Euro. Jnl of Applied Mathematics (2011), *vol.* 22, *pp.* 613–629.
doi:10.1017/S0956792511000325

A Survey in Mathematics for Industry

Two-timing and matched asymptotic expansions for singular perturbation problems

R. E. O'MALLEY, Jr. and E. KIRKINIS

Department of Applied Mathematics, University of Washington, Seattle, WA 98195-2420, USA
email: omalley@amath.washington.edu, kirkinis@amath.washington.edu

(*Received 5 October 2010; revised 1 July 2011; accepted 16 July 2011; first published online 12 August 2011*)

Following the derivation of amplitude equations through a new two-time-scale method [O'Malley, R. E., Jr. & Kirkinis, E (2010) A combined renormalization group-multiple scale method for singularly perturbed problems. *Stud. Appl. Math.* **124**, 383–410], we show that a multi-scale method may often be preferable for solving singularly perturbed problems than the method of matched asymptotic expansions. We illustrate this approach with 10 singularly perturbed ordinary and partial differential equations.

Key words: Asymptotic methods; Perturbation theory; Boundary layers; Multiple scales; Amplitude equations

1 Introduction

The most popular and best understood technique to solve singularly perturbed boundary value problems is matched asymptotic expansions, based on the pioneering work of Prandtl and countless subsequent international contributions since 1904, as surveyed in Van Dyke [28], Il′in [11], O'Malley [25], Vasil′eva *et al.* [29], Lagerstrom [19] and elsewhere. Multi-scale methods were developed much later, based on the independent work of Kuzmak [18], Kevorkian [13], Mahony [22] and Cochran [4]. These are described in the texts of Cole [5], Nayfeh [23], Smith [27], de Jager and Jiang [7], Kevorkian and Cole [14] and Johnson [12], amongst other sources. However, with the advent of more recent techniques such as the Renormalization Group (or RG) method of Goldenfeld, Oono and co-workers (cf., e.g. [2, 3, 10]), it became clear that a simpler framework may exist for the solution of the aforementioned problems that avoids some of the cumbersome inner workings of the method of matched expansions, and that may possibly provide even more accurate results for a wider range of parameter values.

In this paper, we demonstrate that with a refined version of the two-time (or multiple-scale) method much progress can be made regarding the solution of singularly perturbed problems. One possible disadvantage of the present technique is the need to introduce an *a-priori* slow temporal or spatial scale. This, however, is the standard route followed in applying multiple scales. Our skill in selecting this variable can be expected to improve with experience or by incorporating ideas from the RG method.

We refer to Olver [24] for the definition of an asymptotic approximation and its use.

2 Example 1

Lomov [21] considered the linear initial value problem

$$\epsilon u' + \frac{2}{1+x^2}u = 2\frac{3+\arctan^2(x)}{1+x^2}, \quad u(0) = 1 \tag{2.1}$$

for $x \geqslant 0$ in the limit that the small positive parameter ϵ tends to 0.

Two-timing suggests that the asymptotic solution be a function of both the given independent variable x and the stretched (fast) variable

$$\eta = \frac{1}{\epsilon}\int_0^x \frac{2}{1+s^2}ds = \frac{2}{\epsilon}\arctan(x), \tag{2.2}$$

which varies monotonically from zero to infinity in an $O(\epsilon)$-thick initial layer near $x = 0$. This variable occurs naturally since e^{η} is the integrating factor for (2.1). Indeed, linearity shows that the solution is additively decomposed in the form

$$u(x,\epsilon) = U(x,\epsilon) + e^{-\frac{2}{\epsilon}\arctan(x)}(1 - U(0,\epsilon)) \tag{2.3}$$

for an outer expansion

$$U(x,\epsilon) = U_0(x) + \epsilon U_1(x) + \dots \tag{2.4}$$

that is a power series expansion in ϵ that provides the asymptotic solution for $x > 0$, with the supplemental initial layer correction providing non-uniform convergence in a narrow boundary layer.

Regular perturbation methods readily determine the terms of $U(x,\epsilon)$ uniquely. Since coefficients of successive powers of ϵ in the differential equation (2.1) must be zero

$$U_0(x) = 3 + \arctan^2(x), \tag{2.5}$$

while $U_k(x) = -\frac{1}{2}(1+x^2)U'_{k-1}(x)$, for each $k \geqslant 1$. Thus,

$$U_1(x) = -\arctan(x), \quad U_2(x) = \frac{1}{2} \quad \text{and} \quad U_k(x) = 0 \text{ for } k \geqslant 3. \tag{2.6}$$

This three-term outer expansion is an exact solution of the differential equation. The boundary-layer correction is then simply the complementary solution determined by the prescribed initial value. The sum (2.3) is the exact solution to the initial value problem for all $\epsilon > 0$. It features as initial layer for small values of ϵ.

Matched expansions (and the related boundary-layer correction and boundary function method techniques) would instead provide a composite solution in the additive form

$$u(x,\epsilon) = U(x,\epsilon) + \xi(\tau,\epsilon^2), \tag{2.7}$$

with U being the same three-term outer solution (2.4) and where $\xi(\tau,\epsilon^2) \sim \sum_{j\geqslant 0}\xi_j(\tau)\epsilon^{2j} \to 0$ as $\tau = x/\epsilon \to \infty$. The stretched homogeneous equation gives rise to the initial value problems

$$\frac{d\xi_0}{d\tau} + 2\xi_0 = 0, \ \xi_0(0) = -2, \tag{2.8}$$

https://doi.org/10.1017/S0956792511000325 Published online by Cambridge University Press

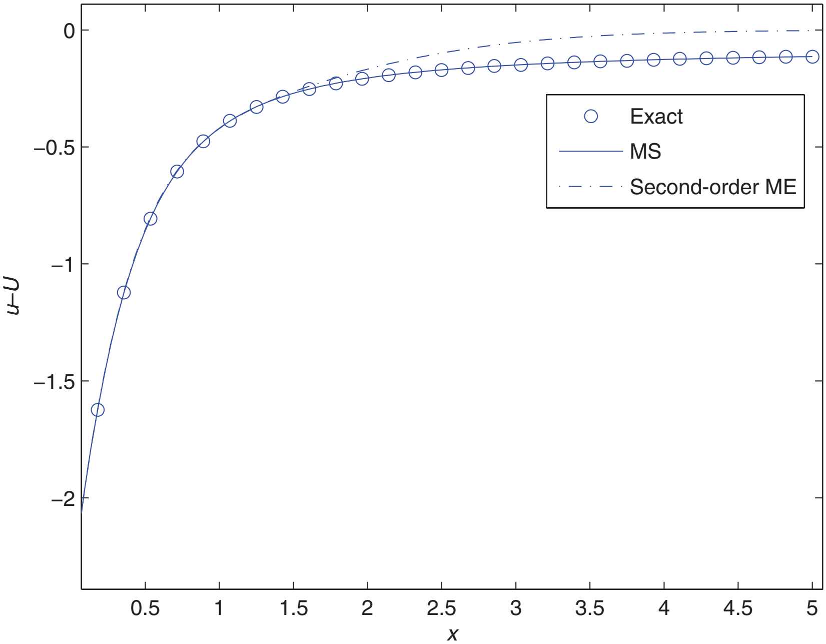

FIGURE 1. (Colour online) Plot of the exact expression $u - U$, multiple-scale (2.3) and the matched-expansions $\xi \sim \xi_0 + \epsilon^2 \xi_1$ for $\epsilon = 0.9$ in the layer region.

