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INCREASING CONVEX ORDER ON GENERALIZED AGGREGATION OF SAI RANDOM VARIABLES WITH APPLICATIONS

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Abstract

In this paper we study general aggregation of stochastic arrangement increasing random variables, including both the generalized linear combination and the standard aggregation as special cases. In terms of monotonicity, supermodularity, and convexity of the kernel function, we develop several sufficient conditions for the increasing convex order on the generalized aggregations. Some applications in reliability and risks are also presented.

Keywords: Arrangement increasing; coverage limit; deductible; generalized linear combination; majorization; submodular; weighted k-out-of-n system

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1. Introduction

The linear combination of independent random variables has drawn much attention from researchers and many interesting studies have been completed in the past twenty years, enriching the literature on this topic quite considerably. For example, for independent and identically distributed (i.i.d.) random variables $X_1, \dots, X_n$ with X_i^p having a log-concave density for some $p \in (0, 1)$, Karlin and Rinott (1983) showed that $\sum_{i=1}^n b_i X_i \le_{\text{st}} \sum_{i=1}^n a_i X_i$ whenever $(b_1^q, \dots, b_n^q) \preceq_{\text{m}} (a_1^q, \dots, a_n^q)$ for $p^{-1} + q^{-1}$ with $q < 0$ (see Section 2 for the usual stochastic order '$\le_{\text{st}}$' and majorization '$\preceq_{\text{m}}$'). Yu (2011) proved that, for $X_1, \dots, X_n$ nonnegative and i.i.d., $(\ln b_1, \dots, \ln b_n) \preceq_{\text{m}} (\ln a_1, \dots, \ln a_n)$ implies that $\sum_{i=1}^n b_i X_i \le_{\text{st}} \sum_{i=1}^n a_i X_i$ whenever $\ln X_i$ is of a log-concave density. Xu and Hu (2011) extended this ordering result to independent but not necessarily identically distributed random variables with some regularity conditions. For more on ordering properties of standard linear combinations of independent random variables, we refer the reader to Bock *et al.* (1987), Kijima and Ohnishi (1996), Ma (2000), Hu and Lin (2001), Khaledi and Kochar (2001), Korwar (2002), Khaledi and Kochar (2004), Nadarajah and Kotz (2005), Manesh and Khaledi (2008), Amiri *et al.* (2011), Kochar and Xu (2010), (2011), Zhao (2011), Zhao *et al.* (2011), Manesh and Khaledi (2015) and the references therein.

Due to the recurring interest in accumulation and aggregation in insurance, finance, operations research, reliability and many other areas, some authors have studied various generalizations of linear combinations of random variables. For example, for independent $X_1, \dots, X_n$ ascendingly arrayed in the likelihood ratio order, twice continuously differentiable, and strictly

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monotone functions $\xi, \gamma\colon \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that $\xi''(x) \geq 0$ and $\xi''(x)\gamma''(y)\xi(x)\gamma(y) \geq [\xi'(x)\gamma'(y)]^2$ for all $(x, y) \in \mathbb{R}_+^2$, Pan *et al.* (2013) proved that the majorization $(b_1, \dots, b_n) \preceq_{\mathrm{m}} (a_1, \dots, a_n)$ implies that

(i) $\sum_{i=1}^n \xi(b_{(i)})\gamma(X_i) \leq_{\mathrm{st}} \sum_{i=1}^n \xi(a_{(i)})\gamma(X_i)$ for decreasing ξ and increasing γ, and

(ii) $\sum_{i=1}^n \xi(b_{(n-i+1)})\gamma(X_i) \leq_{\mathrm{st}} \sum_{i=1}^n \xi(a_{(n-i+1)})\gamma(X_i)$ for increasing ξ, γ, and the X_i with log-concave densities, where $a_{(1)} \leq \dots \leq a_{(n)}$ is the increasing rearrangement of $a_1, \dots, a_n$.

Also, Mao *et al.* (2013) obtained the result parallel to that of Pan *et al.* (2013) for $X_1, \dots, X_n$ mutually independent and ascendingly arrayed in the hazard rate order.

On the other hand, some recent research has been devoted to ordering generalized linear combinations of dependent random variables $\sum_{i=1}^n \phi(X_i, d_i)$, where $(X_1, \dots, X_n)$ usually represents potential risks, $(d_1, \dots, d_n)$ denotes the corresponding parameters such as deductibles, coverage limits, allocated capitals, and so on, and $\phi(x, d)$ is a bivariate function measuring the consequence of matching d to x; see, for example, Cheung and Yang (2004), Cheung (2006), Li and You (2015), and Manesh and Khaledi (2015).

In reliability theory, for engineering systems subject to shocks arriving at random and producing decay random damages, the engineer has to study the aggregated damage suffered by all system components at time $t > 0$ so as to approximate system reliability. For example, the accumulated damage $\sum_{i=1}^{N(t)} W_i \mathrm{e}^{-\delta_i(t-S_i)}$ of a system subject to shocks arriving according to a renewal process $N(t)$ with renewal times $S_1, S_2, \dots$ and the sequence of corresponding annealing damages $W_1, W_2, \dots$. Also, the weighted k-out-of-n system has the cumulative weight $\sum_{i=1}^n w_i \mathbf{1}(X_i > t)$ at time t, where the X_i are component lifetimes and the w_i are corresponding weights. It is plain that the reliability of the above two systems is determined by the stochastic behavior of the aggregated damage and the accumulated weight, respectively. For more on these two systems, we refer the reader to Nakagawa (2007) and Li *et al.* (2016). On the other hand, in the area of risk management, actuaries and financial engineers have to handle the aggregated potential losses of risks covered by an insurance policy or are concerned with a financial portfolio in order to control it at a satisfactory level. For example, following the work of Cheung (2007) on optimization problems on $\sum_{i=1}^n (X_i - d_i)_+ \mathrm{e}^{-\delta T_i}$ and $\sum_{i=1}^n (X_i \wedge d_i)_+ \mathrm{e}^{-\delta T_i}$, where the X_i are random losses with corresponding occurrence frequencies T_i, the research was further generalized by Hua and Cheung (2008b), Zhuang *et al.* (2009), Hu and Wang (2009), Lu and Meng (2011), and Li and You (2012) for policy risks with comonotone distribution, exchangeable distribution, and dependent distribution with Archimedean copulas. Remarkably, Cai and Wei (2014), (2015) developed ordering results on the optimal deductibles for stochastic arrangement increasing potential losses along with stochastically arrangement decreasing and Li and You (2015) investigated such generalized linear combinations for stochastically arrangement occurrence frequencies during their pioneering study on stochastic arrangement increasing random vectors and some nice weak variants.

In this paper we deal with the generalized linear combination $\sum_{i=1}^n W_i \phi(X_i, d_i)$, where $(W_1, \dots, W_n)$ and $(X_1, \dots, X_n)$ are mutually independent random vectors, and $(d_1, \dots, d_n)$ is a response vector from engineers, actuaries, investors, and so on. The paper is organized as follows. In Section 2 we review some basic concepts and several important facts to be used in formulating our main theoretical results. In Section 3 we build several technical lemmas, which are useful in developing our main theoretical results. The main results on the increasing convex order of the generalized aggregations are presented in Section 4. In Section 5 we present several applications of the ordering results in reliability and actuarial risk management as illustrations.

Throughout this paper, the terms *increasing* and *decreasing* stand for nondecreasing and nonincreasing, respectively, and all expectations are implicitly assumed to be finite.

2. Some preliminaries

For ease of reference, in this section we recall some important notions concerned with our study on the generalized aggregation, including majorization, Schur-convexity, Schur-concavity, submodular function and supermodular function, some stochastic orders, the arrangement increasing function, and the stochastic arrangement increasing random vectors.

Definition 2.1. For two real vectors $\boldsymbol{x} = (x_1, \dots, x_n)$ and $\boldsymbol{y} = (y_1, \dots, y_n)$, $\boldsymbol{y}$ is said to

(i) *majorize* $\boldsymbol{x}$, denoted by $\boldsymbol{x} \preceq_{\mathrm{m}} \boldsymbol{y}$, if $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$ and $\sum_{i=1}^j x_{(i)} \geq \sum_{i=1}^j y_{(i)}$ for $j = 1, \dots, n-1$;

(ii) *weakly submajorize* $\boldsymbol{x}$, denoted by $\boldsymbol{x} \preceq_{\mathrm{w}} \boldsymbol{y}$, if $\sum_{i=1}^j x_{(i)} \geq \sum_{i=1}^j y_{(i)}$ for $j = 1, \dots, n$.

For two nonnegative real vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, it is clear that $\boldsymbol{y} \preceq_{\mathrm{m}} \boldsymbol{x}$ implies that $\boldsymbol{y} \preceq_{\mathrm{w}} \boldsymbol{x}$, but the reverse is not necessarily true. As partial orders on the diversity of the components of real vectors, the notions of majorization are useful in risk management, operations research, and reliability theory.

