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# BEHAVIOR OF DICKEY–FULLER *t*-TESTS WHEN THERE IS A BREAK UNDER THE ALTERNATIVE HYPOTHESIS

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This paper analyzes the limiting behavior of Dickey–Fuller *t*-tests when the true generating model is stationary around a broken linear trend. The cases of a break in level and a break in slope are considered separately and found to generate qualitatively different outcomes. In the asymptotic analysis, appropriate normalizations are applied to the break sizes. This leads to theoretical results that generate interesting predictions for sample sizes and break amounts of practical interest. Simulation evidence confirms the value of this approach to an asymptotic theory.

## 1. INTRODUCTION

Following the seminal work of Perron (1989) there has been considerable theoretical and empirical interest in the impact of a structural break on Dickey–Fuller tests for unit autoregressive roots. Perron demonstrated that, in the presence of a break under the trend stationary alternative, the test could frequently fail to reject the unit root null hypothesis when allowance for the break was not made. It was further demonstrated for previously analyzed data sets that dramatic test result reversals could occur when such allowance was made. Leybourne, Mills, and Newbold (1998) and Leybourne and Newbold (1998) have demonstrated the "converse Perron phenomenon"—that spurious rejections of the null hypothesis can occur when the true generating process has a unit autoregressive root, but with a relatively early break.

In this paper we consider the case where the true generating process is stationary around a broken linear trend, as analyzed by Perron (1989) and more recently by Montanes and Reyes (1998, 1999). The latter authors derived limiting values for the Dickey–Fuller $T(\hat{\rho} - 1)$ statistic and for the *t*-ratio variant of the test. Here we shall concentrate on the *t*-ratio statistic, which is by far the more frequently used variant in practical applications. Our approach differs from that of Montanes and Reyes in that in our asymptotics we employ what we claim are more appropriate and revealing normalizations for the break size. Spe-

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cifically, for a break in level we let the break size be proportional to $T^{1/2}$, whereas for a break in slope we consider a break size proportional to $T^{-1/2}$, where $T$ is the number of observations in the time series. We view these normalizations as appropriate because, as shown by Leybourne and Newbold (1998), their imposition in the case of a break under the unit root null hypothesis leads to limiting distributions for the test statistics that differ from the Dickey–Fuller distribution, predicting the "converse Perron phenomenon" noted by Leybourne et al. (1998). Our asymptotic analysis generates predictions of what might be found in practice for sample sizes and break magnitudes of interest, and these predictions are assessed through simulation experiments.

Our generating model for a time series $y_t$ is

$$y_t = d_t + \nu_t,$$

$$\nu_t = \phi \nu_{t-1} + \varepsilon_t, \qquad t = 1, \ldots, T, \tag{1}$$

where $|\phi| < 1$ and the $\varepsilon_t$ are independently and identically distributed disturbances with mean zero and standard deviation $\sigma$. For a model incorporating both a break in level and a break in slope we define $d_t$ as

$$d_t = d_{1t} + d_{2t}, \tag{2}$$

where $d_{1t}$ is the level break component given by

$$\begin{aligned} d_{1t} &= 0, & t &\le \tau T, \\ &= \sigma k_1 T^{1/2}, & t &> \tau T \end{aligned}$$

and $d_{2t}$ is the slope break component given by

$$\begin{aligned} d_{2t} &= 0, & t &\le \tau T, \\ &= \sigma k_2 T^{-1/2}(t - \tau T), & t &> \tau T, \end{aligned}$$

with $\tau$ representing the break fraction.

## 2. ASYMPTOTIC BEHAVIOR OF THE DICKEY–FULLER TEST

Let $e_t$ denote the residuals from regressing $y_t$ on an intercept and time trend. Then, the Dickey–Fuller statistic is the $t$-ratio associated with the regression of $e_t$ on $e_{t-1}$. We denote this statistic as $DF$. The following theorem gives the probability limit of $DF$.