$$\frac{d\xi_1}{d\tau} + 2\xi_1 = 2\tau^2 \xi_0, \; \xi_1(0) = -\frac{1}{2}, \tag{2.9}$$

etc., so

$$\xi_0(\tau) = -2e^{-2\tau}, \quad \xi_1(\tau) = -\left(\frac{1}{2} + \frac{4}{3}\tau^3\right) e^{-2\tau}, \ldots.$$

Both results (2.3) and (2.7) are asymptotically correct, but numerical checks using a stiff integrator verify that the two-timing solution is more accurate in the initial layer for moderate values of ϵ (cf. Figure 1).

3 Example 2

The solution of the general linear first-order singularly perturbed differential equation (cf. Wong [32])

$$\epsilon u' + a(x)u + b(x) = 0, \quad u(0) = 1 \tag{3.1}$$

should likewise have a two-scale asymptotic representation

$$u(x, \epsilon) = U(x, \epsilon) + e^{-\frac{1}{\epsilon}\int_0^x a(s)ds}(1 - U(0, \epsilon)) \tag{3.2}$$

on any bounded interval of $x \geqslant 0$ as $\epsilon \to 0$, provided $a(x) > 0$ holds there. Again, the terms of the outer expansion $U(x, \epsilon) = U_0(x) + \epsilon U_1(x) + \ldots$ will be uniquely determined by regular perturbation techniques, while the complementary initial layer correction decays to zero as the stretched variable

$$\eta = \frac{1}{\epsilon}\int_0^x a(s)ds \to \infty. \tag{3.3}$$

In particular, we will need $a(x)U_0 + b(x) = 0$ and $a(x)U_k + U'_{k-1} = 0$ for all $k \geqslant 1$. The unique asymptotic expansion (3.2) can also be obtained by integrating the exact solution

$$u(x,\epsilon) = e^{-\frac{1}{\epsilon}\int_0^x a(s)ds}u(0) - \frac{1}{\epsilon}\int_0^x e^{-\frac{1}{\epsilon}\int_t^x a(s)ds}b(t)dt \tag{3.4}$$

repeatedly by parts, when ϵ is small. First,

$$u(x,\epsilon) = -\frac{b(x)}{a(x)} + e^{-\frac{1}{\epsilon}\int_0^x a(s)ds}\left(u(0) + \frac{b(0)}{a(0)}\right) - \int_0^x e^{-\frac{1}{\epsilon}\int_t^x a(s)ds}\frac{d}{dt}\left(\frac{b(t)}{a(t)}\right)dt.$$

Because the new integral is $O(\epsilon)$, this shows that the leading term asymptotic approximation is asymptotically correct throughout the x interval. Next, we integrate by parts again to obtain

$$u(x,\epsilon) = -\frac{b(x)}{a(x)} + \frac{\epsilon}{a(x)}\frac{d}{dx}\left(\frac{b(x)}{a(x)}\right) + e^{-\frac{1}{\epsilon}\int_0^x a(s)ds}\left[u(0) + \frac{b(0)}{a(0)} - \frac{\epsilon}{a(0)}\frac{d}{dx}\left(\frac{b(x)}{a(x)}\right)_{x=0}\right] + O(\epsilon^2), \tag{3.5}$$

on bounded x intervals. Although the limiting behaviour of this asymptotic solution is now clear, that is not so transparent from the exact solution itself.

A direct procedure to obtain the asymptotic solution would be to simply guess that it has the form

$$u(x,\epsilon) = U(x,\epsilon) + e^{-\frac{1}{\epsilon}\int_0^x a(s)ds}B(\epsilon) \tag{3.6}$$

for a smooth outer expansion U and a constant coefficient $B(\epsilon) = (1 - U(0,\epsilon))$ determined by it. Note that the second term of (3.6) exactly satisfies the homogeneous differential equation for all ϵ.

4 Example 3

The canard phenomenon of delayed bifurcation (cf. [31]) illustrates one important advantage in using the optimal stretched variable anticipated by our multiple-scale method (in contrast to matched asymptotics whose range of validity is generally more restricted).

Consider the initial value problem

$$\epsilon y' - xp(x)y = 0 \tag{4.1}$$

on $x \geqslant -1$ with $y(-1) = 1$, assuming $p(x) > 0$ is smooth. The exact solution

$$y(x,\epsilon) = e^{-\eta} \tag{4.2}$$

is expressed simply in terms of only the stretched (fast) variable

$$\eta = -\frac{1}{\epsilon}\int_{-1}^x sp(s)ds. \tag{4.3}$$

Because $\eta \to \infty$ as $\epsilon \to 0$ for $-1 < x < 0$, the resulting limiting solution is trivial away from an $O(\epsilon)$ thick initial layer near $x = -1$ when η is finite. The limiting solution however loses stability for $x > 0$ since $-xp(x)$ changes sign at $x = 0$. Nonetheless, the limit of y remains trivial until reaching the jump point $X > 0$ where

$$\int_{-1}^{X} sp(s)ds = 0. \tag{4.4}$$

Clearly, $y(X, \epsilon) = 1$ and y blows up exponentially beyond X since $\eta \to -\infty$ there. Knowing this, we could construct the uniform asymptotic solution for $-1 \leqslant x \leqslant X$, i.e. $y \sim e^{-(x+1)p(-1)/\epsilon} + e^{-(X-x)p(X)/\epsilon}$ using the end-point stretchings $(x+1)/\epsilon$ and $(X-x)/\epsilon$ or, preferably,

$$y(x, \epsilon) \sim e^{\frac{1}{\epsilon}\int_{-1}^{x} sp(s)ds} + e^{-\frac{1}{\epsilon}\int_{x}^{X} sp(s)ds}. \tag{4.5}$$

Typical stiff integrators track the stability of the limiting solution so they must be expected to be unreliable for numerical use beyond the turning point at $x = 0$ (cf. Dahlquist *et al.* [6]). A naive matching procedure would break down at the turning point, in contrast to (4.2). Note that typical matching techniques would be incapable of providing the jump location X.