A real function ϕ on a set $A \subseteq \mathbb{R}^n$ is called *Schur-concave* (*Schur-convex*) on A if $\boldsymbol{x} \succeq_{\mathrm{m}} \boldsymbol{y}$ implies that $\phi(\boldsymbol{x})(\leq (\geq))\phi(\boldsymbol{y})$ for any $\boldsymbol{x}, \boldsymbol{y} \in A$, and ϕ is called log-concave on $A \subset \mathbb{R}^n$ if A is a convex set and $\phi(\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y}) \geq [\phi(\boldsymbol{x})]^{\alpha}[\phi(\boldsymbol{y})]^{1-\alpha}$ for any $\boldsymbol{x}, \boldsymbol{y} \in A$ and $\alpha \in [0, 1]$. For a comprehensive review on majorization and Schur-convexity with applications, we refer the reader to Marshall *et al.* (2011).

Definition 2.2. A real-valued function $\varphi \colon \mathbb{R}^n \to \mathbb{R}$ is said to be *supermodular* (*submodular*) if $\varphi(\boldsymbol{x} \vee \boldsymbol{y}) + \varphi(\boldsymbol{x} \wedge \boldsymbol{y})(\geq (\leq))\varphi(\boldsymbol{x}) + \varphi(\boldsymbol{y})$ for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, where $\vee$ and $\wedge$ denote the componentwise maximum and the componentwise minimum, respectively.

A function φ with finite second partial derivatives on $\mathbb{R}^n$ is supermodular (submodular) if and only if $\partial^2/(\partial x_i \partial x_j)\varphi(\boldsymbol{x})(\geq (\leq))0$ for all $1 \leq i \neq j \leq n$ and $\boldsymbol{x} \in \mathbb{R}^n$. Here are some supermodular or submodular functions:

(i) $\varphi(\boldsymbol{x}, \boldsymbol{a}) = \sum_{i=1}^n \max\{a_i, x_i\}$ is submodular,

(ii) $\varphi(\boldsymbol{x}, \boldsymbol{a}) = \sum_{i=1}^n \sup\{c_1 a_i + c_2 x_i \colon (c_1, c_2) \in C\}$ is submodular for $C \subseteq \mathbb{R}^2$,

(iii) $\varphi(\boldsymbol{x}, \boldsymbol{a}) = \sum_{i=1}^n \phi(x_i - a_i)$ is submodular (supermodular) if ϕ is convex (concave), and

(iv) $\varphi(\boldsymbol{x}, \boldsymbol{a}) = \sum_{i=1}^n \phi_1(a_i)\phi_2(x_i)$ is submodular (supermodular) if ϕ_1 is increasing (decreasing) and ϕ_2 is increasing.

For other such functions, we refer the reader to Marshall *et al.* (2011).

Let X and Y be two random variables with probability density functions f and g (when they are absolutely continuous).

Definition 2.3. We say that X is smaller than Y in the

(i) *likelihood ratio* order, denoted by $X \leq_{\mathrm{lr}} Y$, if $g(t)/f(t)$ increases in t;

(ii) *hazard rate* order, denoted by $X \leq_{\mathrm{hr}} Y$, if $\mathbb{P}(Y > t)/\mathbb{P}(X > t)$ is increasing in t;

(iii) *usual stochastic* order, denoted by $X \leq_{\mathrm{st}} Y$, if $\mathbb{P}(X > t) \leq \mathbb{P}(Y > t)$ for all t;

(iv) *increasing convex* order, denoted by $X \le_{\text{icx}} Y$, if $\int_t^{\infty} \mathbb{P}(X > x)\,\mathrm{d}x \le \int_t^{\infty} \mathbb{P}(Y > x)\,\mathrm{d}x$ for all t.

For more on stochastic orders, we refer the reader to Müller and Stoyan (2002), Shaked and Shanthikumar (2007), and Li and Li (2013).

For $1 \le i \ne j \le n$, let the permutation $\pi_{ij}(\boldsymbol{x}) = (\pi_{ij}(x_1), \dots, \pi_{ij}(x_n))$ such that $\pi_{ij}(x_i) = x_j$, $\pi_{ij}(x_j) = x_i$, and $\pi_{ij}(x_k) = x_k$ for $k \notin \{i, j\}$. A real-valued function g on $\mathbb{R}^n$ is said to be *symmetric* if $g(\boldsymbol{x}) = g(\pi_{ij}(\boldsymbol{x}))$ for any $\boldsymbol{x}$ and (i, j) with $1 \le i < j \le n$. The g is said to be *arrangement increasing* (AI) on $\mathbb{R}^n$ if $(x_i - x_j)[g(\boldsymbol{x}) - g(\pi_{ij}(\boldsymbol{x}))] \le 0$ for any $\boldsymbol{x}$ and (i, j) with $1 \le i < j \le n$, and g is said to be *arrangement decreasing* (AD) when the inequality is reversed.

AI functions and some related versions were utilized to introduce bivariate characterizations for stochastic orders of two independent random variables in Shanthikumar and Yao (1991) and Righter and Shanthikumar (1992). Recently, Belzunce *et al.* (2013) proposed the joint stochastic orders for dependent random variables and successfully applied them to redundant component allocation in reliability. In addition, Cai and Wei (2014), (2015) independently introduced the following notion characterizing the monotonicity of dependent random variables and discussed its application in deductible and limit allocations for actuarial risks.

Definition 2.4. A random vector $\boldsymbol{X} = (X_1, \dots, X_n)$ is said to be *stochastic arrangement increasing* (SAI) if $\mathbb{E}[g(\boldsymbol{X})] \ge \mathbb{E}[g(\pi_{ij}(\boldsymbol{X}))]$ for any AI function g and (i, j) such that $1 \le i < j \le n$.

As a dual, $\boldsymbol{X}$ is said to be *stochastic arrangement decreasing* (SAD) if $(X_n, \dots, X_1)$ is SAI.

The following sufficient conditions for a SAI random vector $(X_1, \dots, X_n)$ were summarized in Marshall *et al.* (2011):

(i) they are i.i.d.,

(ii) they are exchangeable, and

(iii) they are independent and X_i has probability density $h(\lambda_i, x_i)$ TP_2 (total positivity of order two) in λ_i and x_i, $i = 1, \dots, n$.

The following characterization was pointed out in Shanthikumar and Yao (1991) and further remarked upon in Cai and Wei (2014) and Pan *et al.* (2015).

Lemma 2.1. *An absolutely continuous $\boldsymbol{X}$ is SAI (SAD) if and only if it has AI (AD) probability density.*

The next technical lemma can be proved in a similar manner to Proposition 3.3(iii) of Cai and Wei (2014), and, thus, for brevity it is stated below without the proof.

Lemma 2.2. *It holds that $(h(X_1), \dots, h(X_n))$ is SAI if $(X_1, \dots, X_n)$ is SAD and $h(x)$ is decreasing.*

3. Technical lemmas

For a univariate function $g(x)$, a bivariate function $\phi(x, d)$, and real vectors $\boldsymbol{x} = (x, y)$, $\boldsymbol{w} = (w_1, w_2)$, $\boldsymbol{a} = (a_1, a_2)$, $\boldsymbol{b} = (b_1, b_2)$ on $\mathbb{R}^2$, set permutation $\tau(x, y) = (y, x)$. Let $\eta_i(\boldsymbol{x}, \boldsymbol{a}) = w_i\phi(x, a_2) + w_{2/i}\phi(y, a_1)$ for $i = 1, 2$, $\Delta g_1(\boldsymbol{x}, \boldsymbol{a}) = g(\eta_1(\boldsymbol{x}, \boldsymbol{a})) - g(\eta_1(\boldsymbol{x}, \tau(\boldsymbol{a})))$, and

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$\Delta g_i(\boldsymbol{x}, \boldsymbol{a}, \boldsymbol{b}) = g(\eta_{i-1}(\boldsymbol{x}, \boldsymbol{a})) - g(\eta_{i-1}(\boldsymbol{x}, \boldsymbol{b}))$ for $i = 2, 3$. Denote

$$\begin{aligned}
\mathcal{G}_1 &= \{r(\boldsymbol{x}, \boldsymbol{a}, \boldsymbol{b})\colon r(\boldsymbol{x}, \boldsymbol{a}, \boldsymbol{b}) \geq 0 \text{ for all } y \geq x,\ \boldsymbol{a} \succeq_{\mathrm{m}} \boldsymbol{b}\},\\
\mathcal{G}_2 &= \{r(\boldsymbol{x}, \boldsymbol{a}, \boldsymbol{b})\colon r(\boldsymbol{x}, \boldsymbol{a}, \boldsymbol{b}) + r(\tau(\boldsymbol{x}), \boldsymbol{a}, \boldsymbol{b}) \geq 0 \text{ for all } y \geq x,\ \boldsymbol{a} \succeq_{\mathrm{m}} \boldsymbol{b}\},\\
\mathcal{G}_1^+ &= \{r(\boldsymbol{x}, \boldsymbol{a}, \boldsymbol{b})\colon r(\boldsymbol{x}, \boldsymbol{a}, \boldsymbol{b}) \geq 0 \text{ for all } y \geq x \geq 0,\ \boldsymbol{a} \succeq_{\mathrm{m}} \boldsymbol{b}\},\\
\mathcal{G}_2^+ &= \{r(\boldsymbol{x}, \boldsymbol{a}, \boldsymbol{b})\colon r(\boldsymbol{x}, \boldsymbol{a}, \boldsymbol{b}) + r(\tau(\boldsymbol{x}), \boldsymbol{a}, \boldsymbol{b}) \geq 0 \text{ for all } y \geq x \geq 0,\ \boldsymbol{a} \succeq_{\mathrm{m}} \boldsymbol{b}\}.
\end{aligned}$$

Let us first develop several lemmas on inequalities concerned with Δg_i on the area above the main diagonal, $i = 1, 2, 3$. The proofs are relegated to Appendix A.

