Journal of the Inst. of Math. Jussieu (2003) **2**(3), 409–481 © Cambridge University Press 409
DOI:10.1017/S1474748003000124 Printed in the United Kingdom

GERMS OF CHARACTERS OF ADMISSIBLE REPRESENTATIONS OF p-ADIC GENERAL LINEAR GROUPS

FIONA MURNAGHAN

Department of Mathematics, University of Toronto, Toronto M5S 3G3, Canada (fiona@math.toronto.edu)

(Received 9 October 2002; accepted 11 November 2002)

Abstract Let $G = GL_n(F)$, where F is a p-adic field of characteristic zero and residual characteristic p. Assuming that $p > 2n$, we compare germs of characters of irreducible admissible representations of G with germs of characters of unipotent representations of direct products of general linear groups over finite extensions of F. We show that the character of an irreducible admissible representation has an s-asymptotic germ expansion, for some semisimple s in the Lie algebra of G. Furthermore, this expansion matches with the 0-asymptotic expansion (that is, the local character expansion) of the character of a unipotent representation of the centralizer of s in G.

Keywords: character; admissible representation; Fourier transform; orbital integral; Hecke algebra

AMS 2000 *Mathematics subject classification:* Primary 22E50; 22E35

1. Introduction

Let F be a p-adic field of characteristic zero and residual characteristic p. Let $G = GL_n(F)$, where n is an integer such that $n \geqslant 2$. Suppose that π belongs to the set $\mathcal{E}(G)$ of irreducible admissible representations of G. Let $\mathfrak{g}_{0+}$ (respectively, $\mathfrak{g}_{\mathrm{reg}}$) be the set of topologically nilpotent (respectively, regular) elements in the Lie algebra $\mathfrak{g}$ of G. If Θ_π is the character of π, we will refer to the function $X \mapsto \Theta_\pi(1+X)$, $X \in \mathfrak{g}_{0+} \cap \mathfrak{g}_{\mathrm{reg}}$, as the germ of Θ_π.

If $s \in \mathfrak{g}$ is semisimple, let $\Omega_G(s)$ be the set of G-orbits in $\mathfrak{g}$ whose closures contain s. If $\mathcal{O} \in \Omega_G(s)$, let $\hat{\mu}_{\mathcal{O}}$ be the function representing the Fourier transform of the distribution given by integration over $\mathcal{O}$ with respect to a G-invariant measure on $\mathcal{O}$. We will say that the germ of Θ_π is s-asymptotic (on $\mathcal{V}$) if there exist constants $c_{\mathcal{O}}(\pi)$, one for each $\mathcal{O} \in \Omega_G(s)$, and an open neighbourhood $\mathcal{V} \subset \mathfrak{g}_{0+}$ of zero in $\mathfrak{g}$, such that

$$\Theta_\pi(1+X) = \sum_{\mathcal{O} \in \Omega_G(s)} c_{\mathcal{O}}(\pi)\hat{\mu}_{\mathcal{O}}(X), \quad X \in \mathfrak{g}_{\mathrm{reg}} \cap \mathcal{V}.$$

If the germ of Θ_π is s-asymptotic on some $\mathcal{V}$, the expression on the right will be referred to as an s-asymptotic expansion of the germ of Θ_π (on $\mathcal{V}$). If the Fourier transforms $\hat{\mu}_{\mathcal{O}}$, $\mathcal{O} \in \Omega_G(s)$, remain linearly independent upon restriction to any open neighbourhood of zero intersected with $\mathfrak{g}_{\mathrm{reg}}$ (for example, this is the case whenever s belongs to an elliptic Cartan subgroup of $\mathfrak{g}$; see Corollary 11.11), then the coefficients $c_{\mathcal{O}}(\pi)$, $\mathcal{O} \in \Omega_G(s)$, are unique and we refer to the above as the s-asymptotic expansion of the germ of Θ_π.

It is known [**11**, **13**] that the germ of Θ_π is 0-asymptotic on some open neighbourhood of zero. Recall that the depth $\rho(\pi)$ of π, as defined in [**29**, **30**] (see § 4), is a non-negative rational number. Waldspurger [**42**] and DeBacker [**9**] have shown, in the case $\rho(\pi) \in \mathbb{Z}$ and the case $\rho(\pi)$ arbitrary, respectively, that if p is sufficiently large, then the germ of Θ_π is 0-asymptotic on the set $\mathfrak{g}_{\rho(\pi)^+}$ defined in [**8**, **9**] (refer to § 11 of this paper for the definition of $\mathfrak{g}_{\rho(\pi)^+}$).

If $p > 2n$, given $\pi \in \mathcal{E}(G)$, we prove (see Theorem 14.5) that there exists a semisimple s_π in $\mathfrak{g}$ and an irreducible unipotent representation π_H of the centralizer H of s_π in G such that the germ of Θ_π is s_π-asymptotic on $\mathfrak{g}_{\rho(\pi)}$ (respectively, $\mathfrak{g}_{0^+}$) if $\rho(\pi) > 0$ (respectively, if $\rho(\pi) = 0$) and some s_π-asymptotic expansion of the germ of Θ_π matches the 0-asymptotic expansion of the germ of Θ_{π_H} in the following sense. Given $\mathcal{O}_H \in \Omega_H(0)$, let $c_{\mathcal{O}_H}(\pi_H)$ be the coefficient of $\hat{\mu}_{\mathcal{O}_H}$ in the 0-asymptotic expansion of the germ of Θ_{π_H}. The map $\mathcal{O}_H \mapsto G\cdot(s_\pi + \mathcal{O}_H)$ is a bijection from $\Omega_H(0)$ to $\Omega_G(s_\pi)$. If measures on orbits in $\Omega_H(0)$ and $\Omega_G(s_\pi)$ are chosen to be compatible (as described in § 12), then there exists an s_π-asymptotic expansion of the germ of Θ_π for which the coefficients $c_{G\cdot(s_\pi+\mathcal{O}_H)}(\pi)$ of the Fourier transforms $\hat{\mu}_{G\cdot(s_\pi+\mathcal{O}_H)}$ satisfy

$$c_{G\cdot(s_\pi+\mathcal{O}_H)}(\pi) = \lambda c_{\mathcal{O}_H}(\pi_H), \quad \mathcal{O}_H \in \Omega_H(0),$$

where λ is a constant depending on normalizations of Haar measures on G and H.

In an earlier version of this paper, Theorem 14.5 was proved subject to validity of a hypothesis concerning linear independence of the restrictions of the nilpotent orbital integrals to the space spanned by the characteristic functions of certain lattices in $\mathfrak{g}$. The hypothesis has not been proven in general, though is has been verified in some cases (see Proposition 11.6). Here, in order to avoid assuming the hypothesis, we apply a special case of a result of [**22**] concerning s-asymptotic expansions of germs of characters and unrefined minimal K-types (see Theorem 11.8).

In [**17**], Howe and Moy showed that if $p > n$, there exist certain representations of parahoric subgroups of G, called refined minimal K-types, having the property that every $\pi \in \mathcal{E}(G)$ contains a refined minimal K-type. In addition, the Hecke algebra attached to a refined minimal K-type is naturally isomorphic to the Iwahori Hecke algebra of the centralizer G'' of some semisimple element in $\mathfrak{g}$. Suppose that τ is a refined minimal K-type that is contained in some discrete series representation, or, equivalently, $G'' \simeq GL_a(L)$, where L is an extension of F and $a[L:F] = n$. Then, as we show in Theorem 13.2, the character χ_τ of τ satisfies a Kirillov-type character formula. This can be described as follows. The refined minimal K-type τ is a representation of some parahoric subgroup B, with corresponding parahoric subalgebra $\mathfrak{b}$. Suppose that ψ is a non-trivial character of F, h is an integer such that $n \leqslant h \leqslant p$ and $\mathfrak{e}_h(X) = \sum_{i=0}^{h-1} X^i/i!$, $X \in \mathfrak{g}$. There exists a semisimple element $s_{\tau,h}$ such that $L = F(s_{\tau,h})$, and the function $\chi_\tau \circ \mathfrak{e}_h$ coincides with the $\mathrm{Ad}\, B$-orbit of the linear functional $\psi(\mathrm{tr}(s_{\tau,h}\cdot))$ on the set of $X \in \mathfrak{b}$ such that X^h is sufficiently small.

In one of the main results of this paper (Theorem 14.1), we show that if $\pi \in \mathcal{E}(G)$ contains a refined minimal K-type τ of the above form, then the s_π of Theorem 14.5 may be taken equal to $s_{\tau,h}$ (for any h such that $n \leqslant h \leqslant p$) and $H = G''$. Furthermore, π_H is the unipotent representation of G'' corresponding to π via the Hecke algebra isomorphism of Howe and Moy. Also, the constant λ above is equal to $\upsilon_G(B)^{-1}\upsilon_{G''}(B \cap G'')(\dim \tau)$, where $\upsilon_G(B)$ and $\upsilon_{G''}(B \cap G'')$ are the measures of B and $B \cap G''$ relative to Haar measures on G and G'', respectively. If π is essentially square integrable, a formula for the value of each of the coefficients $c_{\mathcal{O}}(\pi)$, $\mathcal{O} \in \Omega_G(s_\pi)$, is given in Theorem 14.4.

A partial analogue of Theorem 14.1 was proved for supercuspidal representations of $GL_n(F)$ (in the case $p > n$) in [**33**]. There we derived a Kirillov-type character formula for the inducing data for π, and, via methods different from those used in this paper, proved that the germ of Θ_π is s_π-asymptotic on some unspecified open neighbourhood of zero, where s_π is a regular elliptic element appearing in the Kirillov-type character formula. The relation between the results of this paper and those of [**33**] is discussed in more detail in § 14. Recently, Adler and DeBacker [**2**] have extended the main result of [**33**] and shown that the s_π-asymptotic expansion of Θ_π holds on $\mathfrak{g}_{\rho(\pi)+}$ if $\rho(\pi) > 0$, and on $\mathfrak{g}_{0+}$ if $\rho(\pi) = 0$. Their methods are refinements of the methods used in [**33**], do not involve Hecke algebra isomorphisms and do not apply to non-supercuspidal representations.

The irreducible complex characters of a finite general linear group have Jordan decompositions, expressed in terms of particular semisimple characters of the group and unipotent characters of centralizers of semisimple elements. Our results, particularly Theorem 14.1, suggest that when p is large, the Hecke algebra isomorphisms of Howe and Moy (attached to refined minimal K-types) realize some kind of Jordan decomposition for characters of irreducible admissible representations of p-adic general linear groups.

In [**39**] and [**7**], formal degrees of discrete series representations of $GL_n(F)$ have been computed using properties of Hecke algebra isomorphisms. Up to sign and division by the formal degree of the Steinberg representation, the formal degree of a discrete series representation is the term corresponding to the trivial orbit in the 0-asymptotic expansion of the germ of the character. The results of this paper are the first in which entire germs of characters of irreducible admissible representations are compared via Hecke algebra isomorphisms.

In [**32**,**34**], analogues of the results of [**33**] (and hence of Theorem 14.1) were proved for many supercuspidal representations of classical groups (for p sufficiently large). However, as discussed in § 4 of [**32**] and § 11 of [**34**], there exist supercuspidal representations having the property that the germ of the character cannot be s-asymptotic for any semisimple element s whose centralizer in the group is compact modulo the centre of the group. Hence if the results of this paper have analogues for characters of admissible representations of other reductive groups, either those analogues have a different form (perhaps in some cases involving germs of stable sums of characters, as results of [**34**] suggest), or they only have analogues for some admissible representations. Preliminary investigations, using the Hecke algebra isomorphisms of [**26**,**27**], show that for some discrete series representations of $GSp_4(F)$, analogues of Theorem 14.1 do hold. In the future, we will investigate the possibility of using the Hecke algebra isomorphisms constructed by Kim [**19**–**21**] to obtain results similar to Theorems 14.1 and 14.5 for classical groups.

The homogeneity results of Waldspurger and DeBacker concerning germs of characters require that the residual characteristic of F be sufficiently large. If these results could be extended to small p, then possibly the methods of this paper could be adapted to use the types and Hecke algebra isomorphisms of Bushnell and Kutzko [**4**] to study germs of characters of admissible representations of $GL_n(F)$ when $p \leqslant n$.

A special case of Theorem 14.1 was announced in [**35**]. In another paper [**36**], some of the results of this paper are used to compute the coefficients in the 0-asymptotic expansions of germs of characters of certain discrete series representations of $GL_n(F)$ (when $p > 2n$).

We now discuss the contents of the paper. Most of our notation is defined as we need it, but § 2 contains a summary of some notation that appears throughout the paper. Let $\mathfrak{o}_F$ be the ring of integers in F. In § 3, we list basic facts about filtrations of parahoric subgroups of G and parahoric $\mathfrak{o}_F$-subalgebras of $\mathfrak{g}$ associated to periodic lattice flags in F^n.

Section 4 starts with a review of the definition of the minimal K-types defined by Howe and Moy [**16**] (these we refer to as standard minimal K-types) and a discussion of their relation to the unrefined minimal K-types of Moy and Prasad. Next we recall the definition of depth $\rho(\pi)$ for $\pi \in \mathcal{E}(G)$, and indicate how $\rho(\pi)$ can be detected from properties of any standard minimal K-type contained in π. There are two sorts of standard minimal K-types, called pure and separated by Howe and Moy. It is straightforward to show that if $\pi \in \mathcal{E}(G)$ is not properly parabolically induced, then π contains a pure minimal K-type.

Much of this paper is devoted to proving results (see § 12) about germs of characters of representations that contain pure minimal K-types. If $\pi \in \mathcal{E}(G)$ happens to contain a refined minimal K-type that occurs in some discrete series representation, then, as seen in [**17**], this refined minimal K-type is defined inductively via a finite sequence of pure minimal K-types for general linear groups over extensions of F. This allows us to argue by induction on depth, using the results of § 12 in the main induction step, to obtain Theorem 14.1. For arbitrary $\pi \in \mathcal{E}(G)$, we show (see Proposition 17.3) that π is parabolically induced from an irreducible admissible representation π_M of a Levi subgroup M, where π_M is a tensor product of irreducible admissible representations (of general linear groups), each of which contains a pure minimal K-type. Theorem 14.5 is then proved by induction on depth, using both the results of § 12 and Proposition 17.3.

Sections 5–10 and 12 are concerned with questions related to pure minimal K-types and the representations $\pi \in \mathcal{E}(G)$ that contain a pure minimal K-type. Suppose that π is such a representation. Then π contains a particular sort of pure minimal K-type (see Lemma 6.1). Attached to that pure minimal K-type there is an extension E/F, along with an $s \in E$ that generates E/F. Let G' be the centralizer s in G. As discussed in § 3, there is a family of parahoric subgroups of G having the property that the corresponding filtrations intersect G' in filtrations of parahoric subgroups of G' attached to lattice flags in $E^{n'}$, $n' = n/[E : F]$. Some more properties of these parahoric filtrations, relative to

G' and its Lie algebra $\mathfrak{g}'$, and the element s, are given in § 5. One of the key results here, Proposition 5.6, due to Waldspurger [**41**], is a descent property for orbital integrals. It can be stated roughly as follows. Suppose that $Z \in \mathfrak{g}'$ is such that $s^{-1}Z$ is topologically nilpotent. If B belongs to the above family of parahoric subgroups and i is the smallest non-negative integer such that the G-orbit $\mathcal{O}_G(s)$ intersects $\mathfrak{b}_{-i}$, then the integral of the characteristic function of $\mathfrak{b}_{-i}$ over the orbit $\mathcal{O}_G(s+Z)$ is equal to a non-zero multiple of the integral of the characteristic function of $\mathfrak{b}_{-i} \cap \mathfrak{g}'$ over the orbit $\mathcal{O}_{G'}(s+Z)$. Here, $\mathfrak{b}_{-i}$ is the $(-i)$th power of the pronilpotent radical of the parahoric $\mathfrak{o}_F$-subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ that corresponds to B. If B is a parahoric subgroup that is not conjugate to a parahoric in the above family, there is a uniquely determined non-negative integer i depending on s and B (but that is not defined the same way as for parahorics that belong to the above family), and $\mathcal{O}_G(s+Z)$ does not intersect $\mathfrak{b}_{-i}$.

Howe and Moy [**15**, **17**] showed that the above-mentioned pure minimal K-type gives rise to an isomorphism of Hecke algebras, which we call η, via which π corresponds to a representation $\pi' \in \mathcal{E}(G')$. Let V_π and $V_{\pi'}$ be the spaces of π and π', respectively. Given a parahoric subgroup B, let i be the non-negative integer discussed above. The aim of §§ 6–8 and 10 is to express $\dim(V_\pi^{B_{i+1}})$ as an explicit multiple of $\dim(V_{\pi'}^{B_{i+1}\cap G'})$ whenever B belongs to the above family of parahoric subgroups, and to show that if B is not conjugate to one of those subgroups, then $V_\pi^{B_{i+1}} = \{0\}$. See Propositions 8.6 and 10.8 for precise statements in the cases $\rho(\pi) > 0$ and $\rho(\pi) = 0$, respectively.

The case $\rho(\pi) > 0$ is dealt with in §§ 6–8. In § 6, we define a set of pure minimal K-types (depending on the element s and on the set of nilpotent G'-orbits in $\mathfrak{g}'$). There is a natural bijection between this set of K-types and a set of pure minimal K-types of G'. If χ is one of these K-types, and χ' is the corresponding K-type of G', Proposition 8.4 expresses the multiplicity $m_\chi(\pi)$ of χ in π as an explicit multiple of the multiplicity $m_{\chi'}(\pi')$ of χ' in π'. If B is a parahoric subgroup of G and i is as above, $\dim(V_\pi^{B_{i+1}})$ is a linear combination of the multiplicities $m_\chi(\pi)$ for χ ranging over the given set of pure minimal K-types. A similar statement holds for $\dim(V_{\pi'}^{B'_i})$ and the multiplicities $m_{\chi'}(\pi')$, when B' is a parahoric subgroup of G'. This allows us to deduce Proposition 8.6 (see above) from Proposition 8.4. The Hecke algebra isomorphism η matches certain isotypic subspaces in V_π and $V_{\pi'}$ in a very simple way. These subspaces, and the resulting relations between multiplicities, are described in § 7. In order to prove Proposition 8.4, it is necessary to relate the isotypic subspaces occurring in § 7 and the χ-isotypic subspaces of V_π and the χ'-isotypic subspaces of $V_{\pi'}$. This is done in the first part of § 8.

In the proofs of Theorems 14.1 and 14.5, the induction step may involve a Hecke algebra isomorphism $\dot\eta$ that is slightly different from η. In order to be able to apply the results of § 12 (which relate the germs of Θ_π and $\Theta_{\pi'}$) in later sections, we need to know that the representation $\dot\pi' \in \mathcal{E}(G')$ corresponding to π via $\dot\eta$ is equivalent to π'. This is proved in § 9.

In § 10, we consider the case where $\rho(\pi) = 0$ (and π contains a pure minimal K-type). In this case, E/F is unramified and the Hecke algebra isomorphism η gives rise to an isomorphism η_0 between a Hecke algebra of $\mathcal{G} = GL_n(\mathbb{F}_q)$ and one of $\mathcal{G}' = GL_{n'}(\mathbb{F}_{q^d})$, $d = [E : F]$. Here, q is the cardinality of the residue class field of F. In order to prove

Proposition 10.8 (see above), it is necessary to compare the dimensions of certain isotypic subspaces of V_π and V'_π using η_0. These particular isotypic subspaces can be viewed as representations of $\mathcal{G}$ and $\mathcal{G}'$, respectively, and one step in this process involves relating η_0 and the twisted induction map $R^{\mathcal{G}}_{\mathcal{G}'}$ of [**23**], which takes virtual representations of $\mathcal{G}'$ to virtual representations of $\mathcal{G}$. Having done this, it is a simple matter to compare the dimensions of the relevant isotypic subspaces using properties of $R^{\mathcal{G}}_{\mathcal{G}'}$. At the end of the section, we derive a Kirillov-type character formula for the character of a pure minimal K-type contained in π.

Section 11 contains statements of homogeneity properties of orbital integrals and germs of characters (due to Waldspurger [**42**] and DeBacker [**9**]), as well as a particular case of a result from [**22**]. These results will be applied in later sections. We also prove some results concerning the behaviour of (the functions representing) Fourier transforms of orbital integrals, on neighbourhoods of zero in $\mathfrak{g}$.

In § 12, Propositions 5.6, 8.4 and 10.8, homogeneity results for orbital integrals and germs of characters, and, if E/F is partly ramified, the hypothesis of § 11, are applied to derive relations between the germs of Θ_π and $\Theta_{\pi'}$.

Section 13 is devoted to proving Theorem 13.2, which gives a Kirillov-type character formula for the character of a refined minimal K-type that is contained in a discrete series representation.

Complete statements of the main theorems appear in § 14. In addition to the results mentioned previously, we give a relation between the wavefront set of a representation $\pi \in \mathcal{E}(G)$, which is one of the ones considered in Theorem 14.1 and the wavefront set of the corresponding unipotent representation of G''. With the exception of Theorem 14.1 (1) (which is Theorem 13.2), proofs of the results stated in § 14 appear in §§ 15–17.