5 Example 4

Next, consider the linear second-order two-point problem

$$\epsilon y'' + a(x)y' + b(x)y = c(x) \tag{5.1}$$

on $0 \leqslant x \leqslant 1$ with prescribed end values $y(0)$ and $y(1)$, again presuming $a(x) > 0$ holds throughout the interval. We can expect (as two-timing will show) that the asymptotic solution will have the two-variable form

$$y(x, \epsilon) = A(x, \epsilon) + e^{-\frac{1}{\epsilon}\int_0^x a(s)ds} B(x, \epsilon), \tag{5.2}$$

where both the outer expansion A and the factor B (determining the initial layer correction $y - A$) have smooth power series expansions in ϵ with respect to x. The second term will indeed decay to zero as $\epsilon \to 0$ since the stretched variable

$$\eta = \frac{1}{\epsilon}\int_0^x a(s)ds \to \infty \tag{5.3}$$

(outside an $O(\epsilon)$-thick initial layer).

We differentiate the ansatz (5.2) twice and substitute into the governing equation (5.1). Linear independence of 1 and $\exp[-\frac{1}{\epsilon}\int_0^x a(s)ds]$ requires the outer solution A to satisfy the amplitude equation

$$aA' + bA = c - \epsilon A'' \tag{5.4}$$

as a power series in ϵ on $0 \leqslant x \leqslant 1$ and B to likewise satisfy the decoupled amplitude equation

$$aB' + (a' - b)B = \epsilon B'' \tag{5.5}$$

there. Because $\exp[-\frac{1}{\epsilon}\int_0^1 a(s)ds]$ is asymptotically negligible, the boundary conditions for y imply that

$$A(1,\epsilon) \sim y(1) \text{ while } B(0,\epsilon) = y(0) - A(0,\epsilon). \tag{5.6}$$

A unique power series expansion $A(x,\epsilon) \sim \sum A_j(x)\epsilon^j$ for the solution of the resulting terminal value problem, (5.4) with $A(1,\epsilon) = y(1)$, can be obtained termwise. For example,

$$A_0(x) = e^{-\int_x^1 \frac{b(s)}{a(s)}ds} y(1) - \int_x^1 \frac{c(t)}{a(t)} e^{-\int_x^t \frac{b(s)}{a(s)}ds} dt. \tag{5.7}$$

A similar power series solution $B(x,\epsilon) \sim \sum B_j(x)\epsilon^j$ for (5.5) with $B(0,\epsilon) = y(0) - A(0,\epsilon)$ can be generated termwise. When $c \equiv 0$ one can readily find linearly independent asymptotic solutions of the differential equation in the form $A(x,\epsilon)$ and $e^{-\frac{1}{\epsilon}\int_0^x a(s)ds}B(x,\epsilon)$. One could do so by eliminating the y' term in (5.1) and then resort to a WKB procedure, following papers of Wentzel, Kramers and Brillouin from 1926, (cf. Olver [24]) or one could rely on the classical results in Wasow [30] or Lomov [21]. The non-homogeneous equation could then be solved asymptotically using variation of parameters.

By contrast, classical two-timing instead directly seeks an asymptotic solution of (5.1) in the form

$$y(x,\epsilon) = y(x,\eta,\epsilon) = y_0(x,\eta) + \epsilon y_1(x,\eta) + \dots \tag{5.8}$$

for a stretched variable

$$\eta = \frac{\phi(x)}{\epsilon}, \tag{5.9}$$

such that $\phi(0) = 0$ and $\phi'(x) > 0$ for $x > 0$. Then the chain rule converts (5.1) into the partial differential equation (PDE)

$$\frac{(\phi')^2}{\epsilon}\left[y_{\eta\eta} + \frac{a}{\phi'}y_\eta\right] + [2\phi' y_{\eta x} + \phi'' y_\eta + a y_x + by] + \epsilon y_{xx} = c(x). \tag{5.10}$$

The leading term $y_{0\eta\eta} + \frac{a}{\phi'}y_{0\eta} = 0$ implies that

$$y_0(x,\eta) = A_0(x) + e^{-\eta}B_0(x) \tag{5.11}$$

with undetermined x-dependent coefficients A_0 and B_0 if we take $\phi' = a$ to get

$$\eta = \frac{1}{\epsilon}\int_0^x a(s)ds, \tag{5.12}$$

as anticipated. The limiting boundary conditions require that

$$A_0(1) \sim y(1), \quad B_0(0) = y(0) - A_0(0). \tag{5.13}$$

The coefficient of ϵ^0 in the PDE (5.10) then requires that

$$a^2(y_{1\eta\eta} + y_{1\eta}) + (-aB_0' + (-a' + b)B_0)e^{-\eta} + aA_0' + bA_0 - c = 0. \tag{5.14}$$

https://doi.org/10.1017/S0956792511000325 Published online by Cambridge University Press

To eliminate secular terms in y like η^k or $\eta^j e^{-\eta}$, we more generally require that $y = A + Be^{-\eta}$, where $A(x, \epsilon)$ satisfies the amplitude equation

$$aA' + bA = c - \epsilon A'' \tag{5.15}$$

as a power series in ϵ and $B(x, \epsilon)$ to likewise satisfies the decoupled amplitude equation

$$aB' + (a' - b)B = \epsilon B''. \tag{5.16}$$

Again, $aA_0' + bA_0 = c$ and $aB_0' + (a' - b)B_0 = 0$, so y_1 has the form

$$y_1(x, \eta) = A_1(x) + e^{-\eta} B_1(x) \tag{5.17}$$

for undetermined x-dependent coefficients A_1 and B_1 to be found (by eliminating secular terms in y_2 or, more directly, by power series methods) as solutions of $aA_1' + bA_1 + A_0'' = 0$ and $aB_1' + (a' - b)B_1 + B_0'' = 0$, with the implied boundary values.