Lemma 3.1. *Both $g(\eta_1(\boldsymbol{x}, \boldsymbol{a}))$ and $g(\eta_1(\boldsymbol{x}, \boldsymbol{a})) + g(\eta_1(\tau(\boldsymbol{x}), \boldsymbol{a}))$ are AI in $\boldsymbol{a}$ for $w_2 \geq w_1 \geq 0$ and $x \leq y$ if $g(x)$ is increasing and convex, and $\phi(x, d)$ is submodular, decreasing in d for any x and increasing in x for any d.*

Lemma 3.2. *We have $\Delta g_2(\boldsymbol{x}, \boldsymbol{a}, \boldsymbol{b}) \in \mathcal{G}_1 \cap \mathcal{G}_2$ for $w_2 \geq w_1 \geq 0$ if $g(x)$ is increasing and convex, and $\phi(x, d)$ is submodular, decreasing and convex in d for any x and increasing in x for any d.*

Lemma 3.3. *We have $\Delta g_2(\boldsymbol{x}, \boldsymbol{a}, \boldsymbol{b}) + \Delta g_3(\boldsymbol{x}, \boldsymbol{a}, \boldsymbol{b}) \in \mathcal{G}_1 \cap \mathcal{G}_2$ for $w_2 \geq w_1 \geq 0$ if $g(x)$ is increasing and convex, and $\phi(x, d)$ is submodular, decreasing and convex in d for any x and increasing in x for any d.*

For the aforementioned functions $g(x)$ and $\phi(x, d)$, a random vector (X_1, X_2), and real vectors $\boldsymbol{w}, \boldsymbol{a}, \boldsymbol{b}$ on $\mathbb{R}^2$, let $\zeta(\boldsymbol{w}, \boldsymbol{a}) = \mathbb{E}[g(w_1\phi(X_1, a_2) + w_2\phi(X_2, a_1))]$ and denote

$$\Delta\zeta_1(\boldsymbol{w}, \boldsymbol{a}) = \zeta(\boldsymbol{w}, \boldsymbol{a}) - \zeta(\boldsymbol{w}, \tau(\boldsymbol{a})), \qquad \Delta\zeta_2(\boldsymbol{w}, \boldsymbol{a}, \boldsymbol{b}) = \zeta(\boldsymbol{w}, \boldsymbol{a}) - \zeta(\boldsymbol{w}, \boldsymbol{b}).$$

Lemma 3.4. *Both $\zeta_1(\boldsymbol{w}, \boldsymbol{a})$ and $\zeta_1(\boldsymbol{w}, \boldsymbol{a}) + \zeta_1(\tau(\boldsymbol{w}), \boldsymbol{a})$ are AI in $\boldsymbol{a}$ for any $w_2 \geq w_1 \geq 0$ if (X_1, X_2) is SAI and absolutely continuous, $g(x)$ is increasing and convex, and submodular $\phi(x, d)$ is decreasing in d for any x and increasing in x for any d.*

Lemma 3.5. *We have $\Delta\zeta_2(\boldsymbol{w}, \boldsymbol{a}, \boldsymbol{b}) \in \mathcal{G}_1^+ \cap \mathcal{G}_2^+$ if (X_1, X_2) is SAI and absolutely continuous, $g(x)$ is increasing and convex, and submodular $\phi(x, d)$ is decreasing and convex in d for any x and increasing in x for any d.*

4. Main results

The theoretical results to be developed here are sufficient conditions for the increasing convex orders on the generalized aggregation of SAI random variables and they are based on the convexity, supermodularity, and monotonicity of the kernel function $\phi(x, d)$. We treat the sufficient conditions as the matched allocations of parameters $(d_1, \ldots, d_n)$ and the majorized ones in two subsections.

4.1. Matched allocations

To investigate how the AI parameters $(d_1, \ldots, d_n)$ play a role in forming the increasing convex order on the generalized aggregation, we state two theorems.

Theorem 4.1. *Suppose that $(X_1, \ldots, X_n)$ and nonnegative $(W_1, \ldots, W_n)$ are both SAI and independent of each other. For any $(d_1, \ldots, d_n) \in \mathbb{R}^n$ and $\phi(x, d)$ increasing in x for any d,*

(i) $\sum_{i=1}^n W_i\phi(X_i, d_{(n-i+1)}) \geq_{\mathrm{icx}} \sum_{i=1}^n W_i\phi(X_i, d_i)$ *if $\phi(x, d)$ is submodular and decreasing in d for any x, and*

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(ii) $\sum_{i=1}^n W_i\phi(X_i, d_{(i)}) \geq_{\rm icx} \sum_{i=1}^n W_i\phi(X_i, d_i)$ *if* $\phi(x, d)$ *is supermodular and increasing in* d *for any* x.

Proof. We will only prove (i) as the proof of (ii) is similar.

Due to the transitivity of the increasing convex order, it suffices to show that

$$\begin{aligned} &W_i\phi(X_i, d_i \vee d_j) + W_j\phi(X_j, d_i \wedge d_j) + \mathcal{L}_{i,j} \\ &\quad \geq_{\rm icx} W_i\phi(X_i, d_i \wedge d_j) + W_j\phi(X_j, d_i \vee d_j) + \mathcal{L}_{i,j} \end{aligned} \tag{4.1}$$

for any $(d_1, \dots, d_n)$ and (i, j) with $1 \leq i < j \leq n$, where $\mathcal{L}_{i,j} = \sum_{k\neq i,j}^n W_k\phi(X_k, d_k)$. According to Proposition 3.4 of Cai and Wei (2014), $(X_1, \dots, X_n)$ is SAI if and only if $[(X_i, X_j) \mid X_k = x_k,\ k \neq i, j]$ is SAI for any x_k with $k \neq i, j$ and any $1 \leq i < j \leq n$. So we can prove (4.1) through achieving the following two steps.

- Prove the bivariate case of $[(X_i, X_j) \mid X_k = x_k,\ k \neq i, j]$ and $[(W_i, W_j) \mid W_k = w_k,\ k \neq i, j]$ for any x_k, w_k with $k \neq i, j$.
- Apply the total expectation to obtain (4.1).

Since the second step is routine, in the following we only detail the first step, which is just equivalent to the $n = 2$ case.

Set $d_1 \leq d_2$. Evidently, it is sufficient to show that, for any increasing and convex function g,

$$\begin{aligned} &\mathbb{E}[\Delta\zeta_1(W_1, W_2;\ d_1, d_2)] \\ &\quad = \mathbb{E}[g(W_1\phi(X_1, d_2) + W_2\phi(X_2, d_1))] - \mathbb{E}[g(W_1\phi(X_1, d_1) + W_2\phi(X_2, d_2))] \\ &\quad \geq 0. \end{aligned}$$

Denote by $h(w_1, w_2)$ the probability density of (W_1, W_2). Since (W_1, W_2) is SAI, using the notation of Lemma 3.1 and Lemma 3.4, we have

$$\begin{aligned} &\mathbb{E}[\Delta\zeta_1(W_1, W_2;\ d_1, d_2)] \\ &\quad = \iint_{0\leq w_1\leq w_2} \Delta\zeta_1(w_1, w_2;\ d_1, d_2)h(\boldsymbol{w})\,\mathrm{d}w_1\,\mathrm{d}w_2 \\ &\qquad + \iint_{0\leq w_2\leq w_1} \Delta\zeta_1(w_1, w_2;\ d_1, d_2)h(\boldsymbol{w})\,\mathrm{d}w_1\,\mathrm{d}w_2 \\ &\quad = \iint_{0\leq w_1\leq w_2} [\Delta\zeta_1(w_1, w_2;\ d_1, d_2)h(\boldsymbol{w}) + \Delta\zeta_1(w_2, w_1;\ d_1, d_2)h(\tau(\boldsymbol{w}))]\,\mathrm{d}w_1\,\mathrm{d}w_2 \\ &\quad \geq \iint_{0\leq w_1\leq w_2} [\Delta\zeta_1(w_1, w_2;\ d_1, d_2) + \Delta\zeta_1(w_2, w_1;\ d_1, d_2)]h(\tau(\boldsymbol{w}))\,\mathrm{d}w_1\,\mathrm{d}w_2 \\ &\quad \geq 0, \end{aligned}$$

where the last two inequalities are due to Lemma 2.1 and Lemma 3.4, respectively. □

4.2. Majorized allocations

In the following two theorems we are concerned with how the majorization of parameters $(d_1, \dots, d_n)$ impacts the increasing convex order on the generalized aggregation.

Theorem 4.2. *Suppose that* $(X_1, \dots, X_n)$ *and nonnegative* $(W_1, \dots, W_n)$ *are SAI and independent of each other. For* $(a_1, \dots, a_n) \succeq_{\rm m} (b_1, \dots, b_n)$ *and* $\phi(x, d)$ *increasing in* x *for any* d,

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(i) $\sum_{i=1}^n W_i\phi(X_i, a_{(n-i+1)}) \geq_{\text{icx}} \sum_{i=1}^n W_i\phi(X_i, b_{(n-i+1)})$ *if* $\phi(x, d)$ *is submodular, and decreasing and convex in* d *for any* x*, and*

(ii) $\sum_{i=1}^n W_i\phi(X_i, a_{(i)}) \geq_{\text{icx}} \sum_{i=1}^n W_i\phi(X_i, b_{(i)})$ *if* $\phi(x, d)$ *is supermodular, and increasing and convex in* d *for any* x*.*

Proof. We only prove (i) for $n = 2$. The proofs for $n > 2$ and (ii) are similar to the proof of Theorem 4.1 and, hence, are omitted.