2. Notation and conventions

Let F be a p-adic field of characteristic zero with ring of integers $\mathfrak{o} = \mathfrak{o}_F$ and maximal ideal $\mathfrak{p} = \mathfrak{p}_F$ in $\mathfrak{o}_F$. Let p and q be the characteristic and cardinality, respectively, of the residue class field $\mathfrak{o}_F/\mathfrak{p}_F$. If m is a positive integer, $\mathbb{F}_{p^m}$ denotes the finite field of order p^m. Let $\varpi = \varpi_F$ be a uniformizer. Normalize the absolute value $|\cdot| = |\cdot|_F$ by $|\varpi| = q^{-1}$. If E/F is a finite extension of E, we denote the corresponding objects relative to E by $\mathfrak{o}_E$, $\mathfrak{p}_E$, p, q_E and ϖ_E.

We will define much of our notation for $G = GL_n(F)$. The obvious analogues of this notation will be used without comment for groups of the form $\prod_{1 \leqslant i \leqslant r} GL_{n_i}(E_i)$, where $n_i \geqslant 1$ and E_i/F is a tamely ramified extension of finite degree, $1 \leqslant i \leqslant r$.

Let $\mathfrak{g}$ be the Lie algebra of G and let $C_{\mathrm{c}}^\infty(\mathfrak{g})$ be the space of complex-valued locally constant compactly supported functions on $\mathfrak{g}$. If $\mathcal{L}$ is a lattice in $\mathfrak{g}$, $C_{\mathrm{c}}(\mathfrak{g}/\mathcal{L})$ denotes the subspace of functions in $C_{\mathrm{c}}^\infty(\mathfrak{g})$ that are invariant under translation by $\mathcal{L}$. The sets of semisimple elements, nilpotent elements and regular (semisimple) elements in $\mathfrak{g}$ will be denoted by $\mathfrak{g}_{\mathrm{ss}}$, $\mathfrak{g}_{\mathrm{nil}}$ and $\mathfrak{g}_{\mathrm{reg}}$, respectively.

The notation $\upsilon_{\mathcal{X}}(S)$ will be used for the volume of a subset S of a space $\mathcal{X}$ (relative to a specific measure on $\mathcal{X}$). If $\mathcal{X} = G$ or $\mathfrak{g}$, the measure is assumed to be Haar measure. Haar measure on a compact group K will be normalized so that $\upsilon_K(K) = 1$.

Fix a character ψ of F such that ψ is trivial on $\mathfrak{p}$ and non-trivial on $\mathfrak{o}$. Throughout the paper, ψ will be the character of F used in the definition of K-types and in the Fourier transform on the Lie algebra. Haar measure on $\mathfrak{g}$ will be normalized to be self-dual with respect to Fourier transform, so $\hat{\hat{f}}(X) = f(-X)$ for all $f \in C_c^\infty(\mathfrak{g})$ and $X \in \mathfrak{g}$. If E/F is a tamely ramified finite extension of F, K-types on $GL_h(E)$, $h \geqslant 1$, and the Fourier transform on $\mathfrak{gl}_h(E)$ will be taken relative to the character $\psi_E = \psi \circ \mathrm{tr}_{E/F}$ of E (note that ψ_E is trivial on $\mathfrak{p}_E$ and non-trivial on $\mathfrak{o}_E$).

Given a representation π of G, V_π denotes the space of π. If K is a compact open subgroup of G, V_π^K denotes the subspace of $\pi(K)$-invariant vectors in V_π. If κ is a non-trivial finite-dimensional representation of K, $V_\pi^{(K,\kappa)}$ denotes the κ-isotypic subspace of V_π (viewed as a K-space). The notation $\mathcal{E}(G)$ will be used for the set of (equivalence classes of) irreducible admissible representations of G. The character of π, viewed either as a distribution or as a locally integrable function on G, will be denoted by Θ_π.

If $X \in \mathfrak{g}$, $\mathcal{O}_G(X)$ denotes the G-orbit of X and $C_G(X)$ denotes the centralizer of X in G. If $S \subset \mathfrak{g}$, $G \cdot S = \{\mathrm{Ad}\, g(X) \mid g \in G,\ X \in S\}$. Given $s \in \mathfrak{g}_{\mathrm{ss}}$, let $\Omega_G(s)$ denote the set of G-orbits $\mathcal{O}$ such that s belongs to the closure of $\mathcal{O}$. Note that $\Omega_G(0)$ is the set of nilpotent orbits.

If $X \in \mathfrak{g}$ and $\mathcal{O} = \mathcal{O}_G(X)$, $\mu_{\mathcal{O}}$ denotes the distribution on $C_c^\infty(\mathfrak{g})$ given by integration over the orbit $\mathcal{O}$, relative to a G-invariant measure on $\mathcal{O}$ (see § 12 for comments on normalizations of measures). As shown in [**11**], the Fourier transform $\hat{\mu}_{\mathcal{O}}$ of the distribution $\mu_{\mathcal{O}}$ is represented by a locally integrable function, also denoted $\hat{\mu}_{\mathcal{O}}$, on $\mathfrak{g}$.

Given a positive integer h, denote the set of partitions of h by $\mathcal{P}(h)$. Each $\alpha \in \mathcal{P}(h)$ is a finite sequence $\alpha = (\alpha_1, \dots, \alpha_r)$ of positive integers α_i such that $\sum_{1 \leqslant i \leqslant r} \alpha_i = h$. When referring to α without mentioning the α_i, we will use the notation $r(\alpha)$ for r. The set $\mathcal{P}^0(h)$ of ordered partitions of h consists of those $\alpha \in \mathcal{P}(h)$ such that $\alpha_1 \geqslant \cdots \geqslant \alpha_r$.

If n_j is a positive integer and $\alpha^{(j)} \in \mathcal{P}(n_j)$, $1 \leqslant j \leqslant m$, let

$$\alpha^{(1)} \cup \cdots \cup \alpha^{(m)} = (\alpha_1^{(1)}, \dots, \alpha_{r(\alpha^{(1)})}^{(1)}, \dots, \alpha_1^{(m)}, \dots, \alpha_{r(\alpha^{(m)})}^{(m)}) \in \mathcal{P}(n_1 + \cdots + n_m).$$

Let i be a positive integer and $\alpha \in \mathcal{P}(h)$. Define $i\alpha$ and $\alpha^i \in \mathcal{P}(ih)$ by

$$i\alpha = (i\alpha_1, \dots, i\alpha_r) \quad \text{and} \quad \alpha^i = \underbrace{\alpha \cup \cdots \cup \alpha}_{i \text{ times}}.$$

If i and j are positive integers, set $u_j(q^i) = \prod_{1 \leqslant t \leqslant j} (q^{it} - 1)$. If $\alpha = (\alpha_1, \dots, \alpha_r) \in \mathcal{P}(h)$, set $u_\alpha(q^i) = \prod_{1 \leqslant t \leqslant r} u_{\alpha_t}(q^i)$.

Let h be a positive integer and let F_0 be either a finite extension of F or a finite field. Given $\alpha = (\alpha_1, \dots, \alpha_r) \in \mathcal{P}^0(h)$, for $1 \leqslant i \leqslant \alpha_1$, let $\dot{\alpha}_i$ be the number of $j \in \{1, \dots, r\}$ such that $\alpha_j \geqslant i$. Fix a nilpotent element Y_α in $\mathfrak{gl}_h(F_0)$ in Jordan canonical form, with blocks of sizes $\dot{\alpha}_i$, $1 \leqslant i \leqslant \alpha_1$. Then $\alpha \leftrightarrow \mathcal{O}_\alpha = \mathcal{O}_{GL_h(F_0)}(Y_\alpha)$ defines a bijection between $\mathcal{P}^0(h)$ and the set of nilpotent $GL_h(F_0)$-orbits in $\mathfrak{gl}_h(F_0)$. Given α and $\beta \in \mathcal{P}^0(h)$, we write $\alpha \leqslant \beta$ whenever $\sum_{i=1}^{\ell} \alpha_i \leqslant \sum_{i=1}^{\min(\ell, r(\beta))} \beta_i$ for $1 \leqslant \ell \leqslant r(\alpha)$. This corresponds to the partial order on the set of nilpotent orbits $\mathcal{O}_\alpha$, $\alpha \in \mathcal{P}^0(h)$, given by inclusion in closure, since $\alpha \leqslant \beta$ if and only if $\bar{\mathcal{O}}_\alpha \supset \mathcal{O}_\beta$.

Finally, if $t \in \mathbb{R}$, $\lfloor t \rfloor$ and $\lceil t \rceil$ are used to denote the greatest integer less than or equal to t and the smallest integer greater than or equal to t, respectively.

3. Lattice flags and filtrations of parahoric subgroups and subalgebras

A lattice L in F^n is a free $\mathfrak{o}$-submodule of rank n. A *periodic lattice flag* $\mathfrak{L}$ in F^n, of period r, is a sequence $\mathfrak{L} = \{L_i \mid i \in \mathbb{Z}\}$ of lattices $L_i \subset F^n$ such that $L_{i+1} \subsetneq L_i$ and $L_{i+r} = \varpi L_i$, $i \in \mathbb{Z}$. We can define an associated filtration of $\mathfrak{g} = \mathfrak{gl}_n(F)$ by

$$\mathfrak{b}_i = \mathfrak{b}_{\mathfrak{L},i} = \{X \in \mathfrak{g} \mid XL_\ell \subset L_{\ell+i} \ \forall \ell \in \mathbb{Z}\}, \quad i \in \mathbb{Z}.$$

The $\mathfrak{o}$-subalgebra $\mathfrak{b} = \mathfrak{b}_0$ is a *hereditary order* in $\mathfrak{g}$ and $\mathfrak{b}_1$ is the nilradical of $\mathfrak{b}$. We have

$$\mathfrak{b}_i = (\mathfrak{b}_1)^i, \quad i \geqslant 1, \qquad \mathfrak{b}_{i+r\ell} = \varpi^\ell \mathfrak{b}_i, \quad \ell, i \in \mathbb{Z}.$$

Define

$$\mathfrak{b}_i^* = \{X \in \mathfrak{g} \mid \operatorname{tr}(XY) \in \mathfrak{p} \ \forall Y \in \mathfrak{b}_i\}.$$

Then $\mathfrak{b}_i^* = \mathfrak{b}_{1-i}$. Given a parahoric $\mathfrak{o}$-subalgebra of $\mathfrak{g}$, there exists a periodic lattice flag $\mathfrak{L}$ such that $\mathfrak{b} = \mathfrak{b}_{\mathfrak{L}}$, and the ith power of the nilradical of $\mathfrak{b}$ equals $\mathfrak{b}_{\mathfrak{L},i}$, $i \geqslant 1$.

The group $B = \mathfrak{b}^\times$ is a parahoric subgroup of G. Define a filtration of B by compact open normal subgroups of B as follows:

$$B_0 = B, \qquad B_i = 1 + \mathfrak{b}_i, \quad i \geqslant 1.$$

For $0 \leqslant i \leqslant n-1$, define $L_i^{\mathrm{std}} \subset F^n$ by

$$L_i^{\mathrm{std}} = \underbrace{\mathfrak{o} \oplus \mathfrak{o} \oplus \cdots \oplus \mathfrak{o}}_{n-i \text{ times}} \oplus \underbrace{\mathfrak{p} \oplus \mathfrak{p} \oplus \cdots \oplus \mathfrak{p}}_{i \text{ times}}.$$

Set $L_{i+\ell n}^{\mathrm{std}} = \varpi^\ell L_i^{\mathrm{std}}$ for $0 \leqslant i \leqslant n-1$ and $\ell \in \mathbb{Z}$. Given $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_r) \in \mathcal{P}(n)$, define a lattice flag $\mathfrak{L}^{\boldsymbol{\alpha}} = \{L_i^{\boldsymbol{\alpha}} \mid i \in \mathbb{Z}\}$ by

$$L_0^{\boldsymbol{\alpha}} = L_0^{\mathrm{std}}, \quad L_i^{\boldsymbol{\alpha}} = L^{\mathrm{std}}_{\alpha_{r-i+1}+\cdots+\alpha_r}, \quad 1 \leqslant i \leqslant r-1,$$
$$L_{i+\ell r}^{\boldsymbol{\alpha}} = \varpi^\ell L_i^{\boldsymbol{\alpha}}, \quad 0 \leqslant i \leqslant r-1, \quad \ell \in \mathbb{Z}.$$

Set

$$\mathfrak{b}_{\alpha,i} = \mathfrak{b}_{\mathfrak{L}^\alpha,i}, \quad i \in \mathbb{Z}, \qquad B_{\alpha,i} = B_{\mathfrak{L}^\alpha,i}, \quad i \geqslant 0.$$

The hereditary order $\mathfrak{b}_{\boldsymbol{\alpha}}$ can be described as follows:

$$\begin{pmatrix} \mathfrak{gl}_{\alpha_1}(\mathfrak{o}) & M_{\alpha_1\times\alpha_2}(\mathfrak{o}) & \dots & M_{\alpha_1\times\alpha_r}(\mathfrak{o}) \\ M_{\alpha_2\times\alpha_1}(\mathfrak{p}) & \mathfrak{gl}_{\alpha_2}(\mathfrak{o}) & \dots & M_{\alpha_2\times\alpha_r}(\mathfrak{o}) \\ \vdots & \vdots & \ddots & \vdots \\ M_{\alpha_r\times\alpha_1}(\mathfrak{p}) & M_{\alpha_r\times\alpha_2}(\mathfrak{p}) & \dots & \mathfrak{gl}_{\alpha_r}(\mathfrak{o}) \end{pmatrix}.$$

Here, if $S \subset F$, $M_{j\times k}(S)$ denotes the set of $j \times k$ matrices with entries in S. The nilradical $\mathfrak{b}_{\boldsymbol{\alpha},1}$ of $\mathfrak{b}_{\boldsymbol{\alpha}}$ consists of matrices having the block form as above, except with $\mathfrak{gl}_{\alpha_k}(\mathfrak{o})$

replaced by $\mathfrak{gl}_{\alpha_k}(\mathfrak{p})$, $1 \leqslant k \leqslant r$. Given a hereditary order $\mathfrak{b} \subset \mathfrak{g}$ and associated filtration $\mathfrak{b}_i$, there exists an $\alpha \in \mathcal{P}(n)$ such that $\operatorname{Ad} g(\mathfrak{b}_i) = \mathfrak{b}_{\boldsymbol{\alpha},i}$, $i \in \mathbb{Z}$, for some $g \in G$. In fact, if $g \in G$, then $g\mathfrak{L} = \{gL_i \mid i \in \mathbb{Z}\}$ is a periodic lattice flag in F^n and $\mathfrak{b}_{g\mathfrak{L}} = \operatorname{Ad} g^{-1}(\mathfrak{b}_{\mathfrak{L}})$.

From the above descriptions of $\mathfrak{b}_\alpha$ and $\mathfrak{b}_{\alpha,i}$, it is easy to see that, if $\alpha = (\alpha_1, \dots, \alpha_r) \in \mathcal{P}(n)$, then

$$\left.\begin{aligned} B_\alpha/B_{\alpha,1} &\simeq \prod_{1 \leqslant i \leqslant r} GL_{\alpha_i}(\mathfrak{o}/\mathfrak{p}) = \prod_{1 \leqslant i \leqslant r} GL_{\alpha_i}(\mathbb{F}_q) \\ [B_{(1)^n,1} : B_{\alpha,1}] &= [\mathfrak{b}_{(1)^n,1}, \mathfrak{b}_{\alpha,1}] = \prod_{1 \leqslant i \leqslant r} q^{\alpha_i(\alpha_i - 1)/2} \\ [B_\alpha : B_{(1)^n,1}] &= u_\alpha(q), \\ [B_{\alpha,i} : B_{\alpha,i+1}] &= \prod_{1 \leqslant j \leqslant r} q^{\alpha_j \alpha_{j+i}}, \quad i \geqslant 1. \end{aligned}\right\} \tag{3.1}$$

Here, $u_\alpha(q)$ is as defined as in § 2.

Suppose that d is a positive divisor of n. Let E/F be a tamely ramified extension of degree d. Set $f = f(E/F)$, $e = e(E/F)$ and $n' = n/d$. Fix a prime element $\varpi_E \in \mathfrak{p}_E$ such that $\varpi_E^e \varpi_F^{-1}$ is a root of unity of order prime to p. Let $G' = GL_{n'}(E)$ and $\mathfrak{g}' = \mathfrak{gl}_{n'}(E)$.

Choose a basis of E over F in such a way that the corresponding F-linear isomorphism $\xi_E : E \simeq F^d$ has the property $\xi_E(\mathfrak{p}_E^i) = L_i^{(f)^e}$, $i \in \mathbb{Z}$. That is, ξ_E maps the lattice flag $\{\mathfrak{p}_E^i \mid i \in \mathbb{Z}\}$ to the lattice flag $\mathfrak{L}^{(f)^e}$. Then the filtration $\mathfrak{b}_{(f)^e,i}$ of $\mathfrak{gl}_d(F)$ has the property that $E \cap \mathfrak{b}_{(f)^e,i} = \mathfrak{p}_E^i$, $i \in \mathbb{Z}$. (Here, E is embedded in $\mathfrak{gl}_d(F)$ via the above-mentioned basis of E over F). Define an F-linear isomorphism $\xi : E^{n'} \simeq F^n$ by

$$\xi(x_1, \dots, x_{n'}) = (\xi_E(x_1), \dots, \xi_E(x_{n'})), \quad x_i \in E, \quad 1 \leqslant i \leqslant n'.$$

Given a periodic lattice flag $\mathfrak{L}' = \{L_i' \mid i \in \mathbb{Z}\}$ in $E^{n'}$, define a lattice flag $\xi(\mathfrak{L}')$ in F^n by

$$\xi(\mathfrak{L}') = \{\xi(L_i') \mid i \in \mathbb{Z}\}.$$

If we write a matrix $X \in \mathfrak{g}$ in the form $X = (X_{ij})_{1 \leqslant i,j \leqslant n'}$, where $X_{ij} \in \mathfrak{gl}_d(F)$, we can view $\mathfrak{g}'$ as a subalgebra of $\mathfrak{g}$ as follows. Given $X' = (X_{ij}')$, $X_{ij}' \in E$, we identify each X_{ij}' with its image in $\mathfrak{gl}_d(F)$ via the above embedding of E in $\mathfrak{gl}_d(F)$. Similarly, G' will be viewed as a subgroup of G. Given $\mathfrak{L}'$, let $\mathfrak{b}_{\mathfrak{L}'}$ and $B_{\mathfrak{L}'}$ be the parahoric $\mathfrak{o}_E$-subalgebra of $\mathfrak{g}'$ and parahoric subgroup of G' corresponding to $\mathfrak{L}'$. Relative to the above identifications, if $\mathfrak{L} = \xi(\mathfrak{L}')$ and $\mathfrak{L}'$ has period r, we have

$$\mathfrak{b}_{\mathfrak{L},i} \cap \mathfrak{g}' = \mathfrak{b}_{\mathfrak{L}',i}, \quad i \in \mathbb{Z}, \qquad B_{\mathfrak{L},i} \cap G' = B_{\mathfrak{L}',i}, \quad i \geqslant 0, \qquad \varpi_E \mathfrak{b}_{\mathfrak{L},i} = \mathfrak{b}_{\mathfrak{L},i+r}. \tag{3.2}$$

Note that it follows from the last equality and $\varpi_E^e \varpi_F^{-1} \in \mathfrak{o}_E^\times$ that $\mathfrak{L} = \xi(\mathfrak{L}')$ has period er.

Given $\boldsymbol{\alpha} \in \mathcal{P}(n')$, define a periodic lattice flag $\mathfrak{L}'^{\boldsymbol{\alpha}} = \{L_i'^{\boldsymbol{\alpha}} \mid i \in \mathbb{Z}\}$ in $E^{n'}$ in a manner analogous to that of $\mathfrak{L}^\beta$, $\beta \in \mathcal{P}(n)$. Set

$$\mathfrak{b}'_{\boldsymbol{\alpha},i} = \mathfrak{b}_{\mathfrak{L}'^{\boldsymbol{\alpha}},i}, \quad i \in \mathbb{Z}, \qquad B'_{\boldsymbol{\alpha},i} = B_{\mathfrak{L}'^{\boldsymbol{\alpha}},i}, \quad i \geqslant 0.$$

https://doi.org/10.1017/S1474748003000124 Published online by Cambridge University Press

Let $L_i^{E,\mathrm{std}} \subset E^{n'}$, $i \in \mathbb{Z}$, be the analogue of L_i^{std}. There exists a permutation matrix $w \in G$ such that $\xi(L_i^{E,\mathrm{std}}) = w^{-1} L_{if}^{\mathrm{std}}$ for every $i \in \mathbb{Z}$. It follows that $\xi(\mathfrak{L}'^{\alpha}) = w^{-1}\mathfrak{L}^{(f\alpha)^e}$ for any $\alpha \in \mathcal{P}^0(n')$. Thus $\mathfrak{b}_{\xi(\mathfrak{L}'^{\alpha})} = \mathrm{Ad}\, w(\mathfrak{b}_{(f\alpha)^e})$. The explicit form of w is given in (4.12) of [**17**] (their m and a are equal to our e and f, respectively).

