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Time almost periodicity for solutions of Toda lattice equation with almost periodic initial datum

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Abstract. This paper analyzes the initial value problem for the Toda lattice with almost periodic initial data: let $J(t; J_0)$ denote the family of Jacobi matrices which are solutions of the Toda flow equation with initial condition $J(0; J_0) = J_0$, then, the given almost periodic datum J_0 is a discrete linear Schrödinger operator with almost periodic potential, which plays a fundamental role in our considerations. We show that, under some given hypotheses, the spectrum of the Schrödinger operator is pure absolute continuous and homogeneous (measure-theoretically) by establishing exponential asymptotics on the size of spectral gaps. These two conclusions enable us to show the boundedness and almost periodicity in the time of solutions for Toda lattice equation with almost periodic initial data. As a consequence, our result presents a positive answer to the discrete Deift's conjecture [Some open problems in random matrix theory and the theory of integrable systems. *Integrable Systems and Random Matrices (Contemporary Mathematics, 458)*. American Mathematical Society, Providence, RI, 2008, pp. 419–430; Some open problems in random matrix theory and the theory of integrable systems. II. *SIGMA Symmetry Integrability Geom. Methods Appl.* **13** (2017), Paper no. 016].

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1. *Introduction and main results*

The study of the spectrum of an almost periodic Schrödinger operator has been sweeping the world for many years, which, consequently, has been called an ‘almost periodic flu’, see the introduction of [61] for details. Other almost periodic problems include, without limitation, the existence of an almost periodic solution to Hamilton partial differential equations, the existence of almost periodic traveling fronts for Fisher and Kolmogorov–Petrovskii–Piskunov (KPP) lattice equations, and the almost periodic initial value problem of a Korteweg–de Vries (KdV) equation. In [16–18] and [39, 40, 70], Bourgain et al and Geng et al construct the almost periodic solution to infinite dimensional partial differential equations. Moreover, in [49, 54], the authors construct almost periodic traveling fronts for the discrete and continuous Fisher–KPP equations, respectively.

Deift’s conjecture, an open question posed by Deift [33, 34] asks whether for almost periodic initial datum, the solutions to the KdV equation are almost periodic in time? The progress about this conjecture are the papers [27, 68], where, under suitable hypotheses, Tsugawa, and Damanik and Goldstein prove the local and global existences of unique solutions to the KdV equation for a small quasi-periodic analytic initial datum. Moreover, in [13], Binder *et al* show that in the same setting as [27], the solution is almost periodic in time.

Notice that the KdV equation is continuous in space variables, so it is important to know what will happen to the discrete Deift conjecture problem. We, in this work, study the initial datum problem of the Toda lattice equation, which is the discrete version of the KdV equation. To state our work, we give some basic definitions.

Set $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ as the cycle and $\mathbb{T}^\infty = \prod_{j\in\mathbb{Z}} \mathbb{T}^1$ with product topology. We say that a function $f : \mathbb{Z} \to \mathbb{R}$ is Bochner almost periodic if $\{f(\cdot + k) : k \in \mathbb{Z}\}$ has a compact closure in $\ell^\infty(\mathbb{Z})$, where $\ell^\infty(\mathbb{Z})$ denotes the space of all bounded functions defined on $\mathbb{Z}$ with sup-norm. Set $\mathcal{H}(f) = \overline{\{f(\cdot + k) : k \in \mathbb{Z}\}}$, which is the closure of $\{f(\cdot + k) : k \in \mathbb{Z}\}$ in $\ell^\infty(\mathbb{Z})$. We also call $\mathcal{H}(f)$ the hull of the almost periodic function f. Essentially, this hull is a quotient group of $\mathbb{T}^\infty$, so it can be represented by a continuously sampling function on $\mathbb{T}^\infty$ [62]. Thus, any Bochner almost periodic function f has the representation, that is, there is another continuous function F defined on $\mathbb{T}^\infty$ such that $f(n) = F(n\omega)$, where $\omega \in \mathbb{R}^\infty$ is called the frequency of f. In particular, if $\mathbb{T}^\infty$ is replaced by $\mathbb{T}^d$, $d \in \mathbb{N}$, we say f is quasi-periodic. See the discussions in [15, 62] for details.

Define the set

$$\mathbb{Z}_*^\infty \triangleq \left\{k \in \mathbb{Z}^\infty : |k|_\eta \triangleq \sum_{j\in\mathbb{N}} \langle j\rangle^\eta |k_j| < \infty\right\},$$

where $\langle j\rangle = \max\{1, |j|\}$. Moreover, we say that $\alpha \in \mathbb{T}^\infty$ is Diophantine, which we denote as $DC_\infty(\gamma, \tau)$, if there exist $\gamma > 0$ and $\tau > 1$ such that

$$DC_\infty(\gamma, \tau) \triangleq \left\{\alpha \in \mathbb{T}^\infty : \|\langle k, \alpha\rangle\|_{\mathbb{R}/\mathbb{Z}} > \prod_{j\in\mathbb{N}} \frac{\gamma}{(1 + |k_j|^\tau \langle j\rangle^\tau)} \text{ for all } k \in \mathbb{Z}_*^\infty \setminus \{0\}\right\},$$

where, for $k = (k_j)_{j\in\mathbb{Z}} \in \mathbb{Z}_*^\infty$, $\alpha = (\alpha_j)_{j\in\mathbb{Z}} \in \mathbb{T}^\infty$,

$$\langle k, \alpha\rangle = \sum_{j\in\mathbb{Z}} k_j\alpha_j \quad \text{and} \quad \|\langle k, \alpha\rangle\|_{\mathbb{R}/\mathbb{Z}} \triangleq \inf_{p\in\mathbb{Z}}\{|\langle k, \alpha\rangle - p|\}.$$

The set $DC_\infty(\gamma, \tau)$ of Diophantine frequencies was first developed by Bourgain [18]. It is shown that there exists a constant $c(\tau) > 0(\tau > 1)$ such that

$$\text{meas } DC_\infty(\gamma, \tau) \geq 1 - c(\tau)\gamma,$$

see the discussions in [12, 18] for details. In this paper, we will always assume $\gamma > 0$, $\tau > 1$. For a given Banach space X, introduce space $\mathcal{H}_\sigma(\mathbb{T}^\infty, X)$:

$$\mathcal{H}_\sigma(\mathbb{T}^\infty, X) = \bigg\{ f(x) = \sum_{k \in \mathbb{Z}_*^\infty} \widehat{f}(k) e^{i2\pi\langle k, x\rangle},\ |f|_\sigma \triangleq \sum_{k \in \mathbb{Z}_*^\infty} |\widehat{f}(k)|_X e^{2\pi\sigma|k|_\eta} < \infty \bigg\}.$$

1.1. *Toda flow.* The Toda lattice was proposed by Toda in 1967 [67] as a model describing the positions and momenta of a chain, which is given by

$$\begin{cases} \dfrac{d}{dt} a_n(t) = a_n(t)(b_{n+1}(t) - b_n(t)), \\ \\ \dfrac{d}{dt} b_n(t) = 2(a_n^2(t) - a_{n-1}^2(t)), \end{cases} \qquad n \in \mathbb{Z}. \tag{1.1}$$

Identifying $(a_n(t), b_n(t))$ as a doubly infinite dimensional Jacobi matrix $J(t)$ defined by

$$(J(t)u)_n = a_{n-1}(t)u_{n-1} + b_n(t)u_n + a_n(t)u_{n+1}. \tag{1.2}$$

The study of Toda flow using equation (1.1) depends heavily on the Jacobi matrix $J(t)$ defined by (1.2). Teschl [66, Theorem 12.6] shows that any initial condition $(a_n(0), b_n(0)) \in \ell^\infty(\mathbb{Z}) \times \ell^\infty(\mathbb{Z})$ will lead to a unique solution $(a, b) \in C^\infty(\mathbb{R}, \ell^\infty(\mathbb{Z}) \times \ell^\infty(\mathbb{Z}))$ to equation (1.1). Under some hypotheses on the spectrum of operator $J(0)$, in [14], Binder *et al* show the existence and uniqueness of almost periodic solutions to equation (1.1) with almost periodic initial value $(a_n(0), b_n(0))$. Later, Leguil *et al* in [48], by taking the initial value $(a_n(0), b_n(0)) = (1, V(x + n\alpha))$, $n \in \mathbb{Z}$ with $V \in C^\omega(\mathbb{T}, \mathbb{R})$, $\alpha \in \mathbb{T}$, extend the conclusions above to Avila's subcritical regime. (For any spectrum E of Schrödinger operator $H_{V,\alpha,x}$ defined by equation (1.3), the Schrödinger cocycle (α, S_E^V) is subcritical.) In particular, for the most important example of almost Mathieu operators (AMO), that is, $V(\cdot) = 2\lambda \cos 2\pi(\cdot)$, Leguil *et al* prove that there exist almost periodic solutions for all $0 < \lambda < 1$. We will extend the conclusions in [48] from finite dimensional frequencies to infinite dimensional frequencies.

THEOREM 1.1. *Let $\alpha \in DC_\infty(\gamma, \tau)$ and $V \in \mathcal{H}_{\sigma_0}(\mathbb{T}^\infty, \mathbb{R})$ with $\eta \geq 1$. We consider the Toda flow in equation (1.1) with initial condition $(a_n(0), b_n(0)) = (1, V(x + n\alpha))$. Then, there exists $\varepsilon_*(\gamma, \tau, \eta, \sigma_0)$, such that if $|V|_{\sigma_0} \leq \varepsilon_*$, we have:*

(1) *for any $x \in \mathbb{T}^\infty$, equation (1.1) admits a unique solution $(a(t), b(t))$ defined for all $t \in \mathbb{R}$;*
(2) *for every t, the Jacobi matrix $J(t)$ defined by equation (1.2) is almost periodic with constant spectrum $\Sigma_{V,\alpha}$;*
(3) *the solution $(a(t), b(t))$ is almost periodic in t in the sense that there exists a continuous map $M : \mathbb{T}^\mathbb{Z} \to \ell^\infty(\mathbb{Z}) \times \ell^\infty(\mathbb{Z})$, a point $\varphi \in \mathbb{T}^\mathbb{Z}$, and a direction $\omega \in \mathbb{R}^\mathbb{Z}$, such that $(a(t), b(t)) = M(\varphi + \omega t)$.*

Initial datum problems for other systems are in [31, 32], where Damanik, Li, and Xu show the existence and uniqueness of spatially quasi-periodic solutions to the generalized KdV equation and Benjamin–Bona–Mahony equation with quasi-periodic initial data on the real line.

The proof of Theorem 1.1 depends on the discussions in [69], see Theorem 6.2. Given almost periodic initial datum $(a_n(0), b_n(0)) = (1, V(x + n\alpha))$, two ingredients of the hypotheses of Theorem 6.2 are that the spectrum set of Schrödinger operator $H_{V,\alpha,x}$ defined by equation (1.3) is homogeneous and pure absolutely continuous (ac). Thus, to apply this theorem, we, in §§1.2 and 1.3, will give our main results about the pure ac spectrum and homogeneous spectrum.

1.2. *Absolute continuous spectrum.* In this subsection, we will give discussions about the pure ac spectrum of the Schrödinger operator $H_{V,\alpha,x}$, which is defined by

$$(H_{V,\alpha,x}u)_n = u_{n+1} + u_{n-1} + V(x + n\alpha)u_n, \quad n \in \mathbb{Z}. \tag{1.3}$$

A lot of research has been conducted by mathematicians and physicians about the almost periodic operator $H_{V,\alpha,x}$, that is, the potential V is almost periodic, see [10, 11, 46, 52] for details. However, subsequently, much attention has been paid to the quasi-periodic and limit periodic cases and less progress made in the almost periodic case.

We first consider the limit-periodic potential V, which lies in the closure of the space of periodic potentials. That is, there exists $\{V_n\}_{n\in\mathbb{N}}$, where V_n, $n \in \mathbb{N}$, are periodic functions, such that

$$\lim_{n\to\infty} \|V - V_n\|_\infty = 0. \tag{1.4}$$

Even though periodic operators always exhibit pure ac spectrum, in [2], Avila shows the possibilities of new phenomena of the limit-periodic potentials, such as the genericity of purely singular continuous spectrum [2, 25] and the denseness of pure point spectrum [30]. Instead of the V defined by equation (1.4), if we restrict the limit-periodic potentials in the perturbative regime, that is, potentials V are with q_n-periodic approximants V_n such that

$$\lim_{n\to\infty} \mathrm{e}^{q_{n+1}b}\|V - V_n\|_\infty = 0 \quad \text{for all } b > 0, \tag{1.5}$$

then in [55, 56] Pastur and Tkachenko show that $H_{V,\alpha,x}$ has pure ac spectrum, which serves as the fundamental work of other related studies.

Now, we consider the quasi-periodic potential V. To study the spectral property of $H_{V,\alpha,x}$, we introduce the coupling constant $\lambda \in \mathbb{R}^+$ in front of the potential V. In the case where λ is sufficiently large, the Lyapuonv exponent of cocycle $(\alpha, S_E^{\lambda V})$ is positive for any $E \in \mathbb{R}$, see [19, 42, 44, 65] and the references therein. Thus, Kotani's theory implies there will be no ac spectrum in the spectrum set. If the coupling constant λ is sufficiently small, the frequency α has to be restricted to the Diophantine frequencies, which we denote by DC for the set of these frequencies. (For any $\gamma > 0$, $\tau > d$, the estimates $\|\langle k, \alpha\rangle\|_{\mathbb{R}/\mathbb{Z}} \geq \gamma|k|^{-\tau}$ for all $0 \neq k \in \mathbb{Z}^d$ hold.) Under hypotheses that λ is small enough and $\alpha \in DC$, Dinaburg and Sinaĭ [35] prove that the ac spectrum set is not empty, and in [36], under the same hypotheses in [35], Eliasson shows that the spectrum is pure ac. In the case where $\alpha \in DC$, Avila and Jitomirskaya, based on a non-perturbative Anderson

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localization result, prove that there exists λ_1 independent on α such that the spectrum set is pure ac for all $\lambda \leq \lambda_1$, see [8] for details. For AMO with subcritical coupling, that is, $0 < \lambda < 1$, Avila and Damanik, in [6], show that the spectrum is pure ac by proving that the integrated density of states is absolutely continuous. Moreover, Avila, Fayad, and Krikorian [7] and Hou and You [45] consider the discrete and continuous Schrödinger operators, respectively. They construct non-standard Kolmogorov–Arnold–Moser (KAM) iterations and show that the set of ac spectrum is not empty for all $\alpha \in \mathbb{T} \setminus \mathbb{Q}$ and small enough λ, and Avila's conclusions in [3, 5] are able to show that the spectrum set is pure ac. The case that λ is neither too large nor too small, which is the so-called global case, is extremely complicated. However, in the one-frequency case, Avila gives the fascinating global theory and shows that for typical analytic one-frequency Schrödinger operator, there is no singular continuous spectrum, see [4] for details. Our pure ac spectrum result is the following.

THEOREM 1.2. *Assume $\alpha \in DC_\infty(\gamma, \tau)$. Then there exists $0 < \varepsilon_*(\gamma, \tau, \eta, \sigma_0) < 1$ such that if $V \in \mathcal{H}_{\sigma_0}(\mathbb{T}^\infty, \mathbb{R})$ with $\eta \geq 1$ and $|V|_{\sigma_0} \leq \varepsilon_*$, the operator $H_{V,\alpha,x}$ has pure absolutely continuous spectrum.*

By the discussions above, we know that the works mentioned above are all with limit periodic or quasi-periodic potentials, and our Theorem 1.2 extends the results above to almost periodic potential cases. Since we consider the infinite dimensional frequency, a new technique is needed to overcome the difficulties brought by infinite dimensional frequencies. See the discussions in §4 for details.

1.3. *Homogeneous spectrum.* In this subsection, we will prove the homogeneity of the spectrum by using the exponential decay of the spectral gaps and Hölder continuity of the integrated density of states. Recall that in [21], the concept of an homogeneous set is introduced by Carleson.

Definition 1.3. Given $\mu > 0$, a closed set $\mathcal{S} \subset \mathbb{R}$ is called μ-homogeneous if

$$|\mathcal{S} \cap (E - \varepsilon, E + \varepsilon)| \geq \mu\varepsilon \quad \text{for all } E \in \mathcal{S}, \text{ for all } 0 < \varepsilon < \operatorname{diam} \mathcal{S}.$$

It is known that the homogeneity of closed subsets of $\mathbb{R}$ is important in inverse spectral theory, see the fundamental works of Sodin and Yuditskii [63, 64]. In particular, under the hypotheses of finite total gap length along with a reflectionless condition, it is shown in [64] that the homogeneity of the spectrum implies the almost periodicity of the associated potentials and Gesztesy and Yuditskii [41], under the same hypotheses in [64], show that the Schrödinger operators have pure ac spectra. Here, we want to mention that the result of Remling and Poltoratski [57, Corollary 2.3] shows that if S is weakly homogeneous and J is reflectionless on S, then S does not support a singular spectrum, see [57] for details.

In [38], Fillman and Lukic consider the Schrödinger operators with limit periodic potential and show the existence of homogeneous spectra by assuming the potential obeys the Pastur–Tkachenko condition, and Fillman [37] proves the spectra of discrete operators with generic potential are homogeneous; here, generic potential is a dense subset of limit periodic potential. Later, Damanik and Fillman in [23] also show the existence of an homogeneous spectrum for the limit periodic potentials defined by (1.5).

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As for the quasi-periodic potential case, we refer to [28], where Damanik, Goldstein, and Lukic consider continuous Schrödinger operators and show the existences of homogeneous spectra with Diophantine frequency and small potential V. In [29], Damanik *et al* consider the discrete Schrödinger operators and show homogeneous spectra under the conditions that frequency is Diophantine and the Lyapunov exponent is positive. Moreover, to ensure the existence of an homogeneous spectrum for supercritical AMO ($\lambda > 1$), [29] shows that the strong Diophantine frequency is needed. Later, Leguil *et al* [48] consider discrete Schrödinger operators with some Liouvillean frequencies and prove that the spectrum is homogeneous if V is small enough. Our result is as follow.