Assume that $a_1 \leq b_1 \leq b_2 \leq a_2$. Then it is sufficient to show that, for any increasing and convex function g,

$$\begin{aligned}
&\mathbb{E}[\Delta\zeta_2(W_1, W_2;\ \boldsymbol{a}, \boldsymbol{b})] \\
&\quad = \mathbb{E}[g(W_1\phi(X_1, a_2) + W_2\phi(X_2, a_1))] - \mathbb{E}[g(W_1\phi(X_1, b_2) + W_2\phi(X_2, b_1))] \\
&\quad \geq 0.
\end{aligned}$$

Denote by h the density of (W_1, W_2). By using the notation of Lemmas 3.2, 3.3, and 3.5, we have

$$\begin{aligned}
&\mathbb{E}[\Delta\zeta_2(W_1, W_2;\ \boldsymbol{a}, \boldsymbol{b})] \\
&\quad = \iint_{0\leq w_1\leq w_2} \Delta\zeta_2(\boldsymbol{w}, \boldsymbol{a}, \boldsymbol{b})h(\boldsymbol{w})\,\mathrm{d}w_1\,\mathrm{d}w_2 + \iint_{0\leq w_2\leq w_1} \Delta\zeta_2(\boldsymbol{w}, \boldsymbol{a}, \boldsymbol{b})h(\boldsymbol{w})\,\mathrm{d}w_1\,\mathrm{d}w_2 \\
&\quad = \iint_{0\leq w_1\leq w_2} [\Delta\zeta_2(\boldsymbol{w}, \boldsymbol{a}, \boldsymbol{b})h(\boldsymbol{w}) + \Delta\zeta_2(\tau(\boldsymbol{w}), \boldsymbol{a}, \boldsymbol{b})h(\tau(\boldsymbol{w}))]\,\mathrm{d}w_1\,\mathrm{d}w_2 \\
&\quad \geq \iint_{0\leq w_1\leq w_2} [\Delta\zeta_2(\boldsymbol{w}, \boldsymbol{a}, \boldsymbol{b}) + \Delta\zeta_2(\tau(\boldsymbol{w}), \boldsymbol{a}, \boldsymbol{b})]h(\tau(\boldsymbol{w}))\,\mathrm{d}w_1\,\mathrm{d}w_2 \\
&\quad \geq 0,
\end{aligned}$$

where the last two inequalities stem from Lemma 2.1 and Lemma 3.5, respectively. □

As a direct consequence of Theorems 4.1 and 4.2, one can easily derive the following result.

Theorem 4.3. *Suppose that* $(X_1, \ldots, X_n)$ *and nonnegative* $(W_1, \ldots, W_n)$ *are SAI and independent of each other. For* $(a_1, \ldots, a_n) \succeq_{\text{m}} (b_1, \ldots, b_n)$ *and* $\phi(x, d)$ *increasing in* x *for any* d*,*

(i) $\sum_{i=1}^n W_i\phi(X_i, a_{(n-i+1)}) \geq_{\text{icx}} \sum_{i=1}^n W_i\phi(X_i, b_i)$ *if* $\phi(x, d)$ *is submodular, and decreasing and convex in* d *for any* x*, and*

(ii) $\sum_{i=1}^n W_i\phi(X_i, a_{(i)}) \geq_{\text{icx}} \sum_{i=1}^n W_i\phi(X_i, b_i)$ *if* $\phi(x, d)$ *is supermodular, and increasing and convex in* d *for any* x*.*

5. Some applications in reliability and risk management

To illustrate the main results developed in Section 4, we present here several applications.

5.1. Degenerate weighted k-out-of-n systems

As a fault tolerant structure, the k-out-of-n system is widely used in reliability engineering. In the past two decades, the binary k-out-of-n structure was generalized to the weighted k-out-of-n system, in which the ith component with lifetime X_i contributes its own weight $\omega_i \geq 0$ to the entire system while functioning ($i = 1, \ldots, n$) and the system itself functions if and

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only if the total weight due to operational components is above some predetermined threshold $k \geq 0$. For more on weighted k-out-of-n systems, we refer the reader to Xie and Pham (2005), Samaniego and Shaked (2008), Eryilmaz (2013), and Li *et al.* (2016) and references therein.

In this subsection we introduce the random decay coefficient Λ_i for the weight d_i of the operational component X_i, $i = 1, \dots, n$. Denote $\boldsymbol{d} = (d_1, \dots, d_n)$ and $\boldsymbol{\Lambda} = (\Lambda_1, \dots, \Lambda_n)$. At time $t \geq 0$ the system retains the weight $V_t(\boldsymbol{d}, \boldsymbol{X}, \boldsymbol{\Lambda}) = \sum_{i=1}^{n} d_i \, \mathbf{1}(X_i > t) \mathrm{e}^{-\Lambda_i t}$ and, thus, the system failure time is represented as

$$T_k(\boldsymbol{d}, \boldsymbol{X}, \boldsymbol{\Lambda}) = \inf\{t\colon V_t(\boldsymbol{d}, \boldsymbol{X}, \boldsymbol{\Lambda}) < k\}.$$

So, the system reliability is

$$\mathbb{P}(T_k(\boldsymbol{d}, \boldsymbol{X}, \boldsymbol{\Lambda}) > t) = \mathbb{P}(V_t(\boldsymbol{d}, \boldsymbol{X}, \boldsymbol{\Lambda}) \geq k) \quad \text{for any } t \geq 0.$$

Obviously, the degenerate weighted k-out-of-n system with $\Lambda_i \equiv 0$ for $i = 1, \dots, n$ is equivalent to the classical version.

Note that $d \, \mathbf{1}(x > t)$ is supermodular, increasing in x for any d, and increasing in d for any x. Directly from Theorem 4.1(ii) we have the following proposition.

Proposition 5.1. *Suppose that* $(X_1, \dots, X_n)$ *is SAI and* $(\Lambda_1, \dots, \Lambda_n)$ *is SAD and independent of* $(X_1, \dots, X_n)$. *Then* $\sum_{i=1}^{n} d_{(i)} \, \mathbf{1}(X_i > t) \mathrm{e}^{-\Lambda_i t} \geq_{\mathrm{icx}} \sum_{i=1}^{n} d_i \, \mathbf{1}(X_i > t) \mathrm{e}^{-\Lambda_i t}$ *for* $t \geq 0$ *and* $\boldsymbol{d} \in \mathbb{R}_{+}^{n}$.

From Proposition 5.1, it follows that *the system with a larger weight assigned to a more reliable component has a longer lifetime in the sense of the increasing and convex order*, and this conforms in spirit with Theorem 3.1 of Li *et al.* (2016).

5.2. Reliability systems subject to shocks with dependent damages

In industrial engineering, it is of practical interest to study systems subject to shocks generating a random amount of damage while arriving according to some renewal process. On the other hand, statisticians also employ such a shock and damage model to generate lifetime distributions with various ageing properties. For an overview of shock and damage models, we refer the reader to Bogdanoff and Kozin (1985) and Nakagawa (2007).

Assume that a reliability system subject to shocks arriving in succession with interarrival times $T_1, T_2, \dots,$ and at $X_i = T_1 + \cdots + T_i$, the arrival of the ith shock produces a random damage W_i decaying at a rate $d_i > 0$ for $i = 1, 2 \dots.$ To avoid unexpected failures, the engineer considers to repair the system at X_{n+1}—the arrival time of the $(n+1)$th shock. Then the total accumulated damage until repair is $\sum_{i=1}^{n} W_i \mathrm{e}^{-d_i (t - X_i)_+}$. Assume that the system fails whenever the aggregated damage reaches a predetermined threshold $c > 0$. Then the probability for the system to survive time t before the repair is $\mathbb{P}(\sum_{i=1}^{n} W_i \mathrm{e}^{-d_i (t - X_i)_+} < c)$.

It is routine to verify that $\phi(x, d) = \mathrm{e}^{-d(t-x)_+}$ is submodular, decreasing and convex in d for any x, and increasing in x for any d. The next proposition follows from Theorem 4.3(ii) directly.

Proposition 5.2. *Suppose that* $(W_1, \dots, W_n)$ *is SAI and* $T_1, \dots, T_n$ *are mutually independent. Then* $(d_1, \dots, d_n) \succeq_{\mathrm{m}} (b_1, \dots, b_n)$ *implies that*

$$\sum_{i=1}^{n} W_i \mathrm{e}^{-d_{(i)}(t - X_i)_+} \geq_{\mathrm{icx}} \sum_{i=1}^{n} W_i \mathrm{e}^{-b_i (t - X_i)_+}.$$

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Intuitively, the slower decay of the damage due to an earlier shock incurs a larger aggregated damage and tends to a weaker system. In fact, Proposition 5.2 confirms this intuition as follows: *the system with increasing decay rates gains larger aggregated damage and, hence, is less reliable in the sense of the increasing and convex order*.

