Glasgow Math. J. **63** (2021) 54–58. © Glasgow Mathematical Journal Trust 2019.
doi:10.1017/S0017089519000508.

ON FINITE-BY-NILPOTENT GROUPS

ELOISA DETOMI
Dipartimento di Ingegneria dell'Informazione - DEI, Università di Padova, Via G. Gradenigo 6/B, 35121 Padova, Italy
e-mail: eloisa.detomi@unipd.it

GURAM DONADZE
Department of Mathematics, University of Brasilia, Brasilia-DF, 70910-900 Brazil and Institute of Cybernetics of the Georgian Technical University, Sandro Euli Str. 5, 0186, Tbilisi, Georgia
e-mail: gdonad@gmail.com

MARTA MORIGI
Dipartimento di Matematica, Università di Bologna, Piazza di Porta San Donato 5, 40126 Bologna, Italy
e-mail: marta.morigi@unibo.it

PAVEL SHUMYATSKY
Department of Mathematics, University of Brasilia, Brasilia-DF, 70910-900 Brazil
e-mail: pavel2040@gmail.com

(Received 3 July 2019; revised 18 November 2019; accepted 28 November 2019; first published online 20 December 2019)

Abstract. Let $\gamma_n = [x_1, \ldots, x_n]$ be the nth lower central word. Denote by X_n the set of γ_n-values in a group G and suppose that there is a number m such that $|g^{X_n}| \leq m$ for each $g \in G$. We prove that $\gamma_{n+1}(G)$ has finite (m, n)-bounded order. This generalizes the much-celebrated theorem of B. H. Neumann that says that the commutator subgroup of a BFC-group is finite.

2010 *Mathematics Subject Classification*. Primary 20E45; Secondary 20F12, 20F24.

1. Introduction. Given a group G and an element $x \in G$, we write x^G for the conjugacy class containing x. Of course, if the number of elements in x^G is finite, we have $|x^G| = [G : C_G(x)]$. A group is said to be a BFC-group if its conjugacy classes are finite and of bounded size. One of the most famous of B. H. Neumann's theorems says that in a BFC-group the commutator subgroup G' is finite [6]. Later Wiegold showed that if $|x^G| \leq m$ for each $x \in G$, then the order of G' is bounded by a number depending only on m. Moreover, Wiegold found a first explicit bound for the order of G' [10], and the best-known bound was obtained in [5] (see also [7] and [9]).

The recent articles [3] and [2] deal with groups G in which conjugacy classes containing commutators are bounded. Recall that multilinear commutator words are words which are obtained by nesting commutators, but using always different variables. More formally, the group-word $w(x) = x$ in one variable is a multilinear commutator; if u and v are multilinear commutators involving disjoint sets of variables then the word

The first and third authors are members of INDAM. The fourth author was supported by CNPq-Brazil.

$w=[u, v]$ is a multilinear commutator, and all multilinear commutators are obtained in this way. Examples of multilinear commutators include the familiar lower central words $\gamma_n(x_1, \dots, x_n)=[x_1, \dots, x_n]$ and derived words δ_n, on 2^n variables, defined recursively by

$$\delta_0=x_1, \qquad \delta_n=[\delta_{n-1}(x_1, \dots, x_{2^{n-1}}), \delta_{n-1}(x_{2^{n-1}+1}, \dots, x_{2^n})].$$

We let $w(G)$ denote the verbal subgroup of G generated by all w-values. Of course, $\gamma_n(G)$ is the nth term of the lower central series of G while $\delta_n(G)=G^{(n)}$ is the nth term of the derived series.

The following theorem was established in [**2**].

THEOREM 1.1. *Let m be a positive integer and w a multilinear commutator word. Suppose that G is a group in which $|x^G| \leq m$ for any w-value x. Then the order of the commutator subgroup of $w(G)$ is finite and m-bounded.*

Throughout the article, we use the expression "$(a, b, \dots)$-bounded" to mean that a quantity is finite and bounded by a certain number depending only on the parameters $a, b, \dots$.

The present article grew out of the observation that a modification of the techniques developed in [**3**] and [**2**] can be used to deduce that if $|x^{G'}| \leq m$ for each $x \in G$, then $\gamma_3(G)$ has finite m-bounded order. Naturally, one expects that a similar phenomenon holds for other terms of the lower central series of G. This is indeed the case.

