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A NOTE ON SEPARABILITY IN OUTER AUTOMORPHISM GROUPS

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Abstract We give a criterion for separability of subgroups of certain outer automorphism groups. This answers questions of Hagen and Sisto, by strengthening and generalizing a result of theirs on mapping class groups.

Keywords: separable subgroup; outer automorphism; acylindrically hyperbolic; mapping class group

1. Introduction

A subgroup $H \leqslant G$ is called *separable* if for all $g \in G \setminus H$ there exists a finite quotient $q \colon G \to Q$ such that $q(g) \notin q(H)$. A question of Reid [26, Question 3.5] asks whether all convex-cocompact subgroups of mapping class groups (as defined in [9], see also [19] for a characterization) are separable. This was first verified for virtually cyclic subgroups [21], and the full conjecture is known conditionally on the residual finiteness of hyperbolic groups [3]. Hagen and Sisto show in [17] that certain examples of free convex-cocompact subgroups, constructed by Mj [22], are separable. In order to do so, they prove the following criterion.

Theorem ([17, Theorem 1.1]). *Let $g \geqslant 2$, and let $H \leqslant \mathrm{MCG}(\Sigma_g)$. Suppose that H is torsion-free, malnormal, and convex-cocompact. If the preimage of H under the natural quotient map $\mathrm{MCG}(\Sigma_g \backslash \{p\}) \to \mathrm{MCG}(\Sigma_g)$, for some point $p \in \Sigma_g$, is conjugacy separable, then H is separable.*

Remark. The theorem is stated for $g \geqslant 1$; however, in that case, $\mathrm{MCG}(\Sigma_g)$ is virtually free so every finitely generated subgroup is separable. We focus on the case $g \geqslant 2$ for coherence with the results of this note.

A group G is said to be *conjugacy separable* if for all non-conjugate elements $g, h \in G$ there exists a finite quotient $q\colon G \to Q$ such that $q(g)$ and $q(h)$ are non-conjugate. Our first result removes the other hypotheses on H, answering [17, Question 1.4].

Theorem A. *Let $g \geqslant 2$, and let $H \leqslant \mathrm{MCG}(\Sigma_g)$. If the preimage of H under the natural quotient map $\mathrm{MCG}(\Sigma_g \setminus \{p\}) \to \mathrm{MCG}(\Sigma_g)$, for some point $p \in \Sigma_g$, is conjugacy separable, then H is separable.*

We can replace the hypothesis of conjugacy separability by a more geometric one, which goes in the direction of Reid's question [26, Question 3.5]. This is the same way that Hagen and Sisto verify that their criterion holds for the groups constructed by Mj.

Corollary B. *Let $g \geqslant 2$, and let $H \leqslant \mathrm{MCG}(\Sigma_g)$ be a convex-cocompact subgroup. If the preimage of H under the natural quotient map $\mathrm{MCG}(\Sigma_g \setminus \{p\}) \to \mathrm{MCG}(\Sigma_g)$, for some point $p \in \Sigma_g$, acts properly and cocompactly on a $\mathrm{CAT}(0)$ cube complex, then H is separable.*

Notice that Corollary B applies to *all* groups constructed in [22], without needing to modify the construction to ensure malnormality, as in [17, Section 5].

Theorem A will follow from a more general result.

Theorem C. *Let G be a finitely generated group with trivial centre, and let $H \leqslant \mathrm{Out}(G)$. Suppose that $\mathrm{Aut}(G)$ is acylindrically hyperbolic and has no non-trivial finite normal subgroups. If the preimage of H under the natural quotient map $\mathrm{Aut}(G) \to \mathrm{Out}(G)$ is conjugacy separable, then H is separable.*

Thanks to recent works proving acylindrical hyperbolicity of automorphism groups [8, 10–13], this can be applied in several contexts. We isolate two instances.