5.3. Deductibles and coverage limits for insurance risks

In the insurance industry, deductibles and coverage limits are often applied to multiple risks. Sometimes the insurer grants a total amount of deductible $\hbar$ or coverage limit ℓ and allows the policyholder to allocate deductibles $\boldsymbol{d} = (d_1, \dots, d_n)$ or coverage limits $\boldsymbol{l} = (l_1, \dots, l_n)$ to risks $\boldsymbol{X} = (X_1, \dots, X_n)$ covered by a policy. Let $\delta > 0$ be the discount rate and $\boldsymbol{T} = (T_1, \dots, T_n)$ be the vector of the corresponding occurrence times of those risks. Denote by $\mathcal{A}_\hbar$ all admissible allocation vectors such that $\sum_{i=1}^n d_i = \hbar$ and $d_i \geq 0$ for all $i = 1, \dots, n$. Then the policyholder retains the total potential loss $\sum_{i=1}^n \mathrm{e}^{-\delta T_i}(X_i - (X_i - d_i)_+) = \sum_{i=1}^n \mathrm{e}^{-\delta T_i}(X_i \wedge d_i)$ for $\boldsymbol{d} \in \mathcal{A}_\hbar$ and $\sum_{i=1}^n \mathrm{e}^{-\delta T_i}[X_i - (X_i \wedge l_i)] = \sum_{i=1}^n \mathrm{e}^{-\delta T_i}(X_i - l_i)_+$ for any $\boldsymbol{l} \in \mathcal{A}_\ell$. So it is of interest for the policyholder to consider the following two optimization problems based on utility theory:

$$\min_{\boldsymbol{l} \in \mathcal{A}_\ell} \mathbb{E}\bigg[u\bigg(\sum_{i=1}^n \mathrm{e}^{-\delta T_i}(X_i - l_i)_+\bigg)\bigg] \tag{5.1}$$

subject to increasing and convex u, and $\boldsymbol{X}$ independent of $\boldsymbol{T}$,

$$\min_{\boldsymbol{d} \in \mathcal{A}_\hbar} \mathbb{E}\bigg[u\bigg(\sum_{i=1}^n \mathrm{e}^{-\delta T_i}(X_i \wedge d_i)\bigg)\bigg] \tag{5.2}$$

subject to increasing and convex u, and $\boldsymbol{X}$ independent of $\boldsymbol{T}$.

Denote by $\boldsymbol{l}^* = (l_1^*, \dots, l_n^*)$ and $\boldsymbol{d}^* = (d_1^*, \dots, d_n^*)$ the solutions to the above problems (5.1) and (5.2), respectively. Hua and Cheung (2008a) were among the first to study the above two problems, and they showed, in the context of comonotone $\boldsymbol{X}$ with mutually independent $\boldsymbol{T}$, that $l_i^* \leq l_j^*$ and $d_i^* \geq d_j^*$ if $T_j \leq_{\mathrm{lr}} T_i$ and $X_i \leq_{\mathrm{st}} X_j$ for any $1 \leq i \neq j \leq n$. Following this, in the context of the comonotone severity $\boldsymbol{X}$ with $\boldsymbol{T}$ having some Archimedean copula, Li and You (2012) proved that it is least favorable for the risk-averse policyholder to allocate a smaller coverage limit and a larger deductible to the loss with higher severity and frequency.

According to Lemma 2.2, $(\mathrm{e}^{-\delta T_1}, \dots, \mathrm{e}^{-\delta T_n})$ is SAI whenever $\boldsymbol{T}$ is SAD. Note that $(x-d)_+$ is submodular, decreasing in d for any x and increasing in x for any d. As an immediate consequence of Theorem 4.1(i), we obtain the proposition on deductibles.

Proposition 5.3. *Suppose that* $(X_1, \dots, X_n)$ *is SAI and* $(T_1, \dots, T_n)$ *is SAD and independent of* $(X_1, \dots, X_n)$. *Then* $\sum_{i=1}^n (X_i - l_{(n-i+1)})\mathrm{e}^{-\delta T_i} \geq_{\mathrm{icx}} \sum_{i=1}^n (X_i - l_i)\mathrm{e}^{-\delta T_i}$ *for any* $\boldsymbol{l} \in \mathbb{R}_+^n$.

Also, since $x \wedge d$ is supermodular, increasing in d for any x, and increasing in x for any d, by Theorem 4.1(ii), we also have the following result on coverage limits.

Proposition 5.4. *Suppose that* $(X_1, \dots, X_n)$ *is SAI and* $(T_1, \dots, T_n)$ *is SAD and independent of* $(X_1, \dots, X_n)$. *Then* $\sum_{i=1}^n (X_i \wedge d_{(i)})\mathrm{e}^{-\delta T_i} \geq_{\mathrm{icx}} \sum_{i=1}^n (X_i \wedge d_i)\mathrm{e}^{-\delta T_i}$ *for any* $\boldsymbol{d} \in \mathbb{R}_+^n$.

Based on Propositions 5.3 and 5.4, we can draw the following conclusions for SAI severities $\boldsymbol{X}$ with SAD occurrence frequencies $\boldsymbol{T}$:

(i) the optimal allocation of coverage limits $\boldsymbol{l}^*$ must satisfy $l_1^* \leq \cdots \leq l_n^*$, and

(ii) the optimal allocation of deductibles $\boldsymbol{d}^*$ must satisfy $d_1^* \geq \cdots \geq d_n^*$.

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According to Propositions 5.2 and 5.5 of Cai and Wei (2014), $\boldsymbol{X}$ with comonotone coordinates is SAI if and only if $X_i \leq_{\mathrm{st}} X_{i+1}$ for $i = 1, \ldots, n-1$, and $\boldsymbol{T}$ with mutually independent coordinates is SAI if and only if $T_i \leq_{\mathrm{lr}} T_{i+1}$ for $i = 1, \ldots, n-1$. Therefore, the above two conclusions serve as substantial extensions to the corresponding ones due to Hua and Cheung (2008a).

5.4. Generalized linear combinations of dependent random variables

Linear combinations of independent random variables have been extensively studied in the literature; see, for example, Xu and Hu (2011) and Yu (2011). In the past several years, some authors have investigated the various generalized weighted sum $\sum_{i=1}^{n}\phi(X_i, a_i)$. For instance, Pan *et al.* (2013) considered the case of $\phi(X, a) = \alpha(a)\beta(X)$ with twice continuously differentiable and strictly monotone α and β, and Mao *et al.* (2013) discussed the case of supermodular and submodular $\phi(x, a)$. Recently, You and Li (2015) studied the generalized weighted sum with SAI random variables and proved the following proposition.

Proposition 5.5. (You and Li (2015, Proposition 4.2).) *Suppose that $\boldsymbol{X}$ is SAI and $\phi(x, d)$ is convex with respect to d for any x. Then $\boldsymbol{b} \preceq_{\mathrm{m}} \boldsymbol{a}$ implies that $\sum_{i=1}^{n}\phi(X_i, a_{(n-i+1)}) \geq_{\mathrm{icx}} \sum_{i=1}^{n}\phi(X_i, b_i)$ for submodular $\phi(x, d)$ and $\sum_{i=1}^{n}\phi(X_i, a_{(i)}) \geq_{\mathrm{icx}} \sum_{i=1}^{n}\phi(X_i, b_i)$ for supermodular $\phi(x, d)$.*

Clearly, as a natural extension of the above proposition, Theorem 4.3 introduces the random weights $W_1, \ldots, W_n$ at the cost of assuming monotone properties for $\phi(x, d)$.

Appendix A

Proof of Lemma 3.1. (i) Due to the submodularity of ϕ, it holds that $\phi(y, a_1) - \phi(y, a_2) \geq \phi(x, a_1) - \phi(x, a_2)$ for any $a_1 \leq a_2$ and $x \leq y$. Since $\phi(x, d)$ is decreasing in d for any fixed x, we have $\phi(x, a_1) - \phi(x, a_2) \geq 0$ and, hence, $w_1\phi(x, a_2) + w_2\phi(y, a_1) \geq w_1\phi(x, a_1) + w_2\phi(y, a_2)$ for any $0 \leq w_1 \leq w_2$, i.e. $\eta_1(\boldsymbol{x}, \boldsymbol{a}) \geq \eta_1(\boldsymbol{x}, \tau(\boldsymbol{a}))$. Taking the increasing g into account, we reach $\Delta g_1(\boldsymbol{x}, \boldsymbol{a}) \geq 0$ for any $x \leq y$.

