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Dromions of flexural-gravity waves

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Here we show that weakly nonlinear flexural-gravity wave packets, such as those propagating on the surface of ice-covered waters, admit three-dimensional fully localized solutions that travel with a constant speed without dispersion or dissipation. These solutions, that are formed at the intersection of line-soliton mean-flow tracks, have exponentially decaying tails in all directions and are called *dromions* in contrast to *lumps* that decay only algebraically. We derive, by asymptotic expansion and assuming multiple scales for spatial and temporal variations, the three-dimensional weakly nonlinear governing equations that describe the coupled motion of the wavepacket envelope and the underlying mean current. We show that in the limit of long waves and strong flexural rigidity these equations reduce to a system of nonlinear elliptic–hyperbolic partial differential equations similar to the Davey–Stewartson I (DSI) equation, but with major differences in the coefficients. Specifically, and contrary to DSI equations, the elliptic and hyperbolic operators in the flexural-gravity equations are not canonical resulting in complications in analytical considerations. Furthermore, standard computational techniques encounter difficulties in obtaining the dromion solution to these equations owing to the presence of a spatial hyperbolic operator whose solution does not decay at infinity. Here, we present a direct (iterative) numerical scheme that uses pseudo-spectral expansion and pseudo-time integration to find the dromion solution to the flexural-gravity wave equation. Details of this direct simulation technique are discussed and properties of the solution are elaborated through an illustrative case study. Dromions may play an important role in transporting energy over the ice cover in the Arctic, resulting in the ice breaking far away from the ice edge, and also posing danger to icebreaker ships. In fact we found that, contrary to DSI dromions that only exist in water depths of less than 5 mm, flexural-gravity dromions exist for a broad range of ice thicknesses and water depths including values that may be realized in polar oceans.

Key words: sea ice, solitary waves, surface gravity waves

1. Introduction

Dromions are spatially localized (i.e. hump-like) surface structures that can travel with a constant speed without changing form, and hence can transport mass, momentum and more importantly energy over long distances. They are formed at the intersection of underlying mean-flow line solitons (the so-called ghost solitons)

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and take their name from the Greek word *dromos* which means tracks (Fokas & Santini 1990). These surface structures are three-dimensional counterparts of the well-known two-dimensional solitons, and similar to the two-dimensional solitons decay exponentially fast along horizontal coordinates. The only other known three-dimensional localized solution of water waves is the 'lump' solution that decays algebraically over distance. A number of three-dimensional systems are known to admit lumps, amongst them are the Kadomtsev–Petviashvili equation (e.g. Ablowitz & Segur 1979), the Benney–Luke equation (e.g. Berger & Milewsky 2000), the full Euler equation (e.g. Groves & Sun 2008), and the Davey–Stewartson equation (e.g. Kim & Akylas 2005, with algebraically decaying tails). However physically relevant dromion solutions are limited so far to the surface-tension-dominated regime of the Davey–Stewartson equation.

The Davey–Stewartson (DS) equation (Davey & Stewartson 1974) is a three-dimensional extension of the nonlinear Schrödinger equation (NLS). It is a set of two partial differential equations for the wavepacket envelope amplitude and the mean underlying flow. The DS equation in its general form has been a subject of extensive research (e.g. Champagne & Winternitz 1988; Clarkson & Hood 1994; Hizel, Turgay & Guldogan 2009). Of our particular interest are, however, the fully (spatially) localized surface solutions. Utilizing Bäcklund transformation Boiti *et al.* (1988) found the first spatially exponentially decaying localized solutions to the shallow-water DS equation in the limit of surface-tension-dominated regimes (the so-called DSI equation, see Djordjevic & Redekopp 1977 for a derivation). This solution was then rederived, extended and generalized by a number of techniques such as the inverse scattering transform (Fokas & Santini 1990, who first proposed the name *dromion*), the bilinear/direct method (Gilson & Nimmo 1991) and the Wronskian formulation (Hietarinta & Hirota 1990). These solutions are all, as stated above, for the surface-tension-dominated regime that requires (for water waves) a water depth of less than 5 mm, and hence are of limited practical applications.

We are interested in the possibility of existence of dromions, and the conditions under which they may appear, in flexural-gravity wave systems. This interest is motivated by the observations of large-amplitude waves penetrating far into the ice-covered areas of polar waters. Liu & Mollo-Christensen (1988), for instance, cite an observation of $\sim$1 m tall wave in the Weddell sea in the solid ice pack 560 km from the ice edge that resulted in the breakup of the ice pack (see also Marko 2003 for a more recent observation of similar phenomenon in the Sea of Okhotsk). To travel such distances, three-dimensional effects clearly play a significant role. Nevertheless, while linear and nonlinear two-dimensional flexural-gravity waves have received much attention (e.g. Forbes 1986) the literature on the three-dimensional problem is sparse and is focused mainly on the problem of moving pressure (i.e. load) on the ice (e.g. Miles & Sneyd 2003; Parau & Vanden-Broeck 2011).

