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THE POWER CONCAVITY OF SOLUTIONS OF SOME SEMILINEAR ELLIPTIC BOUNDARY-VALUE PROBLEMS

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Let Ω be a bounded convex domain in $\mathbb{R}^2$ with a smooth boundary. Let $0 < \gamma < 1$. Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ be a solution, positive in Ω , of

$$-\Delta u = u^\gamma \quad \text{in} \quad \Omega ,$$
$$u = 0 \quad \text{on} \quad \partial\Omega .$$

Then the function u^α is concave for $\alpha = (1-\gamma)/2$.

Let Ω be a bounded convex domain in $\mathbb{R}^2$ with a smooth boundary. To avoid some minor technicalities, assume that the curvature on $\partial\Omega$ is uniformly bounded away from zero. We give a new proof of the following theorem, using techniques which generalise those of Makar-Limanov [*6*] $(\gamma = 0)$ and of Acker, Payne and Philipin [*1*] $(\gamma = 1)$.

Let u be any positive function on Ω . The function u is said to be α-*concave*, for $\alpha > 0$, if u^α is concave. The function u is said to be 0-*concave*, or log-*concave*, if $\log u$ is concave.

THEOREM 1. *Let* $0 \le \gamma \le 1$. *Let* $u \in C^2(\Omega) \cap C(\bar{\Omega})$ *be a solution, positive in* Ω *, of*

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$$\text{(P)} \qquad \begin{cases} -\Delta u = u^\gamma & \text{in } \Omega , \\ u = 0 & \text{on } \partial\Omega . \end{cases}$$

Then the function u *is* α*-concave for* $0 \le \alpha \le (1-\gamma)/2$.

Concerning the interior regularity needed in the proof, it is known that any solution $u \in C^2(\Omega)$ is in $C^\infty(\Omega)$.

Theorem 1 was first proved in a University of Adelaide PhD thesis by Kennington [4]. Kennington's proof actually establishes the theorem in $\mathbb{R}^n$ (with $n \ge 2$) . Kennington's techniques are clever extensions of those of Korevaar [5]. Korevaar used his techniques to establish the result in the case $\gamma = 1$. A similar proof, again for $\gamma = 1$, appears in Caffarelli and Spruck [2].

The proof of Theorem 1 below is just one application of the Maximum Principle (Protter and Weinberger [7], Sperb [8]) in the following form.

MAXIMUM PRINCIPLE. *Let* $\Theta \in C(\Omega)$ *with* $\Theta \ge 0$. *Let*

$$Z = \{z \in \Omega \mid \Theta(z) = 0\} .$$

Let $I \in C^2(\Omega)$ *satisfy*

$$-\Delta I + \frac{\mathbf{A}\cdot\nabla I}{\Theta} + A_0 I = 0 ,$$

$$\text{(1.1)} \qquad I > 0 \text{ in a neighbourhood of } \partial\Omega ,$$

where $\mathbf{A}$ *and* A_0 *belong to* $C(\Omega)$ *with* $A_0 \ge 0$. *Suppose that* $I > 0$ *at points of* Z . *Then* $I \ge 0$ *in* Ω .

Note that the only use of the hypothesis on the curvature of $\partial\Omega$ is to guarantee (1.1). In the application to Problem (P), with $\gamma > 0$, the boundary condition is

$$I(z) \to +\infty \text{ as } z \to \partial\Omega .$$

(Note also that the coefficient A_0 is singular at the boundary.)

The quantity I is the most obvious generalisation of that used in [6] and [1], namely $I = I_\alpha$ with $\alpha = (1-\gamma)/2$. Here, if $0 < \alpha < 1$, (1.2) $$I_\alpha = \frac{u^2 \text{Hessian}(u^\alpha)}{\alpha^2(1-\alpha)} = \frac{u^2\left[(u^\alpha)_{xx}(u^\alpha)_{yy}-\left[(u^\alpha)_{xy}\right]^2\right]}{\alpha^2(1-\alpha)} .$$

If $\alpha = 0$,

$$I_0 = u^2 \text{ Hessian}(\log u) .$$

For a positive superharmonic function u , establishing that $I_\alpha \geq 0$ in Ω is establishing that u is α-concave. (In the case $\gamma = 0$ our notation is exactly as in [6], that is $I = I_{\frac{1}{2}}$. In the case $\gamma = 1$ our $I = I_0$ is $\Phi/2$ where Φ is defined by equation (2.1) of [1].)

Define

(1.3) $$P_2 = |\nabla u|^2 + \frac{2u^{\gamma+1}}{1+\gamma} .$$

(The notation is that of Sperb [8].) The explicit formulae for the coefficients A_0, A and Θ are as follows:

$$\mathsf{A} = \mathsf{A}_1 + A_2\nabla I ,$$

$$A_2 = -8u^{1+\gamma}(1+\gamma) ,$$

$$A_0 = \frac{|\nabla u|^2}{u^2}\gamma(1-\gamma) ,$$

$$\mathsf{A}_1 = 2\gamma(1+\gamma)\frac{P_2}{u}\{2u\nabla P_2-(1+\gamma)P_2\nabla u\}$$

and

$$\Theta = \left(4uu_{xx}+2u^{1+\gamma}-(1+\gamma)\left(u_x^2-u_y^2\right)\right)^2 + 4\left(2uu_{xy}-(1+\gamma)u_xu_y\right)^2$$
$$= -8u^{1+\gamma}(1+\gamma)I + P_2^2(1+\gamma)^2 .$$

The form of the equation was discovered using the earlier results of [6] and [1] as a guide.

The coefficients were determined in the order, first A_2 , then A_0

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and finally A_1 . The only important detail is the sign of A_0 . Further details are given in the research report, Keady [3]. The calculations were sufficiently intricate that the computer algebra system, REDUCE, was used. The REDUCE programs are given in Keady [3].

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