4. Standard minimal *K*-types and depth

A *K-type* is a pair (K, κ) consisting of a compact open subgroup of G and an irreducible representation κ of K. When there is no need to specify the subgroup K, we denote the K-type by κ. An admissible representation π is said to contain the K-type (K, κ) if κ is a constituent of the restriction of π to the subgroup K.

This section begins with a recollection of the definitions of and some basic properties of certain families of K-types attached to filtrations of parahoric subgroups associated to periodic lattice flags. Following that, we recall some properties of the depth of a representation $\pi \in \mathcal{E}(G)$ and show how the depth is determined by any one of the above K-types contained in π. Finally, we give the definition of pure and separated minimal K-types (in the sense of [**17**]) and show that any $\pi \in \mathcal{E}(G)$ that does not contain a pure minimal K-type is properly parabolically induced.

For $GL_n(F)$, the notion of minimal K-type was first defined by Moy [**28**]. Let $B = \mathfrak{b}^{\times}$ be a parahoric subgroup associated to a periodic lattice flag $\mathcal{L} = \{L_i \mid i \in \mathbb{Z}\}$, with associated filtration B_i as described in § 3. The structure of B_i/B_{i+1} and the representations of B_i that are trivial in B_{i+1} can be described as follows.

(i) Set $\alpha_i = \dim_{\mathbb{F}_q}(L_i/L_{i+1})$, $1 \leqslant i \leqslant r$, where r is the period of $\mathcal{L}$. If $i = 0$, then

$$B_0/B_1 \simeq \prod_{1 \leqslant i \leqslant r} GL_{\alpha_i}(\mathbb{F}_q).$$

Hence an irreducible representation of B which is trivial on B_1 is the inflation of an irreducible representation of the finite reductive group $\prod_{1 \leqslant i \leqslant r} GL_{\alpha_i}(\mathbb{F}_q)$.

(ii) If $i > 0$, then B_i/B_{i+1} is abelian and the map $X \mapsto 1+X$ from $\mathfrak{b}_i$ to B_i factors to an isomorphism between $\mathfrak{b}_i/\mathfrak{b}_{i+1}$ and B_i/B_{i+1}. Via this map, the group of characters of B_i/B_{i+1} is realized as the cosets $\mathfrak{b}_{-i}/\mathfrak{b}_{-i+1}$. As in § 2, let ψ be a character of F with conductor $\mathfrak{p}$. The coset $\varXi = X + \mathfrak{b}_{-i+1}$, $X \in \mathfrak{b}_{-i}$, is identified with the character

$$\chi_{\varXi}(y) = \psi(\mathrm{tr}(X(y-1))), \quad y \in B_i.$$

We will often use the notation χ_X for the character $\chi_{\varXi}$.

A coset $\varXi = X + \mathfrak{b}_{-i+1}$ in $\mathfrak{b}_{-i}$ is said to be *non-degenerate* if $\varXi$ does not contain any nilpotent elements. A *standard minimal K-type* is a pair (B_i, κ), where κ is an irreducible representation of B_i that is trivial on B_{i+1} and such that the following hold.

(i) If $i = 0$, κ is the inflation of a cuspidal representation of B/B_1.

(ii) If $i > 0$, $\kappa = \chi_{\varXi}$ for some non-degenerate coset $\varXi = X + \mathfrak{b}_{-i+1}$ of $\mathfrak{b}_{-i+1}$ in $\mathfrak{b}_{-i}$.

https://doi.org/10.1017/S1474748003000124 Published online by Cambridge University Press

We remark that in [**28**] Moy calls the above representations minimal K-types. We refer to them as standard minimal K-types in order to distinguish them from other sorts of minimal K-types (see below).

Two standard minimal K-types (B_i, κ) and $(B'_{i'}, \kappa')$ are said to be *associates* if one of the following holds.

(i) $i = i' = 0$, $B/B_1 \simeq B'/B'_1$, and κ and κ' are inflations of equivalent cuspidal representations.

(ii) $i > 0$, $i' > 0$ and $\Xi \cap \operatorname{Ad} G(\Xi') \neq \emptyset$.

Theorem 4.1. *[cf. [**3**, **16**]] Let $\pi \in \mathcal{E}(G)$. Then we have the following.*

(1) *π contains a standard minimal K-type.*

(2) *Any two standard minimal K-types contained in π are associates.*

In [**29**, **30**], Moy and Prasad defined families of K-types, called *unrefined minimal K-types*, for connected reductive p-adic groups. If x belongs to the Bruhat–Tits building $\mathcal{B}(G)$ of G, then $G_{x,t}$, $t \geqslant 0$, and $\mathfrak{g}_{x,t}$, $t \in \mathbb{R}$, denote the filtration subgroups of G_x and $\mathfrak{o}_F$-subalgebras of $\mathfrak{g}$, respectively, defined by Moy and Prasad [**29**, **30**]. Set $\mathfrak{g}_{x,t^+} = \bigcup_{s>t} \mathfrak{g}_{x,s}$ and, if $t \geqslant 0$, set $G_{x,t^+} = \bigcup_{s>t} G_{x,s}$. If $\mathfrak{g} = \mathfrak{gl}_n(F)$, the Moy–Prasad filtrations $\mathfrak{g}_{x,t}$, $t \in \mathbb{R}$, include the filtrations of $\mathfrak{g}$ associated to periodic lattice flags. Suppose that $\mathfrak{L} \subset F^n$ is a periodic lattice flag, $\mathfrak{b}_i = \mathfrak{b}_{\mathfrak{L},i}$, $i \in \mathbb{Z}$, and $B = \mathfrak{b}_{\mathfrak{L}}^{\times}$. Then there exists $x \in \mathcal{B}(G)$ (which, for our purposes, need not be specified) such that $G_x = B$ and the filtrations $\mathfrak{g}_{x,t}$ and $\mathfrak{b}_j$ are the same in the sense that

$$\mathfrak{g}_{x,t} = \mathfrak{b}_{\lceil tr \rceil} \quad \text{and} \quad \mathfrak{g}_{x,t^+} = \mathfrak{b}_{\lfloor tr \rfloor + 1}, \quad t \in \mathbb{R}.$$

As $G_{x,t} = 1 + \mathfrak{g}_{x,t}$ for $t > 0$, it follows that $B_i = G_{x,(i/r)}$ and $B_{i+1} = G_{x,(i/r)^+}$.

It follows from the definition of the depth $\rho(\pi)$ of a representation $\pi \in \mathcal{E}(G)$ [**29**, **30**] that $\rho(\pi)$ is the smallest non-negative rational number such that the set of $G_{x,\rho(\pi)^+}$-fixed vectors in the space of π is non-zero for some $x \in \mathcal{B}(G)$. If $y \in \mathcal{B}(G)$ and the subspace $V_\pi^{G_{y,\rho(\pi)^+}}$ of $G_{y,\rho(\pi)^+}$-fixed vectors in the space V_π of π is non-zero, then the representation of $G_{y,\rho(\pi)}$ given by the action of $G_{y,\rho(\pi)}$ on $V_\pi^{G_{y,\rho(\pi)^+}}$ contains an unrefined minimal K-type. Although $\rho(\pi)$ is defined in terms of the Moy–Prasad filtrations, it is actually determined by any standard minimal K-type contained in π, as the following result shows.

Lemma 4.2. *Let $G = GL_n(F)$ and let (B_i, κ) be a standard minimal K-type.*

(1) *(B_i, κ) is an unrefined minimal K-type.*

(2) *Let r be the period of the lattice chain $\mathcal{L}$ to which the filtration $\{B_j\}_{j \geqslant 0}$ is attached. If $\pi \in \mathcal{E}(G)$ contains (B_i, κ), then $\rho(\pi) = i/r$.*

Proof. If $i = 0$, then the above cuspidality condition on κ is the same as that of Moy and Prasad [**30**, p. 105].

If $i > 0$, we can use the trace map to identify the coset Ξ that corresponds to κ with a coset Ξ^* in the F-dual $\mathfrak{g}^*$ of $\mathfrak{g}$. The non-degeneracy condition on Ξ is equivalent to the non-degeneracy condition on Ξ^* given in [**30**, p. 104]. Hence (1) holds.

For (2), it follows from results of [**29**, **30**] that if π contains an unrefined minimal K-type $(G_{y,t}, \tau)$, then $\rho(\pi) = t$. In view of (1) and the fact that $B_i = G_{x,(i/r)}$, it follows that $\rho(\pi) = i/r$. □

When considering a standard minimal K-type (B_i, κ), the lattice $\mathcal{L}$ can be chosen so that i and the period r of $\mathcal{L}$ are relatively prime [**17**, p. 391]. From now on, we assume that i and r are relatively prime. The cases of pure and separated minimal K-types (see below for the definitions) are treated differently.

Suppose that $i = 0$. Then there exist irreducible cuspidal representations τ_j of $GL_{\alpha_j}(\mathbb{F}_q)$, $1 \leqslant j \leqslant r$, such that κ is the inflation of $\otimes_{1 \leqslant j \leqslant r} \tau_i$ to B. As in [**17**], we say that (B_0, κ) is a *pure* minimal K-type if $\alpha_j = \alpha_k$ and $\tau_j \simeq \tau_k$ for all j and k. Otherwise, (B_0, κ) is a *separated* minimal K-type.

Suppose that $i > 0$. Let $X \in \Xi$. Then $\varpi^i X^r \in \varpi^i \mathfrak{b}_{-ir} = \mathfrak{b}$ and so for each j, $\varpi^i X^r$ induces an $\mathbb{F}_q$-linear map on $L_j/L_{j+1} \simeq \mathbb{F}_q^{\alpha_j}$. As this map is independent of the choice of $X \in \Xi$, we denote it by $T_{\Xi,j}$. Let $f_j(t)$ be the characteristic polynomial of $T_{\Xi,j}$. As discussed in [**17**, § 2], there are two possibilities.

(i) There exists j such that $f_j(t)$ is the product of two relatively prime polynomials of positive degree in $\mathbb{F}_q[t]$. In this case, (B_i, κ) is a *separated* minimal K-type.

(ii) Each $f_j(t)$ is a power of an irreducible polynomial in $\mathbb{F}_q[t]$. In this case, (B_i, κ) is a *pure* minimal K-type.

In much of this paper, we will study characters of representations $\pi \in \mathcal{E}(G)$ that contain pure minimal K-types. Recall that if π arises via parabolic induction from a $\pi_M \in \mathcal{E}(M)$, where M is the Levi component of a parabolic subgroup of G, van Dijk [**38**] gives a formula expressing the character of π in terms of the character of π_M. Hence the proposition below shows that if π does not contain a pure minimal K-type, then the study of the character of π reduces to the study of the character of an irreducible admissible representation of a proper Levi subgroup of G. As a standard minimal K-type must be pure or separated, any such representation π must contain a separated minimal K-type. If π contains a pure minimal K-type, it may be the case that π does not arise via parabolic induction (from a proper Levi subgroup); for example, if π is a discrete series representation. Given a proper parabolic subgroup P of G, we will use the notation Ind_P^G to denote normalized parabolic induction (that is, induction taking unitary representations of the Levi component of P to unitary representations of G). A refinement of the following result will be proved in § 17.

Proposition 4.3. *Suppose that $\pi \in \mathcal{E}(G)$ and π does not contain a pure minimal K-type. Then there exists a proper parabolic subgroup $P = MN$ of G and a $\pi_M \in \mathcal{E}(M)$ such that $\pi = \mathrm{Ind}_P^G \pi_M$.*

https://doi.org/10.1017/S1474748003000124 Published online by Cambridge University Press

Before proving the proposition, we show that certain representations must contain a pure minimal K-type. Recall [**43**] that the support of a representation $\pi \in \mathcal{E}(G)$ is defined as follows. There exists a parabolic subgroup $P = MN$ of G and an irreducible supercuspidal representation τ of the Levi component M of P such that π is a subquotient of $\mathrm{Ind}_P^G\,\tau$. There exists a partition $\alpha = (\alpha_1, \dots, \alpha_r) \in \mathcal{P}(n)$ such that $M \simeq \prod_{1 \leqslant i \leqslant r} GL_{\alpha_i}(F)$ and irreducible supercuspidal representations τ_i of $GL_{\alpha_i}(F)$ such that $\tau \simeq \otimes_{1 \leqslant i \leqslant r} \tau_i$. The set $\{\tau_1, \dots, \tau_r\}$ (taken in any order) is the support of π.

Lemma 4.4. *Suppose that there exists $d \mid n$ and an irreducible supercuspidal representation τ of $GL_d(F)$ such that the support of π is a subset of $\{|\det(\cdot)|^\ell \tau \mid \ell \in \mathbb{Z}\}$. Then π contains a pure minimal K-type.*

Proof. By assumption, there exist integers b_j, $1 \leqslant j \leqslant n/d$, such that π is an irreducible subquotient of $\mathrm{Ind}_P^G(\otimes_{1 \leqslant j \leqslant n/d} |\det(\cdot)|^{b_j} \tau)$, where the Levi component M of P is the direct product of n/d copies of $GL_d(F)$. Without loss of generality, we can assume that P contains an upper triangular Borel subgroup.

By Theorem 5.1 of [**17**], the supercuspidal representation τ contains a pure minimal K-type (B_i, κ), where $B = \mathfrak{b}_{\mathfrak{L}}^\times$ for some periodic lattice chain $\mathfrak{L} \subset F^d$. After conjugating by an element of $GL_d(F)$, if necessary, we can assume (see [**17**, § 4]) that $\mathfrak{b}_{\mathfrak{L}} = \mathfrak{b}_{(m)^{d/m}}$ for some divisor m of d. Let K (respectively, K^+) be the direct product of n/d copies of B_i (respectively, B_{i+1}) and let $\kappa^{(n/d)}$ be the n/d-fold tensor product of κ with itself. Then $(K, \kappa^{(n/d)})$ is an unrefined minimal K-type contained in $\otimes_{1 \leqslant j \leqslant n/d} |\det(\cdot)|^{b_j} \tau$.

Suppose that $i = 0$. Then $B_{(m)^{n/m}}/B_{(m)^{n/m},1} \simeq K/K^+$, $\kappa^{(n/d)}$ lifts to a representation of $B_{(m)^{n/m}}$ and $(B_{(m)^{n/m}}, \kappa^{(n/d)})$ is a pure minimal K-type of depth zero. By Theorem 5.2 (2) of [**30**], π contains $(B_{(m)^{n/m}}, \kappa^{(n/d)})$.

Suppose that $i > 0$. Let N be the unipotent radical of P, and let N^- be the unipotent radical of the parabolic subgroup opposite to P. There exists a permutation matrix $w \in G$ such that, for any positive integer ℓ,

$$wB_{(m)^{n/m},\ell}w^{-1} = (wB_{(m)^{n/m},\ell}w^{-1} \cap N^-)(wB_{(m)^{n/m},\ell}w^{-1} \cap M)(wB_{(m)^{n/m},\ell}w^{-1} \cap N),$$

$wB_{(m)^{n/m},ni/d}w^{-1} \cap M = K$ and $wB_{(m)^{n/m},(ni/d)+1}w^{-1} \cap M = K^+$, so $\kappa^{(n/d)}$ extends to a character of $wB_{(m)^{n/m},ni/d}w^{-1}$, which is trivial on $wB_{(m)^{n/m},ni/d}w^{-1} \cap N$ and $wB_{(m)^{n/m},ni/d}w^{-1} \cap N^-$, and is a pure minimal K-type. By Theorem 4.5 of [**30**], this pure minimal K-type is contained in π. □

Proof of Proposition 4.3. By assumption, the support of π cannot be of the form given in Lemma 4.4. Results of [**43**] imply that π has the desired form. □

5. Conjugacy and filtrations of parahoric subalgebras

Here we summarize some results that will be used later in the paper for comparing unrefined minimal K-types and germs of characters.

Let E/F be a tamely ramified extension of degree d, where d is a positive divisor of n. As in § 3, set $n' = n/d$, $e = e(E/F)$, $f = f(E/F)$, $\mathfrak{g}' = \mathfrak{gl}_{n'}(E)$ and $G' = GL_{n'}(E)$.

Choose a prime element ϖ_E in E, and embed E in $\mathfrak{gl}_d(F)$ and $\mathfrak{g}'$ in $\mathfrak{g}$ as in § 3. Fix a positive integer j such that e and j are relatively prime. Let $s \in E$ be such that $s \in \mathfrak{p}_E^{-j} - \mathfrak{p}_E^{-j+1}$ and the image of $\varpi_F^j s^e$ in $\mathfrak{o}_E/\mathfrak{p}_E$ generates $\mathfrak{o}_E/\mathfrak{p}_E$ over $\mathfrak{o}_F/\mathfrak{p}_F$. Note that $E = F(s)$. Viewing E as the centre of $\mathfrak{g}'$, we consider s as an element in $\mathfrak{g}$ whose centralizer is $\mathfrak{g}'$.

Given a periodic lattice flag $\mathcal{L}'$ in $E^{n'}$, let $\xi(\mathcal{L}')$ be the periodic lattice flag in F^n defined in § 3. Notation for parahoric subalgebras and subgroups corresponding to periodic lattice flags will be as in § 3.

Lemma 5.1 (cf. [17]). *Fix a positive divisor u of n'. Let $\mathfrak{L} = \xi(\mathfrak{L}'^{(u)^{n'/u}})$ and $\mathfrak{b} = \mathfrak{b}_{\mathfrak{L}}$. Set $\mathfrak{b}_i'^{\perp} = \mathfrak{b}_i \cap \mathfrak{g}'^{\perp}$, $i \in \mathbb{Z}$, where $\mathfrak{g}'^{\perp}$ is the orthogonal complement of $\mathfrak{g}'$ in $\mathfrak{g}$, relative to the trace map. Then the following hold.*

(1) $\mathfrak{b}_i = \mathfrak{b}_i' \oplus \mathfrak{b}_i'^{\perp}$.

(2) $B_i = B_i'(1 + \mathfrak{b}_i'^{\perp})$, $i \geqslant 1$.

(3) $\operatorname{ad} s : \mathfrak{b}_i'^{\perp}/\mathfrak{b}_{i+1}'^{\perp} \to \mathfrak{b}_{i-n'j/u}'^{\perp}/\mathfrak{b}_{i+1-n'j/u}'^{\perp}$ *is an isomorphism*

Remarks 5.2. Parts (1) and (3) are Lemma 4.4 ($u = 1$) and Lemma 4.8 ($u > 1$) of [**17**]. For (2), note that $B_i = 1 + \mathfrak{b}_i = 1 + \mathfrak{b}_i' + \mathfrak{b}_i'^{\perp}$ by (1). And $(1 + \mathfrak{b}_i')\mathfrak{b}_i'^{\perp} = \mathfrak{b}_i'^{\perp}$, as $\mathfrak{b}_i'^{\perp}$ is stable under left multiplication by $\mathfrak{b}'$. Hence $B_i = (1 + \mathfrak{b}_i')(1 + \mathfrak{b}_i'^{\perp}) = B_i'(1 + \mathfrak{b}_i'^{\perp})$.

Lemma 5.3. *Let $\mathfrak{L}$, $\mathfrak{b}$ and $\mathfrak{b}_i'^{\perp}$ be as in Lemma 5.1. Suppose that $Z \in \mathfrak{b}'_{(1)^{n'},-(n'j)+1}$. Then the following hold.*

(1) $\operatorname{ad}(s + Z) : \mathfrak{b}_i'^{\perp}/\mathfrak{b}_{i+1}'^{\perp} \to \mathfrak{b}_{i-n'j/u}'^{\perp}/\mathfrak{b}_{i+1-n'j/u}'^{\perp}$ *is an isomorphism.*

(2) *Let v be a positive integer. Then*

$$s + Z + \mathfrak{b}_{-(n'j/u)+v} = B_v \cdot (s + Z + \mathfrak{b}'_{-(n'j/u)+v}).$$

Proof. Note that, as $\mathfrak{o}\mathfrak{b}_i'^{\perp} \subset \mathfrak{b}_i'^{\perp}$ and $\mathfrak{p}\mathfrak{b}_i'^{\perp} \subset \mathfrak{b}_{i+1}'^{\perp}$, $V = \mathfrak{b}_i'^{\perp}/\mathfrak{b}_{i+1}'^{\perp}$ is a $\mathbb{F}_q$-vector space. Given $X \in \mathfrak{b}'_{-n'j/u} = \varpi_E^{-j}\mathfrak{b}'$, the map $Y \mapsto \varpi_E^j[X, Y]$ from $\mathfrak{b}_i'^{\perp}$ to $\mathfrak{b}_i'^{\perp}$ induces a linear transformation $T_X : V \to V$. It is easy to see that T_s and T_Z are the semisimple and nilpotent parts of T_{s+Z}, respectively. By Lemma 5.1 (3), as left multiplication by ϖ_E^j induces a vector space isomorphism of $\mathfrak{b}_{i-n'j/u}'^{\perp}/\mathfrak{b}_{i+1-n'j/u}'^{\perp}$ onto $\mathfrak{b}_i'^{\perp}/\mathfrak{b}_{i+1}'^{\perp}$, the map T_s is an isomorphism. As T_s is the semisimple part of T_{s+Z}, T_{s+Z} is also an isomorphism. Hence (1) holds.