Here, we consider the three-dimensional propagation of weakly nonlinear flexural-gravity wave packets similar to those propagating on the ice cover of the Arctic Ocean. We show that the governing equation for the evolution of wave packets in an ice-covered shallow sea reduces to a system of elliptic–hyperbolic Davey–Stewartson equations similar to the DSI equation but with different coefficients for which a closed-form dromion solution does not exist. Specifically, governing equations of flexural-gravity waves are non-canonical (due to elliptic and hyperbolic operators being out of sync) leading to complications in analytical considerations. For instance theorems on the existence and regularity of solutions of the DSI equation (e.g. Hayashi & Hirata 1996) do not apply in (and to our knowledge cannot be extended to) the present context. The elliptic–hyperbolic subfamily of the Davey–Stewartson equation is also not readily amenable to conventional computational techniques (owing to the presence of a spatial hyperbolic operator whose solution does not necessarily decay at infinity) and therefore has been the subject of only a few numerical considerations in the past. These studies generally target the time integration of the initial-value problem and require (non-vanishing) boundary conditions (as functions of time) to be specified *a priori*, say from an analytical solution (White & Weideman 1994; Besse, Mauser & Stimming 2004).

Here we devise a computational scheme to find dromion solutions to the general elliptic–hyperbolic subfamily of the Davey–Stewartson equation. We propose an iterative scheme based on the pseudo-spectral technique for the elliptic equation and pseudo-time integration for the hyperbolic equation to find a steadily translating dromion solution in a co-moving frame of reference whose speed and direction is also determined by our numerical method. Utilizing this approach we are able to find dromion solutions of the flexural-gravity wavepackets. The direct scheme converges very fast (order of minutes on a laptop) and is stable for a broad range of dromion geometries. As a side result, the numerical scheme developed here can also integrate in time the initial-value problem associated with the governing equations, and therefore can be used to study stability and time-evolution of dromions as well as any other initial condition.

The equation derived and results presented here are also relevant to propagation of waves on pontoon-type very large floating structures (VLFSs) such as floating airports (e.g. Japanese mega-float concept, Suzuki 2005), floating bridges, and in general in hydroelasticity (Korobkin, Parau & Vanden-Broeck 2011). The Davey–Stewartson equation has applications in other areas of science such as quantum field theory (Schultz, Ablowitz & Bar Yaacov 1987), ferromagnetism (Leblond 1999), plasma physics (Duan 2003) and nonlinear optics (Leblond 2001) where results and techniques developed here may have implications.

2. Governing equations

We consider the propagation of waves on a uniform thin elastic sheet (e.g. a layer of ice) overlying a fluid of depth h. We assume that the fluid is incompressible (density ρ), and the motion of fluid particles is irrotational so that the potential theory applies. We define a velocity potential ϕ such that $\boldsymbol{\nabla}\phi = \boldsymbol{u}$ where $\boldsymbol{u}$ is the velocity vector. We also define a Cartesian coordinate system with the x, y-axes along the (flat) seafloor, and z-axis positive upward. If $\eta(x, y, t)$ denotes the elevation of the elastic sheet (and hence water surface) from the mean surface level, then the governing equations read

$$\nabla^2\phi = 0, \quad 0 < z < h + \eta(x, y), \tag{2.1a}$$

$$\eta_t + \eta_x\phi_x + \eta_y\phi_y = \phi_z, \quad z = h + \eta(x, y), \tag{2.1b}$$

$$\phi_t + g\eta + \frac{1}{2}|\boldsymbol{\nabla}\phi|^2 + \frac{D}{\rho}\nabla^4\eta = 0, \quad z = h + \eta(x, y), \tag{2.1c}$$

$$\phi_z = 0, \quad z = 0, \tag{2.1d}$$

where $\nabla^4 = \partial_{xxxx} + 2\partial_{xxyy} + \partial_{yyyy}$ is the bi-Laplacian operator and $D = E\mathfrak{t}^3/12(1 - \nu^2)$ is the flexural rigidity of the elastic sheet in which $\mathfrak{t}$ is its thickness and E, ν are respectively its Young's modulus and Poisson's ratio. Note that in deriving (2.1c) we have assumed that propagating waves are much longer that the sheet thickness so that the thin-plate approximation is justified (see e.g. Strathdee, Robinson & Haines 1991, for a detailed discussion). We define the following dimensionless variables:

$$x^* = \frac{x}{\lambda}, \quad y^* = \frac{y}{\lambda}, \quad z^* = \frac{z}{h}, \quad \eta^* = \frac{\eta}{a}, \quad t^* = \frac{t\sqrt{gh}}{\lambda}, \quad \phi^* = \frac{\phi h}{\lambda a \sqrt{gh}}, \tag{2.2}$$

where λ and a are respectively the typical wavelength and amplitude of the surface wave. Using (2.2), (2.1) after dropping asterisks (for notational simplicity) becomes

$$\delta^2(\phi_{xx} + \phi_{yy}) + \phi_{zz} = 0, \quad 0 < z < 1 + \epsilon\eta, \tag{2.3a}$$