Corollary D. *Let G be a torsion-free hyperbolic group, and let $H \leqslant \mathrm{Out}(G)$. If the preimage of H under the natural quotient map $\mathrm{Aut}(G) \to \mathrm{Out}(G)$ is conjugacy separable, then H is separable.*

Corollary E. *Let Γ be a finite simplicial graph that does not decompose as a join of two non-empty subgraphs. Let A_Γ be the corresponding right-angled Artin group, and let $H \leqslant \mathrm{Out}(A_\Gamma)$. If the preimage of H under the natural quotient map $\mathrm{Aut}(A_\Gamma) \to \mathrm{Out}(A_\Gamma)$ is conjugacy separable, then H is separable.*

Both of these corollaries apply to $\mathrm{Out}(F_n)$. In this case, there are notions of convex-cocompact subgroups [6, 7, 16]; the corresponding preimage in $\mathrm{Aut}(F_n)$ is hyperbolic [7], but we do not know of instances in which conjugacy separability of such groups is known (besides the case of virtually cyclic subgroups).

Question F. Are there examples of subgroups $H \leqslant \mathrm{Out}(F_n)$ that are separable, not virtually cyclic, and satisfy some version of convex-cocompactness?

We remark that free-by-cyclic groups $F_n \rtimes \mathbb{Z}$ often have non-separable subgroups [20].

The proof of Theorem C involves an element of G that recognises non-inner automorphisms. The existence of such an element relies on the acylindrical hyperbolicity of

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Aut(G), and is what allows to strengthen and generalise the criterion of Hagen and Sisto, with a shorter proof. It partially answers [17, Questions 1.5] (Proposition 2.6) and [17, Question 1.6] (Proposition 2.8).

2. Proofs

We will use π to denote the quotient map $\pi\colon \operatorname{Aut}(G) \to \operatorname{Out}(G)$, and for a subgroup $H \leqslant \operatorname{Out}(G)$ we denote $\widetilde{H}:=\pi^{-1}H \leqslant \operatorname{Aut}(G)$. The starting point is the following criterion for separability, which in turn is based on Grossman's criterion for residual finiteness [14].

Proposition 2.1. ([17, Proposition 2.5]). *Let G be a finitely generated group with trivial centre. Let $H \leqslant \operatorname{Out}(G)$ and $\alpha \in \operatorname{Aut}(G)$. Suppose that:*

1. *There exists $x \in G$ such that $\alpha(x) \neq h(x)$ for all $h \in \widetilde{H}$;*
2. *$\widetilde{H}$ is conjugacy separable.*

Then there exists a finite quotient $q\colon \operatorname{Out}(G) \to Q$ such that $q(\pi(\alpha)) \notin q(H)$.

Let us formulate the special case that we will use:

Corollary 2.2. *Let G be a finitely generated group with trivial centre and let $H \leqslant \operatorname{Out}(G)$. Suppose that:*

1. *There exists $\gamma \in \operatorname{Inn}(G)$ such that the centraliser of γ in $\operatorname{Aut}(G)$ is $\langle\gamma\rangle$;*
2. *$\widetilde{H}$ is conjugacy separable.*

Then H is separable.

Proof. We show that Proposition 2.1 holds for H and an arbitrary $\alpha \in \operatorname{Aut}(G) \setminus \widetilde{H}$. Choose $x \in G$ such that the corresponding inner automorphism γ_x is as in the first assumption of the corollary. Suppose that $\alpha(x) = h(x)$ for some $h \in \widetilde{H}$. Then $h^{-1}\alpha(x) = x$, and so $h^{-1}\alpha$ belongs to the centraliser of γ_x in $\operatorname{Aut}(G)$. By the choice of x, we have $h^{-1}\alpha \in \langle\gamma_x\rangle \leqslant \operatorname{Inn}(G) \leqslant \widetilde{H}$, which contradicts $\alpha \notin \widetilde{H}$. □

Acylindrical hyperbolicity of Aut(G) ensures that the first item holds.

Lemma 2.3. *Let G be a group such that $\operatorname{Inn}(G)$ is infinite and $\operatorname{Aut}(G)$ is acylindrically hyperbolic and has no non-trivial finite normal subgroups. Then there exists $\gamma \in \operatorname{Inn}(G)$ such that the centraliser of γ in $\operatorname{Aut}(G)$ is $\langle\gamma\rangle$.*

In fact, such a γ can be found by performing a simple random walk on G: see Proposition 2.8 and its proof.