5.1 First illustration

The equation

$$\epsilon y'' + (1 + x)y' - y = 0, \tag{5.18}$$

with $y(0) = 0$ and $y(1) = 2$ has the exact solution

$$y(x, \epsilon) = (1 + x)\left(1 - \left[\frac{\int_x^1 \frac{e^{-\frac{1}{\epsilon}\int_0^s (1+t)dt}}{(1+s)^2} ds}{\int_0^1 \frac{e^{-\frac{1}{\epsilon}\int_0^s (1+t)dt}}{(1+s)^2} ds}\right]\right) \tag{5.19}$$

by reduction of order. Note that Laplace's method (cf. Olver [24]) implies that its asymptotic solution has the form

$$y(x, \epsilon) = A(x, \epsilon) + e^{-\frac{1}{\epsilon}\int_0^x (1+t)dt} B(x, \epsilon), \tag{5.20}$$

where

$$(1 + x)A' - A = -\epsilon A'', \quad A(1, \epsilon) = 2 \tag{5.21}$$

determines the exact (one-term) outer solution

$$A(x, \epsilon) = 1 + x \tag{5.22}$$

and where

$$(1 + x)B' + 2B = \epsilon B'', \quad B(0, \epsilon) = -1 \tag{5.23}$$

for a series $B(x, \epsilon) = B_0(x) + \epsilon B_1(x) + \dots$ determines the boundary-layer correction factor

$$B(x, \epsilon) = -\frac{1}{(1 + x)^2} + \frac{3\epsilon}{(1 + x)^4}(1 - (1 + x)^2) + \dots \tag{5.24}$$

termwise. The result is again preferable, for moderate values of ϵ, to that which we would obtain by matched asymptotic expansions as the composite

$$y(x,\epsilon) = 1 + x + \xi(\tau,\epsilon) \tag{5.25}$$

for $\tau = x/\epsilon$. The correction term ξ to lowest order is $\xi_0(\tau) = -e^{-\tau}$, in contrast to the richer two-scale result $B_0(x)e^{-\eta} = -\frac{1}{(1+x)^2}e^{-\frac{x}{\epsilon}}e^{-\frac{x^2}{2\epsilon}}$.

5.2 Second illustration

Bender & Orszag [1] consider the singular linear equation

$$\epsilon x y'' - y' - xy = 0 \tag{5.26}$$

with

$$y(0) = y(1) = 1. \tag{5.27}$$

Because the coefficient $a(x) = -\frac{1}{x}$ of the first-order derivative in (5.1) is negative, we might anticipate a terminal layer described by $e^{-\eta}$ for $\eta = \frac{1}{\epsilon}\int_x^1 \frac{ds}{s} = -\frac{\ln x}{\epsilon}$, i.e. $e^{-\eta} = x^{1/\epsilon}$. Following [26], we introduce the ansatz

$$y(x,\epsilon) = A(x,\epsilon) + B(x,\epsilon)x^{1/\epsilon}. \tag{5.28}$$

The analytic expressions for the slowly-varying amplitudes are

$$A(x,\epsilon) = e^{-x^2/2}\left(1 + \frac{\epsilon}{4}(x^4 - 2x^2) + \dots\right) \tag{5.29}$$

and

$$B(x,\epsilon) = xe^{\frac{x^2-1}{2}}\left[(1 - e^{-1/2}) + \frac{\epsilon}{30}(23 - 18x^5 - 5x^6 + e^{-1/2}(-14 + 18x^5 + 5x^6)) + \dots\right]. \tag{5.30}$$

Note that the behaviour of the solution near $x = 0$ corresponds to that anticipated by the method of Frobenius.

6 Example 5

More generally, if we consider the nonlinear problem

$$\epsilon y'' + a(x)y' + b(x,y) = 0 \tag{6.1}$$

with a and b smooth and $a(x) > 0$, we would naturally seek a multi-scale solution of the two-point problem in the form

$$y(x,\eta,\epsilon) \sim \sum_{j\geqslant 0} y_j(x,\eta)\epsilon^j \tag{6.2}$$

with

$$\eta = \frac{1}{\epsilon}\int_0^x a(s)ds. \tag{6.3}$$

https://doi.org/10.1017/S0956792511000325 Published online by Cambridge University Press

This would require

$$a^2(x)\left(\frac{\partial^2 y}{\partial \eta^2}+\frac{\partial y}{\partial \eta}\right)+\epsilon\left(2a(x)\frac{\partial^2 y}{\partial \eta \partial x}+a'(x)\frac{\partial y}{\partial \eta}+a(x)\frac{\partial y}{\partial x}+b(x,y)\right)+\epsilon^2\frac{\partial^2 y}{\partial x^2}=0. \quad (6.4)$$

Again, we obtain

$$y_0(x,\eta)=A_0(x)+B_0(x)e^{-\eta} \quad (6.5)$$

and y_1 will need to satisfy

$$a^2(x)\left(\frac{\partial^2 y_1}{\partial \eta^2}+\frac{\partial y_1}{\partial \eta}\right)=-a(x)A_0'+(a(x)B_0'+a'(x)B_0)e^{-\eta}-b(x,A_0+B_0e^{-\eta}). \quad (6.6)$$

Considering the right-hand side as a power series in $e^{-\eta}$, we will eliminate secular terms in y_1 if we require the coefficients of 1 and $e^{-\eta}$ to vanish. Thus, A_0 must satisfy the nonlinear terminal value problem

$$a(x)A_0'+b(x,A_0)=0, \quad A_0(1)=y(1) \quad (6.7)$$

throughout $0 \leqslant x \leqslant 1$, and B_0 must satisfy the linear initial value problem

$$a(x)B_0'+(a'(x)-b_y(x,A_0))B_0=0, \quad B_0(0)=y(0)-A_0(0). \quad (6.8)$$

(We cannot expect to proceed in cases where the outer limit A_0 blows up anywhere). This determines y_0 completely and allows y_1 to be determined up to a solution of the homogeneous problem by using undetermined coefficients, i.e. y_1 has the form $A_1(x)+B_1(x)e^{-\eta}+e^{-2\eta}C_1(x,e^{-\eta})$ where C_1 is known. Note that the numerical implementation of such results could be very valuable (cf. [9]).