Part (2) follows from (1) by a standard type of argument (see, for example, the proof of Lemma 3.2 of [**17**]). □

Lemma 5.4. *Let $Z \in \mathfrak{b}'_{(1)^{n'},-(n'j)+1}$.*

(1) *Let $\alpha \in \mathcal{P}(n)$. If $\mathcal{O}_G(s + Z) \cap \mathfrak{b}_{\alpha,-\lfloor jr(\alpha)/e\rfloor} \neq \emptyset$, then $\alpha = (f\beta)^e$ for some $\beta \in \mathcal{P}(n')$.*

(2) *Let $\beta \in \mathcal{P}(n')$. If $\operatorname{Ad} g(s + Z) \in \mathfrak{b}_{\xi(\mathfrak{L}'^{\beta}),-jr(\beta)}$, then $g \in B_{\xi(\mathfrak{L}'^{\beta})}G'$.*

Proof. Let $g \in G$. Set $X = \operatorname{Ad} g(s+Z)$. Note that $s+Z = s(1+s^{-1}Z)$ and $s^{-1}\varpi_E^{-j}$ is a root of unity in $\mathfrak{o}_E^\times$, so $s^{-1}\varpi_E^{-j} \in B'_{(1)^{n'}}$ and $s^{-1}Z \in \mathfrak{b}'_{(1)^{n'},1}$. Hence $1+s^{-1}Z \in B'_{(1)^{n'},1}$ and

$$X^e \in \operatorname{Ad} g(s+Z)^e = \operatorname{Ad} g(s^e(1+s^{-1}Z)^e) \in g(\varpi_F^{-j}\mathfrak{o}_E^\times B'_{(1)^{n'},1})g^{-1},$$

which implies $\det(\varpi_F^j X^e) \in \mathfrak{o}^\times$.

Suppose that e does not divide $r(\alpha)$ and $X \in \mathfrak{b}_{\alpha,-\lfloor jr(\alpha)/e\rfloor}$. Let $m = \lfloor jr(\alpha)/e\rfloor$. Then

$$\varpi_F^j X^e \in \varpi_F^j \mathfrak{b}_{\alpha,-me} = \mathfrak{b}_{\alpha,jr(\alpha)-me} \subset \mathfrak{b}_{\alpha,1},$$

which, as $\det(\mathfrak{b}_{\alpha,1}) \subset \mathfrak{p}_F$, is a contradiction. Therefore, $\mathcal{O}_G(s+Z) \cap \mathfrak{b}_{\alpha,-\lfloor jr(\alpha)/e\rfloor} = \emptyset$ whenever e does not divide $r(\alpha)$.

Suppose that $\alpha \in \mathcal{P}(n)$ and e divides $r(\alpha)$. Set $r = r(\alpha)$. Given $Y \in \mathfrak{b}_m$, $m \in \mathbb{Z}$, the image of $Y + \mathfrak{b}_{m+1}$ in $\mathfrak{b}_m/\mathfrak{b}_{m+1}$ is determined by an r-tuple $(\bar{Y}_1, \dots, \bar{Y}_r)$, where $\bar{Y}_i \in \operatorname{Hom}_{\mathbb{F}_q}(L_{i-1}^\alpha/L_i^\alpha, L_{i-1+m}^\alpha/L_{i+m}^\alpha)$, $1 \leqslant i \leqslant r$. For convenience, we will treat subscripts on the α_i and the $\bar{Y}_i$ as integers modulo r. Let g and X be as above, and set $Y = X^e$. Then

$$\bar{Y}_i = \bar{X}_{i-(e-1)(jr/e)} \circ \cdots \circ \bar{X}_{i-(jr/e)} \circ \bar{X}_i, \quad 1 \leqslant i \leqslant r.$$

As $X \in \mathfrak{b}_{-jr/e}$, $\varpi_F^j X^e \in \varpi_F^j \mathfrak{b}_{-jr} = \mathfrak{b}_{\alpha,0}$. So $\varpi_F^j X^e$ is an element of $\mathfrak{b}_{\alpha,0}$ whose determinant lies in $\mathfrak{o}^\times$. Hence $\varpi_F^j X^e \in B_{\alpha,0}$. This implies that each $\bar{Y}_i$ is an isomorphism. From the above expression for $\bar{Y}_i$, it follows that $\alpha_i = \alpha_{i-(jr/e)} = \cdots = \alpha_{i-(e-1)(jr/e)}$. Using the fact that e and j are relatively prime, we have $\alpha_i = \alpha_{i+(r/e)} = \cdots = \alpha_{i+(e-1)(r/e)}$, which is equivalent to $\gamma = (\alpha_1, \dots, \alpha_{r/e}) \in \mathcal{P}(n/e)$ and $\alpha = \gamma^e$.

Suppose that $\alpha = \gamma^e$ for some $\gamma \in \mathcal{P}(n/e)$. Let $g \in G$. Suppose that $\operatorname{Ad} g(s+Z) \in \mathfrak{b}_{\gamma^e,-jer(\gamma)}$. Because

$$\det(\varpi_E^j \operatorname{Ad} g(s+Z)) \in \det(\varpi_E^j g \varpi_E^{-j}\mathfrak{o}_E^\times B'_{(1)^{n'}} g^{-1}) \subset \mathfrak{o}^\times \quad \text{and} \quad \varpi_E^j \mathfrak{b}_{\gamma^e,-jer(\gamma)} = \mathfrak{b}_{\gamma^e,0},$$

we have $\varpi_E^j \operatorname{Ad} g(s+Z) \in B_{\gamma^e}$. By definition, s is E/F-cuspidal in the sense of [**41**, § VI.2]. Also, $1+s^{-1}Z \in \mathfrak{b}'_{(1)^{n'}}$, and $\varpi_E^{-j} B_{\gamma^e}$ is G-conjugate to the subgroup denoted $\zeta_G^{nd/e} I_{\gamma^e,G}$ in [**41**]. Hence the statement of the lemma for $\alpha = \gamma^e$ is equivalent to that of Lemma VI.3 of [**41**]. □

The following is a restatement of Lemma 8 of [**14**].

Lemma 5.5 (cf. [14]). *Let $Z_1, Z_2 \in \mathfrak{b}'_{(1)^{n'},-(n'j)+1}$. Then $\operatorname{Ad} g(s+Z_1) = s+Z_2$ implies $g \in G'$.*

Given parahoric subalgebras $\mathfrak{b} \subset \mathfrak{g}$ and $\mathfrak{b}' \subset \mathfrak{g}'$, and an integer i, let $[\mathfrak{b}_i]$ and $[\mathfrak{b}'_i]$ be the characteristic functions of $\mathfrak{b}_i$ and $\mathfrak{b}'_i$, respectively. The following proposition is a consequence of Lemmas 5.4 and 5.5. Let

$$Z \in \mathfrak{b}'_{(1)^{n'},-(n'j)+1}.$$

By Lemma 5.5, we have $C_G(s+Z) = C_{G'}(Z)$. Fix left Haar measures on G, G' and $C_{G'}(Z)$. These determine a G-invariant measure on $\mathcal{O}_G(s+Z) \simeq G/C_{G'}(Z)$ and a G'-invariant measure on $\mathcal{O}_{G'}(Z) \simeq G'/C_{G'}(Z)$.

https://doi.org/10.1017/S1474748003000124 Published online by Cambridge University Press

Proposition 5.6. *Let $Z \in \varpi_E^{-j}\mathfrak{b}'_{(1)^{n'},1}$. Let the measures on $\mathcal{O}_G(s+Z)$ and $\mathcal{O}_{G'}(Z)$ be as above. Then the following hold.*

(1) *If $\alpha \in \mathcal{P}(n)$, $\mu_{\mathcal{O}_G(s+Z)}([\mathfrak{b}_{\alpha,-\lfloor jr(\alpha)/e\rfloor}]) = 0$ unless $\alpha = (f\beta)^e$ for some $\beta \in \mathcal{P}(n')$.*

(2) *If $\beta \in \mathcal{P}(n')$,*

$$\mu_{\mathcal{O}_G(s+Z)}([\mathfrak{b}_{(f\beta)^e,-jr(\beta)}]) = \upsilon_G(B_{(n)})\upsilon_{G'}(B'_{(n')})^{-1}u_n(q)^{-1}u_{n'}(q^f) \times u_{(f\beta)^e}(q)u_\beta(q^f)^{-1}\mu_{\mathcal{O}_{G'}(Z)}([\mathfrak{b}'_{\beta,-jr(\beta)}]).$$

Proof. Part (1) is immediate from Lemma 5.4 (1). For (2), assume that $\alpha = (f\beta)^e$ for some $\beta \in \mathcal{P}(n')$. Let $\mathrm{d}g^*$ denote the G-invariant measure on G/G' determined by the chosen Haar measures on G and on G'. Let $\mathrm{d}x'^*$ denote the G'-invariant measure on $G'/C_{G'}(Z)$ determined by the Haar measures on G' and on $C_{G'}(Z)$. As $\mathfrak{b}_{(f\beta)^e}$ is G-conjugate to $\mathfrak{b}_{\xi(\mathfrak{L}'^\beta)}$, we may replace $\mathfrak{b}_{(f\beta)^e,-jr(\beta)}$ with $\mathfrak{b}_{\xi(\mathfrak{L}'^\beta),-jr(\beta)}$. Then

$$\mu_{\mathcal{O}_G(s+Z)}([\mathfrak{b}_{\xi(\mathfrak{L}'^\beta),-jr(\beta)}]) = \int_{G/G'}\int_{G'/C_{G'}(Z)}[\mathfrak{b}_{\xi(\mathfrak{L}'^\beta),-jr(\beta)}](\mathrm{Ad}(gx')(s+Z))\,\mathrm{d}x'^*\,\mathrm{d}g^*.$$

Let $g \in G$ and $x' \in G'$. Suppose that $\mathrm{Ad}(gx')(s+Z) \in \mathfrak{b}_{\xi(\mathfrak{L}'^\beta),-jr(\beta)}$. By Lemma 5.4, $gx' \in B_{\xi(\mathfrak{L}'^\beta)}G'$. Let $g = kg'$, $k \in B_{\xi(\mathfrak{L}'^\beta)}$, $g' \in G'$. Then, since both ϖ_E^{-j} and $B_{\xi(\mathfrak{L}'^\beta)}$ normalize $\mathfrak{b}_{\xi(\mathfrak{L}'^\beta),i}$, $i \in \mathbb{Z}$,

$$s + \mathrm{Ad}(g'x')(Z) = \mathrm{Ad}(g'x')(s+Z) \in \mathrm{Ad}\,k^{-1}(\mathfrak{b}_{\xi(\mathfrak{L}'^\beta),-jr(\beta)}) = \mathfrak{b}_{\xi(\mathfrak{L}'^\beta),-jr(\beta)}.$$

Since $s \in \mathfrak{b}'_{\beta,-jr(\beta)}$ and $\mathrm{Ad}(g'x')(Z) \in \mathfrak{g}'$, it follows that $\mathrm{Ad}(g'x')(Z) \in \mathfrak{b}'_{\beta,-jr(\beta)}$. Therefore, we may rewrite the above integral as

$$\begin{aligned}\mu_{\mathcal{O}_G(s+Z)}&([\mathfrak{b}_{\xi(\mathfrak{L}'^\beta),-jr(\beta)}])\\ &= \upsilon_{G/G'}(B_{\xi(\mathfrak{L}'^\beta)}G')\int_{G'/C_{G'}(Z)}[\mathfrak{b}'_{\beta,-jr(\beta)}](\mathrm{Ad}\,x'(Z))\,\mathrm{d}x'^*\\ &= \upsilon_G(B_{\xi(\mathfrak{L}'^\beta)})\upsilon_{G'}(B'_\beta)^{-1}\int_{G'/C_{G'}(Z)}[\mathfrak{b}'_{\beta,-jr(\beta)}](\mathrm{Ad}\,x'(Z))\,\mathrm{d}x'^*\\ &= \upsilon_G(B_{(n)})\upsilon_{G'}(B'_{(n')})^{-1}[B_{(n)} : B_{(f\beta)^e}]^{-1}[B'_{(n')} : B'_\beta]\\ &\qquad\times\int_{G'/C_{G'}(Z)}[\mathfrak{b}'_{\beta,-jr(\beta)}](\mathrm{Ad}\,x'(Z))\,\mathrm{d}x'^*.\end{aligned}$$

To finish the proof, note that (3.1) can be used to evaluate the above group indices. □

6. Pure minimal K-types of positive depth

Let E, d, n', $\mathfrak{g}'$, G', etc., be as in § 3. We will use the notation of § 3 concerning periodic lattice flags, parahoric subgroups and subalgebras and the associated filtrations.

Given $\pi \in \mathcal{E}(G)$ such that $\rho(\pi) > 0$ and π contains a pure minimal K-type, there is a finite family of pure minimal K-types whose multiplicities in π determine the germ

https://doi.org/10.1017/S1474748003000124 Published online by Cambridge University Press

of the character Θ_π of π. This will follow from results of §§ 8 and 12. In this section, after defining some notation, we recall a result of Howe and Moy which says that any π as above must contain a particular kind of pure minimal K-type. Then we describe the above-mentioned family of pure minimal K-types, and give the decomposition of a particular subspace of the space of π in terms of these K-types.

Let $\mathfrak{L}'^1 = \mathfrak{L}'^{(1)^{n'}}$ and $\mathfrak{L}'^2 = \mathfrak{L}'^{(n')}$ be the periodic lattice flags in $E^{n'}$ attached to the partitions $(1)^{n'}$ and (n'), respectively. Let $\mathfrak{L}^1 = \xi(\mathfrak{L}'^1)$ and $\mathfrak{L}^2 = \xi(\mathfrak{L}'^2)$ be the associated periodic lattice flags in F^n. Attached to $\mathfrak{L}'^h$ and $\mathfrak{L}^h$, $h = 1, 2$, we have parahoric subgroups in G' and G and parahoric subalgebras in $\mathfrak{g}'$ and $\mathfrak{g}$, respectively. Set

$$\mathfrak{q}_i = \mathfrak{b}_{\mathfrak{L}^1,i}, \quad \mathfrak{r}_i = \mathfrak{b}_{\mathfrak{L}^2,i}, \quad i \in \mathbb{Z},$$
$$Q_i = B_{\mathfrak{L}^1,i}, \quad R_i = B_{\mathfrak{L}^2,i}, \quad i \geqslant 0,$$

If $i = 0$ in any of the above filtrations, we may suppress the subscript; for example, we may write $\mathfrak{r}$ for $\mathfrak{r}_0$. The intersection of any of the above with $\mathfrak{g}'$ will be denoted by the same notation with a prime added. By (3.2),

$$\mathfrak{q}'_i = \mathfrak{q}_i \cap \mathfrak{g}' = \mathfrak{b}'_{(1)^{n'},i}, \quad \mathfrak{r}'_i = \mathfrak{r}_i \cap \mathfrak{g}' = \mathfrak{b}'_{(n'),i}, \quad \varpi_E \mathfrak{q}_i = \mathfrak{q}_{i+n'}, \quad \varpi_E \mathfrak{r}_i = \mathfrak{r}_{i+1}, \quad i \in \mathbb{Z}.$$

Fix a positive integer j that is relatively prime to $e = e(E/F)$. Let $s \in E$ be defined as at the beginning of § 5. Note that

$$s + \mathfrak{q}_{-n'j+1} \subset \varpi_E^{-j}(\mathfrak{o}_E^\times + \mathfrak{q}_1) \subset \varpi_E^{-j} Q,$$

so the coset consists of invertible matrices and thus does not contain any nilpotent elements. Hence the character χ_s of $Q_{n'j}$ defined by

$$\chi_s(x) = \psi(\mathrm{tr}(s(x-1))), \quad x \in Q_{n'j}$$

is an standard minimal K-type. It is easy to see that the K-type is pure (see § 4).

Recall from § 4 that we are denoting the depth of a representation $\pi \in \mathcal{E}(G)$ by $\rho(\pi)$.

Lemma 6.1 (cf. § 4 of [17]). *Suppose that $\pi \in \mathcal{E}(G)$ is such that $\rho(\pi) > 0$ and π contains a pure minimal K-type. Then π contains $(Q_{n'j}, \chi_s)$ for some E, n', j and s as above.*

Suppose that $\pi \in \mathcal{E}(G)$ contains $(Q_{n'j}, \chi_s)$.

Note that every $\mathfrak{o}_E$-lattice occurring in $\mathfrak{L}'^2$ occurs in $\mathfrak{L}'^1$. This implies that each $\mathfrak{o}_F$-lattice occurring in $\mathfrak{L}^2$ also occurs in $\mathfrak{L}^1$. Hence $\mathfrak{q} = \mathfrak{b}_{\mathfrak{L}^1} \subset \mathfrak{b}_{\mathfrak{L}^2} = \mathfrak{r}$ and $\mathfrak{q}_1 \supset \mathfrak{r}_1$. Together with the above information regarding the effect of left multiplication by ϖ_E on $\mathfrak{q}_i$ and $\mathfrak{r}_i$, this implies

$$R_{j+1} \subset Q_{n'j+1} \subset Q_{n'j} \subset R_j.$$

Hence

$$V_\pi^{R_{j+1}} \supset V_\pi^{Q_{n'j+1}} \neq \{0\}.$$

https://doi.org/10.1017/S1474748003000124 Published online by Cambridge University Press

Let $\kappa(\pi, R_j)$ be the representation of R_j given by the action of $\pi|_{R_j}$ on $V_\pi^{R_{j+1}}$. Recall that, since the lattice flag $\mathfrak{L}'^2$ has period 1, $\mathfrak{L}^2$ has period e (see remarks following (3.2)). As mentioned in § 4, there exists an $x \in \mathcal{B}(G)$ for which

$$R_j = G_{x,j/e} \supsetneq G_{x,(j/e)^+} = R_{j+1},$$

By Lemma 4.2, since the period of the lattice flag $\mathfrak{L}^1$ is $n'e$ and π contains $(Q_{n'j}, \chi_s)$, it follows that $\rho(\pi) = (n'j)/(n'e) = j/e$. On the other hand, since π contains the trivial representation of $R_{j+1} = G_{x,(j/e)^+}$, Theorem 5.2 of [**30**] implies that $\kappa(\pi, R_j)$ is a finite direct sum of unrefined minimal K-types. Because the filtration $\mathfrak{r}_i$ is the standard one coming from powers of the nilradical of $\mathfrak{r}$, these unrefined minimal K-types are standard minimal K-types. If (χ_X, R_j) is an standard minimal K-type, let $m(\kappa(\pi, R_j), \chi_X)$ denote the multiplicity of (χ_X, R_j) in $\kappa(\pi, R_j)$.

Given $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathcal{P}^0(n')$, let Y_α be the upper triangular nilpotent matrix in $\mathfrak{g}'$ defined as in § 2. It is well known that the set $\{Y_\alpha \mid \alpha \in \mathcal{P}^0(n')\}$ corresponds bijectively to the set $\Omega_{G'}(0)$ of nilpotent G'-orbits in $\mathfrak{g}'$. Since $R'/R'_1 \simeq GL_{n'}(\mathbb{F}_{q^f})$, the set $\{Y_\alpha \mid \alpha \in \mathcal{P}^0(n')\}$ also corresponds bijectively to the set of nilpotent R'/R'_1-orbits in $\mathfrak{r}'/\mathfrak{r}'_1$.