$$\delta^2[\eta_t + \epsilon(\eta_x\phi_x + \eta_y\phi_y)] = \phi_z, \quad z = 1 + \epsilon\eta, \tag{2.3b}$$

$$\phi_t + \eta + \frac{1}{2}\epsilon\left(\phi_x^2 + \phi_y^2 + \frac{1}{\delta^2}\phi_z^2\right) + H(\eta_{xxxx} + 2\eta_{xxyy} + \eta_{yyyy}) = 0, \quad z = 1 + \epsilon\eta, \tag{2.3c}$$

$$\phi_z = 0, \quad z = 0, \tag{2.3d}$$

where

$$H = \frac{D}{\rho g \lambda^4}, \quad \epsilon = \frac{a}{h}, \quad \delta = \frac{h}{\lambda}. \tag{2.4}$$

We are interested in a harmonic wave of wavenumber k with a slowly varying amplitude in both x, y directions. To achieve this solution we assume $\epsilon \ll O(1)$ and introduce the following different-scale variables:

$$\xi = x - c_p t, \quad \zeta = \epsilon(x - c_g t), \quad Y = \epsilon y, \quad \tau = \epsilon^2 t, \tag{2.5}$$

where $c_p(k), c_g(k)$ are to be determined (at this stage we leave $c_p(k), c_g(k)$ as arbitrary unknowns, but later will show that they correspond to the phase and group velocities). We further assume that the solution to the governing equations (2.3) can be expressed by a convergent series in terms of our small parameter ϵ. In terms of new variables (2.5) we suggest the form

$$\phi(\xi, \zeta, Y, z, \tau) = f_0(\zeta, Y, \tau) + \sum_{n=0}^{\infty} \epsilon^n \left\{ \sum_{m=0}^{n+1} F_{nm}(z, \zeta, Y, \tau) E^m + \text{c.c.} \right\}, \tag{2.6}$$

$$\eta(\xi, \zeta, Y, \tau) = \sum_{n=0}^{\infty} \epsilon^n \left\{ \sum_{m=0}^{n+1} A_{nm}(\zeta, Y, \tau) E^m + \text{c.c.} \right\}, \tag{2.7}$$

where $E = \exp(ik\xi)$, $A_{00} = 0$ and c.c. represents complex conjugates. The aim is to find the equation governing the evolution of $A_{01}(\zeta, Y, \tau)$. The procedure of getting this equation is algebraically tedious, but standard (e.g. Johnson 1997) and we will only highlight major steps.

At the leading order, i.e. $O(\epsilon^0) = O(1)$, a progressive wave solution with amplitude $A_{01}(\zeta, Y, \tau)$ results in an expression for the unknown $c_p = \omega/k$:

$$c_p^2 = \frac{\tanh \delta k}{\delta k}\beta \tag{2.8}$$

with $\beta = 1 + Hk^4$. Equation (2.8) is called the dispersion relation of the system (2.3) and reduces to the gravity-wave dispersion relation at the limit of $H = 0$. At the first order, i.e. $O(\epsilon^1)$, c_g is obtained in the form

$$c_g = \frac{1}{2}c_p\left(1 + \frac{2\delta k}{\sinh 2\delta k} + \frac{2\alpha}{\beta}\right), \tag{2.9}$$

where $\alpha = 2Hk^4$. It can be shown that $c_g = \mathrm{d}\omega/\mathrm{d}k$, and is in fact the group velocity of the system (2.3). Proceeding to the second order, i.e. $O(\epsilon^2)$, the governing equation for $A_0 \equiv A_{01}(\zeta, Y, \tau)$ is obtained in the form of two coupled partial differential equations:

$$-2\mathrm{i}kc_pA_{0,\tau} - kc_p\omega''(k)A_{0,\zeta\zeta} - c_pc_gA_{0,YY} + k^2(2c_p + \beta c_g \mathrm{sech}^2\delta k)A_0f_{0,\zeta} + \frac{k^2}{2c_p^2}\Gamma A_0 |A_0|^2 = 0, \tag{2.10a}$$

$$(1 - c_g^2)f_{0,\zeta\zeta} + f_{0,YY} = -\frac{\beta^2}{c_p^2}\left(\frac{2c_p}{\beta} + \frac{c_g}{\cosh^2\delta k}\right)\left(|A_0|^2\right)_{,\zeta} \tag{2.10b}$$

where $q\Gamma = p_1(\tilde{h}) + p_2$ in which, if we define $\tilde{h} \equiv Hk^4$ and $\sigma \equiv \tanh\delta k$, then $q = (-15 + \sigma^2)\tilde{h} + \sigma^2$ and

$$p_1(\tilde{h}) = (-62\sigma^2 + 34\sigma^4 - 2\sigma^6 + 30)\tilde{h}^4 + (115\sigma^4 - 8\sigma^6 + 159 - 318\sigma^2)\tilde{h}^3 + (-462\sigma^2 + 141\sigma^4 - 12\sigma^6 + 237)\tilde{h}^2 + (73\sigma^4 - 218\sigma^2 + 117 - 8\sigma^6)\tilde{h}, \tag{2.11}$$