THEOREM 1.4. *Let $\alpha \in DC_\infty(\gamma, \tau)$ and set $V \in \mathcal{H}_{\sigma_0}(\mathbb{T}^\infty, \mathbb{R})$, $\eta \geq 1$. There exists $0 < \varepsilon_*(\gamma, \tau, \eta, \sigma_0) < 1$ such that if $|V|_{\sigma_0} \leq \varepsilon_*$, then the spectrum set of $H_{V,\alpha,x}$ is μ-homogeneous for some $\mu \in (0, 1)$.*

In Theorem 1.4, we extend the results above from limit periodic and quasi-periodic potentials to almost periodic potentials. In [9], the authors construct the Schrödinger operator with one frequency whose spectrum is not homogeneous. Thus, the hypotheses in Theorem 1.4 are not restrictive.

2. *Preliminary*

In this section, we give some basic definitions and conclusions. Even though the discussions are given in $\mathbb{T}^d$ topology with $d \in \mathbb{N}$, the conclusions will also hold in our $\mathbb{T}^\infty$ topology since we equip $\mathbb{T}^\infty$ with the product topology of $\mathbb{T}$.

2.1. *Jacobi operator and Schrödinger operator.*

For $a = (a_n)_{n\in\mathbb{Z}}, b = (b_n)_{n\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z}, \mathbb{R})$, we define the Jacobi operator J associate with a, b by

$$(Ju)_n = a_{n-1}u_{n-1} + b_n u_n + a_n u_{n+1}. \tag{2.1}$$

The Jacobi operator J arises naturally in the context of the spectral theorem, which says any bounded self-adjoint operator A with a cyclic vector is unitarily equivalent to a Jacobi operator on a half-line. It is self-adjoint since we restrict $a_n, b_n \in \mathbb{R}$ for all $n \in \mathbb{Z}$. We restrict ourselves to the non-singular case, where $a_n > 0$ for all $n \in \mathbb{Z}$.

Let Σ be the spectrum of the self-adjoint Jacobi matrices J defined by equation (2.1). Given any $z \notin \Sigma$, the Green's function of J is the integral kernel of $(J - z)^{-1}$:

$$G_J(m, n; z) = \langle e_n, (J - z)^{-1} e_m\rangle. \tag{2.2}$$

Definition 2.1. [13, 22, 64] Let $\Sigma \in \mathbb{R}$. A Jacobi operator J is said to be reflectionless on Σ if $\mathrm{Re}(G_J(n, n; E + i0)) = 0$ for all $n \in \mathbb{N}$ and Lebesgue almost every (a.e.) $E \in \Sigma$.

Given any $z \in \mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}\ z > 0\}$, the difference equation $Ju = zu$ has two solutions $u^\pm$ (defined up to normalization) with $u_0^\pm \neq 0$, which are in $\ell^2(\mathbb{Z}^\pm)$, respectively. Let $m_J^\pm = \mp(u_{\pm1}^\pm / a_0 u_0^\pm)$. Then, m_J^+ and m_J^- are Herglotz functions, i.e., they map $\mathbb{H}$ holomorphically into itself. For almost every $E \in \mathbb{R}$, the non-tangential limits $\lim_{\varepsilon\to0^+} m_J^\pm(E + i\varepsilon)$ exist. Then, we have

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$$G_J(0,0;z) = \frac{-1}{a_0^2(m_J^+(z)+m_J^-(z))}, \quad z \in \mathbb{H}. \tag{2.3}$$

Consider the linear Schrödinger operator $H_{V,\alpha,x}$ defined by

$$(H_{V,\alpha,x}u)_n = u_{n+1} + u_{n-1} + V(x+n\alpha)u_n, \quad n \in \mathbb{Z}, \tag{2.4}$$

where $x \in \mathbb{T}^d$, $d \in \mathbb{N} \cup \{\infty\}$, is called the phase, $\alpha \in \mathbb{T}^d$ is called the frequency, and V is called the potential.

The Schrödinger operator $H_{V,\alpha,x}$ defined by equation (2.4) will be self-adjoint if we assume V is real-valued and it is a special case of Jacobi operator $J(t)$ defined by equation (2.1) with $a_n \equiv 1$ for all $n \in \mathbb{Z}$. Moreover, since we assume $\alpha \in DC_\infty(\gamma,\tau)$, $V(x+n\alpha)$ is almost periodic in $n \in \mathbb{Z}$, and we call $H_{V,\alpha,x}$ the almost periodic Schrödinger operator.

2.2. *Cocycle.* Set $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ as the cycle (the base) and $SL(2,\mathbb{R})$ as the set of 2 by 2 real matrices with determinant 1 (the fiber). Thus, the smooth cocycles are diffeomorphisms on the product $\mathbb{T}^d \times \mathbb{C}^2$ of the form

$$(\alpha, A) : \mathbb{T}^d \times \mathbb{C}^2 \to \mathbb{T}^d \times \mathbb{C}^2,$$
$$(x, y) \mapsto (x+\alpha, A(x)y),$$

where $\alpha \in \mathbb{T}^d$ and $A \in C^\infty(\mathbb{T}^d, SL(2,\mathbb{R}))$, $d \in \mathbb{N} \cup \{\infty\}$.

Now, we turn back to Schrödinger operator $H_{V,\alpha,x}$ defined by equation (2.4). Note any formal solution $u = (u_n)_{n\in\mathbb{Z}}$ of $H_{V,\alpha,x}u = Eu$ can be rewritten as

$$\begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = S_E^V(x+n\alpha) \begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix},$$

where

$$S_E^V(x) = \begin{pmatrix} E - V(x) & -1 \\ 1 & 0 \end{pmatrix}.$$

We call $(\alpha, S_E^V(x))$ the Schrödinger cocycle. The iterations of $(\alpha, S_E^V(\cdot))$ are of the form $(\alpha, S_E^V(\cdot))^n = (n\alpha, S_{E,n}^V(\cdot))$, where $S_{E,n}^V(\cdot)$ is called the transfer matrix and defined by

$$S_{E,n}^V(\cdot) := \begin{cases} S_{E,V}(\cdot + (n-1)\alpha) \cdots S_{E,V}(\cdot+\alpha)S_{E,V}(\cdot), & n \geq 0, \\ S_{E,V}^{-1}(\cdot + n\alpha)S_{E,V}^{-1}(\cdot + (n+1)\alpha) \cdots S_{E,V}^{-1}(\cdot - \alpha), & n < 0, \end{cases}$$

then, we have

$$\begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = S_{E,n}^V(x) \begin{pmatrix} u_1 \\ u_0 \end{pmatrix}.$$

2.3. *Integrated density of states and Lyapunov exponent.* The integrated density of states (IDS) $N_V : \mathbb{R} \to [0,1]$ of $H_{V,\alpha,x}$ is defined as

$$N_V(E) = \int_{\mathbb{T}^d} \mu_{V,x}(-\infty, E]\, dx,$$

where $\mu_{V,x}$ is the spectral measure of $H_{V,\alpha,x}$.

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Define the finite Lyapunov exponent as

$$L_n(\alpha, S_E^V) = \frac{1}{n}\int_{\mathbb{T}^d} \ln \|S_{E,n}^V(x)\|\, dx,$$

then by Kingman's subadditive ergodic theorem, the *Lyapunov exponent* of (α, S_E^V) is defined as

$$L(\alpha, S_E^V) = \lim_{n\to\infty} L_n(\alpha, S_E^V) = \inf_{n>0} L_n(\alpha, S_E^V) \geq 0. \tag{2.5}$$

Note that in our case, $d = \infty$. It turns out that if $\alpha \in DC_\infty(\gamma, \tau)$ (thus, α is uniquely ergodic), then

$$L(\alpha, S_E^V) = \lim_{n\to\infty} \frac{1}{n} \ln \|S_{E,n}^V(x)\|, \quad \text{a.e. } x \in \mathbb{T}^\infty.$$

By the *Thouless formula*, the relation between the IDS and the Lyapunov exponent defined by equation (2.5) is

$$L(\alpha, S_E^V) = \int \ln |E - E'|\, dN_V(E').$$

2.4. *Rotation number.* Assume that $A \in \mathcal{H}_\sigma(\mathbb{T}^d, SL(2,\mathbb{R}))$ is homotopic to identity and introduce the map:

$$F : \mathbb{T}^d \times S^1 \to \mathbb{T}^d \times S^1, \quad (x, v) \mapsto \left(x + \alpha, \frac{A(x)v}{\|A(x)v\|}\right),$$

which admits a continuous lift $\widetilde{F} : \mathbb{T}^d \times \mathbb{R} \to \mathbb{T}^d \times \mathbb{R}$ of the form $\widetilde{F}(x, y) = (x + \alpha, y + f(x, y))$ such that $f(x, y + 1) = f(x, y)$ and $\pi(y + f(x, y)) = A(x)\pi(y)/\|A(x)\pi(y)\|$. We call that $\widetilde{F}$ is a lift for (α, A). Since $x \mapsto x + \alpha$ is uniquely ergodic on $\mathbb{T}^d$, we can invoke a theorem by Herman [44] and Johnson and Moser [46]: for every $(x, y) \in \mathbb{T}^d \times \mathbb{R}$, the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(\widetilde{F}^k(x, y))$$

exists, is independent of (x, y), and the convergence is uniform in (x, y); the class of this number in $\mathbb{T}$ (which is independent of the chosen lift) is called the *fibered* rotation number of (α, A), which is denoted by $\rho(\alpha, A)$. Moreover, the rotation number $\rho_f(\alpha, A)$ relates density of states N_V as follows:

$$N_V(E) = 1 - 2\rho(\alpha, A). \tag{2.6}$$

For any $C \in SL(2,\mathbb{R})$, it is immediate from the definition that

$$|\rho(\alpha, A) - \rho(\alpha, C)| \leq \|A(x) - C\|_{C^0}^{1/2}. \tag{2.7}$$

In addition to the conclusion given by equation (2.7), we also have two conclusions below.

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LEMMA 2.2. [47] *The rotation number is invariant under the conjugation map which is homotopic to the identity. More precisely, if* $A, B : \mathbb{T}^d \to SL(2, \mathbb{R})$ *is continuous and homotopic to the identity, then*

$$\rho(\alpha, B(\cdot + \alpha)^{-1} A(\cdot) B(\cdot)) = \rho(\alpha, A).$$

PROPOSITION 2.3. *If* $A : \mathbb{T}^\infty \to SL(2, \mathbb{R})$ *is continuous and homotopic to the identity, and* $E : 2\mathbb{T}^\infty \to SL(2, \mathbb{R})$ *is defined by*

$$E(x) = \begin{pmatrix} \cos(\pi \langle r, x \rangle) & -\sin(\pi \langle r, x \rangle) \\ \sin(\pi \langle r, x \rangle) & \cos(\pi \langle r, x \rangle) \end{pmatrix},$$

then

$$\rho((0, E) \circ (\alpha, A) \circ (0, E^{-1})) = \rho(\alpha, A) + \frac{\langle r, \alpha \rangle}{2} \bmod 1.$$

Proof. It is known that $\rho((0, B) \circ (\alpha, A) \circ (0, B^{-1})) = \rho(\alpha, A) \bmod 1$ if $B : 2\mathbb{T}^\infty \to SL(2, \mathbb{R})$ is homotopic to the identity. In this case, the lift of $(\alpha, E(x + \alpha) A(x) E^{-1}(x))$ is given by

$$\widetilde{G}(x, y) = \bigg(x + \alpha, y + f\bigg(x, y - \frac{\langle r, x \rangle}{2}\bigg) + \frac{\langle r, \alpha \rangle}{2}\bigg).$$

Define

$$g(x, y) = f\bigg(x, y - \frac{\langle r, x \rangle}{2}\bigg) + \frac{\langle r, \alpha \rangle}{2},$$

$$g_k(x, y) = \sum_{i=0}^{k-1} g(\widetilde{G}^i(x, y)), \;\; f_k(x, y) = \sum_{i=0}^{k-1} f(\widetilde{F}^i(x, y)).$$

Using mathematical induction, then, $g_k(x, y) = f_k(x, y - \langle r, x \rangle/2) + k\langle r, \alpha \rangle/2$. Note the fact of the convergence of the Birkhoff means

$$\rho((0, E) \circ (\alpha, A) \circ (0, E^{-1})) = \rho(\alpha, A) + \frac{\langle r, \alpha \rangle}{2} \bmod 1. \qquad \square$$

3. *Almost reducibility*

In this section, we will establish our main KAM induction and then give the basic quantitative estimates in the case of reducibility and almost reducibility. These estimates will be applied to control the growth of corresponding Schrödinger cocycles.

3.1. *Decomposition along resonances.*

In this subsection, parameters ρ, ε, N, will be fixed. Define the resonant case as that where $k_* \in \mathbb{Z}_*^\infty$ with $0 < |k_*|_\eta \le N$, such that

$$\|2\rho - \langle k_*, \alpha \rangle\|_{\mathbb{R}/\mathbb{Z}} < \varepsilon^{1/12}.$$

The vector k_* will be referred to as a 'resonant site'.

After here, the constant c values are bounded uniformly but are different in different places. Moreover, for any $K > 0$, define the truncating operators $\mathcal{T}_K$ on $\mathcal{H}_\sigma(\mathbb{T}^\infty, X)$ as

$$(\mathcal{T}_K f)(x) = \sum_{k\in\mathbb{Z}_*^\infty, |k|_\eta < K} \widehat{f}(k)e^{i2\pi\langle k,x\rangle},$$

and projection operator $\mathcal{R}_K$ as

$$(\mathcal{R}_K f)(x) = \sum_{k\in\mathbb{Z}_*^\infty, |k|_\eta \geq K} \widehat{f}(k)e^{i2\pi\langle k,x\rangle}.$$

Decomposition of the space $\mathcal{H}_\sigma(\mathbb{T}^\infty, su(1,1))$ is defined as follows: for any given $\xi > 0$, $\alpha \in \mathbb{T}^\infty$, $A \in SU(1,1)$, we decompose $\mathcal{B}_\sigma = \mathcal{H}_\sigma(\mathbb{T}^\infty, su(1,1)) = \mathcal{B}_\sigma^{nre}(\xi) \oplus \mathcal{B}_\sigma^{re}(\xi)$ in such a way that for any $Y \in \mathcal{B}_\sigma^{nre}(\xi)$,

$$A^{-1}Y(x+\alpha)A \in \mathcal{B}_\sigma^{nre}(\xi), \quad |A^{-1}Y(x+\alpha)A - Y(x)|_\sigma \geq \xi|Y(x)|_\sigma. \tag{3.1}$$

Let $\mathcal{P}_{nre}$, $\mathcal{P}_{re}$ be the standard projections from $\mathcal{B}_\sigma$ onto $\mathcal{B}_\sigma^{nre}(\xi)$ and $\mathcal{B}_\sigma^{re}(\xi)$.

LEMMA 3.1. [20] *Assume that $\varepsilon \leq (4\|A\|)^{-4}$, $\xi \geq 13\|A\|^2\varepsilon^{1/2}$. For any $f \in \mathcal{B}_\sigma$ with $|f|_\sigma \leq \varepsilon$, there exists $Y \in \mathcal{B}_\sigma$, $f^{re} \in \mathcal{B}_\sigma^{re}(\xi)$ such that*

$$e^{Y(x+\alpha)}(Ae^{f(x)})e^{-Y(x)} = Ae^{f^{re}(x)}, \quad |Y|_\sigma \leq \varepsilon^{1/2}, \quad |f^{re}|_\sigma \leq 2\varepsilon.$$

Referring to [20, Remark 3.1], Lemma 3.1 sets up the fact that $\mathcal{B}_\sigma$ is a Banach space. Before giving the induction proposition, we introduce the following well-known results, which control the estimate of the small divisor.

LEMMA 3.2. [51] *Let $\mu_1, \mu_2 \geq 1$. We have the following estimate for $N \gg 1$:*

$$\sup_{k\in\mathbb{Z}_*^\infty, |k|_\eta \leq N} \prod_{j\in\mathbb{N}} (1 + |k_j|^{\mu_1}\langle j\rangle^{\mu_2}) \leq (1+N)^{c(\eta,\mu_1,\mu_2)N^{1/(1+\eta)}} \tag{3.2}$$

for some constant $C(\eta, \mu_1, \mu_2) > 0$.

For the given c, assume that $D(c)$ is the smallest one such that

$$(\ln D(c))^{\eta/(4+4\eta)} \geq c \ln\ln D(c). \tag{3.3}$$

PROPOSITION 3.3. *Let $\alpha \in DC_\infty(\gamma, \tau)$, $\gamma > 0$, $0 < \sigma < 1/10$, $\tau > 1$, $\eta > 0$. Consider the cocycle $(\alpha, Ae^{f(x)})$, where $A \in SU(1,1)$ with eigenvalues $\{e^{i2\pi\rho}, e^{-i2\pi\rho}\}$ and $f \in \mathcal{H}_\sigma(\mathbb{T}^\infty, su(1,1))$ with the estimate*

$$|f|_\sigma \leq \varepsilon \leq D(c)^{-1} \exp\{-[(\sigma-\delta)\pi]^{(-2(2+\eta))/\eta}\}, \tag{3.4}$$

where $\delta \in (0,\sigma)$ and $D(c)$ is the one defined by equation (3.3) with $c = c(\gamma, \tau, \eta, \|A\|)$. Then, there exists $B \in \mathcal{H}_\delta(2\mathbb{T}^\infty, SU(1,1))$, $f_+ \in \mathcal{H}_\delta(\mathbb{T}^\infty, su(1,1))$, and $A_+ \in SU(1,1)$, such that

$$B(x+\alpha)(Ae^{f(x)})B(x)^{-1} = A_+e^{f_+(x)}. \tag{3.5}$$

Moreover, let $N = \{(\sigma - \delta)\pi\}^{-1} \ln \varepsilon^{-1}$, *then we distinguish the conclusions into two cases. Non-resonant case: if for any* $0 < |k|_\eta \le N$, $k \in \mathbb{Z}_*^\infty$,

$$\|2\rho - \langle k, \alpha\rangle\|_{\mathbb{R}/\mathbb{Z}} \ge \varepsilon^{1/12}, \tag{3.6}$$

then

$$|B - \mathrm{Id}|_\delta \le \varepsilon^{1/2}, \quad |f_+|_\delta \le 4\varepsilon^3, \quad \|A_+ - A\| \le 2\varepsilon. \tag{3.7}$$

Resonant case: if there exists $k_* \in \mathbb{Z}_*^\infty$, $0 < |k_*|_\eta \le N$, *such that*

$$\|2\rho - \langle k_*, \alpha\rangle\|_{\mathbb{R}/\mathbb{Z}} < \varepsilon^{1/12},$$

then

$$\begin{gathered} |B|_\delta \le e^{\pi\delta|k_*|_\eta}\varepsilon^{-1/480}, \quad \|B\| \le \varepsilon^{-1/480}, \quad |f_+(x)|_\delta \ll \varepsilon^{10}, \\ \rho(\alpha, A_+e^{f_+(x)}) = \rho(\alpha, Ae^{f(x)}) + \frac{\langle k_*, \alpha\rangle}{2}, \end{gathered} \tag{3.8}$$

and $A_+ = e^{A''_+}$, $\|A''_+\| < 4\varepsilon^{1/12}$,

$$A''_+ = \begin{pmatrix} it_+ & v_+ \\ \bar{v}_+ & -it_+ \end{pmatrix}$$

with $|t_+| \le 4\varepsilon^{1/12}$, $|v_+| \le 2\varepsilon^{1-1/240}e^{-2\pi|k_*|_\eta\sigma}$.