7 Example 6

Consider the first-order linear equation [33]

$$\epsilon u'+(x-1)^2u=1, \quad u(0)=0. \quad (7.1)$$

The exact solution is

$$u(x,\epsilon)=\frac{1}{\epsilon}\int_0^x e^{-\frac{(x-1)^3}{3\epsilon}}e^{\frac{(s-1)^3}{3\epsilon}}\,ds. \quad (7.2)$$

Up to the turning point at $x=1$, the preceding theory implies that u has the asymptotic form

$$u(x,\epsilon)=A(x,\epsilon)+e^{-\frac{1}{\epsilon}\int_0^x(s-1)^2ds}B(x,\epsilon)\equiv A(x,\epsilon)+e^{-\frac{[(x-1)^3-1]}{3\epsilon}}B(x,\epsilon), \quad (7.3)$$

where A and B satisfy the amplitude equations $\epsilon A'+(x-1)^2A=1$ and $B'=0$ so $B(x,\epsilon)=-A(0,\epsilon)$. This also follows from integrating the exact solution by parts. The need for an initial layer is clear since the limiting equation is inconsistent with the initial condition. Note that the second term in (7.3) exactly satisfies the homogeneous differential equation. As we would anticipate, the outer solution A is inappropriate at the turning point $x=1$. Indeed power series methods require $(x-1)^2A_0=1, (x-1)^2A_1+A_0'=0,$

https://doi.org/10.1017/S0956792511000325 Published online by Cambridge University Press

etc., so

$$A(x,\epsilon) = \frac{1}{(x-1)^2} + \frac{2\epsilon}{(x-1)^5} + \dots \tag{7.4}$$

while $u(1,\epsilon) = \frac{1}{\epsilon}\int_0^1 e^{-\frac{(1-s)^3}{3\epsilon}}ds$ is $O(\epsilon^{-2/3})$. Beyond $x = 1$, however, we would again expect the outer asymptotic solution $A(x,\epsilon)$ to apply.

8 Example 7

Consider the two-point nonlinear Neumann problem

$$\epsilon^2 y'' + f(x,y) = 0 \tag{8.1}$$

on $0 \leqslant x \leqslant 1$ with $y'(0)$ and $y'(1)$ prescribed, assuming that the limiting equation

$$f(x, A_0(x)) = 0 \tag{8.2}$$

has a stable root $A_0(x)$ such that $f_y(x, A_0(x)) < 0$ holds throughout the interval. To meet the boundary conditions, we must expect y' to converge non-uniformly at both endpoints. Here, we naturally introduce the stretched variables

$$\eta = \frac{1}{\epsilon}\int_0^x \sqrt{-f_y(s, A(s,\epsilon))}ds \tag{8.3}$$

and

$$\zeta = \frac{1}{\epsilon}\int_x^1 \sqrt{-f_y(s, A(s,\epsilon))}ds \tag{8.4}$$

and seek a uniform asymptotic solution in the three-variable form

$$y(x,\eta,\zeta,\epsilon) = A(x,\epsilon) + \epsilon(B(x,\epsilon)e^{-\eta} + C(x,\epsilon)e^{-\zeta}) + \epsilon^2(D(x,\epsilon)e^{-2\eta} + E(x,\epsilon)e^{-2\zeta}) + \dots. \tag{8.5}$$

Such a choice can be motivated by doing a dominant balance (cf. [1]). If $y \sim A_0 + B_0 e^{-\eta}$ near $x = 0$ with $\eta = \psi/\epsilon$, the dominant terms in the resulting PDE will be $\epsilon(\psi_x^2 + f_y(x, A_0))B_0 e^{-\eta} \sim 0$ requiring us to select $\psi \sim \int_0^x \sqrt{-f_y(s, A_0(s))}ds$. Differentiating the ansatz (8.5) suitably, substituting into (8.1) and equating coefficients of $1, e^{-\eta}, e^{-2\eta}, e^{-\zeta}, e^{-2\zeta}$ requires the outer expansion A to satisfy the given nonlinear amplitude equation

$$\epsilon^2 A'' + f(x, A) = 0; \tag{8.6}$$

B to satisfy the initial value problem for the linear amplitude equation

$$\epsilon^2 B'' - 2\sqrt{-f_y(x,A)}B' - (\sqrt{-f_y(x,A)})'B = 0 \tag{8.7}$$

with the implied initial condition $\sqrt{-f_y(0, A(0,\epsilon))}B(0,\epsilon) = A'(0,\epsilon) - y'(0) + \epsilon B'(0,\epsilon) - 2\epsilon\sqrt{-f_y(0, A(0,\epsilon))}D(0,\epsilon) + \dots$; C to satisfy the linear terminal value problem

$$\epsilon^2 C'' + 2\sqrt{-f_y(x,A)}C' + (\sqrt{-f_y(x,A)})'C = 0 \tag{8.8}$$

https://doi.org/10.1017/S0956792511000325 Published online by Cambridge University Press

with $\sqrt{-f_y(1, A(1,\epsilon))}C(1,\epsilon) = -A'(1,\epsilon) + y'(1) - \epsilon C'(1,\epsilon) - 2\epsilon\sqrt{-f_y(1, A(1,\epsilon))}E(1,\epsilon) + \ldots$; and D and E to satisfy the linear equations

$$\epsilon^2 D'' - 4\epsilon\sqrt{-f_y(x,A)}D' - 2\epsilon(\sqrt{-f_y(x,A)})'D - 3f_y(x,A)D + \frac{1}{2}f_{yy}(x,A)B^2 = 0 \tag{8.9}$$

and

$$\epsilon^2 E'' + 4\epsilon\sqrt{-f_y(x,A)}E' + 2\epsilon(\sqrt{-f_y(x,A)})'E - 3f_y(x,A)E + \frac{1}{2}f_{yy}(x,A)C^2 = 0. \tag{8.10}$$

Terms in the expansions for D and E follow in a straightforward manner in terms of A, B and C. Note that O'Malley and Kirkinis [26] considered the related linear problem $\epsilon^2 y'' - a^2(x)y = b(x), 0 \leqslant x \leqslant 1$, with $y(0)$ and $y(1)$ prescribed. For these problems, it has been quite easy to determine the stretched variable(s) and the form of the asymptotic solutions. One could also prove that the expansions are asymptotic, since they will agree with the results one would obtain using, for example, averaging.

9 Example 8

Zauderer [33] considers the first-order Cauchy problem

$$\epsilon(u_t + u_x) + u = \sin t, \quad t \geqslant 0 \tag{9.1}$$

where a subscript implies partial differentiation, with $u(x,0) = f(x)$ on $-\infty < x < \infty$, for which we would anticipate an asymptotic solution of the form

$$u(x,t) = A(x,t,\epsilon) + B(x,t,\epsilon)e^{-t/\epsilon}, \tag{9.2}$$

i.e. with an $O(\epsilon)$ thick initial layer in time. Then the outer solution must satisfy

$$\epsilon(A_t + A_x) + A = \sin t \tag{9.3}$$

as a power series expansion in ϵ. The explicit solution is

$$A(x,t,\epsilon) = \frac{\sin t - \epsilon\cos t}{1+\epsilon^2}, \tag{9.4}$$

as can be directly checked. Linearity then implies that B must satisfy

$$B_t + B_x = 0, \quad B(x,0,\epsilon) = f(x) + \frac{\epsilon}{1+\epsilon^2}. \tag{9.5}$$