$$p_2 = 9 - 12\sigma^2 + 13\sigma^4 - 2\sigma^6. \tag{2.12}$$

We can further simplify (2.10) by defining new variables

$$\left.\begin{array}{l} \tau^\dagger = \tau\dfrac{c_g}{2k}, \quad \zeta^\dagger = \dfrac{\zeta}{\sqrt{|c_g^2 - 1|}}, \quad A^\dagger = A\sqrt{\dfrac{k^2|\Gamma|}{8c_p^3c_g}}, \\ f_0^\dagger = f_0\dfrac{B}{2c_pc_g\sqrt{|c_g^2 - 1|}}, \quad Y^\dagger = Y, \end{array}\right\} \tag{2.13}$$

where $B = k^2(2c_p + \beta c_g\mathrm{sech}^2\delta k)$. Dropping daggers, (2.10) yields

$$\mathrm{i}A_{0,\tau} + \frac{k\omega''(k)s_1}{c_g(c_g^2 - 1)}A_{0,\zeta\zeta} + A_{0,YY} - 2A_0f_{0,\zeta} - 4s_2A_0|A_0|^2 = 0, \tag{2.14a}$$

$$s_1f_{0,\zeta\zeta} - f_{0,YY} - 4\mathscr{E}s_1s_2\left(|A_0|^2\right)_{,\zeta} = 0, \tag{2.14b}$$

where $\mathscr{E} = \beta B^2/[\Gamma k^4(c_g^2 - 1)]$, $s_1 = \mathrm{sign}(c_g^2 - 1)$ and $s_2 = \mathrm{sign}\ \Gamma$. In the absence of flexural compliance ($H = 0$) (2.14*a*) and (2.14*b*) reduce respectively to (2.15) and (2.14) of Davey & Stewartson (1974).

Note that here we used a nonlinear-flow/linear-plate model equation (similar to e.g. Hărăguş-Courcelle & Il'ichev 1998; Parau & Vanden-Broeck 2011) that for (relatively) large-amplitude deformation of the elastic sheet may be physically restrictive. To account for nonlinearities in higher deflections a number of modification may be incorporated such as using a nonlinear Kirchhoff–Love model (Milewski, Vanden-Broeck & Wang 2011), or von Kármán's theory (that considers in-plane forces but assumes small slopes, e.g. Chen *et al.* 2003). Incorporation of these models results in modified coefficients in (2.10). The numerical technique of § 3 is for general values of coefficients and can be used to look for localized surface solutions of these modified equation as well.

3. Numerical scheme for a dromion solution

Dromions are three-dimensional fully localized and spatially exponentially decaying surface structures that can move with a constant speed without any change in their

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form. In the context of water waves it is known that the surface-tension-dominated regime of the shallow-water Davey–Stewartson equation, the so-called DSI limit, admits a dromion solution. In this section we show that the governing equations for flexural-gravity wavepackets can also admit dromion solutions. We devise a pseudo-spectral iterative scheme along with a pseudo-time integration method to find the exact form of these solutions in the co-moving frame of reference.

Let us consider a supercritical case ($c_g > 1$) of (2.14), therefore $s_1 = +1$. In this case and to get a canonical form of the governing equation we further introduce the following change of variables:

$$v = f_{0,\zeta} - 4\mathscr{E} s_2 \left|A_0\right|^2, \quad u = A_0\sqrt{|1+2\mathscr{E}|}, \tag{3.1}$$

by which we obtain from (2.14):

$$\mathrm{i}u_\tau + p u_{\zeta\zeta} + u_{YY} - 2uv - 4 s_2 s_3 u \left|u\right|^2 = 0, \tag{3.2a}$$

$$v_{\zeta\zeta} - v_{YY} - 4 q s_2 s_3 \left(|u|^2\right)_{,YY} = 0, \tag{3.2b}$$

where $p = k\omega''(k)/[c_g(c_g^2 - 1)]$, $q = \mathscr{E}/(1+2\mathscr{E})$ and $s_3 = \text{sign } 1 + 2\mathscr{E}$.

We now consider the limit of shallow water ($k\delta = \mu \ll 1$) and further assume $H = O(1/\mu^2)$(note that for stiff materials such as ice H is typically a very large number). Defining $\mathscr{H} \equiv H\mu^2/\delta^4$ the coefficients of (3.2) simplify to

$$p = \frac{1-10\mathscr{H}}{1-5\mathscr{H}}, \quad q = \frac{1-15\mathscr{H}}{1-25\mathscr{H}}, \tag{3.3}$$

and $s_1 = \text{sign}(5\mathscr{H} - 1)$, $s_2 = \text{sign}(1 - 15\mathscr{H})$, $s_3 = \text{sign} - (1 - 25\mathscr{H})/(1 - 5\mathscr{H})$. Clearly if $s_1 = +1$ (as assumed before), then $s_2 = s_3 = -1$ and (3.2) turns into

$$\mathrm{i}u_\tau + p u_{\zeta\zeta} + u_{YY} - 2uv - 4u\left|u\right|^2 = 0, \tag{3.4a}$$