Proof. Non-resonant case. Before giving the proof, we give the estimate of small divisor in the following claim.

CLAIM 3.4. *For any* $\alpha \in DC_\infty(\gamma, \tau)$, *and parameters* $N, \eta, \tau, \varepsilon$ *given above, the following estimates hold:*

$$\|\langle k, \alpha\rangle\|_{\mathbb{R}/\mathbb{Z}} \ge \gamma(1+N)^{-c(\eta,\tau)N^{1/(1+\eta)}} > 2\varepsilon^{1/240}. \tag{3.9}$$

Proof. The fact that $\alpha \in DC_\infty(\gamma, \tau)$ and equation (3.2) yield, for any $0 < |k|_\eta \le N$,

$$\|\langle k, \alpha\rangle\|_{\mathbb{R}/\mathbb{Z}} \ge \gamma \prod_{j\in\mathbb{N}} \frac{1}{(1+|k_j|^\tau\langle j\rangle^\tau)} \ge \gamma(1+N)^{-c(\eta,\tau)N^{1/(1+\eta)}}.$$

Now, we give the proof of the last inequality in equation (3.9). First, equation (3.4) shows

$$\ln \varepsilon^{-1} \ge \max\{[(\sigma-\delta)\pi]^{-2(2+\eta)/\eta}, [c \ln\ln \varepsilon^{-1}]^{2(1+\eta)/\eta}\}, \tag{3.10}$$

where c depends on the parameters η, τ, γ. The inequality above yields

$$(\ln \varepsilon^{-1})^{\eta/(1+\eta)} = (\ln \varepsilon^{-1})^{\eta/(2(1+\eta))}(\ln \varepsilon^{-1})^{\eta/(2(1+\eta))} > c[(\sigma-\delta)\pi]^{-1/(1+\eta)} \ln\ln \varepsilon^{-1}. \tag{3.11}$$

Then,

$$\begin{aligned} \ln \varepsilon^{-1} &> (\ln \varepsilon^{-1})^{1/(1+\eta)} c[(\sigma-\delta)\pi]^{-1/(1+\eta)} \ln\ln \varepsilon^{-1} \\ &> c\{[(\sigma-\delta)\pi]^{-1} \ln \varepsilon^{-1}\}^{1/(1+\eta)}(\ln[(\sigma-\delta)\pi]^{-1} + \ln\ln \varepsilon^{-1}), \end{aligned} \tag{3.12}$$

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where the first and second inequalities are from equations (3.11) and (3.10), respectively. The last inequality in equation (3.12) and the fact $N = \{(\sigma-\delta)\pi\}^{-1}\ln\varepsilon^{-1}$ yield

$$\ln\varepsilon^{-1} > cN^{1/(1+\eta)}\ln N,$$

which yields

$$\varepsilon^{-1/240} > 2\gamma^{-1}(1+N)^{c(\eta,\tau)N^{1/(1+\eta)}}. \qquad \square$$

Define

$$\Lambda_N = \bigg\{ f \in \mathcal{H}_\sigma(\mathbb{T}^\infty, su(1,1)) \,|\, f(x) = \sum_{k\in\mathbb{Z}_*^\infty, 0<|k|_\eta\le N} \widehat{f}(k)e^{i2\pi\langle k,x\rangle} \bigg\}. \tag{3.13}$$

By equations (3.6) and (3.9), a simple computation shows that if $Y \in \Lambda_N$, $A \in SU(1,1)$, then

$$|A^{-1}Y(x+\alpha)A - Y(x)|_\sigma \gtrsim \varepsilon^{1/4}|Y(x)|_\sigma.$$

Thus, $\Lambda_N \subset \mathcal{B}_\sigma^{nre}(\varepsilon^{1/4})$. Note $\varepsilon \le (4\|A\|)^{-4}$ and $\varepsilon^{1/4} > 13\|A\|^2\varepsilon^{1/2}$, then by Lemma 3.1 with $\varepsilon^{1/4}$ in place of ξ, we know that there exist $Y \in \mathcal{B}_\sigma$, $f^{re} \in \mathcal{B}_\sigma^{re}(\varepsilon^{1/4})$ such that

$$e^{Y(x+\alpha)}(Ae^{f(x)})e^{-Y(x)} = Ae^{f^{re}(x)}, \tag{3.14}$$

where $|Y|_\sigma \le \varepsilon^{1/2}$, $|f^{re}|_\sigma \le 2\varepsilon$. Notice $f^{re} \in \mathcal{B}_\sigma^{re}(\varepsilon^{1/4})$, then by equation (3.13),

$$f^{re}(x) = \widehat{f}^{re}(0) + \mathcal{R}_N f^{re}(x), \quad \|\widehat{f}^{re}(0)\| \le 2\varepsilon. \tag{3.15}$$

Moreover, we also have

$$\begin{aligned} |\mathcal{R}_N f^{re}(x)|_\delta &= \sum_{k\in\mathbb{Z}_*^\infty, |k|_\eta>N} \|\hat{f}^{re}(k)\|e^{2\pi|k|_\eta\delta} \\ &\le \|f^{re}\|_\sigma \sup_{|k|_\eta>N} e^{-2\pi|k|_\eta(\sigma-\delta)} \le e^{-2\pi N(\sigma-\delta)}2\varepsilon \le 2\varepsilon^3. \end{aligned} \tag{3.16}$$

Set

$$e^{f_+(x)} := e^{-\widehat{f}^{re}(0)}e^{f^{re}(x)}, \quad A_+ := Ae^{\widehat{f}^{re}(0)}, \quad B(x) := e^{Y(x)}.$$

Thus, the cocycle $(\alpha, Ae^{f^{re}(x)})$ can be written as $(\alpha, A_+e^{f_+(x)})$ and $(0, B)$ changes $(\alpha, Ae^{f(x)})$ to $(\alpha, A_+e^{f_+(x)})$. Notice that $e^Ae^E = e^{A+E+D}$, where D is a sum of terms of order at least 2 in A, E, then by equations (3.15) and (3.16), we get

$$|f_+(x)|_\delta \le 4\varepsilon^3, \quad \|A_+ - A\| \le \|A\|\|\mathrm{Id} - e^{\widehat{f}^{re}(0)}\| \le 2\varepsilon, \quad |B - \mathrm{Id}|_\delta \le \varepsilon^{1/2}.$$

Resonant case. Note that we only need to consider the case in which $A \in SU(1,1)$ is elliptic with eigenvalues $\{e^{i2\pi\rho}, e^{-i2\pi\rho}\}$ for $\rho \in \mathbb{R}\setminus\{0\}$, because if $\rho = ib$, with $b \in \mathbb{R}$, then equation (3.9) implies $\|i2b - \langle k_*, \alpha\rangle\|_{\mathbb{R}/\mathbb{Z}} > 2\varepsilon^{1/240}$.

CLAIM 3.5. *Assume that k_* is the resonant site with*

$$0 < |k_*|_\eta \le N,$$

then there is no other resonant site k_*' with $|k_*'| \le N^{1+(\eta/2)}$.

Proof. Assume that there exists $k'_* \neq k_*$ with $|k'_*|_\eta \leq N^{1+(\eta/2)}$, satisfying $\|2\rho - \langle k'_*, \alpha\rangle\|_{\mathbb{R}/\mathbb{Z}} < \varepsilon^{1/12}$, then by the Diophantine condition of α, we have

$$\gamma \prod_{j\in\mathbb{N}} \frac{1}{(1+|k'_{*j} - k_{*j}|^\tau \langle j\rangle^\tau)} \leq \|\langle k'_*, \alpha\rangle - \langle k_*, \alpha\rangle\|_{\mathbb{R}/\mathbb{Z}} < 2\varepsilon^{1/12}.$$

Moreover, note $|k'_* - k_*|_\eta \leq 2N^{1+(\eta/2)}$, then by equation (3.2) and the inequalities above,

$$\frac{\gamma}{2}\varepsilon^{-1/12} < \prod_{j\in\mathbb{N}}(1+|k'_{*j} - k_{*j}|^\tau \langle j\rangle^\tau) \leq (1+2N^{1+(\eta/2)})^{c(\eta,\tau)(2N^{1+(\eta/2)})^{1/(1+\eta)}},$$

which, together with the fact $N = \{(\sigma-\delta)\pi\}^{-1} \ln \varepsilon^{-1}$, implies

$$(\ln \varepsilon^{-1})^{\eta/(2+2\eta)} < c(\gamma, \tau, \eta)[(\sigma-\delta)\pi]^{-(2+\eta)/(2+2\eta)} \ln\ln \varepsilon^{-1}. \tag{3.17}$$

However, equation (3.10) implies $[(\sigma-\delta)\pi]^{-(2+\eta)/(2+2\eta)} < (\ln \varepsilon^{-1})^{\eta/(4+4\eta)}$, which, together with equation (3.17), implies

$$(\ln \varepsilon^{-1})^{\eta/(4+4\eta)} < c(\gamma, \tau, \eta) \ln\ln \varepsilon^{-1}.$$

The inequality above is contradictory to equations (3.4) and (3.3). □

Note that tr $A = e^{i2\pi\rho} + e^{-i2\pi\rho} = 2\cos 2\pi\rho > -2$, so there exists $A' \in su(1,1)$ such that $A = e^{A'}$ with spec $(A') = \{i2\pi\rho, -i2\pi\rho\}$, $\rho \in (0, \frac{1}{2})$. In this resonant case, the fact that there exist k_* with $|k_*|_\eta < N$ such that

$$\|2\rho - \langle k_*, \alpha\rangle\|_{\mathbb{R}/\mathbb{Z}} < \varepsilon^{1/12}$$

together with similar calculations above yield

$$|\rho| \geq \varepsilon^{1/252}. \tag{3.18}$$

Moreover, [45, Lemma 8.1] implies that there exists $P \in SU(1,1)$ such that

$$PAP^{-1} = \operatorname{diag}(e^{i2\pi\rho}, e^{-i2\pi\rho}) := A_*,$$

where

$$\|P\| \leq 2(\|A'\| |2\pi\rho|^{-1})^{1/2} \leq \varepsilon^{-1/480}, \tag{3.19}$$

here, the second inequality is by equation (3.18). Furthermore, set $h(x) = Pf(x)P^{-1}$, then the cocycle $(\alpha, Ae^{f(x)})$ is changed into $(\alpha, A_* e^{h(x)})$ with

$$|h|_\sigma \leq \varepsilon^{-1/240} \cdot \varepsilon \triangleq \varepsilon'. \tag{3.20}$$

The estimate $\varepsilon^{1/12} \geq 13\|A_*\|^2 \varepsilon'^{1/2}$ enables us to apply Lemma 3.1 to the cocycle $(\alpha, A_* e^{h(x)})$ to remove all the non-resonant terms of h, that is, there exists $Y \in \mathcal{B}_\sigma$, $h^{re} \in \mathcal{B}_\sigma^{re}(\varepsilon^{1/12})$ such that

$$e^{Y(x+\alpha)}(A_* e^{h(x)})e^{-Y(x)} = A_* e^{h^{re}(x)},$$

with $|Y|_\sigma \leq \varepsilon'^{1/2}$, $|h^{re}|_\sigma \leq 2\varepsilon'$. Thus, we get the cocycle $(\alpha, A_* e^{h^{re}(x)})$.

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Now define

$$\Lambda_1(\varepsilon^{1/12}) = \{k \in \mathbb{Z}_*^\infty : \|\langle k, \alpha\rangle\|_{\mathbb{R}/\mathbb{Z}} \geq \varepsilon^{1/12}\},$$
$$\Lambda_2(\varepsilon^{1/12}) = \{k \in \mathbb{Z}_*^\infty : \|2\rho - \langle k, \alpha\rangle\|_{\mathbb{R}/\mathbb{Z}} \geq \varepsilon^{1/12}\},$$

and define $\mathcal{B}_\sigma^{nre}(\varepsilon^{1/12})$ as in equation (3.1) with A being substituted by A_*. Then, we can compute that any $Y \in \mathcal{B}_\sigma^{nre}(\varepsilon^{1/12})$ takes the precise form:

$$\begin{aligned} Y(x) &= \begin{pmatrix} it(x) & v(x) \\ \bar{v}(x) & -it(x) \end{pmatrix} \\ &= \sum_{k\in\Lambda_1(\varepsilon^{1/12})} \begin{pmatrix} i\widehat{t}(k) & 0 \\ 0 & -i\widehat{t}(k) \end{pmatrix} e^{i2\pi\langle k,x\rangle} + \sum_{k\in\Lambda_2(\varepsilon^{1/12})} \begin{pmatrix} 0 & \widehat{v}(k)e^{i2\pi\langle k,x\rangle} \\ \bar{\widehat{v}}(k)e^{-i2\pi\langle k,x\rangle} & 0 \end{pmatrix}, \end{aligned}$$

where $t(x) \in \mathbb{R}$, $v(x) \in \mathbb{C}$.

Combining with the fact that $\alpha \in DC_\infty(\gamma, \tau)$ and the Claim 3.5, we have

$$\{\mathbb{Z}_*^\infty \setminus \Lambda_1(\varepsilon^{1/12})\} \cap \{k \in \mathbb{Z}_*^\infty : |k|_\eta \leq N'\} = \{0\},$$
$$\{\mathbb{Z}_*^\infty \setminus \Lambda_2(\varepsilon^{1/12})\} \cap \{k \in \mathbb{Z}_*^\infty : |k|_\eta \leq N'\} = \{k_*\},$$

where $N' \triangleq 2N^{1+(\eta/2)} - N$. Thus, $h^{re}(x) \in \mathcal{B}_\sigma^{re}(\varepsilon^{1/12})$ can be rewritten as

$$\begin{aligned} h^{re}(x) &= h_0^{re}(x) + h_1^{re}(x) + h_2^{re}(x) \\ &= \begin{pmatrix} i\widehat{t}(0) & 0 \\ 0 & -i\widehat{t}(0) \end{pmatrix} + \begin{pmatrix} 0 & \widehat{v}(k_*)e^{i2\pi\langle k_*,x\rangle} \\ \bar{\widehat{v}}(k_*)e^{-i2\pi\langle k_*,x\rangle} & 0 \end{pmatrix} + \sum_{|k|_\eta > N'} \widehat{h}^{re}(k)e^{i2\pi\langle k,x\rangle} \end{aligned}$$

with

$$\|h_j^{re}\|_\sigma \leq \|h^{re}\|_\sigma \leq 2\varepsilon', \quad j = 0, 1, 2. \tag{3.21}$$

Define $Q : 2\mathbb{T}^\infty \to SU(1, 1)$ as

$$Q(x) = \begin{pmatrix} e^{-i\pi\langle k_*,x\rangle} & 0 \\ 0 & e^{i\pi\langle k_*,x\rangle} \end{pmatrix},$$

so we have

$$|Q(x)|_\delta \leq e^{\pi\delta|k_*|_\eta} \leq \varepsilon^{-\delta/(\sigma-\delta)}, \tag{3.22}$$

where the last inequality is from $|k_*|_\eta \leq N = \{(\sigma - \delta)\pi\}^{-1} \ln \varepsilon^{-1}$. One can also show

$$Q(x + \alpha)(A_* e^{h^{re}(x)})Q(x)^{-1} = A'_* e^{h'^{re}(x)},$$

where

$$A'_* = Q(x + \alpha)A_* Q(x)^{-1} = \begin{pmatrix} e^{i2\pi(\rho-\langle k_*,\alpha\rangle/2)} & 0 \\ 0 & e^{-i2\pi(\rho-\langle k_*,\alpha\rangle/2)} \end{pmatrix},$$

and

$$h'^{re}(x) = Q(x)h^{re}(x)Q(x)^{-1} = \sum_{j=0}^{2} Q(x)h_j^{re}(x)Q(x)^{-1}.$$

Moreover,

$$Q(x)h_0^{re}(x)Q(x)^{-1} = h_0^{re}(x), \qquad Q(x)h_1^{re}(x)Q(x)^{-1} = \begin{pmatrix} 0 & \widehat{v}(k_*) \\ \bar{\widehat{v}}(k_*) & 0 \end{pmatrix},$$

$$M^{-1}Q(x)M = \begin{pmatrix} \cos\pi\langle k_*, x\rangle & -\sin\pi\langle k_*, x\rangle \\ \sin\pi\langle k_*, x\rangle & \cos\pi\langle k_*, x\rangle \end{pmatrix}.$$

Denote

$$\begin{aligned} L &= Q(x)h_0^{re}(x)Q(x)^{-1} + Q(x)h_1^{re}(x)Q(x)^{-1}, \\ F &= Q(x)h_2^{re}(x)Q(x)^{-1}, \\ B &= Q(x)e^{Y(x)}P : 2\mathbb{T}^\infty \to SU(1,1). \end{aligned}$$

Then, by the discussions above, we have

$$B(x+\alpha)(Ae^{f(x)})B(x)^{-1} = A'_*e^{h'^{re}(x)} := A_+e^{f_+(x)},$$

where

$$\begin{aligned} A_+ &= A'_*e^L := e^{A''_+} \\ &= \begin{pmatrix} e^{i2\pi(\rho-\langle k_*,\alpha\rangle/2)} & 0 \\ 0 & e^{-i2\pi(\rho-\langle k_*,\alpha\rangle/2)} \end{pmatrix} \exp\begin{pmatrix} i\widehat{t}(0) & \widehat{v}(k_*) \\ \bar{\widehat{v}}(k_*) & -i\widehat{t}(0) \end{pmatrix}, \\ e^{f_+(x)} &= e^{-L}e^{h'^{re}(x)} = e^{-L}e^{L+F}. \end{aligned} \tag{3.23}$$