THEOREM 1.2. *Let m, n be positive integers and G a group. If $|x^{\gamma_n(G)}| \leq m$ for any $x \in G$, then $\gamma_{n+1}(G)$ has finite (m, n)-bounded order.*

Using the concept of verbal conjugacy classes, introduced in [**4**], one can obtain a generalization of Theorem 1.2. Let $X_n=X_n(G)$ denote the set of γ_n-values in a group G. It was shown in [**1**] that if $|x^{X_n}| \leq m$ for each $x \in G$, then $|x^{\gamma_n(G)}|$ is (m, n)-bounded. Hence, we have

COROLLARY 1.3. *Let m, n be positive integers and G a group. If $|x^{X_n(G)}| \leq m$ for any $x \in G$, then $\gamma_{n+1}(G)$ has finite (m, n)-bounded order.*

Observe that Neumann's theorem can be obtained from Corollary 1.3 by specializing $n=1$. Another result which is straightforward from Corollary 1.3 is the following characterization of finite-by-nilpotent groups.

THEOREM 1.4. *A group G is finite-by-nilpotent if and only if there are positive integers m, n such that $|x^{X_n}| \leq m$ for any $x \in G$.*

2. Preliminary results.

Recall that in any group G the following "standard commutator identities" hold, when $x, y, z \in G$.

(1) $[xy, z]=[x, z]^y[y, z]$;

(2) $[x, yz]=[x, z][x, y]^z$;

(3) $[x, y^{-1}, z]^y[y, z^{-1}, x]^z[z, x^{-1}, y]^x=1$ (Hall–Witt identity);

(4) $[x, y, z^x][z, x, y^z][y, z, x^y]=1$.

Note that the fourth identity follows from the third one. Indeed, we have

$$[x^y, y^{-1}, z^y][y^z, z^{-1}, x^z][z^x, x^{-1}, y^x]=1.$$

https://doi.org/10.1017/S0017089519000508 Published online by Cambridge University Press

Since $[x^y, y^{-1}] = [y, x]$, it follows that

$$[y, x, z^y][z, y, x^z][x, z, y^x] = 1.$$

Recall that X_i denote the set of γ_i-values in a group G.

LEMMA 2.1. *Let k, n be integers with $2 \le k \le n$ and let G be a group such that $[\gamma_k(G), \gamma_n(G)]$ is finite and $|x^{\gamma_n(G)}| \le m$ for any $x \in G$. Then for every $g \in X_n$, we have*

$$|g^{\gamma_{k-1}(G)}| \le m^{n-k+2}|[\gamma_k(G), \gamma_n(G)]|.$$

Proof. Let $N = [\gamma_k(G), \gamma_n(G)]$. It is sufficient to prove that in the quotient group G/N, for every integer d with $k - 1 \le d \le n$

$$|(gN)^{\gamma_d(G/N)}| \le m^{n-d+1} \quad \text{for every } \gamma_{n-d+1}\text{-value } gN \in G/N,$$

since this implies that $g^{\gamma_d(G)}$ is contained at most m^{n-d+1} cosets of N, whenever $g \in X_{n-d+1}$.

So in what follows we assume that $N = 1$. The proof is by induction on $n - d$. The case $d = n$ is immediate from the hypotheses.

Let $c = n - d + 1$. Choose $g \in X_c$ and write $g = [x, y]$ with $x \in X_{c-1}$ and $y \in G$. Let $z \in \gamma_d(G)$. We have

$$[x, y, z^x][z, x, y^z][y, z, x^y] = 1.$$

Note that

$$[z, x] \in [\gamma_d(G), \gamma_{c-1}(G)] \le \gamma_{d-1+c}(G) = \gamma_n(G),$$

and

$$[y, z] \in \gamma_{d+1}(G) \le \gamma_k(G),$$

whence $[z, x, y^z] = [z, x, y[y, z]] = [z, x, y]$. Thus,

$$\begin{aligned} 1 = [x, y, z^x][z, x, y^z][y, z, x^y] &= [x, y]^{-1}[x, y]^{z^x}[z, x, y][y, z, x^y] \\ &= [x, y]^{-1}[x, y]^{z^x}(y^{-1})^{[z,x]}y((x^y)^{-1})^{[y,z]}x^y. \end{aligned}$$

It follows that

$$[x, y]^{z^x} = [x, y](x^{-1})^y(x^y)^{[y,z]}y^{-1}y^{[z,x]}.$$

Since $x^y \in X_{c-1}$ and $[y, z] \in \gamma_{d+1}(G)$, by induction

$$|\{(x^y)^{[y,z]} \mid z \in \gamma_d(G)\}| \le m^{n-d-1+1}.$$

Moreover, $[z, x] \in \gamma_n(G)$ and so $|\{y^{[z,x]} \mid z \in \gamma_d(G)\}| \le m$. Thus,

$$|\{[x, y]^{z^x} \mid z \in \gamma_d(G)\}| = |\{[x, y]^z \mid z \in \gamma_d(G)\}| \le mm^{n-d} = m^{n-d+1}$$

as claimed. □

Let H be a group generated by a set X such that $X = X^{-1}$. Given an element $g \in H$, we write $l_X(g)$ for the minimal number l with the property that g can be written as a product of l elements of X. Clearly, $l_X(g) = 0$ if and only if $g = 1$. We call $l_X(g)$ the length of g with respect to X. The following result is Lemma 2.1 in [3].