$$v_{\zeta\zeta} - v_{YY} - 4q\left(|u|^2\right)_{,YY} = 0. \tag{3.4b}$$

If parameters p and q could take the value of unity then (3.4) would turn into the famous Davey–Stewartson I (DSI) equation for which an analytical dromion solution exists. In the case of our interest, and based on the supercritical assumption $c_g > 1$, the ranges of coefficients are $2 \leqslant p \leqslant \infty$ and $1/2 \leqslant q \leqslant 3/5$ for which a closed-form solution to (3.4) is unavailable, and standard direct numerical simulations are very limited due to the complicated nature of these equations. Specifically (3.4) is a set of respectively an elliptic and a hyperbolic nonlinear partial differential equation. Spatial hyperbolicity of (3.4*b*) causes difficulties for many numerical techniques such as finite difference (by making the coefficient matrix poorly conditioned) and spectral methods (by not allowing same wavenumber (i.e. $k_x = k_y$) modes to exist). To make the matter more complicated, the solution to $v(\zeta, Y, t)$ (as we will show later) is not periodic in any direction, and does not decay to zero at infinity, but to non-zero asymptotic *tracks* whose forms are not known *a priori* (White & Weideman 1994; Besse *et al.* 2004).

Here we present an iterative computational scheme based on pseudo-spectral expansion and pseudo-time integration that can converge to the dromion solution of the general form of system (3.4). We first note that the dromion solution to (3.4), if it exists, is stationary in a co-moving frame of reference (whose speed and direction are also to be determined). We therefore re-write governing equations (3.4) in a frame of reference moving with a constant speed along a general velocity vector. Specifically we define new independent variables $\zeta^* = \zeta - c_\zeta\tau$ and $Y^* = Y - c_Y\tau$. We further assume a time-dependent phase for u in the co-moving frame of reference, i.e. $u(\zeta^*, Y^*, \tau) = u^*(\zeta^*, Y^*) \exp(\mathrm{i}\alpha\tau)$. Dropping asterisks, (3.4) now becomes

$$-\alpha u - \mathrm{i}c_\zeta u_\zeta - \mathrm{i}c_Y u_Y + p u_{\zeta\zeta} + u_{YY} - 2uv - 4u\,|u|^2 = 0, \tag{3.5a}$$

$$v_{\zeta\zeta} - v_{YY} - 4q\,(|u|^2)_{,YY} = 0, \tag{3.5b}$$

where u, v are now functions of ζ, Y only.

A dromion solution has a fully localized (exponentially decaying) surface profile u, and two intersecting tracks for v (the hump of the surface elevation profile is at the intersection of tracks). The iterative computational algorithm consists of three steps: (i) starting from an initial guess for (u, v) we solve (3.5*a*) to find an updated (corrected) profile for u; (ii) this new u is substituted in (3.5*b*) and an exact v solution of (3.5*b*) is found; (iii) this new v along with the updated u (obtained in step (i)) are used as new initial conditions for step (i) and the loop is repeated until convergence is achieved.

To solve (3.5*a*), i.e. step (i), a pseudo-spectral method, in which nonlinear terms are calculated in physical space and linear terms in spectral space, can be utilized. Dealing with equation (3.5*b*), i.e. step (ii), is however more involved. Equation (3.5*b*) is a spatial hyperbolic equation with a forcing term (last term on the left-hand side) that decays exponentially to zero as we move away from the centre. Nevertheless, the solution v, in general, does not decay to zero away from the centre and is not periodic in any spatial direction. Therefore the boundary condition for v in a finite domain of our interest, say $(\zeta, Y) \in [-\pi, \pi] \times [-\pi, \pi]$, is also unknown. In our scheme we treat one of spatial coordinates (say Y) in (3.5*b*) as a pseudo-time. As a result (3.5*b*) can be considered a forced wave equation with only one unknown, i.e. the initial condition at $Y = -\pi$, that is, $v(\zeta, -\pi)$. Evolving this equation over the pseudo-time Y and under the effect of external forcing, i.e. $4q\,(|u|^2)_{,YY}$ in (3.5*b*), the solution $v(\zeta, Y)$ is obtained. The updated values of the solution u, v are better approximations (than the initial guess) to (3.5*a*) and are iterated again (step (iii)) until convergence is achieved. Intermediate steps, such as error treatment say by a Newton–Raphson method, may accelerate the convergence rate.