Now, we give the estimates of B and the cocycle $(\alpha, A_+e^{f_+(x)})$. First, Proposition 2.3 shows

$$\rho(\alpha, A_+e^{f_+(x)}) = \rho(\alpha, Ae^{f(x)}) + \frac{\langle k_*, \alpha\rangle}{2},$$

and the estimates in equations (3.19)–(3.22) yield

$$\begin{aligned} &|B|_\delta \le e^{\pi\delta|k_*|_\eta}\varepsilon^{-1/480}, \ \|B\| \le \varepsilon^{-1/480}, \\ &|\widehat{t}(0)| \le 2\varepsilon', \ |\widehat{v}(k_*)| \le 2\varepsilon' e^{-2\pi\sigma|k_*|_\eta}, \\ &|F|_\delta \le \varepsilon^{-2\delta/(\sigma-\delta)}|h_2^{re}|_\delta \le \varepsilon^{-2\delta/(\sigma-\delta)}e^{-2\pi N'(\sigma-\delta)}|h_2^{re}|_\sigma \\ &\qquad \le 2\varepsilon^{1-1/240}\varepsilon^{4N^{\eta/2}}\varepsilon^{-2\sigma/(\sigma-\delta)} \ll \varepsilon^{10}. \end{aligned}$$

The estimates above imply that the B, f_+ and A''_+ defined by equation (3.23) are those we need. □

3.2. *Reducibility of almost-periodic cocycle.* Given any $\sigma \in (0, \sigma_0)$, define

$$\sigma_{j+1} = \sigma_j - \frac{\sigma_0 - \sigma}{12(j+1)^2}, \qquad j \ge 1, \tag{3.24}$$

then,

$$\sigma_\infty = \sigma_0 - \sum_{i=0}^{+\infty}(\sigma_i - \sigma_{i+1}) \ge \frac{5\sigma_0 + \sigma}{6} > \sigma.$$

Let

$$\varepsilon_0 < \varepsilon_* := D(c)^{-1} \exp\{-\{[12(N^*+1)^2][(\sigma_0-\sigma)\pi]^{-1}\}^{2(2+\eta)/\eta}\}, \tag{3.25}$$

where $D(c)$ is the one in equation (3.4) and

$$N^* = (3^{\eta/4(2+\eta)} - 1)^{-1}. \tag{3.26}$$

Moreover, we also define

$$\varepsilon_{j+1} = 4\varepsilon_j^3, \quad N_j = \{(\sigma_j - \sigma_{j+1})\pi\}^{-1} \ln \varepsilon_j^{-1}, \quad j \geq 0. \tag{3.27}$$

Remark 3.6. Assume we arrive at the $(j+1)$th step's cocycle $(\alpha, A_j e^{f_j})$ satisfying the hypotheses of Proposition 3.3 with σ, δ and A, f being replaced by σ_j, σ_{j+1} and A_j, f_j, then we can apply Proposition 3.3 and get a new cocycle $(\alpha, A_{j+1}e^{f_{j+1}})$. The conclusions of Proposition 3.3 show that $|f_{j+1}|_{\sigma_{j+1}} \leq \varepsilon_{j+1}$ for both the resonant and non-resonant cases. The choice of ε_* defined by equation (3.25) with N^* being given by equation (3.26) ensures that

$$\varepsilon_{j+1} \leq D(c)^{-1} \exp\{-[(\sigma_{j+1} - \sigma_{j+2})\pi]^{(-2(2+\eta))/\eta}\}. \tag{3.28}$$

That is, $(\alpha, A_{j+1}e^{f_{j+1}})$ satisfies the hypotheses of Proposition 3.3 with σ, δ and A, f being replaced by $\sigma_{j+1}, \sigma_{j+2}$ and A_{j+1}, f_{j+1}. That is, we can apply Proposition 3.3 to $(\alpha, A_j e^{f_j})$ for all $j \geq 0$.

PROPOSITION 3.7. *Let $\alpha \in DC_\infty(\gamma,\tau)$, $\gamma > 0$, $0 < \sigma_0 < (1/10)$, $\tau > 1$, $\eta > 0$,*

$$K_j = \{E \in \Sigma \mid \text{there exists } k_{j-1} \in \mathbb{Z}_*^\infty, \text{ with } 0 < |k_{j-1}|_\eta \leq N_{j-1} \\ \text{such that } \|2\rho(A_{j-1}) - \langle k_{j-1}, \alpha\rangle\|_{\mathbb{R}/\mathbb{Z}} < \varepsilon_{j-1}^{1/12}\}, \quad j \geq 1. \tag{3.29}$$

Suppose that $A_0 \in SL(2,\mathbb{R})$, $f_0 \in \mathcal{H}_{\sigma_0}(\mathbb{T}^\infty, sl(2,\mathbb{R}))$, and $|f|_{\sigma_0} \leq \varepsilon_0$. Then, the following conclusions hold. (1) The system $(\alpha, A_0 e^{f_0(x)})$ is almost reducible in the strip $|\Im x| < \sigma$. Moreover, there exists $\widetilde{B}_{j-1} \in \mathcal{H}_{\sigma_j}(2\mathbb{T}^\infty, SL(2,\mathbb{R}))$, such that

$$\widetilde{B}_{j-1}(x+\alpha) A_0 e^{f_0(x)} \widetilde{B}_{j-1}(x)^{-1} = A_j e^{f_j(x)}, \quad j \geq 1 \tag{3.30}$$

with estimates

$$\|\widetilde{B}_{j-1}(x)\| \leq \varepsilon_{j-1}^{-1/320}, \quad |f_j|_{\sigma_j} \leq \varepsilon_j, \quad j \geq 1. \tag{3.31}$$

Assume the $(j_\ell + 1)$th KAM step is the ℓth resonant step with $\ell = i, i+1$, then

$$N_{j_i}^{(4+4\eta)/(4+3\eta)} < |k_{j_{i+1}}|_\eta \leq N_{j_{i+1}}. \tag{3.32}$$

(2) There exist unitary matrices $U_j \in SL(2,\mathbb{C})$ such that

$$U_j A_j U_j^{-1} = \begin{pmatrix} e^{i2\pi\rho_j} & c_j \\ 0 & e^{-i2\pi\rho_j} \end{pmatrix} \quad \text{for all } j \in \mathbb{N} \tag{3.33}$$

and

$$\|\widetilde{B}_{j-1}\|^2 \cdot |c_j| \leq 8\|A_0\| \quad \text{for all } j \in \mathbb{N}, \tag{3.34}$$

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where $\widetilde{B}_{j-1}$ is the one in equation (3.30). Moreover, if $E \in K_j$, then

$$\begin{aligned} \rho(\alpha, A_j e^{f_j(x)}) &= \rho(\alpha, A_{j-1}e^{f_{j-1}(x)}) + 2^{-1}\langle k_{j-1}, \alpha\rangle, \\ |c_j| &< 4\varepsilon_{j-1}^{1/12}. \end{aligned} \tag{3.35}$$

In this case, set $A_j = e^{A''_j}$, then

$$MA''_jM^{-1} = \begin{pmatrix} it_j & v_j \\ \bar{v}_j & -it_j \end{pmatrix} \tag{3.36}$$

with

$$|t_j| \le 4\varepsilon_{j-1}^{1/12}, \quad |v_j| \le 2\varepsilon_{j-1}^{1-1/240} e^{-2\pi|k_{j-1}|_\eta \sigma_{j-1}}, \quad \|A''_j\| \le 4\varepsilon_{j-1}^{1/12}. \tag{3.37}$$

(3) If there exists $k \in \mathbb{Z}_*^\infty \setminus \{0\}$, such that

$$\rho(\alpha, A_0 e^{f_0(x)}) = 2^{-1}\langle k, \alpha\rangle, \tag{3.38}$$

and $(\alpha, A_0e^{f_0(x)})$ is not uniformly hyperbolic, then the resonant case defined in Proposition 3.3 only happens finite times. Moreover, there exists $\widetilde{B} \in \mathcal{H}_\sigma(2\mathbb{T}^\infty, SL(2,\mathbb{R}))$ such that

$$\widetilde{B}(x+\alpha)A_0e^{f_0(x)}\widetilde{B}(x)^{-1} = \begin{pmatrix} 1 & \zeta \\ 0 & 1 \end{pmatrix},$$

with estimate $|\zeta| \le 2\varepsilon_{j_n}^{1-1/120} e^{-(25/16)\pi\sigma_0|k|_\eta}$, and $|\widetilde{B}(x)|_\sigma \le 4\varepsilon_{j_n}{}^{-1/320} e^{(8/7)\pi\sigma|k|_\eta}$, where j_n is the index of last resonant site k_{j_n}.

Proof. We give the conclusions in part (1) by induction.

First step. The estimate of ε_0 defined by equation (3.25) enables us to apply Proposition 3.3 to $(\alpha, A_0e^{f_0(x)})$ and obtain the following. There exists $\widetilde{B}_0 \in \mathcal{H}_{\sigma_1}(2\mathbb{T}^\infty, SL(2,\mathbb{R}))$ such that

$$\widetilde{B}_0(x+\alpha)(A_0e^{f_0(x)})\widetilde{B}_0(x)^{-1} = A_1e^{f_1(x)}$$

with the following estimates:

$$|f_1|_{\sigma_1} \le \varepsilon_1, \quad \|\widetilde{B}_0\| \le \varepsilon_0^{-1/320}.$$

That is, we get a new cocycle $(\alpha, A_1e^{f_1(x)})$ with estimates (3.30) and (3.31) with $j = 1$. Moreover, if there exists $k_0 \in \mathbb{Z}_*^\infty$, $0 < |k_0|_\eta \le N_0$, such that $\|2\rho - \langle k_0, \alpha\rangle\|_{\mathbb{R}/\mathbb{Z}} < \varepsilon_0^{1/12}$, then

$$\rho(\alpha, A_1e^{f_1(x)}) = \rho(\alpha, A_0e^{f_0(x)}) + \frac{\langle k_0, \alpha\rangle}{2}.$$

By Remark 3.6, we know that equation (3.28) holds with $j = 0$. That is, $(\alpha, A_1e^{f_1(x)})$ satisfies the hypotheses of Proposition 3.3 with σ, δ and A, f being replaced by σ_1, σ_2 and A_1, f_1, respectively.

Inductive step. Assume that we have completed the lth step and are at the $(l+1)$th KAM step with $l \ge 1$, that is, we already construct $\widetilde{B}_{l-1} \in \mathcal{H}_{\sigma_l}(2\mathbb{T}^\infty, SL(2,\mathbb{R}))$ such that the estimates in equations (3.30) and (3.31) hold with $j = 1, \ldots, l$. Now we consider

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the $(l+1)$th step. By Remark 3.6, we know that equation (3.28) holds with $j=l-1$. Then, by applying Proposition 3.3, we know that there exists $B_l \in \mathcal{H}_{\sigma_{l+1}}(2\mathbb{T}^\infty, SL(2,\mathbb{R}))$, $A_{l+1}\in SL(2,\mathbb{R})$, and $f_{l+1}\in \mathcal{H}_{\sigma_{l+1}}(\mathbb{T}^\infty, sl(2,\mathbb{R}))$ such that

$$B_l(x+\alpha)(A_l e^{f_l(x)})B_l(x)^{-1} = A_{l+1}e^{f_{l+1}(x)}.$$

If $\|2\rho_l - \langle k,\alpha\rangle\|_{\mathbb{R}/\mathbb{Z}} \geq \varepsilon_l^{1/12}$ holds for any $k\in\mathbb{Z}_*^\infty$ with $0<|k|_\eta\leq N_l$, then the conclusions of the *non-resonant case* in Proposition 3.3 show

$$|B_l - \mathrm{Id}|_{\sigma_{l+1}} \leq \varepsilon_j^{1/2}, \quad |f_{l+1}|_{\sigma_{l+1}} \leq 4\varepsilon_l^3 = \varepsilon_{l+1}, \quad \|A_{l+1}-A_l\| \leq 2\varepsilon_l. \tag{3.39}$$

However, if there exists $k_l\in\mathbb{Z}_*^\infty$ with $0<|k_l|_\eta\leq N_l$ such that

$$\|2\rho_l - \langle k_l,\alpha\rangle\|_{\mathbb{R}/\mathbb{Z}} < \varepsilon_l^{1/12}, \tag{3.40}$$

then the conclusions of the *resonant case* in Proposition 3.3 show

$$|B_l|_{\sigma_{l+1}} \leq e^{\pi\sigma_{l+1}|k_l|_\eta}\varepsilon_l{}^{-1/480}, \quad \|B_l\| \leq \varepsilon_l^{-1/480}, \quad |f_{l+1}(x)|_{\sigma_{l+1}} < \varepsilon_{l+1}, \tag{3.41}$$

$$\rho(\alpha, A_{l+1}e^{f_{l+1}(x)}) = \rho(\alpha, A_l e^{f_l(x)}) + \frac{\langle k_l,\alpha\rangle}{2},$$

and

$$A_{l+1} = e^{A''_{l+1}}, \quad \|A''_{l+1}\| < 4\varepsilon_l^{1/12}, \tag{3.42}$$

$$MA''_{l+1}M^{-1} = \begin{pmatrix} it_{l+1} & v_{l+1} \\ \bar{v}_{l+1} & -it_{l+1}\end{pmatrix} \tag{3.43}$$

with

$$|t_{l+1}| \leq 4\varepsilon_l^{1/12}, \quad |v_{l+1}| \leq 2\varepsilon_l^{1-1/240}e^{-2\pi|k_l|_\eta\sigma_l}. \tag{3.44}$$

Let $\widetilde{B}_l = B_l\widetilde{B}_{l-1}$. Then, we get equation (3.30) with $j=l+1$. Moreover, the estimates in equation (3.31) with $j=l$ and equations (3.39), (3.41), yield

$$\|\widetilde{B}_l(x)\| \leq \|B_l(x)\|\|\widetilde{B}_{l-1}(x)\| \leq \varepsilon_l^{-1/320}.$$

That is the estimates in equation (3.31) hold with $j=l+1$.

Inductively, we have proved that estimates in equations (3.30) and (3.31) hold for all $j\in\mathbb{N}$, which imply that $(\alpha, A_0e^{f_0(x)})$ is almost reducible in the strip $|\Im x|<\sigma$ $(\sigma_\infty>\sigma)$.

Now, we will verify equation (3.32). We give the lower bound of $\rho_{j_{i+1}}$ first. Suppose there are two resonance sites k_{j_i}, $k_{j_{i+1}}$, which happen at the KAM steps (j_i+1), $(j_{i+1}+1)$, respectively. Thus, $\|2\rho_{j_{i+1}} - \langle k_{j_{i+1}},\alpha\rangle\|_{\mathbb{R}/\mathbb{Z}} < \varepsilon_{j_{i+1}}^{1/12}$, which yields

$$\begin{aligned} 2\rho_{j_{i+1}} &> \|\langle k_{j_{i+1}},\alpha\rangle\|_{\mathbb{R}/\mathbb{Z}} - \varepsilon_{j_{i+1}}^{1/12} \\ &> 2^{-1}\|\langle k_{j_{i+1}},\alpha\rangle\|_{\mathbb{R}/\mathbb{Z}} > \frac{\gamma}{2}(1+|k_{j_{i+1}}|_\eta)^{-c(\eta,\tau)|k_{j_{i+1}}|_\eta^{1/(1+\eta)}}, \end{aligned} \tag{3.45}$$

where the last two inequalities above are given by equations (3.9) and (3.2).

We consider the upper bound of $\rho_{j_{i+1}}$. According to Proposition 3.3, after the (j_i+1)th KAM step, we get cocycle $(\alpha, A_{j_i+1}e^{f_{j_i+1}})$, then Shur theorem implies that there exists a unitary matrix U_{j_i+1} such that

$$U_{j_i+1}A''_{j_i+1}U^{-1}_{j_i+1} = \begin{pmatrix} i2\pi\rho_{j_i+1} & c''_{j_i+1} \\ 0 & -i2\pi\rho_{j_i+1}\end{pmatrix},$$

which, together with equation (3.42) yields (see the details to obtain equation (3.47))

$$|2\pi\rho_{j_i+1}| \le \|A''_{j_i+1}\| < 4\varepsilon_{j_i}^{1/12}.$$

Moreover, equations (2.7) and (3.39) imply

$$|\rho_{j_{i+1}} - \rho_{j_i+1}| \le \sum_{m=j_i+1}^{j_{i+1}-1} \|A_{m+1} - A_m\|^{1/2} \le c\varepsilon_{j_i}^{3/2}. \tag{3.46}$$

The inequalities (3.9), (3.45), and (3.46) yield

$$\frac{\gamma}{4}(1+|k_{j_{i+1}}|_\eta)^{-c(\eta,\tau)|k_{j_{i+1}}|_\eta^{1/(1+\eta)}} < \rho_{j_{i+1}} < 2\varepsilon_{j_i}^{1/12} < \frac{\gamma}{4}\varepsilon_{j_i}^{1/24}(1+N_{j_i})^{-10c(\eta,\tau)N_{j_i}^{1/(1+\eta)}}.$$

Then, we have $N_{j_i}^{(4+4\eta)/(4+3\eta)} < |k_{j_{i+1}}|_\eta$, and equation (3.32) is proved.

Now, we will verify the conclusions in (2). Obviously, equation (3.33) holds. Now, we assume $E \in K_j$, then the estimates in equations (3.42)–(3.44) with $l = j-1$ yield the conclusions in equations (3.36) and (3.37). Moreover, for the unitary matrix U_j in equation (3.33), we get

$$U_jA''_jU_j^{-1} = \begin{pmatrix} i2\pi\rho_j & c''_j \\ 0 & -i2\pi\rho_j\end{pmatrix}.$$

Thus, the estimates in equation (3.42) with $l = j-1$ yield

$$|c''_j|, |2\pi\rho_j| \le \|A''_j\| < 4\varepsilon_{j-1}^{1/12}. \tag{3.47}$$

Thus,

$$U_jA_jU_j^{-1} = e^{U_jA''_jU_j^{-1}} = \begin{pmatrix} e^{i2\pi\rho_j} & c_j \\ 0 & e^{-i2\pi\rho_j}\end{pmatrix},$$

where $c_j = \sum_{n=0}^{+\infty}(1/(2n+1)!)c''_j(i2\pi\rho_j)^{2n}$. Then, $|c_j| \le |c''_j| < 4\varepsilon_{j-1}^{1/12}$. That is, we get the estimate about c_j in equation (3.35). The equality about the rotation number in equation (3.35) holds obviously, we omit the details.