THEOREM 1. *Under the model* (1) *and* (2),

$$DF \to_p \{k_1^2 + 2(1+\phi)^{-1}\}^{-1/2} q_2^{-1/2} \{q_1 - (1+\phi)^{-1}\},$$

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*where*

$$q_1 = -k_1^2(1-\tau)\{1+6\tau(\tau-\tfrac{1}{2})\} - k_2^2\tau^2(1-\tau)^2(1-2\tau)/2$$
$$+ k_1 k_2 \tau(1-\tau)(1-5\tau+5\tau^2),$$
$$q_2 = k_1^2\tau(1-\tau)\{1-3\tau(1-\tau)\} + k_2^2\tau^3(1-\tau)^3/3$$
$$+ k_1 k_2 \tau^2(1-\tau)^2(2\tau-1).$$

The proof of this result is given in the Appendix.

Montanes and Reyes (1998) also allow for simultaneous changes in level and slope, but their approach differs from ours in two respects. First, setting without loss of generality $\mu_1 = \beta_1 = 0$ in their equation (2), that equation becomes, in our notation,

$$y_t = (\mu_2 + \beta_2 \tau T)DU_t + \beta_2(t-\tau T)DU_t + \nu_t, \quad \textbf{(3)}$$

where $DU_t$ is one if $t > \tau T$ and zero otherwise. Thus, for large $T$, $\mu_2$ in (3) is irrelevant, so that in our terminology asymptotically this generating model represents a level shift of $\tau T$ times the amount of the slope shift. Nevertheless, setting $k_1 = \tau k_2$ in our Theorem 1 does not give the result of Theorem 2 of Montanes and Reyes (1998). This is because of the second distinction between the two approaches. Whereas fixing $\beta_2$ in (3) implies a level shift of $O(T)$ and a slope shift of $O(1)$, fixing $k_1$ and $k_2$ in our approach implies a level shift of $O(T^{1/2})$ and a slope shift of $O(T^{-1/2})$. A consequence of the higher order shifts in Montanes and Reyes is that their Theorem 2 implies a probability limit for the Dickey–Fuller $t$-statistic that depends only on the break fraction and not on either the break magnitude or the autocorrelation structure of $\nu_t$. By contrast, even when $k_1 = \tau k_2$ is substituted in our Theorem 1, the result is a function of both $k_2$ and $\phi$.

Probability limits for the Dickey–Fuller test statistic for any combination of level and slope shifts can be obtained by substituting the corresponding values of $(k_1, k_2)$ in Theorem 1. However, because the impacts of breaks in level and slope turn out to be qualitatively different, it is useful to consider them separately. To analyze the effect of a level break only, we substitute $k_2 = 0$ in Theorem 1.

COROLLARY 1. *In the case where the model contains only a level break*,

$$DF \to_p -[c_1^2(2+c_1^2)\tau(1-\tau)\{1-3\tau(1-\tau)\}]^{-1/2}$$
$$\times [1 + c_1^2(1-\tau)\{1+6\tau(\tau-\tfrac{1}{2})\}]$$

*with* $c_1 = |k_1|(1+\phi)^{1/2}$.

This limiting function is graphed in Figure 1 for $\phi = 0.9$ and different values of $k_1$ and $\tau$. This graph reveals the Perron phenomenon as the parameter $k_1$ determining the break size increases with fixed $\phi$. (More generally, as is clear

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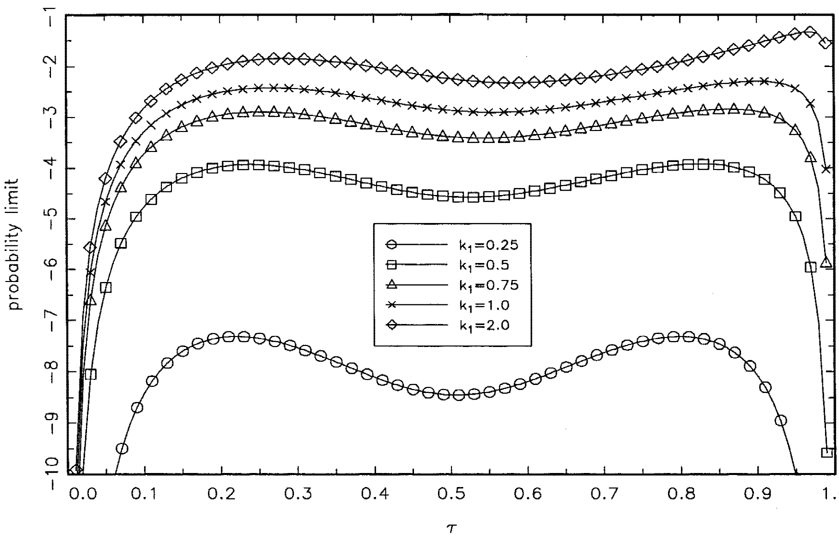