To make our direct computational scheme more efficient, we divide our system (3.5) into the following linear and nonlinear terms:

$$F(u) + \mathscr{F}(u, v) = 0, \quad G(v) + \mathscr{G}(u) = 0, \tag{3.6}$$

where F, G are linear elliptic/hyperbolic operators and $\mathscr{F}, \mathscr{G}$ are nonlinear functions of their arguments (e.g. F contains the first five terms on the left-hand side of (3.5*a*) and so on). We also write $u = u_0 + u_p$, $v = v_0 + v_p$ where u_0, v_0 are base solutions (i.e. an approximate solution) and u_p, v_p are corrections to the base (not necessarily small). Therefore, (3.6) can be written in the form

$$F(u_p) = -\mathscr{F}(u_0 + u_p, v_0 + v_p) - F(u_0), \tag{3.7a}$$

$$G(v_p) = -\mathscr{G}(u_0 + u_p) - G(v_0). \tag{3.7b}$$

We take as the base solution (u_0, v_0), the analytical one-dromion solution of the DSI equation (i.e. (3.4) with $p = q = 1$), which is in terms of our variables (e.g. Gilson & Nimmo 1991)

$$u(\zeta, Y, \tau) = 2\sqrt{\delta\vartheta_r\theta_r}\frac{\varphi\overline{\psi}}{\mathscr{W}}\mathrm{e}^{-\mathrm{i}\alpha\tau}, \quad v(\zeta, Y, \tau) = -2\partial_{YY}\log\mathscr{W}, \tag{3.8}$$

where

$$\mathscr{W} = 1 + c_1 e^{2\varsigma} + c_2 e^{-2\varrho} + c_3 e^{2\varsigma - 2\varrho}, \tag{3.9}$$

$$\left.\begin{aligned} \varsigma &= \frac{\sqrt{2}}{2}\vartheta_r[(\zeta + Y) - (2\sqrt{2}\vartheta_i - c_\zeta - c_Y)\tau], \\ \varrho &= \frac{\sqrt{2}}{2}\theta_r[(-\zeta + Y) - (2\sqrt{2}\theta_i + c_\zeta - c_Y)\tau], \end{aligned}\right\} \tag{3.10}$$

with arbitrary $c_1, c_2, c_3 > 0$ provided that $\delta = c_1 c_2 - c_3 > 0$; also $\vartheta = \vartheta_r + \mathrm{i}\vartheta_i, \theta = \theta_r + \mathrm{i}\theta_i$ and $\alpha = |\vartheta|^2 + |\theta|^2$. The functions

$$\varphi = \exp\left\{\frac{\sqrt{2}}{2}[\vartheta(\zeta + Y) + (\mathrm{i}\vartheta^2\sqrt{2} + \vartheta c_\zeta + \vartheta c_Y)\tau]\right\}, \tag{3.11a}$$

$$\psi = \exp\left\{\frac{-\sqrt{2}}{2}[\theta(-\zeta + Y) + (\mathrm{i}\theta^2\sqrt{2} - \theta c_\zeta + \theta c_Y)\tau]\right\}, \tag{3.11b}$$

are solutions to the Schrödinger equations $\mathrm{i}\varphi_t - \mathrm{i}c_\zeta\varphi_\zeta - \mathrm{i}c_Y\varphi_Y + (1/2)\,(\partial_\zeta + \partial_Y)^2\,\varphi = 0$ and $\mathrm{i}\psi_t - \mathrm{i}c_\zeta\psi_\zeta - \mathrm{i}c_Y\psi_Y - (1/2)\,(\partial_\zeta - \partial_Y)^2\,\psi = 0$. Clearly if we choose $c_\zeta = \sqrt{2}(\vartheta_i - \theta_i)$ and $c_Y = \sqrt{2}(\vartheta_i + \theta_i)$, then v becomes stationary and u only has a periodic phase (i.e. $-\alpha t$) but does not travel/deform over time.The above solution can be obtained from a modified inverse-scattering transform technique (Fokas & Santini 1990) or bilinear method (Gilson & Nimmo 1991).

Before presenting numerical results we would like to comment that the relation between the shallow-water limit of the DS equation and the weakly nonlinear shallow-water models such as the Benney–Luke (BL) (Benney & Luke 1964) or KP equation (Kadomtsev & Petviashvili 1970) is similar to that of the shallow-water NLS versus Korteweg–de Vries (KdV) equations. For the latter, it is straightforward to show that the long-wave limit of NLS matches the short-wave limit of KdV when rewritten for the envelope (e.g. Johnson 1997; note that the DS and NLS family govern the evolution of the envelope of carrier waves whereas KdV, BL and KP govern the primitive waves). In $2 + 1$ dimension, BL and KP further assume a slow (i.e. different scale) transverse variation (cf. Osborne 2009, §§1.4.2 and 2.4.2). If this together with the long-wave assumption is employed then the DS equation can be reduced to an envelope equation derived from the KP equation (Freeman & Davey 1975). Therefore (3.4) may also be obtained if we first derive the KP equation for flexural-gravity waves and calculate the envelope under proper assumptions discussed above, or equivalently the KP equation for flexural-gravity waves can be derived from (3.4). In their original forms, however, shallow-water DS and KP equations are distinct and admit different mathematical properties. For instance the shallow-water limit of the DS equation includes explicit coupling with the mean field whereas KP does not.