Thus,

$$B(x,t,\epsilon) = f(x-t) + \frac{\epsilon}{1+\epsilon^2}, \tag{9.6}$$

i.e. the asymptotic solution is

$$u(x,t) = \frac{1}{1+\epsilon^2}(\sin t - \epsilon\cos t) + \left(f(x-t) + \frac{\epsilon}{1+\epsilon^2}\right)e^{-t/\epsilon}, \tag{9.7}$$

which is exact. Note that the influence of the initial function is short lasting.

https://doi.org/10.1017/S0956792511000325 Published online by Cambridge University Press

10 Example 9

We will next consider the singularly perturbed Cauchy problem for the hyperbolic equation

$$\epsilon(u_{tt} - c^2 u_{xx}) + u_t + a u_x = 0 \quad \text{for} \quad t \geqslant 0, \tag{10.1}$$

with initial conditions

$$u(x,0) = f(x) \quad \text{and} \quad u_t(x,0) = g(x) \tag{10.2}$$

given on $-\infty < x < \infty$. Because the reduced equation is of the first order, we must anticipate an initial layer of $O(\epsilon)$ thickness near $t = 0$. This suggests introducing the fast time t/ϵ and and seeking an asymptotic solution of the form

$$u(x,t) = A(x,t,\epsilon) + \epsilon B(x,t,\epsilon)e^{-t/\epsilon}. \tag{10.3}$$

Then, for example, $u_x = A_x + \epsilon B_x e^{-t/\epsilon}$ and $u_t = A_t + (-B + \epsilon B_t)e^{-t/\epsilon}$, so the PDE (10.1) takes the form

$$\epsilon(A_{tt} - c^2 A_{xx}) + A_t + aA_x + \epsilon[-B_t + aB_x + \epsilon(B_{tt} - c^2 B_{xx})]e^{-t/\epsilon} = 0, \tag{10.4}$$

implying that A must satisfy the outer amplitude equation

$$\epsilon(A_{tt} - c^2 A_{xx}) + A_t + aA_x = 0 \tag{10.5}$$

while B must satisfy

$$\epsilon(B_{tt} - c^2 B_{xx}) - B_t + aB_x = 0. \tag{10.6}$$

We naturally seek solutions as power series

$$A(x,t,\epsilon) = A_0(x,t) + \epsilon A_1(x,t) + \ldots \tag{10.7}$$

and

$$B(x,t,\epsilon) = B_0(x,t) + \epsilon B_1(x,t) + \ldots \tag{10.8}$$

subject to the initial conditions

$$A(x,0,\epsilon) + \epsilon B(x,0,\epsilon) = f(x) \quad \text{and} - B(x,0,\epsilon) + A_t(x,0,\epsilon) + \epsilon B_t(x,0,\epsilon) = g(x). \tag{10.9}$$

Then, A_0, B_0 and A_1 must satisfy the initial value problems

$$A_{0t} + aA_{0x} = 0, \quad A_0(x,0) = f(x), \tag{10.10}$$

$$-B_{0t} + aB_{0x} = 0, \quad B_0(x,0) = -g(x) + A_{0t}(x,0), \tag{10.11}$$

$$A_{1t} + aA_{1x} = c^2 A_{0xx} - A_{0tt}, \quad A_1(x,0) = -B_0(x,0). \tag{10.12}$$

They have the unique solutions

$$A_0(x,t) = f(x - at), \tag{10.13}$$

$$B_0(x,t) = -g(x + at) - af'(x + at) \tag{10.14}$$

https://doi.org/10.1017/S0956792511000325 Published online by Cambridge University Press

and

$$A_1(x,t) = t(c^2 - a^2)f''(x - at) + g(x - at) + af'(x - at). \tag{10.15}$$

The resulting asymptotic approximation

$$u(x,t;\epsilon) = f(x - at) + \epsilon\left(t(c^2 - a^2)f''(x - at) + g(x - at) + af'(x - at)\right) \\ - \epsilon e^{-\frac{t}{\epsilon}}\left(g(x + at) + af'(x + at)\right) + O(\epsilon^2) \tag{10.16}$$

agrees with that obtained by Zauderer [33] using matched expansions. The result is valid for finite t. It breaks down when ϵt is large due to the unbounded secular term ϵA_1. This can be corrected by renormalization (cf. [15–17]). Indeed, if we introduce the spatial operator

$$\mathscr{L} = a\partial_x + \epsilon(a^2 - c^2)\partial_x^2, \tag{10.17}$$

the amplitude equations can be rewritten as

$$A_t = -\mathscr{L}A \quad \text{and} \quad B_t = \mathscr{L}B \tag{10.18}$$

which lead to

$$u(x,t;\epsilon) = e^{-\epsilon t(a^2-c^2)\partial_x^2}A(x - at, 0, \epsilon) + \epsilon e^{-t/\epsilon}e^{\epsilon t(a^2-c^2)\partial_x^2}B(x + at, 0, \epsilon) \tag{10.19}$$

since, for example, $e^{at\partial_x}B(x,0,\epsilon) = B(x + at, 0, \epsilon)$. It can be easily verified that by incorporating the given initial conditions and expanding the exponential operator one recovers (10.16) for finite t. The meaning of the exponentiated differential operators is obtained by expanding the exponentials in their Maclaurin series with differential operator terms.

For the generalized Cauchy problem

$$\epsilon(u_{tt} - c^2\nabla^2 u) + u_t + (\mathbf{a}\cdot\nabla)u = 0, \quad u(\mathbf{x},0) = f(\mathbf{x}), \quad u_t(\mathbf{x},0) = g(\mathbf{x}), \tag{10.20}$$

where $\mathbf{a} = (a_1, a_2, \ldots, a_n)$, we would again seek an asymptotic solution of the form

$$u(\mathbf{x},t) = A(\mathbf{x},t,\epsilon) + \epsilon B(\mathbf{x},t,\epsilon)e^{-t/\epsilon}. \tag{10.21}$$

Proceeding analogously, we get

$$u(\mathbf{x},t) = e^{-\epsilon t[(\mathbf{a}\cdot\nabla)^2 - c^2\nabla^2]}A(\mathbf{x} - \mathbf{a}t, 0, \epsilon) + \epsilon e^{-t/\epsilon}e^{\epsilon t[(\mathbf{a}\cdot\nabla)^2 - c^2\nabla^2]}B(\mathbf{x} + \mathbf{a}t, 0, \epsilon) \tag{10.22}$$

and obtain the initial values for A and B by using power series expansions of the initial conditions with respect to ϵ.