Now, we come to the inequality in equation (3.34). Since the steps between j_i+1, $j_{i+1}+1$ are all non-resonant steps, then equation (3.31) with $j_i = j-1$, equation (3.42) with $l = j_i$, and equation (3.39) yield, for all $j_i+1 < j < j_{i+1}+1$,

$$\begin{aligned} \|A''_j\| &\le \|A''_{j_i+1}\| + \sum_{m=j_i+1}^{j-1}\|A_{m+1}-A_m\|^{1/2} \le 5\varepsilon_{j_i}^{1/12}, \\ |\widetilde{B}_{j-1}(x)|_0 &\le 2\varepsilon_{j_i}^{-1/320}. \end{aligned} \tag{3.48}$$

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The inequality above, together with equation (3.33), yields

$$|c_j| \le 5\varepsilon_{j_i}^{1/12}, \quad j_i+1 < j < j_{i+1}+1. \tag{3.49}$$

The estimates (3.31), (3.35), (3.48), and (3.49) enable us to get the following conclusions.

(I) After each resonant step j, equations (3.31) and (3.35) yield

$$\|\widetilde{B}_{j-1}(x)\|^2|c_j| \le 4\varepsilon_j^{37/480} \le 8\|A_0\|.$$

(II) After each non-resonant step j, we are able to use the estimates of the last resonant step j_i+1 if it exists. Moreover, equations (3.48) and (3.49) imply

$$\|\widetilde{B}_{j-1}(x)\|^2|c_j| \le 20\varepsilon_{j_i}^{37/480} \le 8\|A_0\|.$$

(III) It is possible that no resonant steps happened within the first j steps. In this case, each step is non-resonant and thus we can use the estimate $\|\widetilde{B}_{j-1}\| < 2$. Then, $|c_j| \le \|A_j\| \le 2\|A_0\|$, which implies $\|\widetilde{B}_{j-1}\|^2|c_j| \le 8\|A_0\|$.

The discussions in conclusions (I)–(III) yield equation (3.34).

Finally, we come to part (3). Assume that the resonance occurs at j_l+1th step's cocycle $(\alpha, A_{j_l}e^{f_{j_l}})$, $l \in \mathbb{N}$. For the k in equation (3.38), if there exists $p \in \mathbb{N}$ such that

$$N_{j_p-1} < |k|_\eta \le N_{j_p}, \tag{3.50}$$

then, in the following, we will prove that there is no j_{p+1}.

Assume there exists such j_{p+1} with $0 < |k_{j_{p+1}}|_\eta < \infty$ satisfying

$$\|\langle k_{j_{p+1}}, \alpha\rangle - 2\rho_{j_{p+1}}\|_{\mathbb{R}/\mathbb{Z}} < \varepsilon_{j_{p+1}}^{1/12}.$$

In this case, the estimates in equation (3.47) with $j = j_{p+1}+1$ and equation (2.7) show

$$|\rho(\alpha, A_{j_{p+1}+1})| < \varepsilon_{j_{p+1}}^{1/12}$$

and

$$|2\rho(\alpha, A_{j_{p+1}+1}e^{f_{j_{p+1}+1}}) - 2\rho(\alpha, A_{j_{p+1}+1})| \le 2\varepsilon_{j_{p+1}+1}^{1/2},$$

respectively. The two inequalities above yield

$$|2\rho(\alpha, A_{j_{p+1}+1}e^{f_{j_{p+1}+1}})| \le 4\varepsilon_{j_{p+1}}^{1/12}. \tag{3.51}$$

However, the two inequalities in equation (3.32) imply, for $1 \le m \le p+1$,

$$\sum_{l=1}^{m}|k_{j_l}|_\eta \le \sum_{l=1}^{m-1}|N_{j_l}|_\eta + |k_{j_m}|_\eta \le \sum_{i=0}^{m-1}|k_{j_m}|_\eta^{((4+3\eta)/(4+4\eta))^i} \le \frac{31}{30}|k_{j_m}|_\eta. \tag{3.52}$$

Set $\widetilde{k} := k + \sum_{l=1}^{p+1} k_{j_l}$, then equations (3.52), (3.50), and (3.32) yield

$$\frac{16}{15}N_{j_{p+1}} \ge |\widetilde{k}|_\eta = \left|k_{j_{p+1}} + k + \sum_{l=1}^{p} k_{j_l}\right|_\eta \ge |k_{j_{p+1}}|_\eta - 3N_{j_p} > \frac{14}{15}N_{j_p}. \tag{3.53}$$

Moreover, the estimates in equationsa (3.38) and (3.35) yield

$$2\rho(\alpha, A_{j_{p+1}+1}e^{f_{j_{p+1}+1}}) = \left\langle k + \sum_{l=1}^{p+1} k_{i_l}, \alpha \right\rangle = \langle \widetilde{k}, \alpha\rangle,$$

which, together with equations (3.53) and (3.9), yields

$$2|\rho(\alpha, A_{j_{p+1}+1}e^{f_{j_{p+1}+1}})| = \|\langle \widetilde{k}, \alpha\rangle\|_{\mathbb{R}/\mathbb{Z}} \geq \gamma(1+2N_{j_{p+1}})^{-c(\eta,\tau)N_{j_{p+1}}^{1/(1+\eta)}} > \varepsilon_{j_{p+1}}^{1/120}.$$

The inequality above, together with equation (3.51), implies

$$\varepsilon_{j_{p+1}}^{1/120} < 2\rho(\alpha, A_{j_{p+1}+1}e^{f_{j_{p+1}+1}}) \leq 4\varepsilon_{j_{p+1}}^{1/12},$$

which is a contradiction, so there is no j_{p+1}. Thus, $2\rho(\alpha, A_{j_p+1}e^{f_{j_p+1}}) = 0$, otherwise, there exists the $(p+1)$th resonance.

The discussions above mean that the resonant case occurs only finitely many times in the above almost reducibility procedure. By the estimates of B_j in equation (3.39) and the sequence $(\sigma_j)_{j\in\mathbb{N}}$ given in equation (3.24), we see that the product $\prod_{l=0}^{j} B_l$ converges to some $B \in \mathcal{H}_\sigma(2\mathbb{T}^\infty, SL(2,\mathbb{R}))$ such that $B(x+\alpha)A_0e^{f_0(x)}B(x)^{-1} = A_\infty$, with

$$\rho(\alpha, A_\infty) = \rho(\alpha, A_{j_p+1}e^{f_{j_p+1}}) = 0. \tag{3.54}$$

Assuming that there are actually n times resonant steps, associated with integers vectors

$$k_{j_l} \in \mathbb{Z}_*^\infty \quad \text{with } 0 < |k_{j_l}|_\eta \leq N_{j_l}, l = 1, \ldots, n,$$

then, $k = k_{j_1} + \cdots + k_{j_n}$. In view of inequality (3.52), equation (3.32) with similar calculation of equation (3.53), we get

$$\frac{14}{15}|k_{j_n}|_\eta \leq |k|_\eta \leq \frac{16}{15}|k_{j_n}|_\eta. \tag{3.55}$$

Now we estimate the constant matrix A_∞. The fact that we have assumed that the initial cocycle $(\alpha, A_0e^{f_0(x)})$ is not hyperbolic and equation (3.54) imply A_∞ is a parabolic matrix. As $A_\infty \in SL(2,\mathbb{R})$, we have $A_\infty = e^{A''_\infty}$ with $A''_\infty \in sl(2,\mathbb{R})$ and $\det A''_\infty = 0$. Assume that $A''_\infty = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & -a_{11}\end{pmatrix}$, then there exists $\phi \in \mathbb{T}$ such that $R_\phi A''_\infty R_{-\phi} = \begin{pmatrix} 0 & a_{12}-a_{21} \\ 0 & 0\end{pmatrix}$. Let $\widetilde{B} = R_\phi B$, and $\zeta = a_{12} - a_{21}$, we can see that the cocycle $(\alpha, A_0e^{f_0(x)})$ is conjugated to $A_\infty = \begin{pmatrix} 1 & \zeta \\ 0 & 1\end{pmatrix}$ by $\widetilde{B}(x)$.

To estimate $|\zeta|$, let us focus on $(\alpha, A_{j_n+1}e^{f_{j_n+1}(x)})$, which is obtained by the last resonant step. In the following, we will estimate the constant matrix A_∞. Obviously,

$$A_\infty = e^{A''_\infty} = M^{-1}\exp\begin{pmatrix} i\beta_{11} & \beta_{12} \\ \bar{\beta}_{12} & -i\beta_{11}\end{pmatrix} M,$$

where $\beta_{11} \in \mathbb{R}$, $\beta_{12} \in \mathbb{C}$. Since by equations (3.36) and (3.37) with $j_n + 1$ in place of j, and Proposition 3.3, it follows that

$$\begin{aligned} |\beta_{12}| &= |(M(A''_\infty - A''_{j_n+1})M^{-1})_{12} + (MA''_{j_n+1}M^{-1})_{12}| \\ &\leq 16\varepsilon_{j_n}^3 + 2\varepsilon_{j_n}^{1-1/240}e^{-2\pi|k_{j_n}|_\eta\sigma_{j_n}} \leq \varepsilon_{j_n}^{1-1/120}e^{-2\pi|k_{j_n}|_\eta\sigma_{j_n}}. \end{aligned}$$

Then, we have $|\beta_{11}| \leq \varepsilon_{j_n}^{1-1/120} e^{-2\pi |k_{j_n}|_\eta \sigma_{j_n}}$ since $\det A''_\infty = 0$. Thus, by equation (3.55),

$$|\zeta| = |a_{12} - a_{21}| \leq |\beta_{11}| + |\beta_{12}| \leq 2\varepsilon_{j_n}^{1-1/120} e^{-2\pi |k_{j_n}|_\eta \sigma_{j_n}} \\ \leq 2\varepsilon_{j_n}^{1-1/120} e^{-(25/16)\pi\sigma_0 |k|_\eta} < 2^{-1},$$

where the third inequality is by $|k|_\eta \leq 16/15 |k_{j_n}|_\eta$ in equation (3.55) and $\sigma_\infty > 5\sigma_0/6$. In view of Proposition 3.7, there exists $\widetilde{B}_{j_n} \in \mathcal{H}_{\sigma_{j_n+1}}(2\mathbb{T}^\infty, SL(2, \mathbb{R}))$, such that

$$\widetilde{B}_{j_n}(x+\alpha) A_0 e^{f_0(x)} \widetilde{B}_{j_n}(x)^{-1} = A_{j_n+1} e^{f_{j_n+1}(x)}.$$

By equation (3.8) and Proposition 3.7 with $\sigma_\infty > \sigma$, we get

$$|\widetilde{B}_{j_n}(x)|_\sigma \leq 2 \prod_{i=1}^{n} |B_{j_i}(x)|_\sigma \leq 2\varepsilon_{j_n}{}^{-1/320} e^{(8/7)\pi\sigma |k|_\eta}.$$

Thus, $|\widetilde{B}(x)|_\sigma \leq 4\varepsilon_{j_n}{}^{-1/320} e^{(8/7)\pi\sigma |k|_\eta}$. □

4. *Pure absolutely continuous spectrum*

In this section, we give the proof of Theorem 1.2, whose conclusions show that the operator $H_{V,\alpha,x}$ has purely ac spectrum under some suitable hypotheses. The proof is based on the conclusions in Proposition 3.7.

4.1. *Auxiliary lemmas.*

Before giving the proof of Theorem 1.2, we give some auxiliary lemmas to get some necessary estimates such as the growth of cocycle, the estimates of integrated density of states, Lyapunov exponent, and rotation number. Moreover, we will apply Lemmas 4.6–4.8 without proof in our topology since we equip $\mathbb{T}^\infty$ with the product topology of $\mathbb{T}$, which is similar to the topology in [**1**]. Here and subsequently, we denote $\Sigma_{V,\alpha}$ by the spectrum of $H_{V,\alpha,x}$ and we acquiesce in the equality $S_E^V(x) = A_0 e^{f_0(x)}$.

LEMMA 4.1. *Let K_j, $j \in \mathbb{N}$ be the sets defined by equation (3.29) and set $E \in K_j$. Then, there exists $m \in \mathbb{Z}_*^\infty$ with the estimate $0 < |m|_\eta < 2N_{j-1}$ such that*

$$\|2\rho(\alpha, A_0 e^{f_0}) - \langle m, \alpha\rangle\|_{\mathbb{R}/\mathbb{Z}} \leq \varepsilon_{j-1}^{1/12}. \tag{4.1}$$

Moreover,

$$\sup_{0 \leq n \leq c\varepsilon_{j-1}^{-1/12}} \|S_{E,n}^V\| \leq c\varepsilon_{j-1}^{-1/160}. \tag{4.2}$$

Proof. For $l \in \mathbb{N}$, we assume that the lth resonance occurs at the $(j_l + 1)$th step of KAM iteration. For a given j, there exists $p \in \mathbb{N}$ such that $j_p + 1 = j$. We just consider the case $p \geq 2$ since in the case $p = 1$, we just take $m = k_{j_1}$.

Now, we assume $p \geq 2$. Set $m = k_{j_p} - \sum_{l=1}^{p-1} k_{j_l}$ and note

$$2\rho(\alpha, A_{j_p} e^{f_{j_p}(x)}) = 2\rho(\alpha, A_0 e^{f_0(x)}) + \sum_{l=1}^{p-1} \langle k_{j_l}, \alpha\rangle.$$

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Moreover, equations (2.7), (3.7), and (3.29) yield

$$\|2\rho(\alpha, A_{j_p}e^{f_{j_p}(x)}) - \langle k_{j_p}, \alpha\rangle\|_{\mathbb{R}/\mathbb{Z}} < 2\varepsilon_{j_p}^{1/12}.$$

The inequalities above imply

$$\|2\rho(\alpha, A_0e^{f_0(x)}) - \langle m, \alpha\rangle\|_{\mathbb{R}/\mathbb{Z}} < 2\varepsilon_{j_p}^{1/12}.$$

Moreover, for the m defined above, equation (3.32) yields

$$|m|_\eta \le \sum_{l=1}^{p} |k_{j_l}|_\eta < \sum_{i=0}^{p-1} N_{j_p}^{((4+3\eta)/(4+4\eta))^i} \le 2N_{j_p}.$$

Now we come to equation (4.2). First, equation (3.30) shows that

$$A_0e^{f_0(x)} = \widetilde{B}_{j-1}(x+\alpha)^{-1}A_je^{f_j(x)}\widetilde{B}_{j-1}(x).$$

Moreover, equation (3.37) implies $\|A_je^{f_j(x)}\| \le 1 + \varepsilon_{j-1}^{1/12}$. The discussions above, together with equation (3.31), yield

$$\sup_{0\le n\le c\varepsilon_{j-1}^{-1/12}} \|S_{E,n}^V\| \le (1+\varepsilon_{j-1}^{1/12})^{c\varepsilon_{j-1}^{-1/12}}\varepsilon_{j-1}^{-1/160} \le c\varepsilon_{j-1}^{-1/160}. \qquad \square$$

LEMMA 4.2. *The integrated density of states is* $\frac{1}{2}$*-Hölder for every* $0 < \varepsilon \le \varepsilon_0^{1/4}$. *Moreover, if* $E \in \Sigma_{V,\alpha}$, *then*

$$L(\alpha, S_E^V) = 0. \tag{4.3}$$

Proof. We prove equation (4.3) first, which can be derived from the inequality

$$\|S_{E,n}^V\| \le n^c, \quad n \ge 1, \quad E \in \Sigma_{V,\alpha}, \tag{4.4}$$

where c is a positive constant. We distinguish the proof of equation (4.4) into two cases.

Case 1. If $(\alpha, A_0e^{f_0(x)})$ is reducible, then there exists $\widetilde{B} \in \mathcal{H}_\sigma(2\mathbb{T}^\infty, SL(2,\mathbb{R}))$ such that

$$\widetilde{B}(x+\alpha)A_0e^{f_0(x)}\widetilde{B}(x)^{-1} = A_\infty.$$

Since $E \in \Sigma_{V,\alpha}$, the cocycle $(\alpha, A_0e^{f_0(x)})$ is not uniformly hyperbolic [46], and hence

$$A_\infty = \begin{pmatrix} e^{i2\pi\rho} & c \\ 0 & e^{-i2\pi\rho} \end{pmatrix}$$

with $\rho \in \mathbb{R}$. Then, $\|A_\infty^n\| \le nc + 1$, which implies that

$$\|S_{E,n}^V(x)\| \le \|A_\infty^n\|\|\widetilde{B}(x)\|^2 \le (nc+1)C \le Cn \quad \text{for all } n \ge 1.$$

Thus, equation (4.3) holds.

Case 2. If $(\alpha, A_0e^{f_0(x)})$ is not reducible but almost reducible, we need the following lemma to describe the growth of the cocycle.

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LEMMA 4.3. *Suppose $E \in \Sigma_{V,\alpha}$, then for all $j \geq 1$, there exists $\widetilde{B}'_{j-1}$, such that*

$$\widetilde{B}'_{j-1}(x+\alpha)A_0 e^{f_0(x)}\widetilde{B}'_{j-1}(x)^{-1} = \begin{pmatrix} e^{i2\pi\rho_j} & c_j \\ 0 & e^{-i2\pi\rho_j} \end{pmatrix} + \widetilde{F}_j(x), \tag{4.5}$$

where $\rho_j \in \mathbb{R}$, with

$$\|\widetilde{F}_j(x)\| \leq \varepsilon_j^{1/4}, \quad \|\widetilde{B}'_{j-1}(x)\| \leq \varepsilon_{j-1}^{-1/320}, \quad \|\widetilde{B}'_{j-1}(x)\|^2|c_j| \leq 8\|A_0\|. \tag{4.6}$$

Proof. The conclusions in part (1) of Proposition 3.7 show that there exists $\widetilde{B}_{j-1} \in \mathcal{H}_{\sigma_j}(2\mathbb{T}^\infty, SL(2,\mathbb{R}))$, such that equation (3.30) holds with equation (3.31). Then, there exists unitary matrices U_j such that

$$U_j A_j e^{f_j(x)} U_j^{-1} = \begin{pmatrix} e^{i2\pi\rho_j} & c_j \\ 0 & e^{-i2\pi\rho_j} \end{pmatrix} + F_j(x),$$

where $\rho_j \in \mathbb{R} \cup i\mathbb{R}$ with $|F_j(x)|_{\sigma_j} < \varepsilon_j$ and the estimates in equations (3.31) and (3.34) hold.