**FIGURE 1.** Limit distribution of *DF* under level break: $\phi = 0.9$.

from Corollary 1, the impacts of $k_1$ and $\phi$ are subsumed in the single parameter $c_1$.) As $k_1$ increases, the shape of the graph of probability limits begins to depend quite strongly on the break fraction $\tau$, and one would conjecture that in practice the Perron phenomenon would be least manifest, if at all, for the earliest breaks.

The case of a break only in level is analyzed by Montanes and Reyes (1999). Their approach differs from ours in that they hold the break magnitude fixed, whereas in our analysis that magnitude is $O(T^{1/2})$. Proposition 1 of Montanes and Reyes then shows that, asymptotically the test statistic has the Dickey–Fuller distribution under the null hypothesis and diverges to $-\infty$ under the alternative. Thus, if the break magnitude is $O(1)$ the analysis fails to reveal both the "converse Perron phenomenon" of Leybourne et al. (1998) under the null hypothesis and the Perron phenomenon under the alternative.

Next we turn to the case of a break only in slope, where, as we shall see, the prediction for rejections of the null hypothesis is somewhat different from the previous case. The required result is obtained by setting $k_1 = 0$ in Theorem 1.

COROLLARY 2. *In the case where the model contains only a slope break,*

$$DF \to_p \left(\tfrac{3}{2}\right)^{1/2}\left[\left(\tfrac{1}{2}\right)c_2\{\tau(1-\tau)\}^{1/2}(2\tau-1) - c_2^{-1}\{\tau(1-\tau)\}^{-3/2}\right]$$

*with* $c_2 = |k_2|(1+\phi)^{1/2}$.

The limiting function is graphed in Figure 2 for $\phi = 0.9$ and different values of $k_2$ and $\tau$. It is interesting to compare our Corollary 2 with Theorem 1 of