11 Example 10

Amongst the papers presented at the International Conference on Singular Perturbation Theory and Application in Shanghai in June 2010, Du *et al.* [8] considered the linear singularly perturbed Dirichlet problem $\epsilon(u_{xx}+u_{yy})+a(x,y)u_x+b(x,y)u_y+c(x,y)u = f(x,y)$ on the unit square. They required the coefficients a and b to be positive in order to avoid the $O(\sqrt{\epsilon})$ parabolic boundary layer that would occur if part of the boundary was a

https://doi.org/10.1017/S0956792511000325 Published online by Cambridge University Press

characteristic of the reduced equation (cf., e.g. de Jager and Jiang [7]). We will instead consider the simpler problem

$$\epsilon(u_{xx} + u_{yy}) + u_x + u_y + 2u = 0 \tag{11.1}$$

with continuous boundary data

$$\left.\begin{aligned} u(x,0) &= \phi(x), \\ u(x,1) &= J(x), \\ u(1,y) &= \psi(y), \\ u(0,y) &= K(y), \end{aligned}\right\} \tag{11.2}$$

matching smoothly at the corners of the square.

Comparing the asymptotic behaviour of the corresponding ordinary differential equations, we would expect boundary layers of $O(\epsilon)$ thickness along the edges $x = 0$ and $y = 0$. However, we will also have a 'corner layer' of the same thickness near the origin (cf. Il′in [11]), where the boundary layers intersect (cf. Figure 2).

The limiting behaviour on the upper half triangle $y > x > 0$ will naturally be determined as the solution of the reduced problem

$$A_{0x} + A_{0y} + 2A_0 = 0, \quad A_0(x,1) = J(x). \tag{11.3}$$

We can describe the initial data as $x(s) = s$, $y(s) = 1$ and $A_0(s) = J(s)$ and solving the characteristic equations we obtain $x = t + s$, $y = t + 1$ and $A_0 = e^{-2t}J(s)$, where $t = y - 1, s = x - y + 1$ and

$$A_0(x,y) = e^{2(1-y)}J(x - y + 1). \tag{11.4}$$

In the lower triangle, where $x > y$, the reduced problem must satisfy

$$A_{0x} + A_{0y} + 2A_0 = 0, \quad A_0(1,y) = \psi(y). \tag{11.5}$$

Its solution is

$$A_0(x,y) = e^{2(1-x)}\psi(y - x + 1). \tag{11.6}$$

Note that these two reduced solutions agree on the diagonal of the square since $\psi(1) = J(1)$. Higher-order terms in an outer expansion $A(x,y,\epsilon) = \sum_{j\geqslant 0} A_j(x,y)\epsilon^j$ can likewise be uniquely defined in both triangles by a regular perturbation procedure.

We cannot generally expect $A_0(x,0) = e^{2(1-x)}\psi(1-x) = \phi(x)$ to hold, however, so we must introduce a supplemental additive boundary layer

$$u(x,y,\epsilon) = B(x,y,\epsilon)e^{-y/\epsilon} \tag{11.7}$$

along the edge $y = 0$ where ϕ is prescribed. Linearity then requires B to satisfy the regular perturbation problem

$$\epsilon(B_{xx} + B_{yy}) + (B_x - B_y) + 2B = 0 \tag{11.8}$$

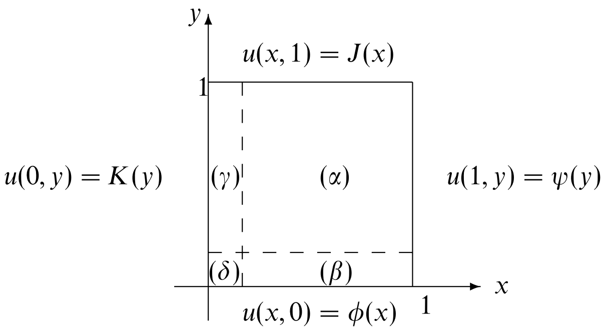

FIGURE 2. Planar geometry of problem (11.1). Regions $(\beta), (\gamma)$ and (δ) correspond to the initial layers $B(x, y, \epsilon)e^{-y/\epsilon}$, $C(x, y, \epsilon)e^{-x/\epsilon}$ and the corner layer $D(\epsilon)e^{-(x+y)/\epsilon}$, respectively. Region (α) corresponds to the outer limits (11.4) and (11.6) above and below the diagonal of the square, respectively.

nearby, with boundary data $B(x, 0, \epsilon) = \phi(x) - A(x, 0, \epsilon)$. In particular, the leading term B_0 will satisfy

$$B_{0x} - B_{0y} + 2B_0 = 0, \quad B_0(x, 0) = \phi(x) - A_0(x, 0). \tag{11.9}$$

Its solution is

$$B_0(x, y) = e^{2y}(\phi(x + y) - A_0(x + y, 0)). \tag{11.10}$$

Later terms in B follow readily. We analogously construct the boundary-layer correction $C(x, y, \epsilon)e^{-x/\epsilon}$ along the edge $x = 0$. The sum $A(x, y, \epsilon) + B(x, y, \epsilon)e^{-y/\epsilon} + C(x, y, \epsilon)e^{-x/\epsilon}$ is an asymptotic solution of the differential equation. At the origin, however, it will have the limiting value $A_0(0, 0) + B_0(0, 0) + C_0(0, 0) = -e^2J(1)$ since we have assumed that $\phi(0) = K(0)$. To meet the prescribed corner value, we now introduce an additive corner solution

$$D(\epsilon)e^{-(x+y)/\epsilon}, \tag{11.11}$$

which satisfies the differential equation for all constants D. This allows us to solve the Dirichlet problem by using the uniformly valid ansatz

$$u(x, y, \epsilon) = A(x, y, \epsilon) + B(x, y, \epsilon)e^{-y/\epsilon} + C(x, y, \epsilon)e^{-x/\epsilon} + D(\epsilon)e^{-(x+y)/\epsilon}. \tag{11.12}$$

Here $\phi(0) = K(0) = A(0, \epsilon) + B(0, \epsilon) + C(0, \epsilon) + D(\epsilon)$ determines $D(\epsilon)$ asymptotically. We note that the need for a corner layer was first recognized by Levinson [20] and a matched solution was given by Zauderer [33].

Following the informal discussion of the present article, readers are urged to apply the formalism to other problems of interest, for example, to those from the texts of Nayfeh [23] and Kevorkian and Cole [14]. A variety of nonlinear problems are solved in ref. [26].

12 Conclusion

The authors demonstrate, using 10 diverse explicit examples, that multiple-scale methods have definite advantages over matched asymptotic expansions. The results are however equivalent as $\epsilon \to 0$, but the multiple-scale approximations will hold for larger ϵ values.

Determining the right stretched variables to use is clear for these examples, but further experience is needed to make such selections in general.