Proof. Recall that a subgroup of an acylindrically hyperbolic group is called *suitable* if it is non-elementary and does not normalise any non-trivial finite normal subgroup. In an acylindrically hyperbolic group with no non-trivial finite normal subgroups, every infinite normal subgroup is suitable [25, Lemma 2.4]. In particular, $\operatorname{Inn}(G) \leqslant \operatorname{Aut}(G)$

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is suitable. Therefore, there exists an inner automorphism $\gamma \in \mathrm{Inn}(G)$ such that the elementary closure of γ in $\mathrm{Aut}(G)$ is reduced to $\langle\gamma\rangle$ [18, Lemma 5.6]. In particular, the centraliser of γ in $\mathrm{Aut}(G)$ is reduced to $\langle\gamma\rangle$ [4, Corollary 6.6]. □

Proof of Theorem C. Combine Lemma 2.3 and Corollary 2.2. □

For the next applications, we will use the following criterion to check that an automorphism group has no non-trivial finite normal subgroups. We say that G has the *unique root property* if $x^n = y^n$ for some $x, y \in G, n \geqslant 1$ implies $x = y$.

Lemma 2.4. *Let G be a group with trivial centre and with the unique root property. Then* $\mathrm{Aut}(G)$ *has no non-trivial finite normal subgroups.*

Proof. Suppose that $N \leqslant \mathrm{Aut}(G)$ is a finite normal subgroup. The action of $G \cong \mathrm{Inn}(G)$ on N by conjugacy has a finite index kernel K. Then every element of K commutes with every element of N, in other words, automorphisms in N fix K pointwise. Now let $\alpha \in N$ and $x \in G$. Let $n \geqslant 1$ be such that $x^n \in K$, so x^n is fixed by α. Then $x^n = \alpha(x^n) = \alpha(x)^n$, so by the unique root property $\alpha(x) = x$. This shows that α fixes every element of G and we conclude. □

Proof of Corollary D. If G is torsion-free elementary hyperbolic, then G is either trivial or isomorphic to $\mathbb{Z}$, and in both cases $\mathrm{Out}(G)$ is finite, so all subgroups are separable.

If G is torsion-free non-elementary hyperbolic, then $\mathrm{Aut}(G)$ is acylindrically hyperbolic [12, Theorem 1.3]. Moreover, G has trivial centre and the unique root property [2, Lemma 2.2] and so $\mathrm{Aut}(G)$ has no non-trivial finite normal subgroups by Lemma 2.4. Therefore, Theorem C applies. □

Proof of Corollary E. Let $G = A_\Gamma$ be as in the statement. If Γ has at most one vertex, then G is either trivial or isomorphic to $\mathbb{Z}$, and in both cases $\mathrm{Out}(G)$ is finite, so all subgroups are separable.

If Γ has at least two vertices, then $\mathrm{Aut}(G)$ is acylindrically hyperbolic [11, Theorem 1.5]. Moreover, G has trivial centre and the unique root property [5, 3-2) and 3-3)] and so $\mathrm{Aut}(G)$ has no non-trivial finite normal subgroups by Lemma 2.4. Therefore, Theorem C applies. □

For the results on mapping class groups, we apply Corollary D to the special case of surface groups.

Proof of Theorem A. Recall the Dehn–Nielsen–Baer Theorem: $\mathrm{MCG}(\Sigma_g)$ can be identified with an index-2 subgroup of $\mathrm{Out}(\pi_1(\Sigma_g))$, and $\mathrm{MCG}(\Sigma_g \setminus \{p\})$ is the corresponding index-2 subgroup of $\mathrm{Aut}(\pi_1(\Sigma_g))$. Under these identifications, given a subgroup $H \leqslant \mathrm{MCG}(\Sigma_g)$, its preimage in $\mathrm{MCG}(\Sigma_g \setminus \{p\})$ is the same as its preimage in $\mathrm{Aut}(\pi_1(\Sigma_g))$. Since $\pi_1(\Sigma_g)$ is torsion-free non-elementary hyperbolic, we can apply Corollary D to get separability of H in $\mathrm{Out}(\pi_1(\Sigma_g))$, which then implies separability in $\mathrm{MCG}(\Sigma_g)$. □

Proof of Corollary B. By Theorem A, it suffices to show that, under the hypotheses, the preimage $\widetilde{H}$ is conjugacy separable. By convex-cocompactness of H, $\widetilde{H}$ is hyperbolic