(ii) Since $\phi(x, d)$ is decreasing in d for any fixed x, we have

$$\begin{aligned}
&\eta_1(\boldsymbol{x}, \boldsymbol{a}) + \eta_1(\tau(\boldsymbol{x}), \boldsymbol{a}) - \eta_1(\tau(\boldsymbol{x}), \tau(\boldsymbol{a})) - \eta_1(\boldsymbol{x}, \tau(\boldsymbol{a})) \\
&\quad = (w_2 - w_1)[\phi(y, a_1) - \phi(y, a_2) + \phi(x, a_1) - \phi(x, a_2)] \\
&\quad \geq 0,
\end{aligned}$$

yielding $\eta_1(\tau(\boldsymbol{x}), \boldsymbol{a}) + \eta_1(\boldsymbol{x}, \boldsymbol{a}) \geq \eta_1(\boldsymbol{x}, \tau(\boldsymbol{a})) + \eta_1(\tau(\boldsymbol{x}), \tau(\boldsymbol{a}))$. On the other hand, one may verify

$$\begin{aligned}
\eta_1(\boldsymbol{x}, \boldsymbol{a}) - \eta_1(\tau(\boldsymbol{x}), \tau(\boldsymbol{a})) &= (w_2 - w_1)[\phi(y, a_1) - \phi(x, a_2)] \\
&= (w_2 - w_1)[\phi(y, a_1) - \phi(y, a_2) + \phi(y, a_2) - \phi(x, a_2)] \\
&\geq 0.
\end{aligned}$$

By (i), we have $\eta_1(\boldsymbol{x}, \boldsymbol{a}) \geq \eta_1(\boldsymbol{x}, \tau(\boldsymbol{a}))$ and, hence,

$$\eta_1(\boldsymbol{x}, \boldsymbol{a}) \geq \max\{\eta_1(\boldsymbol{x}, \tau(\boldsymbol{a})), \eta_1(\tau(\boldsymbol{x}), \tau(\boldsymbol{a}))\}.$$

Consequently, we obtain $(\eta_1(\boldsymbol{x}, \tau(\boldsymbol{a})), \eta_1(\tau(\boldsymbol{x}), \tau(\boldsymbol{a}))) \preceq_{\mathrm{w}} (\eta_1(\boldsymbol{x}, \boldsymbol{a}), \eta_1(\tau(\boldsymbol{x}), \boldsymbol{a}))$. As for increasing and convex $g(x)$, it follows from Theorem C.1.b of Marshall *et al.* (2011) that, for $x \leq y$,

$$\begin{aligned}
&\Delta g_1(\boldsymbol{x}, \boldsymbol{a}) + \Delta g_1(\tau(\boldsymbol{x}), \boldsymbol{a}) \\
&\quad = g(\eta_1(\boldsymbol{x}, \boldsymbol{a})) - g(\eta_1(\boldsymbol{x}, \tau(\boldsymbol{a}))) + g(\eta_1(\tau(\boldsymbol{x}), \boldsymbol{a})) - g(\eta_1(\tau(\boldsymbol{x}), \tau(\boldsymbol{a}))) \\
&\quad \geq 0.
\end{aligned}$$

□

Proof of Lemma 3.2. Noting that $\boldsymbol{a} \succeq_{\mathrm{m}} \boldsymbol{b}$, we assume that $a_1 \leq b_1 \leq b_2 \leq a_2$.

(i) Since $\phi(x, d)$ is submodular and convex with respect to d for fixed x, it holds that

$$\begin{aligned}
&\phi(y, a_1) - \phi(y, b_1) + \phi(x, a_2) - \phi(x, b_2) \\
&\quad = [\phi(x, a_2) - \phi(x, b_2)] - [\phi(x, b_1) - \phi(x, a_1)] \\
&\quad\quad + [\phi(y, a_1) - \phi(x, a_1)] - [\phi(y, b_1) - \phi(x, b_1)] \\
&\quad \geq 0 \quad \text{for any } x \leq y.
\end{aligned}$$

Since $\phi(x, d)$ is decreasing in d for any x, it holds that $\phi(x, b_2) - \phi(x, a_2) \geq 0$ and, hence, $w_2[\phi(y, a_1) - \phi(y, b_1)] \geq w_1[\phi(x, b_2) - \phi(x, a_2)]$ for any $w_2 \geq w_1 \geq 0$. Then we have

$$\eta_1(\boldsymbol{x}, \boldsymbol{a}) = w_1\phi(x, a_2) + w_2\phi(y, a_1) \geq w_1\phi(x, b_2) + w_2\phi(y, b_1) = \eta_1(\boldsymbol{x}, \boldsymbol{b}).$$

Owing to the increasing g, we obtain $\Delta g_2(\boldsymbol{x}, \boldsymbol{a}, \boldsymbol{b}) = g(\eta_1(\boldsymbol{x}, \boldsymbol{a})) - g(\eta_1(\boldsymbol{x}, \boldsymbol{b})) \geq 0$ for any $x \leq y$.

(ii) Since $\phi(x, d)$ is decreasing in d for any x, we have $\phi(x, b_2) - \phi(x, a_2) + \phi(y, b_2) - \phi(y, a_2) \geq 0$. Due to the fact that $\phi(x, d)$ is convex with respect to d for fixed x, we have

$$\begin{aligned}
&[\phi(y, a_1) - \phi(y, b_1) + \phi(x, a_1) - \phi(x, b_1)] - [\phi(x, b_2) + \phi(y, b_2) - \phi(x, a_2) - \phi(y, a_2)] \\
&\quad = [\phi(y, a_1) + \phi(y, a_2) - \phi(y, b_1) - \phi(y, b_2)] \\
&\quad\quad + [\phi(x, a_1) + \phi(x, a_2) - \phi(x, b_1) - \phi(x, b_2)] \\
&\quad \geq 0.
\end{aligned}$$

So

$$\phi(y, a_1) - \phi(y, b_1) + \phi(x, a_1) - \phi(x, b_1) \geq \phi(x, b_2) + \phi(y, b_2) - \phi(x, a_2) - \phi(y, a_2) \geq 0$$

and, thus,

$$\begin{aligned}
&w_2[\phi(y, a_1) - \phi(y, b_1) + \phi(x, a_1) - \phi(x, b_1)] \\
&\quad \geq w_1[\phi(x, b_2) + \phi(y, b_2) - \phi(x, a_2) - \phi(y, a_2)] \quad \text{for any } w_2 \geq w_1 \geq 0,
\end{aligned}$$

which yields

$$\eta_1(\tau(\boldsymbol{x}), \boldsymbol{a}) + \eta_1(\boldsymbol{x}, \boldsymbol{a}) \geq \eta_1(\boldsymbol{x}, \boldsymbol{b}) + \eta_1(\tau(\boldsymbol{x}), \boldsymbol{b}) \quad \text{for any } y \geq x. \tag{A.1}$$

On the other hand, since ϕ is submodular and convex with respect to d for fixed x, we have

$$\begin{aligned}
&\phi(y, a_1) - \phi(x, b_1) - \phi(y, b_2) + \phi(x, a_2) \\
&\quad = [\phi(x, a_2) + \phi(x, a_1) - \phi(x, b_1) - \phi(x, b_2)] \\
&\quad\quad + [\phi(y, a_1) + \phi(x, b_2) - \phi(x, a_1) - \phi(y, b_2)] \\
&\quad \geq 0 \quad \text{for } x \leq y.
\end{aligned}$$

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Since $\phi(x,d)$ is increasing in x for any d and decreasing in d for any x, we have $\phi(y,b_2)-\phi(x,a_2)=\phi(y,b_2)-\phi(x,b_2)+\phi(x,b_2)-\phi(x,a_2)\ge 0$. Thereby, it holds that $\phi(y,a_1)-\phi(x,b_1)\ge\phi(y,b_2)-\phi(x,a_2)\ge 0$ and, thus, for any $w_2\ge w_1\ge 0$,

$$\begin{aligned}\eta_1(\boldsymbol{x},\boldsymbol{a})-\eta_1(\tau(\boldsymbol{x}),\boldsymbol{b})&=w_2[\phi(y,a_1)-\phi(x,b_1)]-w_1[\phi(y,b_2)-\phi(x,a_2)]\\&\ge w_1[\phi(y,a_1)-\phi(x,b_1)-\phi(y,b_2)+\phi(x,a_2)]\\&\ge 0.\end{aligned}$$

Also, from (i), it follows that

$$\eta_1(\boldsymbol{x},\boldsymbol{a})\ge\eta_1(\boldsymbol{x},\boldsymbol{b})\quad\text{and}\quad\eta_1(\boldsymbol{x},\boldsymbol{a})\ge\max\{\eta_1(\boldsymbol{x},\boldsymbol{b}),\eta_1(\tau(\boldsymbol{x}),\boldsymbol{b})\}\quad\text{for } y\ge x.$$

In combination with (A.1), we have

$$(\eta_1(\boldsymbol{x},\boldsymbol{b}),\eta_1(\tau(\boldsymbol{x}),\boldsymbol{b}))\preceq_{\mathrm{w}}(\eta_1(\boldsymbol{x},\boldsymbol{a}),\eta_1(\tau(\boldsymbol{x}),\boldsymbol{a})).$$

As for increasing and convex $g(x)$, from Theorem C.1.b of Marshall *et al.* (2011), it follows that

$$\begin{aligned}&\Delta g_2(\boldsymbol{x},\boldsymbol{a},\boldsymbol{b})+\Delta g_2(\tau(\boldsymbol{x}),\boldsymbol{a},\boldsymbol{b})\\&\quad=g(\eta_1(\boldsymbol{x},\boldsymbol{a}))-g(\eta_1(\boldsymbol{x},\boldsymbol{b}))+g(\eta_1(\tau(\boldsymbol{x}),\boldsymbol{a}))-g(\eta_1(\tau(\boldsymbol{x}),\boldsymbol{b}))\\&\quad\ge 0\quad\text{for any } y\ge x,\end{aligned}$$

and this yields the desired inequality in (ii). □

Proof of Lemma 3.3. Assume that $a_1\le b_1\le b_2\le a_2$ for $(a_1,a_2)\succeq_{\mathrm{m}}(b_1,b_2)$.