LEMMA 2.2. *Let H be a group generated by a set $X = X^{-1}$ and let K be a subgroup of finite index m in H. Then each coset Kb contains an element g such that $l_X(g) \le m - 1$.*

In the sequel, the above lemma will be used in the situation where $H=\gamma_n(G)$ and $X=X_n$ is the set of γ_n-values in G. Therefore, we will write $l(g)$ to denote the smallest number such that the element $g\in\gamma_n(G)$ can be written as a product of as many γ_n-values.

Recall that if G is a group, $a\in G$, and H is a subgroup of G, then $[H, a]$ denotes the subgroup of G generated by all commutators of the form $[h, a]$, where $h\in H$. It is well known that $[H, a]$ is normalized by a and H.

LEMMA 2.3. *Let $k, m, n\geq 2$ and let G be a group in which $|x^{\gamma_n(G)}|\leq m$ for any $x\in G$. Suppose that $[\gamma_k(G), \gamma_n(G)]$ is finite. Then for every $x\in\gamma_{k-1}(G)$, the order of $[\gamma_n(G), x]$ is bounded in terms of m, n, and $|[\gamma_k(G), \gamma_n(G)]|$ only.*

Proof. By Neumann's theorem, $\gamma_n(G)'$ has m-bounded order, so the statement is true for $k\geq n+1$. Therefore, we deal with the case $k\leq n$. Without loss of generality, we can assume that $[\gamma_k(G), \gamma_n(G)]=1$.

Let $x\in\gamma_{k-1}(G)$. Since $|x^{\gamma_n(G)}|\leq m$, the index of $C_{\gamma_n(G)}(x)$ in $\gamma_n(G)$ is at most m and by Lemma 2.2 we can choose elements $y_1,\ldots,y_m\in X_n$ such that $l(y_i)\leq m-1$ and $[\gamma_n(G), x]$ is generated by the commutators $[y_i, x]$. For each $i=1,\ldots,m$, write $y_i=y_{i\,1}\cdots y_{i\,m-1}$, where $y_{i\,j}\in X_n$. The standard commutator identities show that $[y_i, x]$ can be written as a product of conjugates in $\gamma_n(G)$ of the commutators $[y_{ij}, x]$. Since $[y_{ij}, x]\in\gamma_k(G)$, for any $z\in\gamma_n(G)$, we have that

$$[[y_{ij}, x], z]\in[\gamma_k(G), \gamma_n(G)]=1.$$

Therefore, $[y_i, x]$ can be written as a product of the commutators $[y_{ij}, x]$.

Let $T=\langle x, y_{ij}\mid 1\leq i, j\leq m\rangle$. It is clear that $[\gamma_n(G), x]\leq T'$ and so it is sufficient to show that T' has finite (m, n)-bounded order. Observe that $T\leq\gamma_{k-1}(G)$. By Lemma 2.1, $C_{\gamma_{k-1}(G)}(y_{ij})$ has (m, n)-bounded index in $\gamma_{k-1}(G)$. It follows that $C_T(\{y_{ij}\mid 1\leq i, j\leq m\})$ has (m, n)-bounded index in T. Moreover, $T\leq\langle x\rangle\gamma_n(G)$ and $|x^{\gamma_n(G)}|\leq m$, whence $|T : C_T(x)|\leq m$. Therefore, the center of T has (m, n)-bounded index in T. Thus, Schur's theorem [**8**, 10.1.4] tells us that T' has finite (m, n)-bounded order, as required. □

The next lemma can be seen as a development related to Lemma 2.4 in [**3**] and Lemma 4.5 in [**10**]. It plays a central role in our arguments.

LEMMA 2.4. *Let $k, n\geq 2$. Assume that $|x^{\gamma_n(G)}|\leq m$ for any $x\in G$. Suppose that $[\gamma_k(G), \gamma_n(G)]$ is finite. Then the order of $[\gamma_{k-1}(G), \gamma_n(G)]$ is bounded in terms of m, n and $|[\gamma_k(G), \gamma_n(G)]|$ only.*

Proof. Without loss of generality, we can assume that $[\gamma_k(G), \gamma_n(G)]=1$. Let $W=\gamma_n(G)$. Choose an element $a\in X_{k-1}$ such that the number of conjugates of a in W is maximal possible, that is, $r=|a^W|\geq|g^W|$ for all $g\in X_{k-1}$.