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[9, 15]. By assumption, $\widetilde{H}$ acts properly and cocompactly on a CAT(0) cube complex, and so it is virtually compact special [1, 27]. It follows that $\widetilde{H}$ is conjugacy separable [24]. □

Let us end by addressing two questions from [17], which ask for elements that recognise non-inner automorphisms. The observation behind Lemma 2.3 allows to answer both, under the assumptions of Theorem C. The two questions are asked for torsion-free acylindrically hyperbolic groups, with the case of hyperbolic groups being singled out. It is an open question whether the automorphism group of a finitely generated acylindrically hyperbolic group is always acylindrically hyperbolic [10, Question 1.1].

Question 2.5. ([17, Question 1.5]). Let G be a torsion-free acylindrically hyperbolic group, and let $\phi_1, \dots, \phi_n$ be non-inner automorphisms of G. Does there exist $x \in G$ with x and $\phi_i(x)$ non-conjugate for all i?

Proposition 2.6. *Let G be a group such that* $\mathrm{Inn}(G)$ *is infinite and* $\mathrm{Aut}(G)$ *is acylindrically hyperbolic and has no non-trivial finite normal subgroups: for instance, a torsion-free non-elementary hyperbolic group. Then there exists $x \in G$ with the following property: for every non-inner automorphism ϕ, x and $\phi(x)$ are non-conjugate.*

Recall that torsion-free non-elementary hyperbolic groups do indeed satisfy the hypotheses, as we saw in the proof of Corollary D.

Proof. Let x be such that γ_x satisfies the statement of Lemma 2.3. Suppose that $\phi(x) = hxh^{-1}$. Then $\gamma_h^{-1}\phi$ fixes x, so it centralises γ_x. By the choice of x, we have $\gamma_h^{-1}\phi \in \langle \gamma_x \rangle \leqslant \mathrm{Inn}(G)$ and so $\phi \in \mathrm{Inn}(G)$. □

Question 2.7. ([17, Question 1.6]). Let G be a torsion-free acylindrically hyperbolic group, let ϕ be a non-inner automorphism of G, and let (w_n) be a simple random walk on G. Is it true that, with probability going to 1 as n goes to infinity, w_n is not conjugate to $\phi(w_n)$?

Note that finite generation is implicit in this question, as *simple* random walks are not defined over infinitely generated groups.

Proposition 2.8. *Let G be a finitely generated group with trivial centre such that* $\mathrm{Aut}(G)$ *is acylindrically hyperbolic and has no non-trivial finite normal subgroups: for instance, a torsion-free non-elementary hyperbolic group. Let (w_n) be a simple random walk on G. Then, with probability going to 1 as n goes to infinity, w_n has the following property: for every non-inner automorphism ϕ, w_n and $\phi(w_n)$ are non-conjugate.*

Proof. We will use a result from [23], from which we recall some terminology, in the special case we are interested in. We call an element $\alpha \in \mathrm{Aut}(G)$ *asymmetric* if its elementary closure is $\langle \alpha \rangle$. Let μ be a probability distribution on $\mathrm{Aut}(G)$. We say that μ is *admissible* (with respect to a fixed acylindrical action) if the support of μ is bounded and generates a non-elementary subgroup containing an asymmetric element. Let ν be the uniform measure on a finite generating set of G, and let μ be the pushforward of ν under the map $G \to \mathrm{Aut}(G)$. The simple random walk (w_n) is generated by ν, and it induces

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a random walk (γ_{w_n}) generated by μ. Since the support of μ is finite, and it generates $\mathrm{Inn}(G)$ which is non-elementary and contains an asymmetric element (Lemma 2.3), μ is admissible. The cyclic subgroups $\langle\gamma_{w_n}\rangle$ are called *random subgroups* of $\mathrm{Aut}(G)$ (for $k=1$) in the language of [23].

Now we can apply [23, Theorem 2.5], which states that with probability going to 1 as n goes to infinity, γ_{w_n} is an asymmetric element of $\mathrm{Aut}(G)$. This implies that the centraliser of γ_{w_n} is $\langle\gamma_{w_n}\rangle$ [4, Corollary 6.6], and we conclude as in Proposition 2.6. □

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