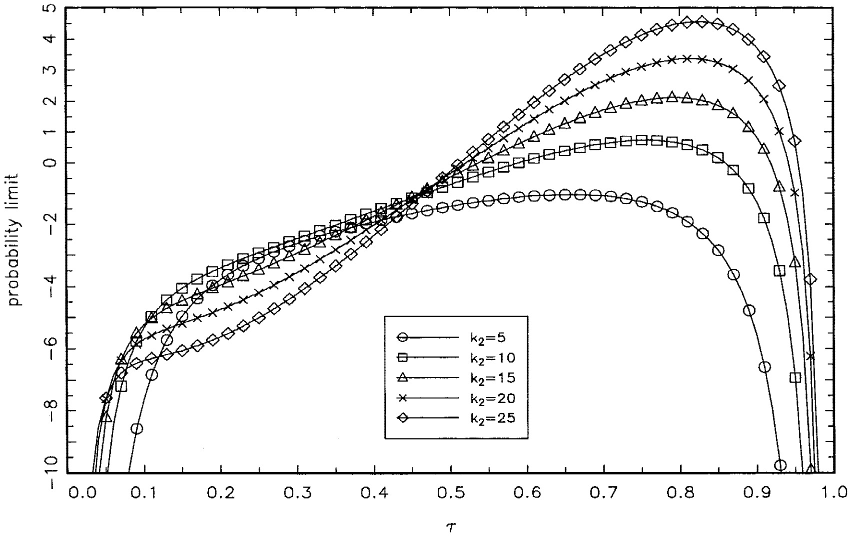


**FIGURE 2.** Limit distribution of *DF* under slope break: $\phi = 0.9$.

Montanes and Reyes (1998), based on the same generating model. The approaches differ in that Montanes and Reyes hold the break size constant with increasing *T*, then finding that $T^{-1/2}DF$ has a finite probability limit, with *DF* diverging to $-\infty$ for $\tau < 0.5$ and to $\infty$ for $\tau > 0.5$. Presumably their result could be interpreted as predicting very frequent rejections when the break is in the first half of the series and very infrequent rejections when the break is in the second half. By contrast, Figure 2 suggests somewhat richer insight. Again the parameters $c_2 = |k_2|(1 + \phi)^{1/2}$ and $\tau$ are crucial. For fixed $\phi$, as $k_2$ increases for a break in the latter part of the series, the probability limit of the test statistic increases dramatically, suggesting that the null hypothesis will virtually never be rejected unless $\tau$ is very close to one, in agreement with Montanes and Reyes (1998). Of course, this is precisely the phenomenon predicted by Perron (1989). However, notice that the curves in Figure 2 cross as $\tau$ falls below 0.5, so that we would expect to find in this region *more* rejections of the null hypothesis as the parameter $k_2$ determining the break size increases. Finally, as $k_2$ increases so does the slope of the probability limit curves in the region $0.2 < \tau < 0.5$. Moreover, in this region, the probability limits of the Dickey–Fuller statistic are quite close to the nominal 5% critical value for the test ($-3.41$, Fuller, 1996, p. 642). We might therefore predict as $\tau$ increases in this region a rapid shift from very many to very few rejections of the null hypothesis. Moreover, this transition is well short of $\tau = 0.5$, unless $k_2$ becomes extremely large—effectively, the special case considered by Montanes and Reyes (1998).

## 3. SIMULATION EVIDENCE

We conducted simulation experiments using the break under the alternative model (1), (2) for series of $T = 200$ observations, with $\varepsilon_t$ generated as standard normal and $\phi = 0.9$. For $\tau$ ranging from 0.01 to 0.99 in steps of 0.01, the empirical rejection frequencies of *DF* at the nominal 5% level, based on 5,000 replications, were recorded. For this case, when there is no break *DF* rejects the null hypothesis for 66% of all series at the 5% level. Thus, our simulations allow scope for detecting values of $\tau$ for which rejection frequencies are either much higher or much lower than this baseline case.

Figure 3 shows the results for the break in level model, with $k_1$ taking the same values as in Figure 1, so that the actual break amounts are $14.14k_1$. As predicted, for almost all values of the break fraction $\tau$, the Perron phenomenon becomes increasingly manifest with growing $k_1$. The only exception is when the break occurs in the first 5% of the series, when the null hypothesis is rejected more frequently than in the no break case. This possibly might be anticipated from the asymmetry of Figure 1.

Figure 4 shows the results for the break in slope model, with $k_2$ taking the same values as in Figure 2, so that the actual break amounts are $0.0707k_2$. As predicted by the theory, this case is quite different and rather more interesting than the break in level case. First note, for breaks in the first half of the series, there is a range in which the rejection frequency is higher than in the no break case. Both the width of that range and the frequency of rejections within the

