$$(-) \times KK_{\mathcal{E}}(\iota_2)\colon KK_{\mathcal{E}}(e_1, \mathfrak{e}_{p,0}) \longrightarrow KK_{\mathcal{E}}(\mathfrak{e}_{p,1}, \mathfrak{e}_{p,0})$$

are surjective homomorphisms. Therefore,

$$\eta_2 = ((-) \times KK_{\mathcal{E}}(\iota_2)) \circ (KK_{\mathcal{E}}(\pi_2) \times (-))$$

is surjective. Similarly, $\theta_2 = \underline{K}_{\mathrm{six}}(\iota_2) \circ \underline{K}_{\mathrm{six}}(\pi_2)$ is surjective. Since $\Gamma_{-,-}$ is natural,

$$\begin{array}{ccc} KK_{\mathcal{E}}(e_1, e_1) & \xrightarrow{\eta_2} & KK_{\mathcal{E}}(\mathfrak{e}_{p,1}, \mathfrak{e}_{p,0}) \\ \downarrow{\scriptstyle \Gamma_{e_1,e_1}} & & \downarrow{\scriptstyle \Gamma_{\mathfrak{e}_{p,1},\mathfrak{e}_{p,0}}} \\ \mathrm{Hom}_{\Lambda}(\underline{K}_{\mathrm{six}}(e_1), \underline{K}_{\mathrm{six}}(e_1)) & \xrightarrow[\theta_2]{} & \mathrm{Hom}_{\Lambda}(\underline{K}_{\mathrm{six}}(\mathfrak{e}_{p,1}), \underline{K}_{\mathrm{six}}(\mathfrak{e}_{p,0})) \end{array}$$

is commutative. By Example 3.3, $\Gamma_{\mathfrak{e}_{p,1},\mathfrak{e}_{p,0}}$ is not surjective. Hence, Γ_{e_1,e_1} is not surjective.

We have just shown that Γ_{e_1,e_1} is neither surjective nor injective. By [15, Proposition 5.4] there is a stable extension $e\colon A_0 \hookrightarrow A_1 \twoheadrightarrow A_2$ in $\mathcal{R}$ such that $K_{\mathrm{six}}(e) \cong K_{\mathrm{six}}(e_1)$. By the UCT of Bonkat, Theorem 1.2, we are able to lift this isomorphism to a $KK_{\mathcal{E}}$-equivalence. Therefore,

$$\begin{array}{ccc} KK_{\mathcal{E}}(e, e) & \xrightarrow{\cong} & KK_{\mathcal{E}}(e_1, e_1) \\ \downarrow{\scriptstyle \Gamma_{e,e}} & & \downarrow{\scriptstyle \Gamma_{e_1,e_1}} \\ \mathrm{Hom}_{\Lambda}(\underline{K}_{\mathrm{six}}(e), \underline{K}_{\mathrm{six}}(e)) & \xrightarrow[\cong]{} & \mathrm{Hom}_{\Lambda}(\underline{K}_{\mathrm{six}}(e_1), \underline{K}_{\mathrm{six}}(e_1)) \end{array}$$

is commutative. Hence, $\Gamma_{e,e}$ is neither injective nor surjective.

Denote the kernel of the surjective map from

$$\mathrm{Hom}_{\Lambda}(\underline{K}_{\mathrm{six}}(e), \underline{K}_{\mathrm{six}}(e)) \text{ to } \mathrm{Hom}_{\mathbb{Z}_6}(K_{\mathrm{six}}(e), K_{\mathrm{six}}(e))$$

by $\mathrm{Ext}_{\mathrm{six}}(K_{\mathrm{six}}(e), K_{\mathrm{six}}(Se))$. Note that if α is an element of $\mathrm{Hom}_{\Lambda}(\underline{K}_{\mathrm{six}}(e), \underline{K}_{\mathrm{six}}(e))$ such that $\alpha|_{K_{\mathrm{six}}(e)}$ is an isomorphism, then α is an isomorphism. Since $\Gamma_{e,e}$ is not surjective and

(4.1)

$$\begin{array}{ccccc} \mathrm{Ext}_{\mathbb{Z}_6}(K_{\mathrm{six}}(e), K_{\mathrm{six}}(Se)) & \hookrightarrow & KK_{\mathcal{E}}(e, e) & \twoheadrightarrow & \mathrm{Hom}_{\mathbb{Z}_6}(K_{\mathrm{six}}(e), K_{\mathrm{six}}(e)) \\ \downarrow{\scriptstyle \Gamma_{e,e}} & & \downarrow{\scriptstyle \Gamma_{e,e}} & & \| \\ \mathrm{Ext}_{\mathrm{six}}(K_{\mathrm{six}}(e), K_{\mathrm{six}}(Se)) & \hookrightarrow & \mathrm{Hom}_{\Lambda}(\underline{K}_{\mathrm{six}}(e), \underline{K}_{\mathrm{six}}(e)) & \twoheadrightarrow & \mathrm{Hom}_{\mathbb{Z}_6}(K_{\mathrm{six}}(e), K_{\mathrm{six}}(e)) \end{array}$$