(i) Since $\phi(x,d)$ is submodular and convex with respect to d for any x,

$$\begin{aligned}&\phi(x,a_2)+\phi(y,a_1)-\phi(x,b_2)-\phi(y,b_1)\\&\quad=[\phi(x,a_2)+\phi(x,a_1)-\phi(x,b_1)-\phi(x,b_2)]\\&\qquad+[\phi(x,b_1)+\phi(y,a_1)-\phi(y,b_1)-\phi(x,a_1)]\\&\quad\ge 0\quad\text{for } y\ge x,\end{aligned}\tag{A.2}$$

and, hence, for $y\ge x$ and $w_2\ge w_1\ge 0$,

$$\begin{aligned}&\eta_1(\boldsymbol{x},\boldsymbol{a})-\eta_1(\boldsymbol{x},\boldsymbol{b})+\eta_2(\boldsymbol{x},\boldsymbol{a})-\eta_2(\boldsymbol{x},\boldsymbol{b})\\&\quad=(w_1+w_2)[\phi(x,a_2)+\phi(y,a_1)-\phi(x,b_2)-\phi(y,b_1)]\end{aligned}\tag{A.3}$$

is nonnegative. On the other hand, $\phi(x,d)$ is decreasing in d for any x and increasing in x for any d, then, for $y\ge x$, we also have $\phi(y,b_1)-\phi(x,a_2)=\phi(y,b_1)-\phi(y,a_2)+\phi(y,a_2)-\phi(x,a_2)\ge 0$ and, thus, $\phi(y,a_1)-\phi(x,b_2)\ge\phi(y,b_1)-\phi(x,a_2)\ge 0$ holds due to (A.2). As a result, for $w_2\ge w_1\ge 0$ and $y\ge x$, it holds that

$$\eta_1(\boldsymbol{x},\boldsymbol{a})-\eta_2(\boldsymbol{x},\boldsymbol{b})=w_2[\phi(y,a_1)-\phi(x,b_2)]-w_1[\phi(y,b_1)-\phi(x,a_2)]\ge 0.$$

In combination with $\eta_1(\boldsymbol{x},\boldsymbol{a})\ge\eta_1(\boldsymbol{x},\boldsymbol{b})$ for $y\ge x$, we reach

$$\eta_1(\boldsymbol{x},\boldsymbol{a})\ge\max\{\eta_1(\boldsymbol{x},\boldsymbol{b}),\eta_2(\boldsymbol{x},\boldsymbol{b})\}\quad\text{for } y\ge x.\tag{A.4}$$

Based on (A.3), (A.4), and by Theorem C.1.b of Marshall *et al.* (2011), we conclude that, for $y\ge x$,

$$\Delta g_2(\boldsymbol{x},\boldsymbol{a},\boldsymbol{b})+\Delta g_3(\boldsymbol{x},\boldsymbol{a},\boldsymbol{b})=g(\eta_1(\boldsymbol{x},\boldsymbol{a}))-g(\eta_1(\boldsymbol{x},\boldsymbol{b}))+g(\eta_2(\boldsymbol{x},\boldsymbol{a}))-g(\eta_2(\boldsymbol{x},\boldsymbol{b}))\ge 0.$$

(ii) Due to $\phi(x, d)$ convex with respect to d for fixed x, it holds that

$$\begin{aligned}
&\eta_1(\boldsymbol{x}, \boldsymbol{a}) + \eta_1(\tau(\boldsymbol{x}), \boldsymbol{a}) + \eta_2(\boldsymbol{x}, \boldsymbol{a}) + \eta_2(\tau(\boldsymbol{x}), \boldsymbol{a}) \\
&\quad - \eta_1(\boldsymbol{x}, \boldsymbol{b}) - \eta_1(\tau(\boldsymbol{x}), \boldsymbol{b}) - \eta_2(\boldsymbol{x}, \boldsymbol{b}) - \eta_2(\tau(\boldsymbol{x}), \boldsymbol{b}) \\
&\qquad = (w_1 + w_2)[\phi(x, a_2) + \phi(x, a_1) - \phi(x, b_1) - \phi(x, b_2) \\
&\qquad\quad + \phi(y, a_2) + \phi(y, a_1) - \phi(y, b_1) - \phi(y, b_2)]
\end{aligned} \tag{A.5}$$

is nonnegative. Also, the submodularity of $\phi(x, d)$ implies that $\phi(x, a_1) - \phi(x, b_1) - \phi(y, a_2) + \phi(y, b_2) \geq 0$ for $y \geq x$. Therefore, we have $\eta_1(\tau(\boldsymbol{x}), \boldsymbol{a}) - \eta_1(\tau(\boldsymbol{x}), \boldsymbol{b}) \geq \eta_2(\tau(\boldsymbol{x}), \boldsymbol{a}) - \eta_2(\tau(\boldsymbol{x}), \boldsymbol{b})$ for $y \geq x$.

- If $\eta_2(\tau(\boldsymbol{x}), \boldsymbol{a}) \geq \eta_2(\tau(\boldsymbol{x}), \boldsymbol{b})$ for $y \geq x$, from the above it follows that

  $$\eta_1(\tau(\boldsymbol{x}), \boldsymbol{a}) - \eta_1(\tau(\boldsymbol{x}), \boldsymbol{b}) \geq \eta_2(\tau(\boldsymbol{x}), \boldsymbol{a}) - \eta_2(\tau(\boldsymbol{x}), \boldsymbol{b}) \geq 0.$$

  Since g increases, we have

  $$\begin{aligned}
  \Delta g_2(\tau(\boldsymbol{x}), \boldsymbol{a}, \boldsymbol{b}) &= g(\eta_1(\tau(\boldsymbol{x}), \boldsymbol{a})) - g(\eta_1(\tau(\boldsymbol{x}), \boldsymbol{b})) \geq 0, \\
  \Delta g_3(\tau(\boldsymbol{x}), \boldsymbol{a}, \boldsymbol{b}) &= g(\eta_2(\tau(\boldsymbol{x}), \boldsymbol{a})) - g(\eta_2(\tau(\boldsymbol{x}), \boldsymbol{b})) \geq 0.
  \end{aligned}$$

  In view of (i), we have, for $y \geq x$,

  $$\Delta g_2(\boldsymbol{x}, \boldsymbol{a}, \boldsymbol{b}) + \Delta g_3(\boldsymbol{x}, \boldsymbol{a}, \boldsymbol{b}) + \Delta g_2(\tau(\boldsymbol{x}), \boldsymbol{a}, \boldsymbol{b}) + \Delta g_3(\tau(\boldsymbol{x}), \boldsymbol{a}, \boldsymbol{b}) \geq 0. \tag{A.6}$$

- If $\eta_2(\tau(\boldsymbol{x}), \boldsymbol{a}) \leq \eta_2(\tau(\boldsymbol{x}), \boldsymbol{b})$ for $y \geq x$, then, by (A.5) we have, for $y \geq x$,

  $$\begin{aligned}
  \theta(\boldsymbol{x}) \equiv \eta_2(\tau(\boldsymbol{x}), \boldsymbol{a}) - [&\eta_2(\tau(\boldsymbol{x}), \boldsymbol{b}) + \eta_1(\boldsymbol{x}, \boldsymbol{b}) + \eta_1(\tau(\boldsymbol{x}), \boldsymbol{b}) \\
  &- \eta_1(\boldsymbol{x}, \boldsymbol{a}) - \eta_1(\tau(\boldsymbol{x}), \boldsymbol{a}) - \eta_2(\boldsymbol{x}, \boldsymbol{a})] \\
  \geq \eta_2(\boldsymbol{x}, \boldsymbol{b}).&
  \end{aligned} \tag{A.7}$$

  Note that symmetric convexity implies schur-convexity. For increasing, convex $g(x)$, and

  $$\begin{aligned}
  &(\eta_1(\boldsymbol{x}, \boldsymbol{a}), \eta_1(\tau(\boldsymbol{x}), \boldsymbol{a}), \eta_2(\boldsymbol{x}, \boldsymbol{a}), \eta_2(\tau(\boldsymbol{x}), \boldsymbol{a})) \\
  &\qquad \succeq_{\mathrm{m}} (\eta_2(\tau(\boldsymbol{x}), \boldsymbol{b}), \eta_1(\boldsymbol{x}, \boldsymbol{a}), \eta_1(\tau(\boldsymbol{x}), \boldsymbol{a}), \theta(\boldsymbol{x})),
  \end{aligned}$$

  it holds that, for $y \geq x$,

  $$\begin{aligned}
  &g(\eta_1(\boldsymbol{x}, \boldsymbol{a})) + g(\eta_1(\tau(\boldsymbol{x}), \boldsymbol{a})) + g(\eta_2(\boldsymbol{x}, \boldsymbol{a})) + g(\eta_2(\tau(\boldsymbol{x}), \boldsymbol{a})) \\
  &\quad \geq g(\eta_1(\boldsymbol{x}, \boldsymbol{b})) + g(\eta_1(\tau(\boldsymbol{x}), \boldsymbol{b})) + g(\eta_2(\tau(\boldsymbol{x}), \boldsymbol{b})) + g(\theta(\boldsymbol{x})) \\
  &\quad \geq g(\eta_1(\boldsymbol{x}, \boldsymbol{b})) + g(\eta_1(\tau(\boldsymbol{x}), \boldsymbol{b})) + g(\eta_2(\tau(\boldsymbol{x}), \boldsymbol{b})) + g(\eta_2(\boldsymbol{x}, \boldsymbol{b})),
  \end{aligned}$$

  where the last inequality is due to (A.7). This invokes (A.6) again. □

Proof of Lemma 3.4. Denote by $f(\boldsymbol{x})$ the probability density of (X_1, X_2). We proceed using the notation of Lemma 3.1.