Lemma 6.2. *Every irreducible component of $\kappa(\pi, R_j)$ is of the form $(R_j, \chi_{\operatorname{Ad} k(s+\varpi_E^{-j} Y_\alpha)})$ for some $k \in R$ and some $\alpha \in \mathcal{P}^0(n')$. Furthermore,*

$$m(\kappa(\pi, R_j), \chi_{\operatorname{Ad} k(s+\varpi_E^{-j} Y_\alpha)}) = m(\kappa(\pi, R_j), \chi_{s+\varpi_E^{-j} Y_\alpha}) \quad \forall k \in R.$$

Proof. Let (R_j, χ_X) be such that $m(\kappa(\pi, R_j), \chi_X) \neq 0$. As π contains $(Q_{n'j}, \chi_s)$ and (R_j, χ_X), by results of [**3**] and [**15**] (see Theorem 4.1 (2)), there exists $g \in G$ such that

$$\operatorname{Ad} g(s + \mathfrak{q}_{-n'j+1}) \cap (X + \mathfrak{r}_{-j+1}) \neq \emptyset.$$

Choose $Y \in \mathfrak{q}_{-n'j+1}$ such that $\operatorname{Ad} g(s+Y) \in X + \mathfrak{r}_{-j+1}$. By Lemma 5.3 (2) (with $u = 1$), there exist $y \in Q_1$ and $Y' \in \mathfrak{q}'_{-n'j+1}$ such that $s + Y = \operatorname{Ad} y(s + Y')$. By Lemma 5.4 (2), $gy \in RG'$. Write $gy = kg'$ with $k \in R$ and $g' \in G'$. We have

$$X \in \operatorname{Ad} k(s + \operatorname{Ad} g'(Y')) + \mathfrak{r}_{-j+1}.$$

Because $\varpi_E^j Y' \in \mathfrak{q}'_1$, we have $(\operatorname{Ad} g'(\varpi_E^j Y'))^m \to 0$ as $m \to \infty$. Combining this with $\operatorname{Ad} g'(\varpi_E^j Y') \in \mathfrak{r}'$, we see that the image of $\operatorname{Ad} g'(\varpi_E^j Y')$ in $\mathfrak{r}'/\mathfrak{r}'_1$ is nilpotent. Hence there exists $k' \in R'$ and $\alpha \in \mathcal{P}^0(n')$ such that

$$\operatorname{Ad} g'(Y') \in \operatorname{Ad} k'^{-1} \varpi_E^{-j} Y_\alpha + \mathfrak{r}'_{-j+1}.$$

It follows that

$$X \in \operatorname{Ad}(kk')(s + \varpi_E^{-j} Y_\alpha) + \mathfrak{r}_{-j+1}.$$

The final statement of the lemma follows from the fact that R_j is normal in R. □

Corollary 6.3. *We have*

$$\kappa(\pi, R_j) = \bigoplus_{\alpha\in\mathcal{P}^0(n')} m(\kappa(\pi,R_j),\chi_{s+\varpi_E^{-j}Y_\alpha}) \bigoplus_{k\in R/C_{R'}(Y_\alpha)R_1} \chi_{\mathrm{Ad}\,k(s+\varpi_E^{-j}Y_\alpha)}.$$

Proof. Fix $\alpha\in\mathcal{P}^0(n')$. Let $k\in R$ be such that $\chi_{\mathrm{Ad}\,k(s+\varpi_E^{-j}Y_\alpha)} = \chi_{s+\varpi_E^{-j}Y_\alpha}$. Then

$$\mathrm{Ad}\,k(s+\varpi_E^{-j}Y_\alpha)\in s+\varpi_E^{-j}Y_\alpha+\mathfrak{r}_{-j+1}.$$

Note that, by definition, $\mathfrak{r}$ is the parahoric $\mathfrak{b}$ of Lemmas 5.1 and 5.3 in the case $u=n'$, and $\varpi_E^{-j}Y_\alpha\in\varpi_E^{-j}\mathfrak{b}'_{(1)^{n'},1}=\mathfrak{q}'_{-n'j+1}$. By Lemma 5.3 (2),

$$R_1\cdot(s+\varpi_E^{-j}Y_\alpha+\mathfrak{r}'_{-j+1}) = s+\varpi_E^{-j}Y_\alpha+\mathfrak{r}_{-j+1}.$$

Let $k_1\in R_1$ and $Z\in\mathfrak{r}'_{-j+1}$ be such that

$$\mathrm{Ad}(k_1^{-1}k)(s+\varpi_E^{-j}Y_\alpha) = s+\varpi_E^{-j}Y_\alpha+Z.$$

Now, since

$$\varpi_E^{-j}Y_\alpha,\ \varpi_E^{-j}Y_\alpha+Z\in\mathfrak{b}'_{(1)^{n'},-n'j+1},$$

we can apply Lemma 5.5 to conclude that $k_1^{-1}k\in G'$. We have $k_1^{-1}k\in G'\cap R=R'$, and therefore $k\in R_1R'=R'R_1$. We now have, setting $k'=k_1^{-1}k$,

$$\mathrm{Ad}\,k(s+\varpi_E^{-j}Y_\alpha)\in(\mathrm{Ad}\,k'(s+\varpi_E^{-j}Y_\alpha)+\mathfrak{r}_{-j+1})\cap(s+\varpi_E^{-j}Y_\alpha+\mathfrak{r}_{-j+1}),$$

which implies that

$$\mathrm{Ad}\,k'(s+\varpi_E^{-j}Y_\alpha)\in s+\varpi_E^{-j}Y_\alpha+\mathfrak{r}'_{-j+1}.$$

Now k' commutes with s, and ϖ_E belongs to the centre of G'. Hence we have $\mathrm{Ad}\,k'(Y_\alpha)\in Y_\alpha+\mathfrak{r}'_1$. That is, $k'\in C_{R'}(Y_\alpha)R'_1$. It follows that $k\in R_1C_{R'}(Y_\alpha)R'_1=C_{R'}(Y_\alpha)R_1$. □

Suppose that $\pi'\in\mathcal{E}(G')$ contains the pure minimal K-type $(Q'_{n'j},\chi'_s)$. The representation $\kappa(\pi',R'_j)$ of R'_j on $V_{\pi'}^{R'_{j+1}}$ decomposes into a direct sum of pure minimal K-types of the form (R'_j,χ'_X), $X\in\mathfrak{r}'_{-j}$. The following lemma and corollary, whose proofs are similar to (but easier than) those of Lemma 6.2 and Corollary 6.3, describe those X that appear and give the decomposition of $\kappa(\pi',R'_j)$.

Lemma 6.4. *Each irreducible component of* $\kappa(\pi',R'_j)$ *is of the form*

$$\Bigl(R'_j,\chi'_{s+\varpi_E^{-j}\,\mathrm{Ad}\,k'(Y_\alpha)}\Bigr),$$

for some $\alpha\in\mathcal{P}^0(n')$ *and some* $k'\in R'$. *Furthermore,*

$$m(\kappa(\pi',R'_j),\chi'_{s+\varpi_E^{-j}Y_\alpha}) = m(\kappa(\pi',R'_j),\chi'_{s+\varpi_E^{-j}\,\mathrm{Ad}\,k'(Y_\alpha)}),\quad k'\in R'.$$

Corollary 6.5. *We have*

$$\kappa(\pi',R'_j) = \bigoplus_{\alpha\in\mathcal{P}^0(n')} m(\kappa(\pi',R'_j),\chi'_{s+\varpi_E^{-j}Y_\alpha}) \bigoplus_{k\in R'/C_{R'}(Y_\alpha)R'_1} \chi_{s+\mathrm{Ad}\,k(\varpi_E^{-j}Y_\alpha)}.$$

https://doi.org/10.1017/S1474748003000124 Published online by Cambridge University Press

7. Hecke algebra isomorphisms and matching of certain K-types

We continue to use notation from § 6. Set

$$\ell = [\tfrac{1}{2}(n'j+1)], \quad m = [\tfrac{1}{2}n'j]+1 \quad \text{and} \quad \mathfrak{q}_i'^{\perp} = \mathfrak{q}_i \cap \mathfrak{g}'^{\perp}, \quad i \in \mathbb{Z}.$$

Let

$$J = 1 + \mathfrak{q}_{n'j} + \mathfrak{q}_\ell'^{\perp},$$
$$J_+ = 1 + \mathfrak{q}_{n'j} + \mathfrak{q}_m'^{\perp}.$$

It follows from Lemma 5.1 (2) that $Q'_\ell J = Q_\ell$ and $Q'_m J_+ = Q_m$. As $\mathfrak{q}_i \cap \mathfrak{g}' = \mathfrak{q}'_i$, we have $J \cap G' = Q'_{n'j}$. Extend χ_s to J_+ by extending trivially across $1 + \mathfrak{q}_m'^{\perp}$. The representation σ of J is defined to be χ_s if $J = J_+$, that is, if $\ell = m$. Otherwise, σ is the unique irreducible component of $\mathrm{Ind}_{J_+}^{J} \chi_s$ [**17**, § 4]. Set $\mathcal{H} = \mathcal{H}(G /\!\!/ J, \tilde{\sigma})$ and $\mathcal{H}' = \mathcal{H}(G' /\!\!/ Q'_{n'j}, \chi'_{-s})$.

Theorem 7.1 (cf. [15, 17]). *There is an isomorphism $\eta : \mathcal{H}' \to \mathcal{H}$ satisfying*

$$\mathrm{supp}(\eta(f')) = J\,\mathrm{supp}(f')J \quad \textit{and} \quad \mathrm{supp}(\eta(f')) \cap G' = \mathrm{supp}(f')$$

for $f' \in \mathcal{H}'$. Furthermore, η is an L^2-isometry for the natural L^2 structures on $\mathcal{H}$ and $\mathcal{H}'$.

Remark 7.2. Howe and Moy state the theorem for $\mathcal{H}(G /\!\!/ J, \sigma)$ and $\mathcal{H}(G' /\!\!/ Q'_{n'j}, \chi'_s)$. The existence of an isomorphism between our $\mathcal{H}$ and $\mathcal{H}'$ having the desired properties is immediate upon noting that $\chi_{-s} = \tilde{\chi}_s$ and if χ_s is replaced by χ_{-s} then the corresponding representation of J is $\tilde{\sigma}$.

Howe and Moy [**17**, Theorem 4.6] also show that if $\pi \in \mathcal{E}(G)$ contains $(Q_{n'j}, \chi_s)$, then π must contain (J, σ). Therefore, via the Hecke algebra isomorphism η, we obtain a bijection between the set of $\pi \in \mathcal{E}(G)$ that contain $(Q_{n'j}, \chi_s)$ and the set of $\pi' \in \mathcal{E}(G')$ that contain $(Q'_{n'j}, \chi'_s)$. When π and π' correspond via η, we will write $\pi' = \eta^*(\pi)$.

We begin with some elementary results giving the decomposition of $V_{\pi'}^{(Q'_{n'j}, \chi'_s)}$ as a Q'_ℓ-space, and a qualitative description of the Q_ℓ-spaces contained in $V_\pi^{(Q_{n'j}, \chi_s)}$. Following that, in Lemma 7.7 and Corollary 7.8, we show how η matches Q'_ℓ-subspaces of $V_{\pi'}^{(Q'_{n'j}, \chi'_s)}$ with Q_ℓ-subspaces of $V_\pi^{(\sigma, J)}$, and compare the dimensions of the matching Q'_ℓ- and Q_ℓ-spaces. Finally, Lemma 7.9 is a technical result that will be used in § 8 to apply results of this section to obtain relations between the multiplicities $m(\kappa(\eta^*(\pi), R'_j), \chi'_{s+\varpi_E^{-j} Y_\alpha}))$ and $m(\kappa(\pi, R_j), \chi_{s+\varpi_E^{-j} Y_\alpha})$ for $\alpha \in \mathcal{P}^0(n')$ (defined as in § 6).

Suppose that $\pi' \in \mathcal{E}(G')$ contains $(Q'_{n'j}, \chi'_s)$. Fix a linear character Λ_s of G' that extends χ'_s. Note that $\pi' \otimes \Lambda_s^{-1}$ contains the trivial representation of $Q'_{n'j}$. As $2\ell \geqslant n'j$, the group of characters of $Q'_\ell / Q'_{n'j} \simeq \mathfrak{q}'_\ell / \mathfrak{q}'_{n'j}$ is isomorphic to $\mathfrak{q}'_{-n'j+1} / \mathfrak{q}'_{-\ell+1}$ via

$$X + \mathfrak{q}'_{-\ell+1} \mapsto \chi'_X, \quad \text{where } \chi'_X(y) = \psi(\mathrm{tr}(X(y-1))), \quad y \in Q'_\ell.$$

Hence $(\Lambda_s^{-1}\pi') \mid Q'_\ell$ acting on $V_{\Lambda_s^{-1}\pi'}^{Q'_{n'j}}$ decomposes into a direct sum of characters of the form (Q'_ℓ, χ'_X), $X \in \mathfrak{q}'_{-n'j+1}$. This implies that $\pi' \mid Q'_\ell$ acting on $V_{\pi'}^{(Q'_{n'j},\chi'_s)}$ is a direct sum of characters of the form $(Q'_\ell, \Lambda_s\chi'_X)$, $X \in \mathfrak{q}'_{-n'j+1}$. Consider the action of $\pi' \mid Q'_1$ on $V_{\pi'}^{(Q'_{n'j},\chi'_s)}$. We have $\pi(k)V_{\pi'}^{(Q'_\ell,\Lambda_s\chi'_X)} = V_{\pi'}^{(Q'_\ell,\Lambda_s\chi'_{\mathrm{Ad}\,k(X)})}$, $k \in Q'_1$. Number the Q'_1-orbits of Q'_ℓ-types occurring in $V_{\pi'}^{(Q'_{n'j},\chi'_s)}$ from 1 to t. To the ith such orbit, attach a (finite) subset S_i of $\mathfrak{q}'_{-n'j+1}/\mathfrak{q}'_{-\ell+1}$ having the property that the orbit coincides with the direct sum $\bigoplus_{X\in S_i} V_{\pi'}^{(Q'_\ell,\Lambda_s\chi'_X)}$. We will often abuse notation and identify a coset $X \in S_i$ with a representative of the coset in $\mathfrak{q}'_{-n'j+1}$.

Lemma 7.3. *With notation and assumptions as above,*

$$V_{\pi'}^{(Q'_{n'j},\chi'_s)} = \bigoplus_{1\leqslant i\leqslant t}\ \bigoplus_{X\in S_i} V_{\pi'}^{(Q'_\ell,\Lambda_s\chi'_X)}.$$

Suppose that $X \in \mathfrak{q}_{-n'j+1}$. As $2\ell \geqslant n'j$, we may define a character of Q_ℓ by

$$\chi_X(y) = \psi(\mathrm{tr}(X(y-1))), \quad y \in Q_\ell.$$

As $\Lambda_s \mid Q'_\ell$ extends (trivially across J_+) to $Q'_\ell J_+$, given $Z \in s + \mathfrak{q}_{-n'j+1}$, $\chi_Z = \Lambda_s\chi_{Z-s} \mid Q'_\ell J_+$ defines a character of $Q'_\ell J_+$. Because $Q_\ell = Q'_\ell J$, Q'_ℓ normalizes J, and $\sigma \mid Q'_\ell \cap J = Q'_{n'j}$ is a multiple of χ_s,

$$\sigma_s \mid J = \sigma, \qquad \sigma_s \mid Q'_\ell = \Lambda_s \cdot 1_{\dim\sigma}$$

defines an irreducible representation of Q_ℓ.

Given $X \in \mathfrak{q}_{-n'j+1}$, let χ_X denote the character of Q_ℓ that corresponds to the coset $X + \mathfrak{q}_{-\ell+1}$. Let $\sigma_{s+X} = \chi_X\sigma_s$.

Lemma 7.4. *Let τ be an irreducible representation of Q_ℓ such that $\tau \mid Q_{n'j}$ is a multiple of $\chi_s \mid Q_{n'j}$. Then there exists $X \in \mathfrak{q}_{-n'j+1}$ such that $\tau = \sigma_{s+X}$.*

Proof. Let τ be as in the statement of the lemma. Let $k \in Q'_\ell J_+$ and $y \in Q_\ell$. Set $X = k-1$ and $Y = y-1$. Then $k^{-1}y^{-1}ky \in 1 + [X,Y] + \mathfrak{q}_{n'j+1}$, which implies

$$\begin{aligned}\tau(k^{-1}y^{-1}ky) &= \chi_s(k^{-1}y^{-1}ky)1_{\dim\tau}\\ &= \psi(\mathrm{tr}(s[X,Y]))1_{\dim\tau}\\ &= \psi(\mathrm{tr}([s,X]Y))1_{\dim\tau}\\ &= 1_{\dim\tau},\end{aligned}$$

since $[s,X]Y \in ([s,\mathfrak{q}'_\ell] + [s,\mathfrak{q}_m])Y = [s,\mathfrak{q}_m]Y \subset \mathfrak{q}_{-n'j+m+\ell} = \mathfrak{q}_1$. Thus $\tau(k)$, $k \in Q'_\ell J_+$ commutes with $\tau(y)$ for all $y \in Q_\ell$, forcing $\tau(k)$ to be scalar by Schur's Lemma. Thus $\tau \mid Q'_\ell J_+$ is a multiple of a character of $Q'_\ell J_+$. Note that this, together with $Q_\ell = Q'_\ell J$, forces $\tau \mid J$ to be irreducible.

Let ω_s be the character of $Q'_\ell J_+$ that is equal to Λ_s on Q'_ℓ, and χ_s on J_+. Since $\tau \mid Q_{n'j}$ is a multiple of χ_s, it follows from above that $\omega_s^{-1}\tau \mid Q'_\ell J_+$ is a multiple of a character

https://doi.org/10.1017/S1474748003000124 Published online by Cambridge University Press

of $Q'_\ell J_+$ which is trivial on $Q_{n'j}$. Note that $Q_\ell \supset Q'_\ell J_+ \supset Q_{n'j} \supset Q_{2\ell}$. Thus there exists $X \in \mathfrak{q}_{-n'j+1}$ such that $\omega_s^{-1}\tau \mid Q'_\ell J_+$ is a multiple of χ_X.

As $\tau \mid J$ is irreducible, $\chi_X^{-1}\tau \mid J$ is irreducible. Also, $\chi_X^{-1}\tau \mid J_+$ is a multiple of χ_s. Hence $\chi_X^{-1}\tau \mid J = \sigma$. Together with the fact that $\chi_X^{-1}\tau \mid Q'_\ell$ is a multiple of Λ_s, this implies that $\chi_X^{-1}\tau = \sigma_s$ (by definition of σ_s). □

Remarks 7.5.

(1) If $X \in \mathfrak{q}_{-n'j+1}$, σ_{s+X} is the unique irreducible representation of Q_ℓ whose restriction to $Q'_\ell Q_m$ is a multiple of $\omega_s \chi_X$.

(2) Lemma 7.4 gives the form of the Q_ℓ-spaces contained in $V_\pi^{(Q_{n'j},\chi_s)}$.

It is easy to show the following.

Lemma 7.6. *Let $X \in \mathfrak{q}_{-n'j+1}$. Then*

$$\pi(k)V_\pi^{(Q_\ell,\sigma_{s+X})} = V_\pi^{(Q_\ell,\sigma_{\mathrm{Ad}\,k(s+X)})}, \quad k \in Q_1.$$

For the rest of this section, assume that $\pi \in \mathcal{E}(G)$ contains $(Q_{n'j}, \chi_s)$.

Lemma 7.7. *Let $X \in \mathfrak{q}'_{-n'j+1}$. Set*

$$\varphi'_{s+X}(x) = \begin{cases} \upsilon_{G'}(Q'_{n'j})^{-1}(\Lambda_s^{-1}\chi_{-X})(x) & \text{if } x \in Q'_\ell, \\ 0 & \text{if } x \in G' - Q'_\ell \end{cases}$$

and

$$\varphi_{s+X}(x) = \begin{cases} \upsilon_G(J)^{-1}\tilde{\sigma}_{s+X}(x) & \text{if } x \in Q_\ell, \\ 0 & \text{if } x \in G - Q_\ell. \end{cases}$$

Then $\varphi'_{s+X} \in \mathcal{H}'$, $\varphi_{s+X} \in \mathcal{H}$ and $\eta(\varphi'_{s+X}) = \varphi_{s+X}$.

Proof. First, note that $X \in \mathfrak{q}'_{-n'j+1}$ implies $\chi_X \mid J \equiv 1$. As $\tilde{\sigma}_{s+X} = \chi_{-X}\tilde{\sigma}_s$, we have $\tilde{\sigma}_{s+X} \mid J = \tilde{\sigma}_s \mid J = \tilde{\sigma}$. Hence it is immediate from the definitions of $\mathcal{H}'$ and $\mathcal{H}$ that $\varphi'_{s+X} \in \mathcal{H}'$ and $\varphi_{s+X} \in \mathcal{H}$.