Case 1: $\rho_j \in \mathbb{R}$. Let $\widetilde{B}'_{j-1}(x) = U_j \widetilde{B}_{j-1}(x)$, $\widetilde{F}_j = F_j$, then

$$\widetilde{B}'_{j-1}(x+\alpha)A_0 e^{f_0(x)}\widetilde{B}'_{j-1}(x)^{-1} = \begin{pmatrix} e^{i2\pi\rho_j} & c_j \\ 0 & e^{-i2\pi\rho_j} \end{pmatrix} + \widetilde{F}_j(x).$$

Moreover, the estimates in equation (4.6) are also satisfied.

Case 2: $\rho_j \in i\mathbb{R}$. We first assume $|\rho_j| > \varepsilon_j^{1/4}$. Let

$$Q_j = \begin{pmatrix} q_j & 0 \\ 0 & q_j^{-1} \end{pmatrix},$$

where $q_j = \|\widetilde{B}_{j-1}\|\varepsilon_j^{1/4}$. Then, we have

$$Q_j \left[\begin{pmatrix} e^{i2\pi\rho_j} & c_j \\ 0 & e^{-i2\pi\rho_j} \end{pmatrix} + F_j(x)\right] Q_j^{-1} = \begin{pmatrix} e^{i2\pi\rho_j} & 0 \\ 0 & e^{-i2\pi\rho_j} \end{pmatrix} + F'_j(x),$$

where

$$F'_j(x) = \begin{pmatrix} 0 & c_j q_j^2 \\ 0 & 0 \end{pmatrix} + Q_j F_j(x) Q_j^{-1}.$$

Moreover, equation (3.34) yields

$$|c_j| q_j^2 = |c_j| \|\widetilde{B}_{j-1}\|^2 \varepsilon_j^{1/2} \leq 8\|A_0\|\varepsilon_j^{1/2}, \quad \|Q_j F_j Q_j^{-1}\| \leq \varepsilon_j^{1/2},$$

then we have $\|F'_j\| \leq c\varepsilon_j^{1/2}$. We will show that this implies that the system is uniformly hyperbolic.

More precisely, given a non-zero vector $\binom{a}{b} \in \mathbb{R}^2$ with $|a| \geq |b|$, let

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = \left[\begin{pmatrix} e^{i2\pi\rho_j} & 0 \\ 0 & e^{-i2\pi\rho_j} \end{pmatrix} + F'_j(x)\right]\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} e^{i2\pi\rho_j} a \\ e^{-i2\pi\rho_j} b \end{pmatrix} + F'_j(x)\begin{pmatrix} a \\ b \end{pmatrix}.$$

Without loss of generality, assume $i2\pi\rho_j > 0$, then

$$|a'| \geq (e^{i2\pi\rho_j} - 2\|F_j'\|)|a|, \quad |b'| \leq e^{-i2\pi\rho_j}|b| + 2\|F_j'\||a|.$$

Therefore, $|a'| - |b'| \geq (i4\pi\rho_j - 4c\varepsilon_j^{1/2})|a| > 0$, which, together with the cone field criterion (compare, e.g., [71]), implies $(\alpha, A_0e^{f_0})$ is uniformly hyperbolic. This conflicts with our assumption that $E \in \Sigma_{V,\alpha}$. So we have $|\rho_j| \leq \varepsilon_j^{1/4}$. We put it into the perturbation to obtain the following:

$$\widetilde{B}_{j-1}'(x+\alpha)A_0e^{f_0(x)}\widetilde{B}_{j-1}'(x)^{-1} = \begin{pmatrix} 1 & c_j \\ 0 & 1 \end{pmatrix} + \widetilde{F}_j(x)$$

with $\|\widetilde{F}_j\| \leq \varepsilon_j^{1/4}$. Thus, we get the new cocycle with $\rho_j = 0$ and the perturbation being given above. So we have $\rho \in \mathbb{R}$. □

To control the growth of the cocycle, we need the following lemma proved by Avila, Fayad, and Krikorian [7].

LEMMA 4.4. *We have that*

$$M_l(\mathrm{id} + \xi_l) \cdots M_0(\mathrm{id} + \xi_0) = M^{(l)}(\mathrm{id} + \xi^{(l)}),$$

where $M^{(l)} = M_l \cdots M_0$ *and*

$$\|\xi^{(l)}\| \leqslant e^{\sum_{k=0}^{l} \|M^{(k)}\|^2 \|\xi_k\|} - 1.$$

Now, we come back to the proof of Case 2 of Lemma 4.2. Let

$$M_k = U_jA_jU_j^{-1} = \begin{pmatrix} e^{i2\pi\rho_j} & c_j \\ 0 & e^{-i2\pi\rho_j} \end{pmatrix}, \quad \xi_k = U_jA_j^{-1}U_j^{-1}\widetilde{F}_j(x + k\alpha).$$

Apply Lemma 4.4 to the M_k given above and equation (4.5), so that we obtain

$$S_{E,n}^V(x) = \widetilde{B}_{j-1}'^{-1}(x + (n+1)\alpha)U_jA_j^nU_j^{-1}(\mathrm{id} + \xi^{(n)})\widetilde{B}_{j-1}'(x+\alpha),$$

where $\|\xi^{(n)}\| \leq e^{\sum_{k=1}^{n} \|U_jA_j^kU_j^{-1}\|^2\|\xi_k\|} - 1$. Since $\rho_j \in \mathbb{R}$, we have $\|U_jA_j^kU_j^{-1}\| \leq 1 + k|c_j|$, which, together with $|\widetilde{F}_j| \leq \varepsilon_j^{1/4}$, yields

$$\begin{aligned} \|S_{E,n}^V\| &\leq \|\widetilde{B}_{j-1}'\|^2(1 + n|c_j|) \cdot e^{\sum_{k=1}^{n}(1+k|c_j|)^2(1+|c_j|)\varepsilon_j^{1/4}} \\ &\leq \|\widetilde{B}_{j-1}'\|^2(1 + n|c_j|)e^{n^3\varepsilon_j^{1/4}}. \end{aligned}$$

The inequality above, together with equation (4.6), implies

$$\sup_{0 \leq n \leq c\varepsilon_{j-1}^{-1/12}} \|S_{E,n}^V\| \leq 2\|\widetilde{B}_{j-1}\|^2(1 + n|c_j|) \leq 2\varepsilon_{j-1}^{-1/160} + 16n. \tag{4.7}$$

Since we are at the almost reducible situation, for any fixed large $n \in \mathbb{N}$, there exists i such that $n \in [\varepsilon_{i-1}^{-1/36}, \varepsilon_{i-1}^{-1/12}]$, then equation (4.7) shows that

$$\sup_{0 \leq n \leq c\varepsilon_{i-1}^{-1/12}} \|S_{E,n}^V\| \leq 2n^{9/40} + 16n \leq 17n.$$

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The above equation yields

$$\lim_{n\to\infty}\frac{1}{n}\ln\|S_{E,n}^V(x)\|\leq\lim_{n\to\infty}\frac{1}{n}\ln(17n)=0.$$

So we get equation (4.3) for $E\in\Sigma_{V,\alpha}$.

Next, we will prove the conclusion of Lemma 4.2 about the integrated density of states. To this end, let $I_j\triangleq(\varepsilon_j^{1/4},\varepsilon_{j-1}^{1/15})$, $j\geq 1$. Then, for any small $0<\varepsilon<\varepsilon_0^{1/15}$, there exists j such that $\varepsilon\in I_j$. Moreover, we also need the lemma below.

LEMMA 4.5. *There exists* $W\in\mathcal{H}_\sigma(2\mathbb{T}^\infty,GL(2,\mathbb{C}))$ *such that*

$$Q(x)=W(x+\alpha)A_0e^{f_0(x)}W(x)^{-1},\tag{4.8}$$

with

$$\|W\|\leq\varepsilon^{-1/4},\quad \|Q\|\leq 1+c\varepsilon^{1/2}.\tag{4.9}$$

Proof. Let $\widetilde{B}'_{j-1}$ and $\widetilde{F}_j$ be those in Lemma 4.3, so

$$\widetilde{B}'_{j-1}(x+\alpha)A_0e^{f_0(x)}\widetilde{B}'_{j-1}(x)^{-1}=\begin{pmatrix}e^{i2\pi\rho_j} & c_j\\ 0 & e^{-i2\pi\rho_j}\end{pmatrix}+\widetilde{F}_j(x).$$

Let $D=\operatorname{diag}\{d,\ d^{-1}\}$, $W(x)=D\widetilde{B}'_{j-1}$ with $d=\varepsilon^{1/4}\|\widetilde{B}'_{j-1}\|$. Then, we have

$$W(x+\alpha)A_0e^{f_0(x)}W(x)^{-1}=\begin{pmatrix}e^{i2\pi\rho_j} & 0\\ 0 & e^{-i2\pi\rho_j}\end{pmatrix}+\widetilde{F}'_j(x),$$

where

$$\widetilde{F}'_j(x)=\begin{pmatrix}0 & \|\widetilde{B}'_{j-1}\|^2\varepsilon^{1/2}c_j\\ 0 & 0\end{pmatrix}+D\widetilde{F}_jD^{-1}.$$

Since $\varepsilon\in I_j$, we have $d<1$, which implies $\|W\|\leq\varepsilon^{-1/4}$. Moreover, equation (4.6) and the fact that $\varepsilon\in I_j$ imply $\|\widetilde{F}'_j(x)\|<c\varepsilon^{1/2}$ and $\rho_j\in\mathbb{R}$. Let

$$Q(x)=W(x+\alpha)A_0e^{f_0(x)}W(x)^{-1},$$

then $\|Q\|\leq 1+c\varepsilon^{1/2}$. □

Now, we prove conclusions about the integrated density of states. Notice

$$L(\alpha,S_E^V)=\lim_{n\to\infty}\frac{1}{n}\ln\|S_{E,n}^V(x)\|.$$

Abbreviate $L(\alpha,S_E^V)$ to $L(E,V)$ and by the Thouless formula, we obtain

$$\begin{aligned}L(E+i\varepsilon,V)-L(E,V)&=\int_{\mathbb{R}}\frac{1}{2}\ln\left(1+\frac{\varepsilon^2}{(E'-E)^2}\right)\mathrm{d}N(E')\\&\geq\int_{E-\varepsilon}^{E+\varepsilon}\left(\frac{1}{2}\ln 2\right)\mathrm{d}N(E'),\\&\geq\frac{1}{2}\ln 2(N(E+\varepsilon)-N(E-\varepsilon)).\end{aligned}\tag{4.10}$$

So it is enough to show that for $0<\varepsilon\leq\varepsilon_0^{1/4}$, $L(E+i\varepsilon)<C\varepsilon^{1/2}$.

Notice that

$$S^V_{E+i\varepsilon}(x) = \begin{pmatrix} E + i\varepsilon - V(x) & -1 \\ 1 & 0 \end{pmatrix} = S^V_E(x) + \begin{pmatrix} i\varepsilon & 0 \\ 0 & 0 \end{pmatrix}$$

and there exists I_j such that $\varepsilon \in I_j$, then by equations (4.8) and (4.9), we get

$$\begin{aligned} W(x+\alpha)S^V_{E+i\varepsilon}(x)W(x)^{-1} &= Q(x) + W(x+\alpha)\begin{pmatrix} i\varepsilon & 0 \\ 0 & 0 \end{pmatrix} W(x)^{-1}, \\ &\triangleq \widetilde{S}(x). \end{aligned}$$

The Lyapunov exponent is clearly invariant under conjugacies, so we have $L(E+i\varepsilon, V) = L(\alpha, S^V_{E+i\varepsilon}) = L(\alpha, \widetilde{S})$. Thus,

$$\|\widetilde{S}(x)\| \le 1 + c\varepsilon^{1/2} + \varepsilon \cdot \varepsilon^{-1/2} = 1 + c\varepsilon^{1/2},$$

which implies

$$L(\alpha, \widetilde{S}) = \lim_{n\to\infty} \frac{1}{n} \ln \|\widetilde{S}_n(x)\| \le \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln \|\widetilde{S}(x + (i-1)\alpha)\| \le c\varepsilon^{1/2},$$

which, together with equation (4.10), implies that integrated density of states is $\frac{1}{2}$- Hölder continuous. □

LEMMA 4.6. [1] *We have* $\mu(E - i\varepsilon, E + i\varepsilon) \le C\varepsilon \sup_{0\le m\le C\varepsilon^{-1}} \|S^V_{E,m}\|^2$, *where* $C > 0$ *is a universal constant.*

LEMMA 4.7. [1] *If* $E \in \Sigma_{V,\alpha}$, *then for* $0 < \varepsilon \le \varepsilon_0^{1/4}$, $N(E+\varepsilon) - N(E-\varepsilon) \ge c\varepsilon^{3/2}$.

Let $\mathcal{B}'$ be the set of $E \in \Sigma_{V,\alpha}$ such that $(\alpha, S^V_E(x))$ is bounded, and $\mathcal{R}'$ be the set of $E \in \Sigma_{V,\alpha}$ such that $(\alpha, S^V_E(x))$ is reducible. Then, we have two lemmas below, which are the basis of the proof of Theorem 1.2.

LEMMA 4.8. [1] *Let* $\mathcal{B}'$ *be the set of* $E \in \mathbb{R}$ *such that the cocycle* (α, S^V_E) *is bounded. Then,* $\mu_{V,\alpha,x}|\mathcal{B}'$ *is absolutely continuous for all* $x \in \mathbb{R}$.

LEMMA 4.9. *If* $E \in \mathcal{R}' \setminus \mathcal{B}'$, *then there exists a unique* $k \in \mathbb{Z}^\infty_*$ *such that* $2\rho(\alpha, S^V_E(x)) = \langle k, \alpha\rangle \bmod 1$.

Proof. If $E \in \mathcal{R}'$, then there exist at most finitely many resonance sites $k_{j_0}, \ldots, k_{j_l}$. Since there exists $B \in \mathcal{H}_\sigma(2\mathbb{T}^\infty, SL(2,\mathbb{R}))$ such that

$$B(x+\alpha)A_0 e^{f_0(x)} B(x)^{-1} = A_\infty,$$

with $\|B\| \le 2\varepsilon_{j_l}^{-1/320}$, then by equation (3.35) and $E \notin \mathcal{B}'$, we have

$$\rho(\alpha, A_\infty) = \rho(\alpha, A_0 e^{f_0(x)}) + \sum_{i=0}^{l} \frac{\langle k_{j_i}, \alpha\rangle}{2} = 0,$$

which yields $2\rho(\alpha, S^V_E(x)) = \langle k, \alpha\rangle \bmod 1$ for some $k \in \mathbb{Z}^\infty_*$. □

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4.2. *Proof of Theorem 1.2.* Once we get the conclusions in §4.1, we are ready to prove Theorem 1.2. First, Lemma 4.8 shows that to prove Theorem 1.2, it is enough to prove $\mu(\Sigma_{V,\alpha} \setminus \mathcal{B}') = 0$. Moreover, Lemma 4.9 shows that $\mathcal{R}' \setminus \mathcal{B}'$ is countable. The definition of $\mathcal{R}'$ implies that if $E \in \mathcal{R}'$, any non-zero solution $H_{V,\alpha,x}u = Eu$ satisfies $\inf_{n\in\mathbb{Z}}\{|u_n|^2 + |u_{n+1}|^2\} > 0$. Thus, if, furthermore, $E \in \mathcal{R}' \setminus \mathcal{B}'$, then $u \notin \ell^2(\mathbb{Z})$. That is, there are no eigenvalues in $\mathcal{R}' \setminus \mathcal{B}'$ and $\mu(\mathcal{R}' \setminus \mathcal{B}') = 0$. So it is enough to prove that $\mu(\Sigma_{V,\alpha} \setminus \mathcal{R}') = 0$.

Actually, $E \in \Sigma_{V,\alpha} \setminus \mathcal{R}'$ and $\mu(\Sigma_{V,\alpha} \setminus \mathcal{R}') = 0$ are equivalent to $E \in \limsup_{j\to\infty} K_j$ and $\mu(\limsup_{j\to\infty} K_j) = 0$, respectively. Moreover, by the Borel–Cantelli lemma, $\mu(\limsup_{j\to\infty} K_j) = 0$ is equivalent to $\sum_j \mu(\overline{K_j}) < \infty$. So, the last thing is to show

$$\sum_j \mu(\overline{K_j}) < \infty. \tag{4.11}$$

We will devote ourselves to proving equation (4.11) in the rest of this section.

Let $J_j(E) = (E - c\varepsilon_{j-1}^{1/18}, E + c\varepsilon_{j-1}^{1/18})$, $j \geq 1$. Lemma 4.6 and $E \in K_j$ yield

$$\mu(J_j(E)) \leq c\varepsilon_{j-1}^{1/18} \sup_{0\leq n\leq c\varepsilon_{j-1}^{-1/18}} \|S_{E,n}^V\|^2 \leq c\varepsilon_{j-1}^{31/720}. \tag{4.12}$$

Thus, $\overline{K_j} \subset \bigcup_{l=0}^r J_j(E_l)$ since $\overline{K_j}$ is compact, where $E_l \in K_j$, $l = 0, \ldots, r$. Refine this subcover if necessary, then we can assume that any $x \in \mathbb{R}$ is contained in at most two different $J_j(E_l)$. Thus, $N(\overline{K_j}) \subset \bigcup_{i=0}^r N(J_j(E_i))$.

We see that $N(E) = 1 - 2\rho(\alpha, S_E^V)$ and equation (4.1) yield

$$\|N(E) - \langle m, \alpha\rangle\|_{\mathbb{R}/\mathbb{Z}} \leq \varepsilon_{j-1}^{1/12} \quad \text{for all } E \in K_j. \tag{4.13}$$

For $m(l) \in \mathbb{Z}_*^\infty$ satisfying equation (4.13), set

$$T_l = (\langle m(l), \alpha\rangle - \varepsilon_{j-1}^{1/12}, \langle m(l), \alpha\rangle + \varepsilon_{j-1}^{1/12}). \tag{4.14}$$

For the fixed T_l defined above, we have

$$|T_l| = 2\varepsilon_{j-1}^{1/12} = 2c^{-1}|J_j(E)|^{3/2} \leq 2c^{-1}|N(J_j(E))| \ \text{ for all } E \in K_j, \tag{4.15}$$

where the last inequality is by Lemma 4.7. For the constant c in equation (4.15), set $N_c = [4(c^{-1} + 1)] + 1$, where $[\cdot]$ denotes the integer part. Then, for the fixed T_l, select $\{E_{l,s}\}_{s=1}^{N_c} \subset K_j$ such that $N(J_j(E_{l,s}))$, $s = 1, \ldots, N_c$, intersects T_l with $J_j(E_{l,s}) \in \{J_j(E_i)\}_{i=0}^r$. Thus,

$$N(\overline{K_j}) \subset \bigcup_{i=0}^r N(J_j(E_i)) \subset \bigcup_{l=0}^{\widetilde{N}} \bigcup_{s=1}^{N_c} N(J_j(E_{l,s})), \tag{4.16}$$

where $\widetilde{N}$ is the number of such $m(l)$ satisfying equation (4.13).