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is commutative, there exists β_1 in $\mathrm{Ext}_{\mathrm{six}}(K_{\mathrm{six}}(e), K_{\mathrm{six}}(Se))$ that is not in the image of $\Gamma_{e,e}$. Since $(\underline{K}_{\mathrm{six}}(\mathrm{id}_e) + \beta_1)|_{K_{\mathrm{six}}(e)} = \underline{K}_{\mathrm{six}}(\mathrm{id}_e)|_{K_{\mathrm{six}}(e)}$, we have that $\underline{K}_{\mathrm{six}}(\mathrm{id}_e) + \beta_1$ is an automorphism of $\underline{K}_{\mathrm{six}}(e)$. Since β_1 is not in the image of $\Gamma_{e,e}$, $\underline{K}_{\mathrm{six}}(\mathrm{id}_e) + \beta_1$ is not in the image of $\Gamma_{e,e}$. Hence, $\underline{K}_{\mathrm{six}}(\mathrm{id}_e) + \beta_1$ is an automorphism of $\underline{K}_{\mathrm{six}}(e)$ that does not lift to an automorphism of e. Consequently,

$$\mathrm{Aut}(e) \longrightarrow \mathrm{Aut}_\Lambda(\underline{K}_{\mathrm{six}}(e)) \longrightarrow \{1\}$$

is not exact.

Since the diagram in (4.1) is commutative and $\Gamma_{e,e}$ is not injective, there exists a nonzero element β_2 of $\mathrm{Ext}_{\mathbb{Z}_6}(K_{\mathrm{six}}(e), K_{\mathrm{six}}(Se))$ such that $\Gamma_{e,e}(\beta_2) = 0$. Therefore, $\beta_2 + KK_{\mathcal{E}}(\mathrm{id}_e)$ is an invertible element in $KK_{\mathcal{E}}(e,e)$ such that $\Gamma_{e,e}(\beta_2) + \underline{K}_{\mathrm{six}}(\mathrm{id}_e) = \underline{K}_{\mathrm{six}}(\mathrm{id}_e)$. By [9, Folgerung 4.3], $\beta_2 + KK_{\mathcal{E}}(\mathrm{id}_e)$ lifts to an automorphism ϕ of e. So $\underline{K}_{\mathrm{six}}(\phi) = \underline{K}_{\mathrm{six}}(\mathrm{id}_e)$ in $\mathrm{Aut}_\Lambda(\underline{K}_{\mathrm{six}}(e))$.

Set

$$\begin{aligned} G &= \mathrm{Hom}\big(KK_{\mathcal{E}}(S\mathfrak{e}_{p,1}, e_1), KK_{\mathcal{E}}(S\mathfrak{e}_{p,1}, e_1)\big) \\ &\quad \oplus \Big(\bigoplus_{i=0}^{2} \mathrm{Hom}(KK_{\mathcal{E}}(\mathfrak{e}_{p,i}, e_1), KK_{\mathcal{E}}(\mathfrak{e}_{p,i}, e_1))\Big) \\ H &= \mathrm{Hom}\big(KK_{\mathcal{E}}(S\mathfrak{e}_{p,1}, e), KK_{\mathcal{E}}(S\mathfrak{e}_{p,1}, e)\big) \\ &\quad \oplus \Big(\bigoplus_{i=0}^{2} \mathrm{Hom}\big(KK_{\mathcal{E}}(\mathfrak{e}_{p,i}, e), KK_{\mathcal{E}}(\mathfrak{e}_{p,i}, e)\big) \Big) \end{aligned}$$

Since e_1 is equal to $S\mathfrak{e}_{p,1} \oplus \mathfrak{e}_{p,1} \oplus \mathfrak{e}_{p,0}$, by Proposition 4.2 the map from $KK_{\mathcal{E}}(e_1, e_1)$ to G given by $x \mapsto (-) \times x$ is an isomorphism. Hence, the map from $KK_{\mathcal{E}}(e,e)$ to H given by $x \mapsto (-) \times x$ is an isomorphism. A computation shows that if ϕ is in $\overline{\mathrm{Inn}}(e)$, then ϕ induces the identity element in H. Therefore, ϕ is not approximately inner. We have just shown that

$$\overline{\mathrm{Inn}}(e) \longrightarrow \mathrm{Aut}(e) \longrightarrow \mathrm{Aut}_\Lambda(\underline{K}_{\mathrm{six}}(e))$$

is not exact. ∎

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