(i) For any $a_1 \leq a_2$ and $0 \leq w_1 \leq w_2$,

$$\begin{aligned}
\Delta\zeta_1(\boldsymbol{w}, \boldsymbol{a}) &= \iint_{\mathbb{R}^2} \Delta g_1(\boldsymbol{x}, \boldsymbol{a}) f(\boldsymbol{x})\, \mathrm{d}x_1\, \mathrm{d}x_2 \\
&= \iint_{x_1 \leq x_2} \Delta g_1(\boldsymbol{x}, \boldsymbol{a}) f(\boldsymbol{x})\, \mathrm{d}x_1\, \mathrm{d}x_2 + \iint_{x_2 \leq x_1} \Delta g_1(\boldsymbol{x}, \boldsymbol{a}) f(\boldsymbol{x})\, \mathrm{d}x_1\, \mathrm{d}x_2
\end{aligned}$$

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$$
\begin{aligned}
&= \iint_{x_1 \le x_2} [\Delta g_1(\boldsymbol{x}, \boldsymbol{a}) f(\boldsymbol{x}) + \Delta g_1(\boldsymbol{x}, \boldsymbol{a}) f(\tau(\boldsymbol{x}))] \, \mathrm{d}x_1 \, \mathrm{d}x_2 \\
&\ge \iint_{x_1 \le x_2} [\Delta g_1(\boldsymbol{x}, \boldsymbol{a}) + \Delta g_1(\boldsymbol{x}, \boldsymbol{a})] f(\tau(\boldsymbol{x})) \, \mathrm{d}x_1 \, \mathrm{d}x_2 \\
&\ge 0,
\end{aligned}
$$

where the first inequality is due to Lemma 2.1 and the second one stems from Lemma 3.1.

(ii) Due to the submodular ϕ, it holds that $\phi(x_2, a_1) - \phi(x_1, a_1) \ge \phi(x_2, a_2) - \phi(x_1, a_2)$ for any $x_1 \le x_2$ and $a_1 \le a_2$.

- Set $x_1 \le x_2$. Then we have $\eta_1(\boldsymbol{x}, \boldsymbol{a}) + \eta_2(\boldsymbol{x}, \boldsymbol{a}) \ge \eta_1(\boldsymbol{x}, \tau(\boldsymbol{a})) + \eta_2(\boldsymbol{x}, \tau(\boldsymbol{a}))$.
- Also, since $\phi(x, d)$ is increasing in x for any fixed d, $\phi(x_2, a_2) - \phi(x_1, a_2) \ge 0$ and then $w_2\phi(x_2, a_1) + w_1\phi(x_1, a_2) \ge w_2\phi(x_1, a_1) + w_1\phi(x_2, a_2)$ for any $w_2 \ge w_1 \ge 0$, i.e. $\eta_1(\boldsymbol{x}, \boldsymbol{a}) \ge \eta_2(\boldsymbol{x}, \tau(\boldsymbol{a}))$. By Lemma 3.1, we obtain

$$\eta_1(\boldsymbol{x}, \boldsymbol{a}) \ge \max\{\eta_2(\boldsymbol{x}, \tau(\boldsymbol{a})), \eta_1(\boldsymbol{x}, \tau(\boldsymbol{a}))\}.$$

In combination with the above two inequalities, we further have, for $x_1 \le x_2$,

$$g(\eta_1(\boldsymbol{x}, \boldsymbol{a})) + g(\eta_2(\boldsymbol{x}, \boldsymbol{a})) \ge g(\eta_1(\boldsymbol{x}, \tau(\boldsymbol{a}))) + g(\eta_2(\boldsymbol{x}, \tau(\boldsymbol{a}))).$$

Therefore, owing to Lemma 2.1, it holds that

$$
\begin{aligned}
&\Delta\zeta_1(\boldsymbol{w}, \boldsymbol{a}) + \Delta\zeta_1(\tau(\boldsymbol{w}), \boldsymbol{a}) \\
&\quad = \iint_{\mathbb{R}^2} [g(\eta_1(\boldsymbol{x}, \boldsymbol{a})) + g(\eta_2(\boldsymbol{x}, \boldsymbol{a})) - g(\eta_1(\boldsymbol{x}, \tau(\boldsymbol{a}))) - g(\eta_2(\boldsymbol{x}, \tau(\boldsymbol{a})))] f(\boldsymbol{x}) \, \mathrm{d}x_1 \, \mathrm{d}x_2 \\
&\quad = \iint_{x_1 \le x_2} [g(\eta_1(\boldsymbol{x}, \boldsymbol{a})) + g(\eta_2(\boldsymbol{x}, \boldsymbol{a})) - g(\eta_1(\boldsymbol{x}, \tau(\boldsymbol{a}))) - g(\eta_2(\boldsymbol{x}, \tau(\boldsymbol{a})))] \\
&\qquad\qquad \times [f(\boldsymbol{a}) - f(\tau(\boldsymbol{x}))] \, \mathrm{d}x_1 \, \mathrm{d}x_2 \\
&\quad \ge 0 \quad \text{for } w_2 \ge w_1 \ge 0.
\end{aligned}
$$

□

Proof of Lemma 3.5. Denote by $f(x_1, x_2)$ the probability density of (X_1, X_2) and let us proceed using the notation of Lemmas 3.2 and 3.3. Again, we assume that $a_1 \le b_1 \le b_2 \le a_2$ without loss of generality.

(i) For any $(a_1, a_2) \succeq_{\mathrm{m}} (b_1, b_2)$ and $w_2 \ge w_1 \ge 0$,

$$
\begin{aligned}
\Delta\zeta_2(\boldsymbol{w}, \boldsymbol{a}, \boldsymbol{b}) &= \iint_{x_1 \le x_2} \Delta g_2(\boldsymbol{x}, \boldsymbol{a}, \boldsymbol{b}) f(\boldsymbol{x}) \, \mathrm{d}x_1 \, \mathrm{d}x_2 + \iint_{x_1 \ge x_2} \Delta g_2(\boldsymbol{x}, \boldsymbol{a}, \boldsymbol{b}) f(\boldsymbol{x}) \, \mathrm{d}x_1 \, \mathrm{d}x_2 \\
&= \iint_{x_1 \le x_2} [\Delta g_2(\boldsymbol{x}, \boldsymbol{a}, \boldsymbol{b}) f(\boldsymbol{x}) + \Delta g_2(\tau(\boldsymbol{x}), \boldsymbol{a}, \boldsymbol{b}) f(\tau(\boldsymbol{x}))] \, \mathrm{d}x_1 \, \mathrm{d}x_2 \\
&\ge \iint_{x_1 \le x_2} [\Delta g_2(\boldsymbol{x}, \boldsymbol{a}, \boldsymbol{b}) + \Delta g_2(\tau(\boldsymbol{x}), \boldsymbol{a}, \boldsymbol{b})] f(\tau(\boldsymbol{x})) \, \mathrm{d}x_1 \, \mathrm{d}x_2 \\
&\ge 0,
\end{aligned}
$$

where the two inequalities follow from Lemma 2.1 and Lemma 3.2(ii), respectively.

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(ii) For any $(a_1, a_2) \succeq_{\mathrm{m}} (b_1, b_2)$ and $w_2 \geq w_1 \geq 0$,

$$
\begin{aligned}
\Delta\zeta_2(\boldsymbol{w}, \boldsymbol{a}, \boldsymbol{b}) &+ \Delta\zeta_2(\tau(\boldsymbol{x}), \boldsymbol{a}, \boldsymbol{b}) \\
&= \iint_{x_1 \leq x_2} [\Delta g_2(\boldsymbol{x}, \boldsymbol{a}, \boldsymbol{b}) + \Delta g_3(\boldsymbol{x}, \boldsymbol{a}, \boldsymbol{b})] f(\boldsymbol{x}) \, \mathrm{d}x_1 \, \mathrm{d}x_2 \\
&\quad + \iint_{x_1 \geq x_2} [\Delta g_2(\boldsymbol{x}, \boldsymbol{a}, \boldsymbol{b}) + \Delta g_3(\boldsymbol{x}, \boldsymbol{a}, \boldsymbol{b})] f(\boldsymbol{x}) \, \mathrm{d}x_1 \, \mathrm{d}x_2 \\
&\geq \iint_{x_1 \leq x_2} [\Delta g_2(\boldsymbol{x}, \boldsymbol{a}, \boldsymbol{b}) + \Delta g_2(\tau(\boldsymbol{x}), \boldsymbol{a}, \boldsymbol{b}) + \Delta g_3(\boldsymbol{x}, \boldsymbol{a}, \boldsymbol{b}) + \Delta g_3(\tau(\boldsymbol{x}), \boldsymbol{a}, \boldsymbol{b})] \\
&\quad \times f(\tau(\boldsymbol{x})) \, \mathrm{d}x_1 \, \mathrm{d}x_2 \\
&\geq 0,
\end{aligned}
$$

where the two inequalities follow from Lemma 2.1 and Lemma 3.3, respectively. □

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