Note for fixed $E_i \in K_j$, there exists $l \in \mathbb{N}$ such that $N(E_i) \in T_l$, and thus $N(J_j(E_i))$ intersects T_l, which, together with the selection of $\{E_{l,s}\}_{s=1}^{N_c}$ above, yields the second relation in equation (4.16).

We, in the lemma below, give the upper bound of $\widetilde{N}$, which is the number of such $m(l)$ satisfying equation (4.13).

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LEMMA 4.10. *$\widetilde{N}$ satisfies the following inequality:*

$$\widetilde{N} < \varepsilon_{j-1}^{-1/40}. \tag{4.17}$$

Proof. For $E \in K_j$, Lemma 4.1 implies that there exists $m \in \mathbb{Z}_*^\infty$ such that equation (4.1) holds (thus, this m satisfies equation (4.13) since $N(E) = 1 - 2\rho(\alpha, S_E^V)$) with

$$|m|_\eta = \sum_{s\in\mathbb{N}} \langle s\rangle^\eta |m_s| < 2N_{j-1}. \tag{4.18}$$

Moreover, we assume j is large enough such that $1 \leq \eta < \sqrt{1 + \log_2 N_{j-1}} - 1$.

For the m satisfying equation (4.18), denote $\widetilde{M}_m$ by the number of its non-zero components. Then, we have the following estimates:

$$2N_{j-1} \geq |m|_\eta = \sum_{j=1}^{\widetilde{M}_m} \langle i_j\rangle^\eta |m_{i_j}| \geq \sum_{j=1}^{\widetilde{M}_m} \langle i_j\rangle^\eta \geq \sum_{j=1}^{\widetilde{M}_m} j^\eta \approxeq \frac{1}{1+\eta}\widetilde{M}_m^{1+\eta},$$

and thus,

$$\widetilde{M}_m \leq (2\eta+2)^{1/(1+\eta)}(2N_{j-1})^{1/(1+\eta)} < 2(2N_{j-1})^{1/(1+\eta)} < (2N_{j-1})^{1/\eta}, \tag{4.19}$$

where the last inequality is by $\eta < \sqrt{1 + \log_2 N_{j-1}} - 1$. The second inequality above shows that $2(2N_{j-1})^{1/(1+\eta)}$ is the uniform bound of the $\widetilde{M}_m$ for all m given by equation (4.13). Set C_n^j, $n \geq j$, to be the notation of combination, which denotes the prescribed size of taking j numbers from the given set $\{1, 2, \ldots, n\}$.

Now, we are ready for proving the inequality in equation (4.17). First, equation (4.18) shows that $m_s = 0$ for $s \geq (2N_{j-1})^{1/\eta}$. Thus, we are restricted to the $(2N_{j-1})^{1/\eta}$-dimensional tori, which, for the fixed $j \in \mathbb{N}$, is finite dimensional. Moreover, equation (4.19) shows that the uniform bound of the number of non-zero components of m satisfying equation (4.13) is $2(2N_{j-1})^{1/(1+\eta)}$. That is, there are at least $(2N_{j-1})^{1/\eta} - 2(2N_{j-1})^{1/(1+\eta)}$ many components being zero. So for these m satisfying equation (4.13), we just need to consider its $2(2N_{j-1})^{1/(1+\eta)}$ many components, which are allowed to be non-zero. Second, equation (4.18) also yields $|m_s| < 2N_{j-1}|s|^{-\eta}$, $s \geq 1$. Thus, for fixed $s \leq (2N_{j-1})^{1/\eta}$, m_s can take any of the values below:

$$\{\pm[2N_{j-1}|s|^{-\eta}], \pm([2N_{j-1}|s|^{-\eta}] - 1), \ldots, \pm 2, \pm 1, 0\}.$$

Set

$$\{n_1, n_2, \ldots, n_{2(2N_{j-1})^{1/(1+\eta)}}\}. \tag{4.20}$$

We consider these m values, which satisfy equation (4.13) and whose indexes set of $2(2N_{j-1})^{1/(1+\eta)}$ many components, which may not be zero, is set above. By the discussions above, we know that the number of these m is less than

$$\prod_{s=1}^{2(2N_{j-1})^{1/(1+\eta)}} (4N_{j-1}\langle n_s\rangle^{-\eta} + 1).$$

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Note the elements of the index set defined in equation (4.20) are taken from the set $\{0, 1, \ldots, (2N_{j-1})^{1/\eta}\}$, and thus we have the estimate

$$\begin{aligned}\widetilde{N} &< C_{(2N_{j-1})^{1/\eta}}^{2(2N_{j-1})^{1/(1+\eta)}} \prod_{s=1}^{2(2N_{j-1})^{1/(1+\eta)}} (4N_{j-1}\langle n_s\rangle^{-\eta}+1)\\ &\le C_{(2N_{j-1})^{1/\eta}}^{2(2N_{j-1})^{1/(1+\eta)}} \prod_{s=0}^{2(2N_{j-1})^{1/(1+\eta)}} (4N_{j-1}\langle s\rangle^{-\eta}+1) < \varepsilon_{j-1}^{-1/40},\end{aligned}$$

where the second inequality is by the fact that $\langle n_s\rangle^{-\eta} \le \langle s\rangle^{-\eta}$ and the last inequality is by the second inequality in equation (3.9). □

Now, we continue the estimation of $\mu(\overline{K_j})$. First, equation (4.16) shows that there are at most $N_c\widetilde{N}$ intervals $J_j(E_i)$. Then equations (4.16) and (4.17) yield

$$\mu(\overline{K_j}) \le \sum_{i=0}^{r} \mu(J_j(E_i)) < N_c\varepsilon_{j-1}^{-1/40} \cdot c\varepsilon_{j-1}^{31/720} \le c\varepsilon_{j-1}^{13/720},$$

where the second inequality is by equation (4.12).

5. *Homogeneous spectrum*

In this section, we will prove our main result about the homogeneous spectrum, that is, Theorem 1.4.

5.1. *Gap estimate.* The proof of Theorem 1.4 is based on the gap estimates via the Moser–Pöschel argument, see Theorem 5.1. Thus, we will give a brief introduction about the gap estimate and the conclusion in our setting.

In [53], Moser and Pöschel consider a continuous quasi-periodic Schrödinger operator with small analytic potential V and show that $|G_k(V)|$ is exponentially small with $|k|$ for large enough $|k|$. Later, in [36, 59], Eliasson and Puig follow the work in [53] and show $G_m(V)$ is at least sub-exponentially small with respect to m under some hypotheses. In [43, 60], Hadj-Amor, and Shi and Yuan consider the discrete Schrödinger operator and the extended Harper model, showing the sub-exponential smallness with respect to m of $G_m(V)$. Damanik and Goldstein, in [26], give a sharper upper bound $2\varepsilon e^{-r_0|m|/2}$ of $|G_k(V)|$. Leguil *et al*, in [48], give the upper bound of $|G_k(V)|$ with the Liouvillean frequency, and, in the Diophantine frequency case, the authors extend the result in [26] by giving a sharper upper bound $\varepsilon^{2/3}e^{-2\pi r|m|}$ for all $r \in (0, r_0)$. In [24, 50], Damanik, and Liu and Yuan prove the gap labeling theorem for ergodic Schrödinger operators and give the estimate of spectral gaps of AMO in the exponential regime. Our result is the following.

THEOREM 5.1. *Let $\alpha \in DC_\infty(\gamma, \tau)$ and $V \in \mathcal{H}_{\sigma_0}(\mathbb{T}^\infty, \mathbb{R})$ with $\eta > 0$. There exists $0 < \varepsilon_0(\gamma, \tau, \eta, \sigma_0) < 1$ such that if $|V|_{\sigma_0} \le \varepsilon_0$, then for the discrete Schrödinger operator $H_{V,\alpha,x}$, we have $|G_k(V)| \le \varepsilon_0^{3/4}e^{-(5/4)\pi\sigma_0|k|_\eta}$.*

For any $r_0 > 0$, under the Diophantine frequency case, the estimate of gap in [48] is $\varepsilon^{2/3}e^{-2\pi r|m|}$ for all $r \in (0, r_0)$. However, the weight in our Theorem 5.1 is $\frac{5}{4}\pi\sigma_0$. The

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main reason that we lose much more analytic radius is that, with the infinite dimensional frequency, the estimate of the resonant site given by equation (3.32) is much worse than that in [48]. Consequently, we can only get much worse estimates about ζ and $\widetilde{B}$ in part (3) of Proposition 3.7.

5.2. *Proof of Theorem 1.4.* We will apply the estimate in Theorem 5.1 to prove Theorem 1.4. Set

$$\underline{E} = \min \Sigma_{V,\alpha}, \quad \overline{E} = \max \Sigma_{V,\alpha}, \quad G_0(V) = (-\infty, \underline{E}) \cup (\overline{E}, +\infty).$$

Moreover, given any $E \in \Sigma_{V,\alpha}$ and any $\varepsilon > 0$, set

$$\mathcal{M} = \mathcal{M}(E, \varepsilon) \triangleq \{m \in \mathbb{Z}_*^\infty \setminus \{0\} \mid G_m(V) \cap (E-\varepsilon, E+\varepsilon) \neq \varnothing\},$$

and let $m_0 \in \mathcal{M}$ be such that $|m_0|_\eta = \min_{m\in\mathcal{M}} |m|_\eta$. Since $E \in \Sigma_{V,\alpha}$, it is obvious that

$$|G_{m_0}(V) \cap (E-\varepsilon, E+\varepsilon)| \leq \varepsilon,$$
$$|(-\infty, \underline{E}) \cap (E-\varepsilon, E+\varepsilon)| \leq \varepsilon,$$
$$|(\overline{E}, +\infty) \cap (E-\varepsilon, E+\varepsilon)| \leq \varepsilon.$$

Moreover, by the definition of $\mathcal{M}$, we know that

$$\operatorname{dist}(G_m(V), G_{m'}(V)) \leq 2\varepsilon \quad \text{for all } m, m' \in \mathcal{M}. \tag{5.1}$$

Now, we assume $0 < \varepsilon \leq \varepsilon_0$ and separate the discussions into the following three cases.

Case 1: $G_0(V) \cap (E-\varepsilon, E+\varepsilon) = \varnothing$. Consider two different gaps $G_m(V)$ and $G_{m'}(V)$. Without loss of generality, we assume that $E_m^+ \leq E_{m'}^-$. Hence,

$$\operatorname{dist}(G_m(V), G_{m'}(V)) = E_{m'}^- - E_m^+.$$

Thus, by Lemma 4.2, the equality above with equation (5.1), we have

$$|N(E_{m'}^-) - N(E_m^+)| \leq c(E_{m'}^- - E_m^+)^{1/2},$$

where c is independent of E. Since $\alpha \in DC_\infty(\gamma, \tau)$, according to equation (2.6) and Lemma 4.9, gap labeling [46], we have

$$|N(E_{m'}^-) - N(E_m^+)| = \|\langle m'-m, \alpha\rangle\|_{\mathbb{R}/\mathbb{Z}} \geq \gamma(1+|m'-m|_\eta)^{-c(\eta,\tau)(|m'-m|_\eta)^{1/(1+\eta)}}.$$

The three inequalities above show, for all $m' \neq m \in \mathbb{Z}_*^\infty \setminus \{0\}$,

$$2\varepsilon \geq \operatorname{dist}(G_m(V), G_{m'}(V)) \geq \frac{\gamma^2}{c^2}(1+|m'-m|_\eta)^{-2c(\eta,\tau)(|m'-m|_\eta)^{1/(1+\eta)}}. \tag{5.2}$$

By equation (5.2) we get, for all $m \in \mathcal{M} \setminus \{m_0\}$,

$$2\varepsilon \geq \operatorname{dist}(G_m(V), G_{m_0}(V)) \geq \frac{\gamma^2}{c^2}(1+|2m|_\eta)^{-2c(\eta,\tau)(|2m|_\eta)^{1/(1+\eta)}}, \tag{5.3}$$

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which, together with $0 < \varepsilon \le \varepsilon_0$, yields $|m|_\eta > c_1 \ln(1/\varepsilon)$ with $c_1 = c_1(\eta, \tau, \gamma)$. Moreover, Theorem 5.1 shows $|G_m(V)| \le \varepsilon_0^{3/4} e^{-(5/4)\pi\sigma_0|m|_\eta}$. Thus, with direct calculations, we get

$$\sum_{m\in\mathcal{M}\setminus\{m_0\}} |G_m(V)\cap(E-\varepsilon, E+\varepsilon)| \le \sum_{m\in\mathcal{M}\setminus\{m_0\}} |G_m(V)| \le \sum_{|m|_\eta > c_1\ln(1/\varepsilon)} \varepsilon_0^{3/4} e^{-(5/4)\pi\sigma_0|m|_\eta} \le \frac{\varepsilon}{4},$$

where the last inequality is by $\varepsilon \le \varepsilon_0$ and with the similar calculations in the proof of Lemma 4.10, we omit the details. Then we have, for any $0 < \varepsilon \le \varepsilon_0$,

$$|\Sigma_{V,\alpha}\cap(E-\varepsilon, E+\varepsilon)| \ge 2\varepsilon - |G_{m_0}(V)\cap(E-\varepsilon, E+\varepsilon)| - \sum_{m\in\mathcal{M}\setminus\{m_0\}} |G_m(V)\cap(E-\varepsilon, E+\varepsilon)| \ge \frac{3}{4}\varepsilon.$$

Case 2: $(-\infty, \underline{E})\cap(E-\varepsilon, E+\varepsilon) \ne \varnothing$. In this case, for any $m \in \mathcal{M}$,

$$|E_m^- - \underline{E}| \le 2\varepsilon. \tag{5.4}$$

Using the same way to get equation (5.3), we also get

$$|E_m^- - \underline{E}| \ge \frac{\gamma^2}{c^2}(1+|m|_\eta)^{-2c(\eta,\tau)(|m|_\eta)^{1/(1+\eta)}}. \tag{5.5}$$

Then, equations (5.4) and (5.5) imply $|m|_\eta > 2c_1 \ln(1/\varepsilon)$. Similarly, we have

$$\sum_{m\in\mathcal{M}} |G_m(V)\cap(E-\varepsilon, E+\varepsilon)| \le \sum_{|m|_\eta > 2c_1\ln(1/\varepsilon)} \varepsilon_0^{3/4} e^{-(5/4)\pi\sigma_0|m|_\eta} \le \frac{\varepsilon}{4},$$

provided $\varepsilon \le \varepsilon_0$. So we have, for any $0 < \varepsilon \le \varepsilon_0$,

$$|\Sigma_{V,\alpha}\cap(E-\varepsilon, E+\varepsilon)| \ge 2\varepsilon - |(-\infty, \underline{E})\cap(E-\varepsilon, E+\varepsilon)| - \sum_{m\in\mathcal{M}} |G_m(V)\cap(E-\varepsilon, E+\varepsilon)| \ge \frac{3}{4}\varepsilon.$$

Case 3: $(\overline{E}, +\infty)\cap(E-\varepsilon, E+\varepsilon) \ne \varnothing$. The proof is similar to that of Case 2, so we omit the details.

Finally, we get

$$|\Sigma_{V,\alpha}\cap(E-\varepsilon, E+\varepsilon)| \ge \tfrac{3}{4}\varepsilon \quad \text{for all } 0 < \varepsilon \le \varepsilon_0, \text{ for all } E \in \Sigma_{V,\alpha}.$$

As for the case $\varepsilon \in (\varepsilon_0, \operatorname{diam}\Sigma_{V,\alpha})$, we have

$$|\Sigma_{V,\alpha}\cap(E-\varepsilon, E+\varepsilon)| \ge \frac{3}{4}\varepsilon_0 \ge \frac{3\varepsilon_0}{4\operatorname{diam}\Sigma_{V,\alpha}}\varepsilon.$$

Choose $\mu = \min\{3/4, 3\varepsilon_0/4\operatorname{diam}\Sigma_{V,\alpha}\}$ and this concludes the proof.

5.3. *Proof of Theorem 5.1.* In this subsection, we give the proof of Theorem 5.1. To this end, we give the lemma below.

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LEMMA 5.2. *Let* $\alpha \in DC_\infty(\gamma, \tau), \kappa \in (0, \frac{1}{4})$, *and* $V \in \mathcal{H}_{\sigma_0}(\mathbb{T}^\infty, \mathbb{R}), \eta > 0$, *be a non-constant function. Assume that* E_k^+ *is a right edge point of the spectral gap* $G_k(V)$, *and there are* $\zeta \in (0, \frac{1}{2})$ *and* $B \in \mathcal{H}_\sigma(2\mathbb{T}^\infty, SL(2, \mathbb{R}))$ *such that*

$$B(x+\alpha)^{-1} S^V_{E_k^+}(x) B(x) = X := \begin{pmatrix} 1 & \zeta \\ 0 & 1 \end{pmatrix}. \tag{5.6}$$

If

$$|B|_\sigma^6 \zeta^\kappa \leq \frac{1}{16\widetilde{C}^2}, \tag{5.7}$$

then $|G_k(V)| \leq \zeta^{1-\kappa}$.

We postpone the proof of Lemma 5.2 to Appendix and now we apply Lemma 5.2 to prove Theorem 5.1. First, according to the conclusions in part (3) of Proposition 3.7, we have

$$\begin{aligned} |B|^6_{\sigma_0/24}\zeta^{1/5} &\leq 4^6(\varepsilon_{jl})^{-3/160} e^{(48/7)\pi(\sigma_0/24)|k|_\eta} \cdot 4^{1/5}\varepsilon_{jl}^{1/5-1/600} e^{-(5/16)\pi\sigma_0|k|_\eta}, \\ &\leq 4^{6+(1/5)}\varepsilon_{jl}^{6/25-49/2400} e^{-(3/112)\pi\sigma_0|k|_\eta} \leq (4\widetilde{C})^{-2}. \end{aligned}$$

Thus, we can apply the conclusion of Lemma 5.2 and get

$$|G_k(V)| \leq \zeta^{4/5} \leq 4^{4/5}\varepsilon_{jl}^{4/5-4s/50} e^{-(5/4)\pi\sigma_0|k|_\eta} \leq \varepsilon_0^{3/4} e^{-(5/4)\pi\sigma_0|k|_\eta}.$$

6. *Deift's conjecture—proof of Theorem 1.1*

To prove Theorem 1.1, we describe the spectral conditions of Jacobi operator induced by the Toda lattice in equation (1.1) based on the discussions in [63, 64, 69].