Acknowledgements

E.K. acknowledges financial support from the Hellenic *State Scholarships Foundation* (Ἵδρυμα Κρατικών Υποτροφιών) through a graduate fellowship. The authors wish to thank the referees for their helpful comments and suggestions.

References

[1] Bender, C. M. & Orszag, S. A. (1978) *Advanced Mathematical Methods for Scientists and Engineers*. International Series in Pure and Applied Mathematics. McGraw-Hill, New York.

[2] Chen, L.-Y., Goldenfeld, N. & Oono, Y. (1994) Renormalization group theory for global asymptotic analysis. *Phys. Rev. Lett.* **73**, 1311–1315.

[3] Chen, L.-Y., Goldenfeld, N. & Oono, Y. (1996) Renormalization group and singular perturbations: Multiple scales, boundary layers, and reductive perturbation theory. *Phys. Rev. E* **54**(1), 376–394.

[4] Cochran, J. A. (1962) *Problems in Singular Perturbation Theory*, PhD thesis, Stanford University, Palo Alto, CA.

[5] Cole, J. D. (1968) *Perturbation Methods in Applied Mathematics*, Blaisdell, Waltham, MA.

[6] Dahlquist, G., Edsberg, L., Skollermo, G. & Soderlind, G. (1982) Are the numerical methods and software satisfactory for Chemical-Kinetics? *Lect. Notes Math.* **968**, 149–164.

[7] de Jager, E. M. & Jiang, F. (1996) *The Theory of Singular Perturbations*. North-Holland Series in Applied Mathematics and Mechanics, Vol. 42. North-Holland, Amsterdam.

[8] Du, L.-L., Wu, X.-H. & Chen, S. (2010) A novel mathematical modelling of multiple scales for a class of two-dimensional singular perturbed problems. Presented during the *International Conference on Singular Perturbation Theory and Application*, June 2010, Shanghai.

[9] Franz, S. & Roos, H.-G. (2011) The capriciousness of numerical methods for singular perturbations. *SIAM Rev.* **53**, 157–173.

[10] Goldenfeld, N., Martin, O. & Oono, Y. (1989) Intermediate asymptotics and renormalization group theory. *J. Sci. Comput.* **4**, 355–372.

[11] Il'in, A. M. (1992) *Matching of Asymptotic Expansions of Solutions of Boundary Value Problems*. Translations of Mathematical Monographs, Vol. 102. American Mathematical Society, Providence, RI. Translated from the Russian by V. Minachin [V. V. Minakhin].

[12] Johnson, R. S. (2005) *Singular Perturbation Theory*. Mathematical and Analytical Techniques with Applications to Engineering. Springer, New York.

[13] Kevorkian, J. (1961) *The Uniformly Valid Asymptotic Approximations to the Solutions of Certain Nonlinear Ordinary Differential Equations*, PhD thesis, California Institute of Technology, Pasadena, CA.

[14] Kevorkian, J. & Cole, J. D. (1996) *Multiple Scale and Singular Perturbation Methods*. Applied Mathematical Sciences, Vol. 114. Springer-Verlag, New York.

[15] Kirkinis, E. (2008a) Reduction of amplitude equations by the renormalization group approach. *Phys. Rev. E* **77**(1), 011105.

[16] Kirkinis, E. (2008b) Secular series and renormalization group for amplitude equations. *Phys. Rev. E* **78**(1), 032104.

[17] Kirkinis, E. (2010) Amplitude equations and asymptotic expansions for multi-scale problems. *Asymptotic Anal.* **67**(1-2), 1–16.

[18] Kuzmak, G. E. (1959) Asymptotic solutions of nonlinear second order differential equations with variable coefficients. *J. Appl. Math. Mech.* **23**, 730–744.

[19] Lagerstrom, P. A. (1988) *Matched Asymptotic Expansions*. Applied Mathematical Sciences, Vol. 76. Springer-Verlag, New York. Ideas and Techniques.

[20] Levinson, N. (1950) The first boundary value problem for $\varepsilon\Delta u + A(x,y)u_x + B(x,y)u_y + C(x,y)u = D(x,y)$ for small ε. *Ann. Math.* **51**(2), 428–445.

[21] Lomov, S. A. (1992) *Introduction to the General Theory of Singular Perturbations*. Translations of Mathematical Monographs, Vol. 112. American Mathematical Society, Providence, RI.

[22] Mahony, J. J. (1961/1962) An expansion method for singular perturbation problems. *J. Austral. Math. Soc.* **2**, 440–463.

[23] Nayfeh, A. H. (1973) *Perturbation Methods*, John Wiley & Sons, New York-London-Sydney. Pure and Applied Mathematics.

[24] Olver, F. W. J. (1974) *Asymptotics and Special Functions*, Academic Press, New York-London. Computer Science and Applied Mathematics.

[25] O'Malley, R. E., Jr. (1991) *Singular Perturbation Methods for Ordinary Differential Equations*. Applied Mathematical Sciences, Vol. 89. Springer-Verlag, New York.

[26] O'Malley, R. E., Jr. & Kirkinis, E. (2010) A combined renormalization group-multiple scale method for singularly perturbed problems. *Stud. Appl. Math.* **124**(4), 383–410.

[27] Smith, D. R. (1985) *Singular-Perturbation Theory. An Introduction with Applications,* Cambridge University Press, Cambridge, England, UK.

[28] Van Dyke, M. (1964) Perturbation Methods in Fluid Mechanics. *Applied Mathematics and Mechanics, Vol. 8*, Academic Press, New York.

[29] Vasil'eva, A. B., Butuzov, V. F. & Kalachev, L. V. (1995) *The Boundary Function Method for Singular Perturbation Problems*. SIAM Studies in Applied Mathematics, Vol. 14. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA. With a foreword by Robert E. O'Malley, Jr.

[30] Wasow, W. (1944) On the asymptotic solution of boundary value problems for ordinary differential equations containing a parameter. *J. Math. Phys. Mass. Inst. Tech.* **23**, 173–183.

[31] Wechselberger, M. (2005) Existence and bifurcation of canards in $\mathbb{R}^3$ in the case of a folded node. *SIAM J. Appl. Dyn. Syst.* **4**(1), 101–139 (electronic).

[32] Wong, R. (2009) A panoramic view of asymptotics. In: *Foundations of Computational Mathematics Hong Kong 2008*. London Mathematical Society Lecture Note Series, Vol. 363. Cambridge University Press, England, UK, pp. 190–235.

[33] Zauderer, E. (1989) *Partial Differential Equations of Applied Mathematics*, Pure and Applied Mathematics, 2nd ed., John Wiley & Sons, New York.