Fix $g \in Q'_\ell$. Set $J' = Q'_{n'j}$. Note that, as $\ell \geqslant 1$ and Q'_1 normalizes J' and J, $J'gJ' = gJ'$ and $JgJ = gJ$. As discussed in [**16**, **17**], there exists a unique $f'_g \in \mathcal{H}'$ that is supported on gJ' and has the property that $f'_g(gy) = \upsilon_{G'}(J')^{-1}\chi'_{-s}(y)$ for all $y \in J'$. As discussed in [**16**, pp. 41–45], for $g \in Q'$, in order to define the function $f_g \in \mathcal{H}$, which is equal to $\eta(f'_g)$, a so-called 'oscillator' representation of Q' is needed. However, the restriction of the oscillator representation to Q'_1 is trivial, and $Q'_\ell \subset Q'_1$, so, for $g \in Q'_\ell$, f_g has a particularly simple form. In fact, f_g is the unique function in $\mathcal{H}$ whose support lies in gJ and that satisfies $f_g(gy) = \upsilon_G(J)^{-1}\tilde{\sigma}(y)$, $y \in J$. (We have included the volumes $\upsilon_{G'}(J')$ and $\upsilon_G(J)$ as we have not assumed that they are equal to 1.) Fix a set of coset representatives $\{x_u\}$ for $Q'_\ell/J' \simeq Q_\ell/J$. Then

$$\eta(\varphi'_{s+X}) = \eta\biggl(\sum_u (\Lambda_s^{-1}\chi_{-X})(x_u) f'_{x_u}\biggr) = \sum_u (\Lambda_s^{-1}\chi_{-X})(x_u) f_{x_u}.$$

As $\tilde{\sigma}_{s+X}$ is the unique irreducible representation of Q_ℓ whose restriction to $Q'_\ell J_+$ is a multiple of $(\Lambda_s\chi_X)\tilde{} \simeq \Lambda_s^{-1}\chi_{-X}$, and $\tilde{\sigma}_{s+X} \mid J = \tilde{\sigma}$, it follows that

$$\tilde{\sigma}_{s+X}(x_u y) = (\Lambda_s^{-1}\chi_{-X})(x_u)\tilde{\sigma}(y) = \upsilon_G(J)(\Lambda_s^{-1}\chi_{-X})(x_u) f_{x_u}(y), \quad y \in J.$$

By the above, this implies that $\eta(\varphi'_{s+X}) = \varphi_{s+X}$. □

Corollary 7.8. *Let $X \in \mathfrak{q}'_{-n'j+1}$. Then*

$$\dim(V_\pi^{(Q_\ell,\sigma_{s+X})}) = (\dim\sigma)\dim(V_{\pi'}^{(Q'_\ell,\Lambda_s\chi'_X)}).$$

Proof. Let φ_{s+X} and φ'_{s+X} be as in Lemma 7.7. By definition of φ_{s+X}, the operator on $(V_\pi^{(J,\sigma)} \otimes \tilde{W})^J$ corresponding to the element φ_{s+X} of $\mathcal{H}$ is projection onto the subspace

$$(V_\pi^{(J,\sigma)} \otimes \tilde{W})^{Q_\ell} \simeq (V_\pi^{(Q_\ell,\sigma_{s+X})} \otimes \tilde{W})^{Q_\ell},$$

where Q_ℓ is acting on $\tilde{W}$ by $\tilde{\sigma}_{s+X}$. And the operator on $(V_{\pi'}^{(Q'_{n'j},\chi'_s)} \otimes \mathbb{C})^{Q'_{n'j}}$ corresponding to the element φ'_{s+X} of $\mathcal{H}'$ is projection onto the subspace

$$(V_{\pi'}^{(Q'_{n'j},\chi'_s)} \otimes \mathbb{C})^{Q'_\ell} \simeq (V_{\pi'}^{(Q'_\ell,\Lambda_s\chi'_X)} \otimes \mathbb{C})^{Q'_\ell},$$

where Q'_ℓ acts on $\mathbb{C}$ by $\Lambda_s^{-1}\chi'_{-X}$. By Lemma 7.7, as $\varphi_{s+X} = \eta(\varphi'_{s+X})$, the dimensions of the images of the two projections are equal. □

Lemma 7.9. *Let $X \in \mathfrak{q}_{-n'j+1}$ be such that $V_\pi^{(Q_\ell,\sigma_{s+X})} \neq 0$. There exist $k \in Q_1$, $i \in \{1,\dots,t\}$, and $X_i \in S_i$ such that the following hold.*

(1) $\operatorname{Ad} k(s+X_i) \in s + X + \mathfrak{q}_{-\ell+1} + \mathfrak{q}'^{\perp}_{-m+1}$.

(2) $V_\pi^{(Q_\ell,\sigma_{s+X})} = \pi(k)V_\pi^{(Q_\ell,\sigma_{s+X_i})}$.

(3) *If $k_0 \in Q_1$ and $Z \in \mathfrak{q}'_{-n'j+1}$ are such that $V_\pi^{(Q_\ell,\sigma_{s+X})} = \pi(k_0)V_\pi^{(Q_\ell,\sigma_{s+Z})}$, then $k_0 \in kQ'_1Q_\ell$ and $Z \in \operatorname{Ad} z(X_i) + \mathfrak{q}'_{-\ell+1}$ for any $z \in Q'_1$ such that $k_0^{-1}k \in zQ_\ell$.*

Proof. By Lemma 5.1 (4) (with $u = 1$, $Z = 0$ and $v = 1$), there exist $k \in Q_1$ and $Y \in \mathfrak{q}'_{-n'j+1}$ such that $s + X = \operatorname{Ad} k(s+Y)$. As $Y \in \mathfrak{q}'_{-n'j+1}$, we have $V_\pi^{(Q'_\ell,\Lambda_s\chi'_Y)} \neq 0$ by Corollary 7.8 and $V_{\pi'}^{(Q'_{n'j},\Lambda_s\chi'_Y)} \subset V_{\pi'}^{(Q'_{n'j},\chi'_s)}$. By Lemma 7.3, there exists $i \in \{1,\dots,t\}$ and $X_i \in S_i$ such that $V_{\pi'}^{(Q'_\ell,\Lambda_s\chi'_Y)} = V_{\pi'}^{(Q'_\ell,\Lambda_s\chi'_{X_i})}$. As $Y \in X_i + \mathfrak{q}'_{-\ell+1}$, it follows that $\chi_Y = \chi_{X_i}$, so $V_\pi^{(Q_\ell,\sigma_{s+Y})} = V_\pi^{(Q_\ell,\sigma_{s+X_i})}$. Hence $V_\pi^{(Q_\ell,\sigma_{s+X})} = V_\pi^{(Q_\ell,\sigma_{\operatorname{Ad} k(s+X_i)})}$.

Since $\sigma_{\operatorname{Ad} k(s+X_i)} = \chi_{\operatorname{Ad} k(s+X_i)-s}\sigma_s$ and $\sigma_{s+X} = \chi_X\sigma_s$, and these two representations coincide, their characters must coincide. From the remarks following Lemma 7.4, this is equivalent to

$$\chi_{\operatorname{Ad} k(s+X_i)-s} \mid Q'_\ell Q_m = \chi_X \mid Q'_\ell Q_m,$$

which, as $Q'_\ell Q_m = 1 + \mathfrak{q}'_\ell + \mathfrak{q}_m$, is equivalent to

$$\operatorname{Ad} k(s+X_i) - s \in X + \mathfrak{q}_{-\ell+1} + \mathfrak{q}'^{\perp}_{-m+1}.$$

Hence (1) and (2) both hold.

https://doi.org/10.1017/S1474748003000124 Published online by Cambridge University Press

For (3), suppose k, i, X_i, Z and k_0 are as in the statement of the lemma. Then

$$V_\pi^{(Q_\ell,\sigma_{s+Z})} = \pi(k_0^{-1}k)V_\pi^{(Q_\ell,\sigma_{s+X_i})}.$$

Arguing as above (with Z playing the role of X), results in

$$\mathrm{Ad}(k_0^{-1}k)(s+X_i) \in s+Z+\mathfrak{q}_{-\ell+1}+\mathfrak{q}'^{\perp}_{-m+1}.$$

As $\mathfrak{q}_{-\ell+1}+\mathfrak{q}'^{\perp}_{-m+1} \subset \mathfrak{q}_{-m+1}$ and $\ell+m=n'j+1$, we may apply Lemma 5.3 (2), with $u=1$ and $v=\ell$, to conclude that there exists $x \in Q_\ell$ such that

$$\mathrm{Ad}(x^{-1}k_0^{-1}k)(s+X_i) \in s+Z+\mathfrak{q}'_{-m+1}.$$

As $Z \in \mathfrak{q}'_{-n'j+1}$ and $X_i+\mathfrak{q}'_{-m+1} \subset \mathfrak{q}'_{-n'j+1}$, Lemma 5.5 implies $x^{-1}k_0^{-1}k \in G'$. Since it is also the case that $x^{-1}k_0^{-1}k \in Q_1$, we have $x^{-1}k_0^{-1}k \in Q'_1$. Hence $k_0 \in kQ'_1Q_\ell$. Let $z = x^{-1}k_0^{-1}k$. Then, as $\mathrm{Ad}\, z^{-1}$ preserves $\mathfrak{q}_{-\ell+1}+\mathfrak{q}'^{\perp}_{-m+1}$, it follows from above that

$$s+\mathrm{Ad}\, z^{-1}(Z) = \mathrm{Ad}\, z^{-1}(s+Z) \in Q_\ell \cdot (s+X_i)+\mathfrak{q}_{-\ell+1}+\mathfrak{q}'^{\perp}_{-m+1}.$$

Note that $\mathfrak{q}_\ell = \mathfrak{q}'_\ell \oplus \mathfrak{q}'^{\perp}_\ell$ and $s \in \mathfrak{q}'_{-n'j}$ commutes with $\mathfrak{g}'$, so $\mathrm{ad}\, s(\mathfrak{q}_\ell) \subset \mathfrak{q}'^{\perp}_{-\ell+n'j+1} = \mathfrak{q}'^{\perp}_{-m+1}$. And $X_i \in \mathfrak{q}_{-n'j+1}$ implies $\mathrm{ad}\, X_i(\mathfrak{q}_\ell) \subset \mathfrak{q}_{-\ell+1}$. Thus

$$Q_\ell \cdot (s+X_i) \subset s+X_i+\mathfrak{q}_{-\ell+1}+\mathfrak{q}'^{\perp}_{-m+1},$$

from which it follows that

$$s+\mathrm{Ad}\, z^{-1}(Z) \in s+X_i+\mathfrak{q}_{-\ell+1}+\mathfrak{q}'^{\perp}_{-m+1}.$$

Because $s+\mathrm{Ad}\, z^{-1}(Z)$, $s+X_i \in \mathfrak{g}'$, and $(\mathfrak{q}_{-\ell+1}+\mathfrak{q}'^{\perp}_{-m+1}) \cap \mathfrak{g}' = \mathfrak{q}'_{-\ell+1}$, this forces $s+\mathrm{Ad}\, z^{-1}(Z) \in s+X_i+\mathfrak{q}'_{-\ell+1}$. □

8. Comparison of multiplicities of K-types

Let notation be as in §§ 6, 7. Throughout this section we will assume that $\pi \in \mathcal{E}(G)$ contains the pure minimal K-type $(Q_{n'j}, \chi_s)$ and $\pi' = \eta^*(\pi)$ is the corresponding representation of G'.

For convenience of notation, given $\alpha \in \mathcal{P}^0(n')$, set

$$\left.\begin{aligned} m_\alpha(\pi) &= m(\kappa(\pi,R_j),\chi_{s+\varpi_E^{-j}Y_\alpha}), \qquad & m_\alpha(\pi') &= m(\kappa(\pi',R'_j),\chi'_{s+\varpi_E^{-j}Y_\alpha}),\\ V_\alpha &= V_\pi^{(R_j,\chi_{s+\varpi_E^{-j}Y_\alpha})}, & V'_\alpha &= V_{\pi'}^{(R'_j,\chi'_{s+\varpi_E^{-j}Y_\alpha})}. \end{aligned}\right\} \tag{8.1}$$

Then $m_\alpha(\pi) = \dim(V_\alpha)$ and $m_\alpha(\pi') = \dim(V'_\alpha)$. One of the main results of this section, Proposition 8.4, gives the relation between $m_\alpha(\pi)$ and $m_\alpha(\pi')$. In Proposition 8.6, this is translated into a form in which it will later be applied to compare the germs of Θ_π and $\Theta_{\pi'}$, namely a result relating dimensions of subspaces of V_π and $V_{\pi'}$ that are invariant under certain compact open subgroups contained in $G_{(j/e)^+}$ and G'_{j^+}, respectively.

https://doi.org/10.1017/S1474748003000124 Published online by Cambridge University Press

Because $R_{j+1} \subset Q_{n'j+1} \subset Q_{n'j} \subset R_j$ and $\varpi_E^{-j} Y_\alpha \in \mathfrak{q}'_{-n'j+1}$, it follows that

$$V_\alpha \subset V_\pi^{(Q_{n'j},\chi_s)} \quad \text{and} \quad V'_\alpha \subset V_{\pi'}^{(Q'_{n'j},\chi'_s)}.$$

As shown in § 7, the Hecke algebra isomorphism η matches Q'_ℓ-types contained in $V_{\pi'}^{(Q'_{n'j},\chi'_s)}$ with Q_ℓ-types contained in $V_\pi^{(J,\sigma)} \subset V_\pi^{(Q_{n'j},\chi_s)}$. In order to compare $m_\alpha(\pi)$ and $m_\alpha(\pi')$, we must express V_α and V'_α in terms of Q_ℓ-types and Q'_ℓ-types, respectively. It is fairly easy to see that if $j \geqslant 2$, $Q_\ell \subset R_1$, which implies that V_α and V'_α are Q_ℓ-stable and Q'_ℓ-stable, respectively. However, if $j = 1$, this is not the case, so it is necessary to work with the smallest Q_ℓ-stable subspace of $V_\pi^{(Q_{n'j},\chi_s)}$ that contains V_α, and the smallest Q'_ℓ-stable subspace of $V_{\pi'}^{(Q'_{n'j},\chi'_s)}$ that contains V'_α. Set

$$\begin{aligned}
U'_\alpha &= \sum_{x \in Q'_\ell/(Q'_\ell \cap R'_1)} \pi'(x) V'_\alpha, \\
U_\alpha &= \sum_{x \in Q_\ell/(Q_\ell \cap R_1)} \pi(x) V_\alpha, \\
S_i^\alpha &= S_i \cap (\varpi_E^{-j} Y_\alpha + \mathfrak{r}'_{-j+1} + \mathfrak{q}'_{-\ell+1}), \quad 1 \leqslant i \leqslant t, \\
W'_\alpha &= \bigoplus_{1 \leqslant i \leqslant t} \bigoplus_{X \in S_i^\alpha} V_{\pi'}^{(Q'_\ell, \chi'_{s+X})}, \\
W_\alpha &= \sum_{y \in R_1/(R_1 \cap Q_\ell)} \pi(y) \bigoplus_{1 \leqslant i \leqslant t} \bigoplus_{X \in S_i^\alpha} V_\pi^{(Q_\ell, \sigma_{s+X})}.
\end{aligned}$$

Lemma 8.1. *Let $\alpha \in \mathcal{P}^0(n')$. Then*

$$W_\alpha = \bigoplus_{y \in R_1/(R_1 \cap Q'_1 Q_\ell)} \pi(y) \bigoplus_{1 \leqslant i \leqslant t} \bigoplus_{X \in S_i^\alpha} V_\pi^{(Q_\ell, \sigma_{s+X})}.$$

Proof. Let

$$W_\alpha^0 = \bigoplus_{1 \leqslant i \leqslant t} \bigoplus_{X \in S_i^\alpha} V_\pi^{(Q_\ell, \sigma_{s+X})}.$$

By definition, W_α^0 is a Q_ℓ-space. Suppose that $k \in R'_1$ and $X \in S_i^\alpha$ for some i. Then

$$\operatorname{Ad} y(X) - \varpi_E^{-j} Y_\alpha \in X - \varpi_E^{-j} Y_\alpha + \mathfrak{r}'_{-j+1},$$

which implies that $\pi(k) V_\pi^{(Q_\ell, \sigma_{s+X_i})} \subset W_\alpha^0$. Hence W_α^0 is $R_1 \cap (Q'_1 Q_\ell)$-stable, as it is both Q_ℓ- and R'_1-stable.

Suppose that $y \in R_1$ is such that $W_\alpha^0 \cap \pi(y) W_\alpha^0 \neq \{0\}$. Then there are $i, h \in \{1, \dots, t\}$, $X_i \in S_i$, and $Z_h \in S_h$, such that

$$V_\pi^{(Q_\ell, \sigma_{s+X_i})} = \pi(y) V_\pi^{(Q_\ell, \sigma_{s+Z_h})}.$$

Applying Lemma 7.9 with $X = X_i$, $Z = Z_h$, $k = 1$ and $k_0 = y$ results in $y \in k Q'_1 Q_\ell = Q'_1 Q_\ell$. Hence $y \in R_1 \cap (Q'_1 Q_\ell)$. □

https://doi.org/10.1017/S1474748003000124 Published online by Cambridge University Press

Lemma 8.2. *Let $\alpha \in \mathcal{P}^0(n')$. Then $U_\alpha = W_\alpha$ and $U'_\alpha = W'_\alpha$.*

Proof. We will omit the proof of $U'_\alpha = W'_\alpha$ as it is similar to, but easier than, that of $U_\alpha = W_\alpha$.

Set $Y = \varpi_E^{-j} Y_\alpha$. Since V_α is a sum of copies of (R_j, χ_{s+Y}), $V_\pi^{(Q_\ell, \sigma_{s+X})} \cap U_\alpha \neq 0$ if and only if (Q_ℓ, σ_{s+X}) is a component of $(\mathrm{Ind}_{R_j}^{Q_\ell R_j} \chi_{s+Y})|_{Q_\ell} \simeq \mathrm{Ind}_{R_j \cap Q_\ell}^{Q_\ell} \chi_{s+Y}$. By Frobenius reciprocity, this will be the case if and only if $\chi_{s+Y}|_{R_j \cap Q_\ell}$ is a component of $\sigma_{s+X}|_{R_j \cap Q_\ell}$. Note that $\sigma_{s+X}|_{R_j \cap Q_\ell}$ is a multiple of $\chi_{s+X}|_{R_j \cap Q_\ell}$. The characters χ_{s+Y} and χ_{s+X} agree on $R_j \cap Q_\ell$ if and only if $X - Y \in \mathfrak{r}_{-j+1} \cap \mathfrak{q}_{-\ell+1}$. Hence, as a Q_ℓ-space, U_α is a sum of those spaces $V_\pi^{(Q_\ell, \sigma_{s+X})}$ where $X \in \mathfrak{q}_{-n'j+1}$ is such that $X \in Y + \mathfrak{r}_{-j+1} + \mathfrak{q}_{-\ell+1}$.

Suppose that $i \in \{1, \dots, t\}$, $X_i \in S_i^\alpha$ and $y \in R_1$. Then, by definition of S_i^α, since $R_1 \cdot (s + Y) \subset s + Y + \mathfrak{r}_{-j+1}$, and $\mathfrak{r}_{-j+1}$ and $\mathfrak{q}_{-\ell+1}$ are R_1-stable,

$$\mathrm{Ad}\, y(s + X_i) \in \mathrm{Ad}\, y(s + Y + \mathfrak{r}'_{-j+1} + \mathfrak{q}'_{-\ell+1}) \subset s + Y + \mathfrak{r}_{-j+1} + \mathfrak{q}_{-\ell+1}.$$

It follows that $\pi(y) V_\pi^{(Q_\ell, \sigma_{s+X_i})} \subset U_\alpha$, which, by definition of W_α, implies $W_\alpha \subset U_\alpha$.

It remains to show that $U_\alpha \subset W_\alpha$. Let $X \in \mathfrak{q}_{-n'j+1}$ be such that $X - Y \in \mathfrak{r}_{-j+1} + \mathfrak{q}_{-\ell+1}$. In light of the above description of U_α as a Q_ℓ-space, it suffices to show that $V_\pi^{(Q_\ell, \sigma_{s+X})} \subset W_\alpha$ for all such X. By Lemma 7.9, there exists $k \in Q_1$, $i \in \{1, \dots, t\}$, and $X_i \in S_i$ such that

$$\mathrm{Ad}\, k(s + X_i) \in s + X + \mathfrak{q}_{-\ell+1} + \mathfrak{q}'^{\perp}_{-m+1} \quad \text{and} \quad V_\pi^{(Q_\ell, \sigma_{s+X})} = \pi(k) V_\pi^{(Q_\ell, \sigma_{s+X_i})}.$$

Adding an element of $\mathfrak{q}_{-\ell+1}$ to X will not affect σ_{s+X}, or the above inclusion and identity. As $X \in Y + \mathfrak{r}_{-j+1} + \mathfrak{q}_{-\ell+1}$, there is therefore no loss of generality in assuming that $X \in Y + \mathfrak{r}_{-j+1}$. By Lemma 5.3 (2), $s + Y + \mathfrak{r}_{-j+1} = R_1 \cdot (s + Y + \mathfrak{r}'_{-j+1})$. Thus there exists $x \in R_1$ and $Y_0 \in \mathfrak{r}'_{-j+1}$ such that $s + X = \mathrm{Ad}\, x(s + Y + Y_0)$. This implies

$$V_\pi^{(Q_\ell, \sigma_{s+X})} = \pi(x) V_\pi^{(Q_\ell, \sigma_{s+Y+Y_0})}.$$

By Lemma 7.9 (3), with $k_0 = x$, if $z \in Q'_1$ is such that $x^{-1}k \in zQ_\ell$, then

$$Y + Y_0 \in \mathrm{Ad}\, z(X_i) + \mathfrak{q}'_{-\ell+1},$$

which, as $Y_0 \in \mathfrak{r}'_{-j+1}$ implies that $\mathrm{Ad}\, z(X_i) + \mathfrak{q}'_{-\ell+1} \in S_i^\alpha$. We have

$$V_\pi^{(Q_\ell, \sigma_{s+X})} = \pi(x)\pi(x^{-1}kQ_\ell) V_\pi^{(Q_\ell, \sigma_{s+X_i})} = \pi(x) V_\pi^{(Q_\ell, \sigma_{s+\mathrm{Ad}\, z(X_i)})} \subset W_\alpha.$$

□

Lemma 8.3. *Let $\alpha \in \mathcal{P}^0(n')$.*

(1) $\dim(U_\alpha) = [Q_\ell : Q_\ell \cap R_1][C_{Q'_\ell}(Y_\alpha) : C_{Q'_\ell}(Y_\alpha) \cap R'_1]^{-1} \dim(V_\alpha)$.