Set

$$(P(t)u)_n := -a_{n-1}(t)u_{n-1} + a_n(t)u_{n+1}, \tag{6.1}$$

and cite $J(t)$ defined by equation (1.2),

$$(J(t)u)_n = a_{n-1}(t)u_{n-1} + b_n(t)u_n + a_n(t)u_{n+1}. \tag{6.2}$$

Then, equation (1.1) can be rewritten as a Lax pair:

$$\frac{d}{dt}J(t) = P(t)J(t) - J(t)P(t). \tag{6.3}$$

Consider the self-adjoint almost periodic Jacobi matrix J defined by equation (6.2) and denote by $J_0 = J(0)$ the Jacobi operator corresponding to the initial data $(a_n, b_n)(0) = (1, V(x+n\alpha))$. The spectrum $\Sigma := \sigma(J_0)$ is compact, and thus, can be denoted as $\Sigma = [\inf \Sigma, \sup \Sigma] \setminus \bigcup_{k\in\mathbb{Z}} G_k(V)$.

We assume Σ is homogeneous, and let J_Σ and $\pi_1(\mathbb{C}\setminus\Sigma)$ be the class of reflectionless Jacobi matrices with spectrum Σ and the fundamental group of $\mathbb{C}\setminus\Sigma$, respectively. By $\pi_*(\mathbb{C}\setminus\Sigma)$, we denote the group of unimodular characters of $\pi_1(\mathbb{C}\setminus\Sigma)$ endowed with the topology induced by the metric

$$d(\omega, \widetilde{\omega}) = \sum_{j\in I} \min\{|\omega_j - \widetilde{\omega}_j|, |G_j|\}, \quad \omega, \widetilde{\omega} \in \pi_*(\mathbb{C}\setminus\Sigma),$$

where $|G_j| = E_j^+ - E_j^-$. Furthermore, we will use the additive form of notation for the compact abelian group $\pi_*(\mathbb{C} \setminus \Sigma)$:

$$\pi_*(\mathbb{C} \setminus \Sigma) = \{K : \pi_1(\mathbb{C} \setminus \Sigma) \to \mathbb{T},\\ K(\gamma_1\gamma_2) = K(\gamma_1)K(\gamma_2),\ \gamma_j \in \pi_1(\mathbb{C} \setminus \Sigma),\ j = 1, 2\}.$$

Note $\pi_1(\mathbb{C} \setminus \Sigma)$ is a free group admitting a set of generators $\{c_k\}_{k\in\mathbb{Z}}$, where c_k is a counterclockwise simple loop intersecting $\mathbb{R}$ at $\inf \Sigma - 1$ and $2^{-1}(E_k^+ + E_k^-)$. Moreover, any $K \in \pi_*(\mathbb{C} \setminus \Sigma)$ is uniquely determined by its action on loops c_k, so it can be written as $K = (K(c_k))_{k\in\mathbb{Z}} = (e^{i2\pi \tilde{K}_k})_{k\in\mathbb{Z}}$. See the discussions in [**14**, **48**, **63**, **64**] for details.

THEOREM 6.1. [**63**] *There is a continuous one-to-one correspondence between almost periodic Jacobi matrices $J \in J_\Sigma$ and characters $\mathcal{K} \in \pi_*(\mathbb{C} \setminus \Sigma)$.*

Now, we consider the Jacobi matrix J given in equation (6.2) with $(a, b) \in \ell^\infty(\mathbb{Z}) \times \ell^\infty(\mathbb{Z})$, then Theorem 6.1 implies that there exists a continuous map $\mathcal{H} : \mathbb{T}^\infty \to \ell^\infty(\mathbb{Z}) \times \ell^\infty(\mathbb{Z})$ such that one can find a unique $\mathcal{K} \in \pi_*(\mathbb{C} \setminus \Sigma)$ with

$$(a, b) = \mathcal{H}((K(c_k))_{k\in\mathbb{Z}}) = \mathcal{H}((e^{-i2\pi \tilde{K}_k})_{k\in\mathbb{Z}}). \tag{6.4}$$

We will consider a more general Lax pair rather than equation (6.3). More concretely, for $\Sigma \subset X$ and any $g \in L^\infty(X, \mathbb{R})$, we define the infinite dimensional matrix $g(J)$ in the sense of standard functional calculus and decompose it into $g^+(J) + g^-(J)$, where $g^+(J)(g^-(J))$ is an upper triangular matrix (a lower triangular matrix). Moreover, set $M_g(J) := g^+(J) - g^-(J)$. Thus, for any almost periodic Jacobi matrix $J_0 \subset J_\Sigma$, we can define the Lax pair

$$\frac{d}{dt}J(t) = M_g(J)(t)J(t) - J(t)M_g(J)(t), \quad J(0) = J_0. \tag{6.5}$$

THEOREM 6.2. [**69**] *Assume that Σ is homogeneous and the almost periodic Jacobi matrix $J_0 \in J_\Sigma$ has pure absolutely continuous spectrum. Given $g \in L^\infty(X, \mathbb{R})$ with $\Sigma \in X$, the following hold.*

(1) *There exists a unique solution $J = J(t)$ of equation (6.5), well defined for all $t \in \mathbb{R}$. Moreover, for every t, $J(t)$ is an almost periodic Jacobi matrix with constant spectrum Σ.*
(2) *For $t \in \mathbb{R}$, let $\mathcal{K}^t \in \pi_*(\mathbb{C} \setminus \Sigma)$ be the character corresponding to $J(t)$. There exists a homomorphism $\xi : \pi_1(\mathbb{C} \setminus \Sigma) \to \mathbb{R}$, depending on g, such that $\mathcal{K}^t(c_k) = \mathcal{K}^0(c_k)e^{-i2\pi t\xi(c_k)}$.*

Now, we give the proof of Theorem 1.1. By setting $g(x) = x$ and assuming that all the diagonal elements of $M_g(J)$ vanish, we get the Lax pair in equation (6.3) or, with further discussions, the Toda flow in equation (1.1). Set $0 < \varepsilon_* \leq \varepsilon_0$, where ε_0 is defined by equation (3.25). Thus, the conclusions of Proposition 3.7 hold. Consequently, the conclusions of Theorems 1.2 and 1.4 hold. That is, the spectrum set $\Sigma_{V,\alpha}$ of operator

$H_{V,\alpha,x}$ is homogeneous and purely absolute continuity. Moreover, $L_{V,\alpha}(E) = 0$, which implies

$$m^{+}_{H_{V,\alpha,x}} = -\overline{m^{-}_{H_{V,\alpha,x}}}. \tag{6.6}$$

Later, in [1, Theorem 2.2], Avila shows that the equality in equation (6.6) holds for all $x \in \mathbb{T}^\infty$, which, together with equation (2.3), implies that $H_{V,\alpha,x}$ is reflectionless for every $x \in \mathbb{T}^\infty$. (In [1], the base point $x \in \mathbb{T}$. However, the proof of [1, Theorem 2.2] is based on Kotani theory, which also holds in the $\mathbb{T}^\infty$ setting. So, the conclusions in Theorem 2.2 will also hold in $\mathbb{T}^\infty$). Thus, Theorems 6.1 and 6.2 are applied. The assertion of part (1) of Theorem 6.2 yields the conclusions (1) and (2) of Theorem 1.1. Moreover, the conclusion of part (2) of Theorem 6.2 and equation (6.4) also yield

$$(a(t), b(t)) = \mathcal{H}((e^{-i2\pi \tilde{K}^t_k})_{k\in\mathbb{Z}}) = \mathcal{H}((e^{-i2\pi[\tilde{K}^0_k + t\xi(c_k)]})_{k\in\mathbb{Z}}), \tag{6.7}$$

which implies the time almost periodicity of solutions of equation (1.1). Thus, the conclusion (3) of Theorem 1.1 is shown with $\mathcal{M} = \mathcal{H}$.

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Appendix. Proof of Lemma 5.2

In this section, we will give the proof of Lemma 5.2. We start from $(\alpha, S^V_{E^+_k}(x))$ and conclusion (3) of Proposition 3.7 shows that the cocycle $(\alpha, S^V_{E^+_k})$ is reducible, that is, there are $B \in \mathcal{H}_\sigma(2\mathbb{T}^\infty, SL(2,\mathbb{R}))$ for some $0 < \sigma < \sigma_0$ and a constant matrix X, such that equation (5.6) holds with $0 \le \zeta < \frac{1}{2}$. Moreover, $\zeta = 0$ if and only if the corresponding gap is collapsed (see [58] for details). We will show that the size of the gap is determined by B and ζ.

To prove the inequality $|G_k(V)| \le \zeta^{1-\kappa}$, we first make some technical preparations. For any $0 < \delta < 1$, a direct calculation and equation (5.6) yield

$$B(x+\alpha)^{-1} S^V_{E^+_k - \delta}(x) B(x) = X - \delta P(x) \tag{A.1}$$

with

$$P(x) := \begin{pmatrix} B_{11}(x)B_{12}(x) - \zeta B^2_{11}(x) & -\zeta B_{11}(x)B_{12}(x) + B^2_{12}(x) \\ -B^2_{11}(x) & -B_{11}(x)B_{12}(x) \end{pmatrix}.$$

Obviously,

$$|P|_{r''} \le (1+\zeta)|B|^2_{r''} < 2|B|^2_{r''} \quad \text{for all } r'' \in (0, \sigma]. \tag{A.2}$$

In fact, moving the energy E from the right end of the gap E^+_k to $E^+_k - \delta$, we can determine the other edge point of the spectral gap according to the variation of the rotation number $\rho(\alpha, B(x+\alpha)^{-1} S^V_{E^+_k-\delta}(x) B(x))$. Note, the rotation number of the constant cocycle (α, X) vanishes since X is parabolic, then we know that if the rotation number of $(\alpha, B(x+\alpha)^{-1}S^V_{E_k^+-\delta}(x)B(x))$ is positive, then $E_k^+-\delta_1$ is beyond the left edge of $G_k(V)$, and thus $|G_k(V)|<\delta_1$. We give some auxiliary lemmas first.

LEMMA A.1. *Given $\alpha\in DC_\infty(\gamma,\tau)$ with $0<\sigma<\sigma_0$, if $0<\delta<(1/2\widetilde{C})|B|_\sigma^{-2}$, $\widetilde{C}\triangleq C(\sigma,\eta,\tau,\gamma)=((c(\sigma,\eta,\tau)/40\gamma^3)+1)$, then there exist $\widetilde{B}\in\mathcal{H}_{\sigma/2}(2\mathbb{T}^\infty, SL(2,\mathbb{R}))$ and $P_1\in\mathcal{H}_{\sigma/2}(\mathbb{T}^\infty, gl(2,\mathbb{R}))$ such that*

$$\widetilde{B}(x+\alpha)^{-1}(X-\delta P(x))\widetilde{B}(x)=e^{b_0-\delta b_1}+\delta^2P_1(x) \tag{A.3}$$

with

$$|\widetilde{B}-\mathrm{Id}|_{\sigma/2}\le 2\widetilde{C}\delta|B|_\sigma^2,\quad |P_1(x)|_{\sigma/2}\le 2\widetilde{C}^2|B|_\sigma^4+(4\delta)^{-1}\zeta^2|B|_\sigma^2, \tag{A.4}$$

where

$$b_0:=\begin{pmatrix}0&\zeta\\0&0\end{pmatrix},\quad b_1:=\begin{pmatrix}[B_{11}B_{12}]-\frac{\zeta}{2}[B_{11}^2] & -\zeta[B_{11}B_{12}]+[B_{12}^2]\\ -[B_{11}^2] & -[B_{11}B_{12}]+\frac{\zeta}{2}[B_{11}^2]\end{pmatrix}.$$

The construction of $\widetilde{B}$ in equation (A.3) and calculations to get the estimates in equation (A.4) are similar to [**48**, the proof of Lemma 6.1], we omit the details. Moreover, $\widetilde{B}$ in equation (A.3) is homotopic to identity by its construction, then we have

$$\rho(\alpha, X-\delta P(x))=\rho(\alpha, e^{b_0-\delta b_1}+\delta^2P_1(x)). \tag{A.5}$$

LEMMA A.2. *For any $B\in\mathcal{H}_\sigma(2\mathbb{T}^\infty, SL(2,\mathbb{R}))$, $[B_{11}^2]\le(2|B|_{\mathbb{T}^\infty})^{-2}$.*

The proof of this lemma is similar the proof of [**48**, Lemma 6.2], we omit the details.

LEMMA A.3. [**48**] *For any $\kappa\in(0,\frac{1}{4})$, if*

$$|B|_\sigma\zeta^{\kappa/2}\le\tfrac{1}{4},$$

then the following hold:

$$0<\frac{[B_{11}^2]}{[B_{11}^2][B_{12}^2]-[B_{11}B_{12}]^2}\le\frac{1}{2}\zeta^{-\kappa}, \tag{A.6}$$

$$[B_{11}^2][B_{12}^2]-[B_{11}B_{12}]^2\ge 8\zeta^{2\kappa}. \tag{A.7}$$

Now, we give the proof $|G_k(V)|\le\zeta^{1-\kappa}$ in Lemma 5.2. Set

$$d(\delta):=\det(b_0-\delta b_1)+\frac{\delta^2\zeta^2}{4}[B_{11}^2]^2. \tag{A.8}$$

As we will show, $d(\delta)$ is the key quantity in determining the gaps. Moreover, fix $\kappa\in(0,\frac{1}{4})$ and let $\delta_1=\zeta^{1-\kappa}$. Here, $1-\kappa\in(\frac{3}{4},1)$, so $\zeta^{1-\kappa}<\zeta^\kappa$. If $\zeta>0$ satisfies equation (5.7), then it is obvious that

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$$\delta_1 = \zeta^{1-\kappa} < \zeta^{\kappa} < (2\widetilde{C})^{-1}|B|_{\sigma}^{-2}. \tag{A.9}$$

Hence, we can apply Lemma A.1 and conjugate $(\alpha, X - \delta_1 P(x))$ to the cocycle $(\alpha, e^{b_0-\delta_1 b_1} + \delta_1^2 P_1(x))$ (that is equation (A.3)) with estimates given in equation (A.4). Recall equation (A.8),

$$\begin{aligned} d(\delta_1) &= -\delta_1[B_{11}^2]\zeta + \delta_1^2([B_{11}^2][B_{12}^2] - [B_{11}B_{12}]^2) \\ &= \delta_1([B_{11}^2][B_{12}^2] - [B_{11}B_{12}]^2)\bigg(\delta_1 - \frac{[B_{11}^2]\zeta}{[B_{11}^2][B_{12}^2] - [B_{11}B_{12}]^2}\bigg). \end{aligned}$$

To prove $|G_k(V)| \le \delta_1$, it is sufficient to show that $\rho(\alpha, e^{b_0-\delta_1 b_1} + \delta_1^2 P_1(x)) > 0$. By equation (5.7), one has $|B|_\sigma \zeta^{\kappa/2} \le \frac{1}{4}$, then Lemma A.3 is applied and yields

$$\frac{[B_{11}^2]\zeta}{[B_{11}^2][B_{12}^2] - [B_{11}B_{12}]^2} \le \frac{1}{2}\delta_1.$$

The two inequalities yield

$$d(\delta_1) \ge \zeta^{1-\kappa} \cdot 8\zeta^{2\kappa} \cdot 2^{-1}\zeta^{1-\kappa} = 4\zeta^2 \tag{A.10}$$

and

$$\det(b_0 - \delta_1 b_1) \ge 4\zeta^2 - 4^{-1}\delta_1^2\zeta^2|B|_\sigma^4 > 3\zeta^2 > 0. \tag{A.11}$$

Following the expressions of b_0 and b_1 in Lemma A.1, we have

$$|b_0 - \delta_1 b_1| \le \zeta + \delta_1(1+\zeta)|B|_\sigma^2 \le \frac{3}{2}\zeta^{1-\kappa}|B|_\sigma^2. \tag{A.12}$$

In view of [**45**, Lemma 8.1], if $\det(b_0 - \delta_1 b_1) > 0$, then there exists $P \in SL(2, \mathbb{R})$ with

$$|P| \le 2\bigg(\frac{|b_0 - \delta_1 b_1|}{\sqrt{\det(b_0 - \delta_1 b_1)}}\bigg)^{1/2} \tag{A.13}$$

such that

$$P^{-1}e^{b_0-\delta b_1}P = \begin{pmatrix} \cos\sqrt{\det(b_0 - \delta_1 b_1)} & \sin\sqrt{\det(b_0 - \delta_1 b_1)} \\ -\sin\sqrt{\det(b_0 - \delta_1 b_1)} & \cos\sqrt{\det(b_0 - \delta_1 b_1)} \end{pmatrix}.$$

Combining equations (A.11)–(A.13), we have

$$4^{-1}|P|^2 \le \frac{|b_0 - \delta_1 b_1|}{\sqrt{\det(b_0 - \delta_1 b_1)}} \le \zeta^{-\kappa}|B|_\sigma^2.$$

The two inequalities above, and equations (A.4) and (2.7), yield

$$\begin{aligned} &|\rho(\alpha, e^{b_0-\delta_1 b_1} + \delta_1^2 P_1(x)) - \sqrt{\det(b_0 - \delta_1 b_1)}| \\ &\quad \le |P^{-1}\{e^{b_0-\delta_1 b_1} + \delta_1^2 P_1(x)\}P - P^{-1}e^{b_0-\delta b_1}P| \\ &\quad = |P^{-1}\delta_1^2 P_1(x)P| \le \delta_1^2|P_1|_0|P|^2 < 16\widetilde{C}^2\zeta^{2-3\kappa}|B|_\sigma^6. \end{aligned}$$

Under the assumption (5.7), combining with equation (A.10), we have

$$16\widetilde{C}^2\zeta^{1-3\kappa}|B|_\sigma^6 < 1,$$

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which, together with the inequality above and equation (A.5), implies that

$$\rho(\alpha, X - \delta P(x)) > \sqrt{\det(b_0 - \delta_1 b_1)} - \zeta > 0.$$

Thus, we finish the proof of the upper bound estimate.

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