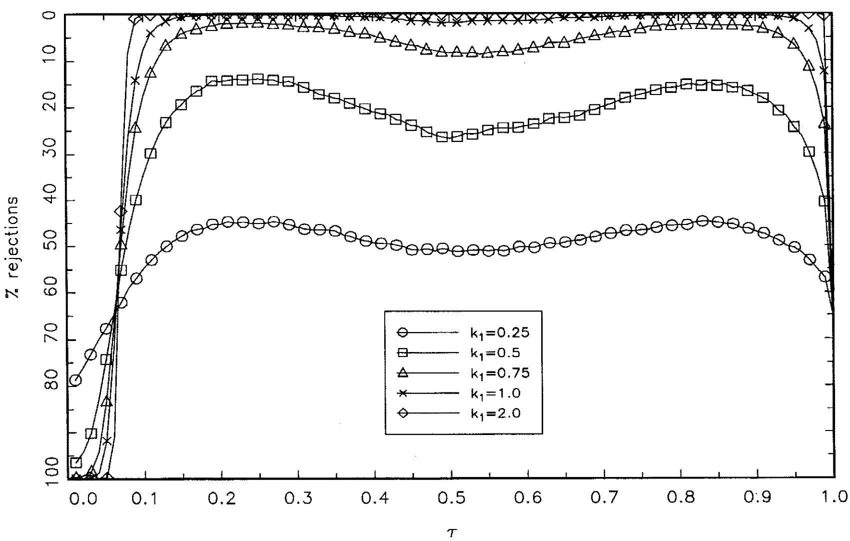


**FIGURE 3.** Rejection frequency of *DF* under level break: $\phi = 0.9$, $T = 200$.

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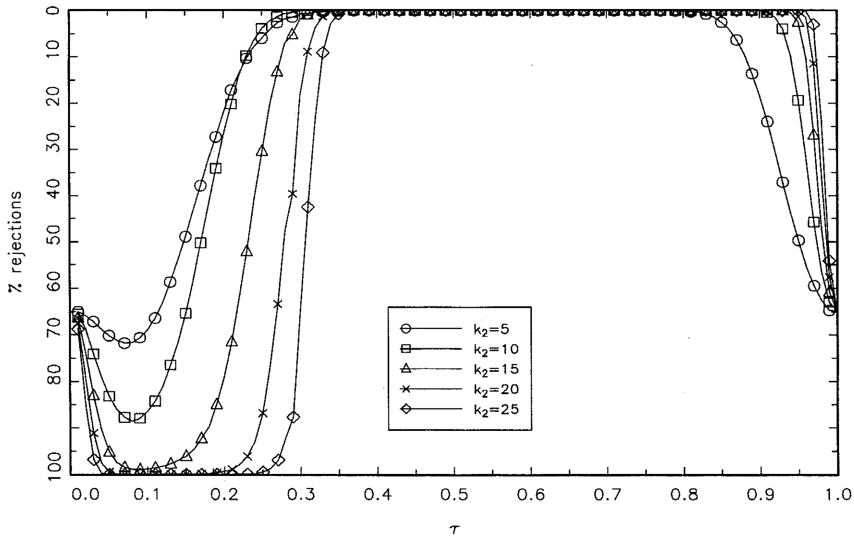


**FIGURE 4.** Rejection frequency of *DF* under slope break: $\phi = 0.9$, $T = 200$.

range increase with $k_2$. However, particularly for the largest values of $k_2$ there is a very steep climb from virtually 100% rejections to almost no rejections. Again, this is not unexpected given the results shown in Figure 2. (Notice also that as the break size increases the part of the upper interval for $\tau$ in which there is an appreciable number of rejections narrows.) Thus, by contrast with the break in level case, we might say that, whereas the Perron phenomenon is manifest for large breaks for break fractions above some amount $\tau^*$, it rapidly evaporates for break fractions below that amount. The precise value of $\tau^*$ depends on the break size and more generally according to our theoretical predictions on $|k_2|(1 + \phi)^{1/2}$. However, for cases of practical interest, $\tau^*$ can clearly be well below 0.5.

## 4. CONCLUSIONS

We have analyzed the Perron phenomenon—that is, the failure to reject the unit root null hypothesis given a generating model that is stationary around a broken trend. The cases of a break in level and break in slope have been analyzed separately, their impacts being shown to be qualitatively quite different. Imposing appropriate normalizations on the break sizes allows an asymptotic theory that makes interesting predictions. Of course, because the theoretical values thereby obtained are probability limits of test statistics, they cannot be expected to perfectly mirror what will be found in practically interesting cases where test statistics have nontrivial finite sample distributions. Taken at face value, the results would predict rejection frequencies of either 0% or 100%, depending on whether the probability limits were above or below the nominal critical values. Nevertheless, the shapes of the curves in Figures 1 and 2 are suggestive of patterns that might be found in moderate-sized samples and so are valuable in prompting simulation experiments. Our experimental results are on the whole unsurprising on the basis of our asymptotic theory, it being possible to anticipate, at least qualitatively, the findings of Figures 3 and 4, given the results of Figures 1 and 2. Overall, the conclusion is that when a break occurs no less than halfway through a series the Perron phenomenon is apparent on the obvious basis—that is, relatively few rejections of the null and, the larger the break, all else equal, the fewer the rejections. This picture is rather different for breaks in the first half of the series, where it is entirely possible that the Perron phenomenon will not be observed. The precise picture here depends on the nature and size of the break.