By Lemma 2.2, we can choose $b_1,\ldots,b_r\in W$ such that $l(b_i)\leq m-1$ and $a^W=\{a^{b_i}|i=1,\ldots,r\}$. Let $K=\gamma_{k-1}(G)$. Set $M=(C_K(\langle b_1,\ldots,b_r\rangle))_K$ (i.e. M is the intersection of all K-conjugates of $C_K(\langle b_1,\ldots,b_r\rangle)$. Since $l(b_i)\leq m-1$ and, by Lemma 2.1, $C_K(x)$ has (m, n)-bounded index in K for each $x\in X_n$, observe that the centralizer $C_K(\langle b_1,\ldots,b_r\rangle)$ has (m, n)-bounded index in K. So also M has (m, n)-bounded index in K.

Let $v\in M$. Note that $(va)^{b_i}=va^{b_i}$ for each $i=1,\ldots,r$. Therefore, the elements va^{b_i} form the conjugacy class $(va)^W$ because they are all different and their number is the allowed maximum. So, for an arbitrary element $h\in W$, there exists $b\in\{b_1,\ldots,b_r\}$ such that $(va)^h=va^b$ and hence $v^ha^h=va^b$. Therefore, $[h, v]=v^{-h}v=a^ha^{-b}$ and so $[h, v]^a=a^{-1}a^ha^{-b}a=[a, h][b, a]\in[W, a]$. Thus, $[W, v]^a\leq[W, a]$ and so $[W, M]\leq[W, a]$.

Let $x_1, \ldots, x_s$ be a set of coset representatives of M in K. As $[W, x_i]$ is normalized by W for each i, it follows that

$$[W, K] \leq [W, x_1] \cdots [W, x_s][W, M] \leq [W, x_1] \cdots [W, x_s][W, a].$$

Since s is (m, n)-bounded and by Lemma 2.3 the orders of all subgroups $[W, x_i]$ and $[W, a]$ are bounded in terms of m and n only, the result follows. □

Proof of Theorem 1.2. Let G be a group in which $|x^{\gamma_n(G)}| \leq m$ for any $x \in G$. We need to show that $\gamma_{n+1}(G)$ has finite (m, n)-bounded order. We will show that the order of $[\gamma_k(G), \gamma_n(G)]$ is finite and (m, n)-bounded for $k = n, n-1, \ldots, 1$. This is sufficient for our purposes since $[\gamma_1(G), \gamma_n(G)] = \gamma_{n+1}(G)$. We argue by backward induction on k. The case $k = n$ is immediate from Neumann's theorem so we assume that $k \leq n-1$ and the order of $[\gamma_{k+1}(G), \gamma_n(G)]$ is finite and (m, n)-bounded. Lemma 2.4 now shows that also the order of $[\gamma_k(G), \gamma_n(G)]$ is finite and (m, n)-bounded, as required. □

Proof of Corollary 1.3. Let G be a group in which $|x^{X_n(G)}| \leq m$ for any $x \in G$. We wish to show that $\gamma_{n+1}(G)$ has finite (m, n)-bounded order. Theorem 1.2 of [**1**] tells us that $|x^{\gamma_n(G)}|$ is (m, n)-bounded. The result is now immediate from Theorem 1.2. □

Proof of Theorem 1.4. In view of Corollary 1.3, the theorem is self-evident since a group G is finite-by-nilpotent if and only if some term of the lower central series of G is finite. □

REFERENCES

1. S. Brazil, A. Krasilnikov and P. Shumyatsky, Groups with bounded verbal conjugacy classes, *J. Group Theory* **9** (2006), 127–137.

2. E. Detomi, M. Morigi and P. Shumyatsky, BFC-theorems for higher commutator subgroups, *Q. J. Math.* **70** (2019), 849–858, doi: 10.1093/qmath/hay068.

3. G. Dierings and P. Shumyatsky, Groups with Boundedly Finite Conjugacy Classes of Commutators, *Q. J. Math.* **69** (2018), 1047–1051, doi: 10.1093/qmath/hay014.

4. S. Franciosi, F. de Giovanni and P. Shumyatsky, On groups with finite verbal conjugacy classes, *Houston J. Math.* **28** (2002), 683–689.

5. R. M. Guralnick and A. Maroti, Average dimension of fixed point spaces with applications, *J. Algebra* **226** (2011), 298–308.

6. B. H. Neumann, Groups covered by permutable subsets, *J. London Math. Soc.* **29** (1954), 236–248.

7. P. M. Neumann and M. R. Vaughan-Lee, An essay on BFC groups, *Proc. Lond. Math. Soc.* **35** (1977), 213–237.

8. D. J. S. Robinson, *A Course in the Theory of Groups*, Graduate Texts in Mathematics, vol. 80, 2nd edition (Springer-Verlag, New York, 1996).

9. D. Segal and A. Shalev, On groups with bounded conjugacy classes, *Quart. J. Math. Oxford* **50** (1999), 505–516.

10. J. Wiegold, Groups with boundedly finite classes of conjugate elements, *Proc. Roy. Soc. London Ser. A* **238** (1957), 389–401.