4. Results and discussion

For illustrating the performance of direct simulation and properties of a flexural-gravity dromion we pick a base solution (3.8) with parameters $c_1 = 2, c_2 = 1, c_3 = 1$ and $\vartheta = 2 + 4\mathrm{i},\ \ \theta = 3 + 0.5\mathrm{i}$. It then follows that if we move to a moving coordinate system with speeds $c_\zeta = 4.95$, $c_Y = 6.36$ the base profile will be stationary, except for a time-dependent phase with the coefficient $\alpha = 29.25$. A dromion solution to (3.5)

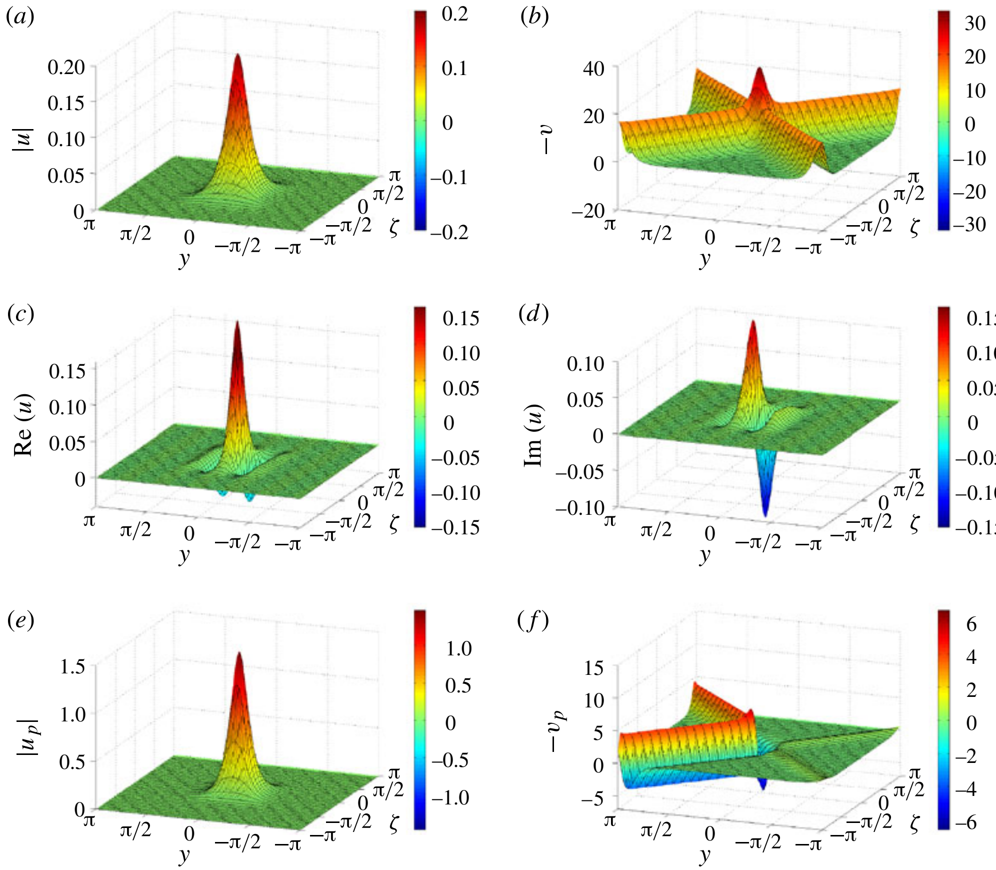

FIGURE 1. A dromion solution of the flexural-gravity wavepackets. This solution is obtained from our numerical iterative scheme near a base dromion of the DSI equation (3.5) with parameters $c_1 = 2, c_2 = c_3 = 1$ and $\vartheta = 2 + 4\mathrm{i}$, $\theta = 3 + 0.5\mathrm{i}$. Computation parameters are $N_x = N_y = 256$. For pseudo-time evolution of hyperbolic equation (3.5*b*) we take $\delta Y/\delta\zeta = 1/4$. The solution has converged with a relative error of the order of computer accuracy, i.e. $O(10^{-15})$. (*a*) The profile of wavepacket envelope u; (*b*) underlying current v (negative values are shown for clearer presentation); (*c,d*) real and imaginary parts of u; (*e,f*) correction values u_p, v_p.

with $p = 2, q = 3/5$ is obtained using this base solution, with initial conditions $u_p = v_p = 0$ and by the numerical scheme formulated above. This dromion solution is shown in figure 1. Specifically, figure 1(*a*) shows the magnitude of the envelope of waves u, figure 1(*b*) shows the underlying tracks (note that v is related to the current below the surface through (3.1)), figure 1(*c,d*) plots real and imaginary parts of u and figure 1(*e,f*) plots the difference between dromions of the flexural-gravity waves and the DSI equation.

The dromion amplitude of flexural-gravity waves is much smaller than that of the DSI equation (figure 1*a,e*) while their general geometry is similar. The difference in the amplitude may reach an order of magnitude or higher for narrower dromions. The correction in v is however relatively small ($\sim$%25, cf. figure 1*b,f*). The converged velocities are $c_\zeta = 6.32$, c_Y= 6.60, and $\alpha = 28.96$. Therefore the direction of motion of this dromion is $\sim 6^\circ$ to the right of the base dromion. Note that the relative value of $|u_p|/|u| > O(1)$, and therefore u_p cannot be treated as a perturbation to the DSI dromion in (3.7). This is not unexpected as small changes in the coefficient of PDEs are known to be able to easily change even the qualitative behaviour of the solution. In fact it was not obvious to us that (3.8) would admit a dromion solution even if the DSI equation has such a solution until our computation proved the existence of such solutions. A broad range of dromion solutions to (3.8) for a variety of parameters can be computed by the proposed scheme. Flexural-gravity dromions differ (sometimes significantly) in size, direction of motion and symmetricity from those of the DSI equation.