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# APPENDIX

**Proof of Theorem 1.** Because the limit of $DF$ is invariant to $\sigma$, we may, without loss of generality, set $\sigma = 1$. Regressing $y_t$ on an intercept and time trend, and denoting the residuals $e_t$, we have

$$e_t = \eta_t + g_t - h_t,$$

where

$$g_t = (d_t - \bar{d}),$$

$$h_t = (t - \bar{t}) \sum_{t=1}^{T} (t - \bar{t}) d_t \left\{ \sum_{t=1}^{T} (t - \bar{t})^2 \right\}^{-1},$$

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and $\eta_t$ is the stationary AR(1) process $\nu_t$ demeaned and detrended. Defining

$$f_0 = \sum_{t=2}^{T} e_t^2, \qquad f_1 = \sum_{t=2}^{T} e_{t-1}^2, \qquad f_2 = \sum_{t=2}^{T} e_t e_{t-1},$$

the two-step variant of the Dickey–Fuller statistic, *DF*, is then given by

$$\begin{aligned} DF &= (\hat{\sigma}^2 f_1^{-1})^{-1/2}(\hat{\rho} - 1) \\ &= \hat{\sigma}^{-1}(T^{-2} f_1)^{-1/2} T^{-1}(f_2 - f_1) \end{aligned} \tag{A.1}$$

with $\hat{\rho} = f_2 f_1^{-1}$ and

$$\hat{\sigma}^2 = T^{-1} \sum_{t=2}^{T} (e_t - \hat{\rho} e_{t-1})^2 = T^{-1} f_0 + \hat{\rho}^2 T^{-1} f_1 - 2\hat{\rho} T^{-1} f_2.$$

First, consider the term

$$\begin{aligned} T^{-1}(f_2 - f_1) &= T^{-1} \sum_{t=2}^{T} e_{t-1} \Delta e_t \\ &= T^{-1} \sum_{t=2}^{T} \eta_{t-1} \Delta \eta_t + T^{-1} \sum_{t=2}^{T} (g_{t-1} - h_{t-1}) \Delta (g_t - h_t) \\ &\quad + T^{-1} \sum_{t=2}^{T} \eta_{t-1} \Delta (g_t - h_t) + T^{-1} \sum_{t=2}^{T} (g_{t-1} - h_{t-1}) \Delta \eta_t. \end{aligned}$$

Under (2) it is straightforward to show that

$$T^{-1} \sum_{t=2}^{T} (g_{t-1} - h_{t-1}) \Delta h_t \to 0, \qquad T^{-1} \sum_{t=2}^{T} \eta_{t-1} \Delta (g_t - h_t) \to_p 0,$$

$$T^{-1} \sum_{t=2}^{T} (g_{t-1} - h_{t-1}) \Delta \eta_t \to_p 0.$$

So,

$$T^{-1}(f_2 - f_1) = T^{-1} \sum_{t=2}^{T} \eta_{t-1} \Delta \eta_t + T^{-1} \sum_{t=2}^{T} (g_{t-1} - h_{t-1}) \Delta g_t + o_p(1). \tag{A.2}$$

Moreover, from the standard properties of stationary AR(1) processes

$$T^{-1} \sum_{t=2}^{T} \eta_{t-1} \Delta \eta_t \to_p -(1 + \phi)^{-1}.$$

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Next, consider the term

$$T^{-2}f_1 = T^{-2}\sum_{t=2}^{T} e_{t-1}^2$$
$$= T^{-2}\sum_{t=2}^{T}\eta_{t-1}^2 + T^{-2}\sum_{t=2}^{T}(g_{t-1}-h_{t-1})^2 + 2T^{-2}\sum_{t=2}^{T}\eta_{t-1}(g_{t-1}-h_{t-1}).$$