(2) $\dim(U'_\alpha) = [Q'_\ell : Q'_\ell \cap R'_1][C_{Q'_\ell}(Y_\alpha) : C_{Q'_\ell}(Y_\alpha) \cap R'_1]^{-1} \dim(V'_\alpha)$.

https://doi.org/10.1017/S1474748003000124 Published online by Cambridge University Press

Proof. For (1), recall that, as shown in the proof of Corollary 6.3, if $k \in R$ and

$$\chi_{\operatorname{Ad} k(s+\varpi_E^{-j} Y_\alpha)} \quad \text{and} \quad \chi_{s+\varpi_E^{-j} Y_\alpha}$$

agree on R_j, then $k \in C_{R'}(Y_\alpha)R_1$. Hence it follows from the definition of U_α that

$$\frac{\dim(U_\alpha)}{\dim(V_\alpha)} = [Q_\ell : Q_\ell \cap (C_{R'}(Y_\alpha)R_1)] = [Q_\ell : C_{Q'_\ell}(Y_\alpha)(Q_\ell \cap R_1)].$$

The proof of (2) is similar. □

Proposition 8.4. *Let $\alpha \in \mathcal{P}^0(n')$ and let $m_\alpha(\pi)$ and $m_\alpha(\pi')$ be as in (8.1). Then*

$$\begin{aligned} m_\alpha(\pi) &= [Q_1 : R_1]^{-1}[Q'_1 : R'_1][Q_1 : Q_\ell][Q'_1 : Q'_\ell]^{-1}(\dim \sigma) m_\alpha(\pi') \\ &= q^{fn'(n-n')(j-1)/2} m_\alpha(\pi'). \end{aligned}$$

Proof. It follows from Lemmas 8.2 and 8.3 that $V'_\alpha = \{0\}$ if and only if $V_\alpha = \{0\}$. Furthermore, if $V'_\alpha \neq \{0\}$, then

$$\frac{m_\alpha(\pi)}{m_\alpha(\pi')} = \frac{[Q_\ell : R_1 \cap Q_\ell]^{-1} \dim(W_\alpha)}{[Q'_\ell : R'_1 \cap Q'_\ell]^{-1} \dim(W'_\alpha)}.$$

As $[R_1 : R_1 \cap (Q'_1 Q_\ell)] = [R_1 : R_1 \cap Q_\ell][R'_1 : R'_1 \cap Q'_\ell]^{-1}$, by Corollary 7.8, the definition of W'_α, and Lemma 8.1,

$$\dim(W_\alpha) = [R_1 : R_1 \cap Q_\ell][R'_1 : R'_1 \cap Q'_\ell]^{-1}(\dim \sigma) \dim(W'_\alpha).$$

Combining this with the above gives

$$\frac{m_\alpha(\pi)}{m_\alpha(\pi')} = [R_1 : R_1 \cap Q_\ell][Q_\ell : Q_\ell \cap R_1]^{-1}[R'_1 : R'_1 \cap Q'_\ell]^{-1}[Q'_\ell : Q'_\ell \cap R'_1](\dim \sigma),$$

which, together with

$$[R_1 : R_1 \cap Q_\ell][Q_\ell : R_1 \cap Q_\ell]^{-1} = [Q_1 : R_1]^{-1}[Q_1 : Q_\ell],$$

yields the first equality in the statement of the proposition.

Note that

$$\begin{aligned} [Q_i : Q_{i+1}] &= [\mathfrak{b}_{(f)^{n'e}} : \mathfrak{b}_{(f)^{n'e},1}] = |\mathfrak{gl}_f(\mathfrak{o}/\mathfrak{p})|^{n'e} = q^{fn}, \\ [Q'_i : Q'_{i+1}] &= [\mathfrak{b}'_{(1)^{n'}} : \mathfrak{b}'_{(1)^{n'},1}] = |\mathfrak{o}_E/\mathfrak{p}_E|^{n'} = q^{fn'}, \quad i \geqslant 1, \end{aligned}$$

and

$$[Q_\ell : Q_m] = [Q'_\ell J : Q'_m(J \cap Q_m)] = [Q'_\ell : Q'_m][J : J \cap Q_m].$$

Hence, as $\dim \sigma = [J : J \cap Q_m]^{1/2}$ and $\ell + m = n'j + 1$, we have

$$[Q_1 : Q_\ell][Q'_1, Q'_\ell]^{-1}(\dim \sigma) = q^{f(n-n')(n'j-1)/2}$$

In view of this, after using (3.1) to evaluate the indices

$$[Q_1 : R_1] = [B_{(f)^{n'e},1} : B_{(fn')^e,1}], \qquad [Q'_1 : R'_1] = [B'_{(1)^{n'},1} : B'_{(n'),1}],$$

we obtain the second equality in the statement of the proposition. □

https://doi.org/10.1017/S1474748003000124 Published online by Cambridge University Press

Recall that $r(\beta)$ denotes the length of a partition β.

Lemma 8.5. *Let $\beta \in \mathcal{P}(n')$ and $\gamma \in \mathcal{P}^0(n')$. If $k \in R$, then*

$$\chi_{\mathrm{Ad}\,k(s+\varpi_E^{-j}Y_\gamma)} \mid B_{\xi(\mathfrak{L}'^\beta),jr(\beta)+1} \equiv 1 \quad \Leftrightarrow \quad k \in B_{\xi(\mathfrak{L}'^\beta)}\{k' \in R' \mid \mathrm{Ad}\,k'(Y_\gamma) \in \mathfrak{b}'_\beta\}.$$

Proof. The restriction of the character to $B_{\xi(\mathfrak{L}'^\beta),jr(\beta)+1}$ is trivial if and only if

$$\mathrm{Ad}\,k(s+\varpi_E^{-j}Y_\gamma) \in (\mathfrak{b}_{\xi(\mathfrak{L}'^\beta),jr(\beta)+1})^* = \mathfrak{b}_{\xi(\mathfrak{L}'^\beta),-jr(\beta)}.$$

As $\varpi_E^{-j}Y_\gamma \in \mathfrak{q}'_{-n'j+1}$, Lemma 5.4 (2) implies that $k \in B_{\xi(\mathfrak{L}'^\beta)}G'$. By assumption, $k \in R$. Thus $k \in B_{\xi(\mathfrak{L}'^\beta)}R'$. Choose $k' \in R'$ such that $kk'^{-1} \in B_{\xi(\mathfrak{L}'^\beta)}$. Then

$$\mathrm{Ad}\,k'(s+\varpi_E^{-j}Y_\gamma) = s+\varpi_E^{-j}\,\mathrm{Ad}\,k'(Y_\gamma) \in \mathfrak{b}_{\xi(\mathfrak{L}'^\beta),jr(\beta)} \cap \mathfrak{g}' = \mathfrak{b}'_{\beta,jr(\beta)}$$

and $s \in \mathfrak{q}'_{-n'j} = \varpi_E^{-j}\mathfrak{b}'_{(1)^{n'}} \subset \varpi_E^{-j}\mathfrak{b}'_\beta$. Hence $\mathrm{Ad}\,k'(Y_\gamma) \in \mathfrak{b}'_\beta$. □

Proposition 8.6. *Let $\alpha \in \mathcal{P}(n)$. Then $\dim(V_\pi^{B_{\alpha,\lfloor jr(\alpha)/e\rfloor+1}})$ is equal to*

$$q^{fn'((n-n')j-e+1)/2}u_{(f\beta)^e}(q)u_\beta(q^f)^{-1}\dim(V_{\pi'}^{B'_{\beta,jr(\beta)+1}})$$

if $\alpha = (f\beta)^e$ for some $\beta \in \mathcal{P}(n')$, and zero otherwise.

Proof. Let $\alpha \in \mathcal{P}(n)$. Recall (see § 4) that there exists x in the Bruhat–Tits building $\mathcal{B}(G)$ such that $\mathfrak{g}_{x,t^+} = \mathfrak{b}_{\alpha,\lfloor tr(\alpha)\rfloor+1}$, $t \in \mathbb{R}$. Suppose that π has non-zero $B_{\alpha,\lfloor jr(\alpha)/e\rfloor+1}$-fixed vectors. Then π has non-zero $G_{x,(\lfloor jr(\alpha)/e\rfloor/r(\alpha))^+}$-fixed vectors. It follows from the definition of depth that $\rho(\pi) \leqslant \lfloor jr(\alpha)/e\rfloor/r(\alpha)$. By Lemma 4.2, $\rho(\pi) = j/e$, so

$$\frac{j}{e} \leqslant \frac{\lfloor jr(\alpha)/e\rfloor}{r(\alpha)}.$$

As j and e are relatively prime, the latter quantity is at most j/e, with equality if and only if e divides $r(\alpha)$. Thus $V_\pi^{B_{\alpha,\lfloor jr(\alpha)/e\rfloor+1}} = \{0\}$ whenever e does not divide $r(\alpha)$.

Assume that e divides $r(\alpha)$ and π has non-zero $B_{\alpha,jr(\alpha)/e+1}$-fixed vectors. As $\rho(\pi) = j/e$, by Theorem 5.2 (2) of [**29**], the action of $B_{\alpha,jr(\alpha)/e}$ on $V_\pi^{B_{\alpha,(jr(\alpha)/e)+1}}$ contains a standard minimal K-type $(B_{jr(\alpha)/e},\chi_X)$, where $X \in \mathfrak{b}_{-jr(\alpha)/e}$. Any two unrefined minimal K-types contained in π are associates of each other (see Theorem 4.1 (2)). This means that

$$G\cdot(s+\mathfrak{q}_{-n'j+1}) \cap (X+\mathfrak{b}_{\alpha,-(jr(\alpha)/e)+1}) \neq \emptyset.$$

As $X \in \mathfrak{b}_{-jr(\alpha)/e}$, this implies

$$G\cdot(s+\mathfrak{q}'_{-n'j+1}) \cap \mathfrak{b}_{\alpha,-jr(\alpha)/e} \neq \emptyset,$$

By Lemma 5.4 (1), $\alpha = (f\beta)^e$ for some $\beta \in \mathcal{P}(n')$.

To finish, assume that $\alpha = (f\beta)^e$ for some $\beta \in \mathcal{P}(n')$. Note that $\lfloor jr(\alpha)/e\rfloor = jr(\beta)$. In the statement of the proposition, we can replace $B_{(f\beta)^e,jr(\beta)+1}$ by $B_{\xi(\mathfrak{L}'^\beta),jr(\beta)+1}$, as these two subgroups are G-conjugate. We know that $V_\pi^{B_{\xi(\mathfrak{L}'^\beta),jr(\beta)+1}}$ is equal to the space of $B_{\xi(\mathfrak{L}'^\beta),jr(\beta)+1}$-fixed vectors in the space $V_\pi^{R_{j+1}}$ of $\kappa(\pi, R_j)$. Let $\gamma \in \mathcal{P}^0(n')$. Set

$$A_{\gamma,\beta} = \{k \in R' \mid \operatorname{Ad} k(Y_\gamma) \in \mathfrak{b}'_\beta\}.$$

It is easy to see that $B_{\xi(\mathfrak{L}'^\beta)} A_{\gamma,\beta}$ is right $C_{R'}(Y_\gamma)R_1$-invariant. Let $u(\gamma,\beta)$ be the cardinality of the image of $A_{\gamma,\beta}$ in $R'/C_{R'}(Y_\gamma)R'_1$. Then the cardinality of the image of $B_{\xi(\mathfrak{L}'^\beta)} A_{\gamma,\beta}$ in $R/C_{R'}(Y_\gamma)R_1$ is equal to $[B_{\xi(\mathfrak{L}'^\beta)} : B'_\beta R_1] u(\gamma,\beta)$.

It now follows from Corollary 6.3 and Lemma 8.5 that

$$\dim(V_\pi^{B_{\xi(\mathfrak{L}'^\beta),jr(\beta)+1}}) = [B_{\xi(\mathfrak{L}'^\beta)} : B'_\beta R_1] \sum_{\gamma \in \mathcal{P}^0(n')} m_\gamma(\pi) u(\gamma,\beta).$$

Similarly, Corollary 6.5 implies

$$\dim(V_{\pi'}^{B'_{jr(\beta)+1}}) = \sum_{\gamma \in \mathcal{P}^0(n')} m_\gamma(\pi') u(\gamma,\beta).$$

After comparing the above two dimensions via Proposition 8.4, we obtain

$$\dim(V_\pi^{B_{\xi(\mathfrak{L}'^\beta),jr(\beta)+1}}) = [B_{\xi(\mathfrak{L}'^\beta)} : B'_\beta R_1] q^{fn'(n-n')(j-1)/2} \dim(V_{\pi'}^{B'_{jr(\beta)+1}}).$$

To complete the proof, note that

$$\begin{aligned}
&[B_{\xi(\mathfrak{L}'^\beta)} : B'_\beta R_1] \\
&\qquad = [B_{\xi(\mathfrak{L}'^\beta)} : B'_\beta B_{\xi(\mathfrak{L}'^{(n')}),1}] \\
&\qquad = [B_{\xi(\mathfrak{L}'^\beta)} : B_{\xi(\mathfrak{L}'^{(n')}),1}][B'_\beta : B'_{(n'),1}]^{-1} \\
&\qquad = [B_{(f\beta)^e} : B_{(1)^n,1}][B'_\beta : B'_{(1)^{n'},1}]^{-1}[B_{(1)^n,1} : B_{(fn')^e,1}][B_{(1)^{n'},1} : B_{(n'),1}]^{-1} \\
&\qquad = u_{(f\beta)^e}(q) u_\beta(q^f)^{-1} q^{fn'(n-n'-e+1)/2} \quad \text{by (3.1).}
\end{aligned}$$

□

9. Varying Hecke algebra isomorphisms

In [**17**], Howe and Moy construct refined minimal K-types and the corresponding Hecke algebra isomorphisms inductively. As a result, in some cases, the Hecke algebra isomorphism η of Theorem 7.1 must be replaced by a slightly different Hecke algebra isomorphism, which we call $\dot\eta$. Suppose that π' and $\dot\pi' \in \mathcal{E}(G')$ correspond via η and $\dot\eta$, respectively, to $\pi \in \mathcal{E}(G)$. In § 12, the results of § 8 will be applied to match the germs of Θ_π and $\Theta_{\pi'}$. The same kinds of arguments cannot be used to match the germs of Θ_π and $\Theta_{\dot\pi'}$, due to the fact that the analogues, relative to $\dot\eta$, of the results of § 8 may not involve enough different K-types to determine the germs of the two characters. An essential step in the proof of Theorem 14.1 requires matching of the germs of Θ_π and $\Theta_{\dot\pi'}$. This can be done using the results of § 12 as a consequence of Proposition 9.2, which says that π' and $\dot\pi'$ are equivalent.

https://doi.org/10.1017/S1474748003000124 Published online by Cambridge University Press

Let f_o be a positive integer such that f divides f_o and f_o divides n/e. Let $\mathfrak{L}'^3 = \mathfrak{L}'^{(f_o/f)^{n/ef_o}}$ be the periodic lattice flag in $E^{n'}$ attached to the partition $(f_o/f)^{n/ef_o}$ of n' (see § 3). Let $\mathfrak{L}^3$ be the periodic lattice flag in F^n defined by $\mathfrak{L}^3 = \xi(\mathfrak{L}'^3)$, where ξ is as in § 3. Attached to $\mathfrak{L}'^3$ and $\mathfrak{L}^3$, we have parahoric subgroups in G' and G and parahoric subalgebras in $\mathfrak{g}'$ and $\mathfrak{g}$. Set

$$\dot{\mathfrak{q}}'_i = \mathfrak{b}_{\mathfrak{L}'^3,i}, \quad \dot{\mathfrak{q}}_i = \mathfrak{b}_{\mathfrak{L}^3,i}, \quad i \in \mathbb{Z},$$
$$\dot{Q}'_i = B_{\mathfrak{L}'^3,i}, \quad \dot{Q}_i = B_{\mathfrak{L}^3,i}, \quad i \geqslant 1.$$

With these definitions, $\dot{\mathfrak{q}}_i \cap \mathfrak{g}' = \dot{\mathfrak{q}}'_i$, $i \in \mathbb{Z}$. Also, $\dot{\mathfrak{q}}$ is G-conjugate to $\mathfrak{b}_{(f_o)^{n/f_o}}$. Since $\mathfrak{L}'^3$ has period n/ef_o, by (3.2),

$$\varpi_E \dot{\mathfrak{q}}'_i = \dot{\mathfrak{q}}'_{i+n/ef_o} \quad \text{and} \quad \varpi_E \dot{\mathfrak{q}}_i = \dot{\mathfrak{q}}_{i+n/ef_o}, \quad i \in \mathbb{Z}.$$

Hence

$$s + \dot{\mathfrak{q}}_{-(nj/ef_o)+1} \subset \varpi_E^{-j}(\mathfrak{o}_E^\times + \dot{\mathfrak{q}}_1) \subset \varpi_E^{-j} \dot{Q},$$

which implies that the coset consists of invertible matrices and therefore cannot contain any nilpotent elements. Thus the character χ_s of $\dot{Q}_{nj/ef_o}$ defined by $\chi_s(x) = \psi(\mathrm{tr}(s(x-1)))$, $x \in \dot{Q}_{nj/ef_o}$ is a standard minimal K-type. The restriction χ'_s of χ_s to $\dot{Q}'_{nj/ef_o}$ is also a standard minimal K-type (of G').

Set $\ell_o = \lfloor(\frac{1}{2}(nj/ef_o)+1)\rfloor$ and $m_o = \lfloor nj/2ef_o \rfloor + 1$ and

$$\dot{J} = 1 + \dot{\mathfrak{q}}'_{nj/ef_o} + \dot{\mathfrak{q}}'^{\perp}_{\ell_o}, \qquad \dot{J}_+ = 1 + \dot{\mathfrak{q}}'_{nj/ef_o} + \dot{\mathfrak{q}}'^{\perp}_{m_o}.$$

The character χ_s of $\dot{Q}_{nj/ef_o}$ extends trivially across $1 + \dot{\mathfrak{q}}'^{\perp}_{m_o}$ to give a character (also denoted by χ_s) of $\dot{J}_+$. Define the representation $\dot{\sigma}$ of $\dot{J}$ to be χ_s if $\dot{J} = \dot{J}_+$ (that is, if $\ell_o = m_o$). Otherwise, $\dot{\sigma}$ is the unique irreducible subrepresentation of $\mathrm{Ind}_{\dot{J}_+}^{\dot{J}} \chi_s$.

Let $\dot{\mathcal{H}} = \mathcal{H}(G /\!\!/ \dot{J}, \tilde{\dot{\sigma}})$ and $\dot{\mathcal{H}}' = \mathcal{H}(G' /\!\!/ \dot{Q}'_{nj/ef_o}, \chi'_{-s})$.

Theorem 9.1 (cf. Theorem 4.9 of [17]). *There exists an isomorphism $\dot{\eta} : \dot{\mathcal{H}}' \to \dot{\mathcal{H}}$ satisfying*

$$\mathrm{supp}(\dot{\eta}(f')) = \dot{J}\,\mathrm{supp}(f')\dot{J} \quad \textit{and} \quad \mathrm{supp}(\dot{\eta}(f')) \cap G' = \mathrm{supp}(f')$$

for $f' \in \dot{\mathcal{H}}'$. Furthermore, $\dot{\eta}$ is an L^2-isometry for the natural L^2-structures on $\dot{\mathcal{H}}$ and $\dot{\mathcal{H}}'$.

We remark that Howe and Moy use the notation ${}^e\eta$ for the isomorphism $\dot{\eta}$, and their e is our n/ef_o. By Theorem 4.10 of [**17**], if $\pi \in \mathcal{E}(G)$ contains $(\dot{Q}_{nj/ef_o}, \chi_s)$, then π contains $(\dot{J}, \dot{\sigma})$. Thus $\dot{\eta}$ gives rise to a bijection between the set of (equivalence classes of) $\pi \in \mathcal{E}(G)$ that contain $(\dot{Q}_{nj/ef_o}, \chi_s)$ and the set of (equivalence classes of) $\dot{\pi}' \in \mathcal{E}(G')$ that contain $(\dot{Q}'_{nj/ef_o}, \chi'_s)$. When π and $\dot{\pi}'$ correspond to each other via $\dot{\eta}$, we write $\dot{\pi}' = \dot{\eta}^*(\pi)$.