Mean-field tracks (that propagate in the water beneath the ice cover) are associated with no elevation in physical space (i.e. ice cover), and hence are sometimes called ghost solitons (Hietarinta 1990; Radha & Lakshmanan 1994). It is to be noted, nevertheless, that these structures correspond to a flow field in the fluid context and therefore carry energy. Underlying mean-field tracks of dromions, mathematically speaking, need to either extend to infinity or to proper boundary conditions in the case where they are in a (semi-) confined space. Clearly, for dromions to appear in a physical space (i.e. in a real ice-covered sea) the mean field does not necessarily need to extend to infinity, but just to an order of magnitude longer than the typical length scale of the problem (e.g. dromion size). For the farther distance the existence/shape of the mean field will be governed by a higher-order equation (cf. Hogan (1985), but the extension has not been pursued here).

Understanding of stability of the dromion solution is important particularly for practical applications (i.e. their being observed). Nevertheless, such analysis is not straightforward (some details follow) and a comprehensive consideration deserves an independent study. The Lyapunov stability analysis (commonly used for instance for the study of localized structures of the NLS equation, e.g. Kuznetsov, Rubenchik & Zakharov 1986) requires the existence of a Hamiltonian. A dromion structure clearly lacks a Hamiltonian due to its tracks extending beyond (and exchanging energy via) finite boundaries. However, an indication of stability can be obtained if the governing equation (3.4) is integrated in time with the initial condition set as its dromion solution (Nishinari & Yajima 1994). We write down the governing equation in a co-moving frame of reference (cf. equation (3.5)) but retain the time-derivative term. A fourth-order Runge–Kutta method is used for the integration in time and $\delta t = 5 \times 10^{-5}$, for which the time-integration is convergent. Other parameters are kept the same as in § 3. Following Nishinari & Yajima (1994) we set the boundary conditions by the exact solution (and keep them unperturbed), and compare the maximum of the amplitude $|u|_{max}$ and the first conserved quantity $I_1 = \int |u|^2 \,\mathrm{d}\zeta \,\mathrm{d}Y$. The solution stays stable to computational perturbations with oscillating but bounded relative errors of $[|u|_{max}(t) - |u|_{max}(0)]/|u|_{max}(0) <$ %0.01 and $[I_1(t) - I_1(0)]/I_1(0) <$ %0.1.

While details of applied aspects of the presented solution are beyond the scope of this paper, we would like to briefly comment on the real-life relevance of assumptions made here. Ice flexural rigidity is of the order of $D \sim 0.9t_{ice}^3$ GPa where t_{ice} is the ice thickness. For this rigidity and for an ice thickness of $t_{ice} \sim 2$ m in water of depth $\sim$3.5 m (measured below the ice) waves with wavelengths $\sim$40 m satisfy assumptions made here and may form a dromion of amplitude $\sim$1 m whose size (i.e. wavelength) is $\sim$300 m. These numbers are consistent with the rough observation report of Liu & Mollo-Christensen (1988), except for the water depth that is missing in their report. Our assumption for the water depth is, however, not unrealistic as the incident occurred $\sim$500 km inside the solid icepack.

5. Conclusion

In this paper we have shown that governing equations for flexural-gravity wavepackets admit the *dromion* solution, a three-dimensional fully localized and spatially exponentially decaying surface structure overlying intersecting mean-flow tracks (ghost solitons). Dromions of flexural-gravity waves exist for a broad range of wavelength and water depth, in contrast to dromions of the DSI equation that only exist for water depth of less than 5 mm.

We showed that the governing equation of the envelope of the wavepacket and the underlying current is a set of two nonlinear partial differential equations that each can be elliptic and/or hyperbolic depending on the values of chosen parameters. Specifically, we showed that in the limit of strong flexural rigidity and long waves, the system becomes an elliptic/hyperbolic set similar to the DSI equation but in a non-canonical form whose closed-form dromion solution is elusive.

We developed an iterative direct simulation scheme, using simultaneously a pseudo-spectral method for the elliptic equation and a pseudo-time integration technique for the hyperbolic equation. The scheme, when initialized near the 'cousin' dromions of the DSI equation, efficiently converges to the dromion solution of flexural-gravity waves whose size and direction can be quite different from the base solution (sometimes by an order of magnitude).

Dromions are efficient mechanisms for transporting mass, momentum and energy over long distances. In the Arctic they may be behind the presence of large-amplitude flexural-gravity waves deep inside the icepack that result in the ice cracking/breaking and in posing dangers to icebreaker ships.

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