Clearly

$$T^{-2}\sum_{t=2}^{T}\eta_{t-1}^2 \to_p 0, \qquad T^{-2}\sum_{t=2}^{T}\eta_{t-1}(g_{t-1}-h_{t-1}) \to_p 0,$$

and so

$$T^{-2}f_1 = T^{-2}\sum_{t=2}^{T}(g_{t-1}-h_{t-1})^2 + o_p(1). \qquad \textbf{(A.3)}$$

Finally, on substituting for $\hat{\rho}$ in the expression for $\hat{\sigma}^2$, and simplifying, we obtain

$$\hat{\sigma}^2 = T^{-1}f_0 - T^{-1}f_2^2 f_1^{-1}$$
$$= T^{-1}f_1 - T^{-1}f_2^2 f_1^{-1} + T^{-1}(f_0 - f_1).$$

Next, note that

$$T^{-1}f_1 - T^{-1}f_2^2 f_1^{-1} = (T^{-2}f_1)^{-1}T^{-3}(f_1^2 - f_2^2)$$
$$= -T^{-1}(f_2 - f_1)\{1 + T^{-2}f_2(T^{-2}f_1)^{-1}\}$$

and because $T^{-1}(f_2 - f_1)$ is $O_p(1)$ in view of (A.2), $T^{-2}f_2(T^{-2}f_1)^{-1} \to_p 1$. So,

$$T^{-1}f_1 - T^{-1}f_2^2 f_1^{-1} = -2T^{-1}(f_2 - f_1) + o_p(1). \qquad \textbf{(A.4)}$$

Also,

$$T^{-1}(f_0 - f_1) = T^{-1}e_T^2 - T^{-1}e_1^2$$
$$= T^{-1}(g_T - h_T)^2 - T^{-1}(g_1 - h_1)^2 + o_p(1). \qquad \textbf{(A.5)}$$

It is straightforward, but tedious, to establish the following limits:

$$T^{-1}\sum_{t=2}^{T}(g_{t-1}-h_{t-1})\Delta g_t \to q_1,$$
$$T^{-2}\sum_{t=2}^{T}(g_{t-1}-h_{t-1})^2 \to q_2,$$
$$T^{-1}(g_T - h_T)^2 \to q_3,$$
$$T^{-1}(g_1 - h_1)^2 \to q_4,$$

where

$$q_1 = -k_1^2(1-\tau)\{1+6\tau(\tau-\tfrac{1}{2})\} - k_2^2\tau^2(1-\tau)^2(1-2\tau)/2$$
$$+ k_1 k_2 \tau(1-\tau)(1-5\tau+5\tau^2),$$
$$q_2 = k_1^2\tau(1-\tau)\{1-3\tau(1-\tau)\} + k_2^2\tau^3(1-\tau)^3/3$$
$$+ k_1 k_2 \tau^2(1-\tau)^2(2\tau-1),$$
$$q_3 = \{k_1\tau(3\tau-2) + k_2\tau^2(1-\tau)\}^2,$$
$$q_4 = \{k_1(1-\tau)(3\tau-1) + k_2\tau(1-\tau)^2\}^2.$$

Using these results, together with (A.2)–(A.5), we find

$$T^{-1}(f_2 - f_1) \to_p q_1 - (1+\phi)^{-1}, \quad \textbf{(A.6)}$$
$$T^{-2}f_1 \to_p q_2,$$
$$\hat{\sigma}^2 \to_p -2q_1 + q_3 - q_4 + 2(1+\phi)^{-1}. \quad \textbf{(A.7)}$$

Simplifying this last expression for $\hat{\sigma}^2$, we obtain

$$\hat{\sigma}^2 \to_p k_1^2 + 2(1+\phi)^{-1}. \quad \textbf{(A.8)}$$

Substituting (A.6)–(A.8) into the expression for $DF$ given by (A.1), and rearranging, gives

$$DF \to_p \{k_1^2 + 2(1+\phi)^{-1}\}^{-1/2} q_2^{-1/2}\{q_1 - (1+\phi)^{-1}\}.$$