Recall (see § 7) that $\mathfrak{L}'^2 = \mathfrak{L}'^{(1)^{n'}}$, $\mathfrak{L}^2 = \xi(\mathfrak{L}'^{(1)^{n'}})$, $\mathfrak{q}_i = \mathfrak{b}_{\mathfrak{L}^2,i}$ $i \in \mathbb{Z}$, and $Q_i = B_{\mathfrak{L}^2,i}$, $i \geqslant 0$. Every $\mathfrak{o}_E$-lattice occurring in $\mathfrak{L}'^2$ occurs in $\mathfrak{L}'^3$. This implies that every $\mathfrak{o}_F$-lattice

https://doi.org/10.1017/S1474748003000124 Published online by Cambridge University Press

occurring in $\mathfrak{L}^2$ occurs in $\mathfrak{L}^3$. Hence $\mathfrak{q} \subset \dot{\mathfrak{q}}$. And $Q_{n'j} = 1 + \varpi_E^j \mathfrak{q} \subset 1 + \varpi_E^j \dot{\mathfrak{q}} = Q_{jn/ef_o}$, so any representation that contains $(\dot{Q}_{nj/ef_o}, \chi_s)$ also contains $(Q_{n'j}, \chi_s)$. Let $\eta : \mathcal{H}' \to \mathcal{H}$ be the Hecke algebra isomorphism of Theorem 7.1.

Proposition 9.2. *Suppose that* $\pi \in \mathcal{E}(G)$ *contains* $(\dot{Q}_{nj/ef_o}, \chi_s)$. *Let* $\pi' = \eta^*(\pi)$ *and* $\dot{\pi}' = \dot{\eta}^*(\pi)$. *Then* $\pi' \simeq \dot{\pi}'$.

Proof. Let $J' = Q'_{n'j}$ and $\dot{J}' = \dot{Q}'_{nj/ef_o}$. By above, $J' \subset \dot{J}'$.

Note that $\dot{J}'$ normalizes J, as $\dot{J}' \subset \dot{Q}'_1 \subset Q'_1$ and Q'_1 normalizes J. Set $\mathcal{J}_0 = \dot{J}'J$. Let σ be the representation of J defined in § 7. Define an extension σ_{ext} of σ to $\mathcal{J}_0$ by $\sigma_{\text{ext}}(xy) = \chi'_s(x)\sigma(y)$, $x \in \dot{J}'$ and $y \in J$. Next, set $\mathcal{J} = \dot{J}J$. Howe and Moy treat three separate cases. This is not really necessary, as we can always define $\mathcal{J}$. In the case where j is even, $\mathcal{J}$ is just $\dot{J}$. In all cases (see the proof of Theorem 4.9 in [**17**]), $\mathrm{Ind}_{\dot{J}}^{\mathcal{J}} \dot{\sigma} \simeq \mathrm{Ind}_{\mathcal{J}_0}^{\mathcal{J}} \sigma_{\text{ext}}$. Hence

$$\dot{\mathcal{H}} \simeq \mathcal{H}(G/\!\!/\mathcal{J}, \mathrm{Ind}_{\dot{J}}^{\mathcal{J}} \dot{\sigma}) \simeq \mathcal{H}(G/\!\!/\mathcal{J}_0, \sigma_{\text{ext}}).$$

For the remainder of the proof we will identify $\dot{\mathcal{H}}$ with the subalgebra $\mathcal{H}(G/\!\!/\mathcal{J}_0, \sigma_{\text{ext}})$ of $\mathcal{H}$.

Given g in the support of $\mathcal{H}'$, let f'_g be the unique function in $\mathcal{H}'$ such that $f'_g(g) = 1$ and f'_g is supported on $J'gJ'$. Set

$$\Lambda' = \upsilon_{G'}(\dot{J}')^{-1} \sum_{g \in \dot{J}'/J'} \chi'_s(g)^{-1} f'_g.$$

This looks slightly different from the formula of Howe and Moy because we are using the Hecke algebras with the contragredient representations $\chi_s'^{-1}$, $\tilde{\sigma}$, etc., and because we have not assumed that Haar measure on G' is normalized so that $\dot{J}'$ has volume one. Since Λ' is equal to $\upsilon_{G'}(\dot{J}')^{-1}\chi_s'^{-1}$ on $\dot{J}'$, and zero elsewhere, Λ' is the identity element of $\dot{\mathcal{H}}'$. Also,

$$\dot{\mathcal{H}}' = \{\Lambda' * f' * \Lambda' \mid f' \in \mathcal{H}'\}.$$

Set $\Lambda = \eta(\Lambda')$. Let $g \in \dot{J}'$. Then $\eta(f'_g) = \upsilon_G(J)^{-1}\upsilon_{G'}(J')f_g$, where $f_g \in \mathcal{H}$ is such that $f_g(g) = 1_{\dim \tilde{\sigma}}$ and f_g is supported on $J'gJ' = gJ'$. We remark that the definition of $\eta(f'_g)$ for more general g can be quite complicated (see [**15**, pp. 42–45]). But $\dot{J}' \subset \dot{Q}' \subset Q'_1$ guarantees that the oscillator representation that normally appears in $\eta(f'_g)$ is trivial for $g \in \dot{J}'$, and thus $\eta(f'_g)$ is as above. Hence, as $\mathcal{J}_0/J \simeq \dot{J}'/J'$,

$$\begin{aligned}
\Lambda &= \upsilon_G(J)^{-1}\upsilon_{G'}(J')\upsilon_{G'}(\dot{J}')^{-1} \sum_{g \in \dot{J}'/J'} \chi'_s(g)^{-1} f_g \\
&= \upsilon_G(\mathcal{J}_0)^{-1} \sum_{g \in \dot{J}'/J'} \tilde{\sigma}_{\text{ext}}(g) f_g \\
&= \begin{cases} \upsilon_G(\mathcal{J}_0)^{-1}\tilde{\sigma}_{\text{ext}} & \text{on } \mathcal{J}_0, \\ 0 & \text{on } G - \mathcal{J}_0. \end{cases}
\end{aligned}$$

https://doi.org/10.1017/S1474748003000124 Published online by Cambridge University Press

That is, Λ is the identity element of $\dot{\mathcal{H}}$. Also, $\dot{\mathcal{H}} = \{\Lambda * f * \Lambda \mid f \in \mathcal{H}\}$ and the Hecke algebra isomorphism $\dot{\eta}$ is the restriction of η to $\dot{\mathcal{H}}$.

Let $\boldsymbol{r}$, $\dot{\boldsymbol{r}}$, $\boldsymbol{r}'$ and $\dot{\boldsymbol{r}}'$ be the representations of $\mathcal{H}$, $\dot{\mathcal{H}}$, $\mathcal{H}'$ and $\dot{\mathcal{H}}'$ associated to π, $\dot{\pi}$, π', and $\dot{\pi}'$, respectively.

Let $\tilde{W}$ be the space of $\tilde{\sigma}_{\text{ext}}$. The spaces of $\boldsymbol{r}$ and $\dot{\boldsymbol{r}}$ are $(V_\pi \otimes \tilde{W})^J \simeq (V_\pi^{(J,\sigma)} \otimes \tilde{W})^J$ and $(V_\pi \otimes \tilde{W})^{\mathcal{J}_0} \simeq (V_\pi^{(\mathcal{J}_0,\sigma_{\text{ext}})} \otimes \tilde{W})^{\mathcal{J}_0}$, respectively, where $\mathcal{J}_0$ acts via $\pi \mid \mathcal{J}_0$ on V_π and by $\tilde{\sigma}_{\text{ext}}$ on $\tilde{W}$. Furthermore, if $f \in \mathcal{H}$, $\dot{\boldsymbol{r}}(\Lambda * f * \Lambda) = \boldsymbol{r}(\Lambda * f * \Lambda) \mid (V_\pi \otimes \tilde{W})^{\mathcal{J}_0}$.

Let $\dot{J}'$ act on $\mathbb{C}$ via $\chi_s'^{-1}$. Then the spaces of $\boldsymbol{r}'$ and $\dot{\boldsymbol{r}}'$ are $(V_{\pi'} \otimes \mathbb{C})^{J'} \simeq V_{\pi'}^{(J',\chi_s')}$ and $(V_{\dot{\pi}'} \otimes \mathbb{C})^{\dot{J}'} \simeq V_{\dot{\pi}'}^{(\dot{J}',\chi_s')}$, respectively.

Since $\pi' = \eta^*(\pi)$ and $\dot{\pi}' = \dot{\eta}^*(\pi)$, there exist isomorphisms

$$T : (V_{\pi'} \otimes \mathbb{C})^{J'} \to (V_\pi \otimes \tilde{W})^J \qquad \text{such that } T \circ \boldsymbol{r}'(f') = \boldsymbol{r}(\eta(f')) \circ T, \quad f' \in \mathcal{H}',$$
$$\dot{T} : (V_{\dot{\pi}'} \otimes \mathbb{C})^{\dot{J}'} \to (V_\pi \otimes \tilde{W})^{\dot{J}'J} \quad \text{such that } \dot{T} \circ \dot{\boldsymbol{r}}'(\dot{f}') = \dot{\boldsymbol{r}}(\dot{\eta}(\dot{f}')) \circ \dot{T}, \quad \dot{f}' \in \dot{\mathcal{H}}'.$$

Set $T' = \dot{T}^{-1} \circ T \mid (V_{\pi'} \otimes \mathbb{C})^{\dot{J}'}$. The representation of $\dot{\mathcal{H}}'$ associated to π' is given by $\dot{f}' \mapsto \boldsymbol{r}'(\dot{f}') \mid (V_{\pi'} \otimes \mathbb{C})^{\dot{J}'}$. To see that this representation is equivalent to $\dot{\boldsymbol{r}}'$, it suffices to show that

$$T' \circ \boldsymbol{r}'(\Lambda' * f' * \Lambda') \mid (V_{\pi'} \otimes \mathbb{C})^{\dot{J}'} = \dot{\boldsymbol{r}}'(\Lambda' * f' * \Lambda') \circ T' \quad \forall f' \in \mathcal{H}',$$

and this follows easily from the intertwining properties of T and $\dot{T}$ and the relation between $\boldsymbol{r} \mid \dot{\mathcal{H}}$ and $\dot{\boldsymbol{r}}$. Thus $\pi' \simeq \dot{\pi}'$. □

10. Multiplicities of *K*-types—the depth-zero pure case

Let $\pi \in \mathcal{E}(G)$ be such that the depth $\rho(\pi)$ of π is zero and such that π contains a pure minimal K-type. In this section, we study the dimension of the subspace $V_\pi^{B_{\alpha,1}}$ of $B_{\alpha,1}$-fixed vectors the space V_π of π, for $\alpha \in \mathcal{P}^0(n)$. Via a Hecke algebra isomorphism, π corresponds to a unipotent representation π' of a general linear group over an unramified extension of F. In Proposition 10.8, we show that if the dimension of $V_\pi^{B_{\alpha,1}}$ is non-zero, it is an explicit multiple of the dimension of the space of $B'_{\beta,1}$-fixed vectors in the space $V_{\pi'}$ for some $\beta \in \mathcal{P}^0(n')$.

There exists a natural number d dividing n, together with an irreducible cuspidal representation σ_0 of $GL_d(\mathbb{F}_q)$, such that, if $n' = n/d$, π contains $(B_{(d)^{n'}}, \sigma)$, where σ is the inflation of the n'-fold tensor product of σ_0 to $B_{(d)^{n'}}$. As $B_{(n),1} \subset B_{(d)^{n'},1}$, it follows that $V_\pi^{B_{(n),1}} \neq 0$. Let $\kappa(\pi, B_{(n)})$ denote the representation of $B_{(n)}$ obtained by restricting the action of $\pi|_{B_{(n)}}$ to $V_\pi^{B_{(n),1}}$.

Lemma 10.1. *The minimal K-type $(B_{(d)^{n'}}, \sigma)$ occurs with positive multiplicity in each irreducible component of $\kappa(\pi, B_{(n)})$.*

Proof. Let $\mathcal{V} = V_\pi^{B_{(n),1}}$ and $\kappa = \kappa(\pi, B_{(n)})$. Let τ be any irreducible component of κ. By Theorem 3.5 of [**30**], τ contains an unrefined minimal K-type of depth zero of a parahoric subgroup $G_{x,0} \subset B_{(n)}$. After replacing $G_{x,0}$ by a $B_{(n)}$-conjugate, we may assume

https://doi.org/10.1017/S1474748003000124 Published online by Cambridge University Press

that $G_{x,0} = B_\alpha$ for some $\alpha \in \mathcal{P}(n)$. Thus τ contains an irreducible representation ξ of B_α which is trivial on $B_{\alpha,1}$ and is the inflation of an irreducible cuspidal representation of $B_\alpha/B_{\alpha,1}$. By Proposition 6.2 of [**30**], all unrefined minimal K-types occurring in κ are associates of (B_α, ξ). Therefore (see § 4 for the definition of associate), $B_\alpha/B_{\alpha,1} \simeq B_{(d)^{n'}}/B_{(d)^{n'},1}$ and ξ and σ are inflations of equivalent cuspidal representations. This implies that $B_\alpha = B_{(n)^{d'}}$ and $\sigma \simeq \xi$. □

Let E/F be the unramified extension of degree d, embedded in $\mathfrak{gl}_n(F)$ as in § 3. Let $G' = GL_{n'}(E)$. In this section, we will be comparing properties of $\kappa(\pi, B_{(n)})$ and $\kappa(\pi', B_{(n')})$ for some irreducible unipotent representation π' of G'. Set

$$\mathcal{H} = \mathcal{H}(G /\!\!/ B_{(d)^{n'}}, \tilde{\sigma}), \qquad \mathcal{H}_0 = \mathcal{H}(B_{(n)} /\!\!/ B_{(d)^{n'}}, \tilde{\sigma}),$$
$$\mathcal{H}' = \mathcal{H}(G' /\!\!/ B'_{(1)^{n'}}), \qquad \mathcal{H}'_0 = \mathcal{H}(B'_{(n')} /\!\!/ B'_{(1)^{n'}}).$$

It is clear that $\mathcal{H}_0$ and $\mathcal{H}'_0$ are subalgebras of $\mathcal{H}$ and $\mathcal{H}'$, respectively. Let $\mathcal{P} = \mathcal{P}_{(d)^{n'}}$, $\mathcal{G}' = B'_{(n')}/B'_{(n'),1} \simeq GL_{n'}(\mathbb{F}_{q^d})$, and $\mathcal{B}' = B_{(1)^{n'}}/B_{(n'),1}$. Then $\mathcal{B}'$ is a Borel subgroup of $\mathcal{G}'$. Note that

$$\mathcal{H}_0 \simeq \mathcal{H}(\mathcal{G} /\!\!/ \mathcal{P}, \tilde{\sigma}) \quad \text{and} \quad \mathcal{H}'_0 \simeq \mathcal{H}(\mathcal{G}' /\!\!/ \mathcal{B}').$$

Theorem 10.2 (cf. Theorem 2.1.2 of [15]). *There is an isomorphism $\eta : \mathcal{H}' \to \mathcal{H}$ satisfying*

$$\operatorname{supp}(\eta(f')) = B_{(d)^{n'}} \operatorname{supp}(f') B_{(d)^{n'}} \quad \textit{and} \quad \operatorname{supp}(\eta(f')) \cap G' = \operatorname{supp}(f')$$

for $f' \in \mathcal{H}'$. The isomorphism η is an L^2-isometry for the natural L^2 structures on $\mathcal{H}$ and $\mathcal{H}'$. Furthermore, the restriction η_0 of η to $\mathcal{H}'_0$ is an isomorphism of $\mathcal{H}'_0$ onto $\mathcal{H}_0$ which preserves the natural L^2-structures on $\mathcal{H}'_0$ and $\mathcal{H}_0$.

Via the Hecke algebra isomorphism η, we obtain a bijection between the set of $\pi \in \mathcal{E}(G)$ that contain $(B_{(d)^{n'}}, \sigma)$ and the set of $\pi' \in \mathcal{E}(G')$ that contain the trivial representation of the Iwahori subgroup $B'_{(1)^{n'}}$ of G'. When π and π' correspond via η, we write $\pi' = \eta^*(\pi)$. Similarly, η_0 gives rise to a bijection between the set of representations of $\mathcal{G}$ having the property that every irreducible component contains $(\mathcal{P}, \sigma)$ and the set of representations of $\mathcal{G}'$ having the property that every irreducible component contains the trivial representation of $\mathcal{B}'$. When τ and τ' correspond via η_0, we write $\tau' = \eta_0^*(\tau)$ or $\tau = \eta_0^{-1*}(\tau')$.

Lemma 10.3. *Suppose that $\pi \in \mathcal{E}(G)$ contains $(B_{(d)^{n'}}, \sigma)$. Let $\pi' = \eta^*(\pi)$. Then*

$$\kappa(\pi', B'_{(n')}) = \eta_0^*(\kappa(\pi, B_{(n)})).$$

Proof. The representation π corresponds to an irreducible representation of $\mathcal{H}$ whose restriction to $\mathcal{H}_0$ corresponds to the representation of $B_{(n)}$ arising via restriction of $\kappa(\pi, B_{(n)})$ to the $(B_{(d)^{n'}}, \sigma)$-isotypic subspace of $V_\pi^{B_{(n),1}}$. By Lemma 10.1, this isotypic subspace is all of $V_\pi^{B_{(n),1}}$. Similarly, the representation of $\mathcal{H}'_0$ obtained by restricting the representation of $\mathcal{H}'$ coming from π' corresponds to $\kappa(\pi', B'_{(n')})$. As $\pi' = \eta^*(\pi)$ and $\eta_0 = \eta|_{\mathcal{H}'_0}$, the lemma follows. □

https://doi.org/10.1017/S1474748003000124 Published online by Cambridge University Press

As the restriction of $\kappa(\pi, B_{(n)})$ to $B_{(n),1}$ is iso-trivial and $B_{(n)}/B_{(n),1} \simeq GL_n(\mathbb{F}_q)$, whenever it is convenient, we will view $\kappa(\pi, B_{(n)})$ as a representation of $\mathcal{G} = GL_n(\mathbb{F}_q)$. Let $\alpha \in \mathcal{P}(n)$. Then

$$B_{(n),1} \subset B_{\alpha,1} \subset B_\alpha \subset B_{(n)}$$

and $\mathcal{P}_\alpha = B_\alpha/B_{(n),1}$ is a standard parabolic subgroup of $\mathcal{G}$ with standard Levi component $\mathcal{M}_\alpha \simeq B_\alpha/B_{\alpha,1}$ and unipotent radical $\mathcal{N}_\alpha \simeq B_{\alpha,1}/B_{(n),1}$. And hence $V_\pi^{B_{\alpha,1}}$ can be viewed as the space of $\mathcal{N}_\alpha$-invariant vectors in the space of the representation $\kappa(\pi, B_{(n)})$ of $\mathcal{G}$. In view of Lemma 10.3, in order to compare multiplicities of trivial representations of pro-unipotent radicals of parahoric subgroups in π and $\eta^*(\pi)$, it suffices to compare multiplicities of trivial representations of finite unipotent radicals in $\kappa(\pi, B_{(n)})$ and $\eta_0^*(\kappa(\pi, B_{(n)}))$. Determining the relation between η_0^* and twisted induction allows us to carry out the comparison.

Suppose that $\mathcal{L} = \mathbb{L}(\mathbb{F}_q)$ is a subgroup of $\mathcal{G}$ such that $\mathbb{L}$ is a Levi factor of a parabolic subgroup of $GL_n(\bar{\mathbb{F}}_q)$ such that $\mathbb{L}$ is defined over $\mathbb{F}_q$ (the parabolic subgroup need not be defined over $\mathbb{F}_q$). That is, $\mathcal{L}$ is the centralizer of some torus in $\mathcal{G}$. The *twisted induction* map $R_{\mathcal{L}}^{\mathcal{G}}$ defined in [**23**] takes virtual representations of $\mathcal{L}$ to virtual representations of $\mathcal{G}$. In the special case where $\mathcal{L}$ is a Levi subgroup of a parabolic subgroup of $\mathcal{G}$, $R_{\mathcal{L}}^{\mathcal{G}}$ coincides with parabolic induction (also known as Harish-Chandra induction). If $\mathcal{L} = \mathcal{T}$ is a Cartan subgroup of $\mathcal{G}$, then $R_{\mathcal{T}}^{\mathcal{G}}$ is known as Deligne–Lusztig induction. Whenever convenient, we will identify a virtual representation of a finite reductive group with the corresponding class function.

Recall that an irreducible representation of a finite general linear group is unipotent if and only if it contains the trivial representation of a Borel subgroup. Let $\hat{\mathcal{G}}'_{\text{unip}}$ denote the set of (equivalence classes of) unipotent representations of $\mathcal{G}'$. Any elliptic Cartan subgroup $\mathcal{T}_0$ of $GL_d(\mathbb{F}_q)$ is isomorphic to $\mathbb{F}_{q^d}^\times$. As σ_0 is an irreducible cuspidal representation of $GL_d(\mathbb{F}_q)$, there exists a character ν of $\mathcal{T}_0 \simeq \mathbb{F}_{q^d}^\times$, which is not fixed by any non-trivial element of $\mathrm{Gal}(\mathbb{F}_{q^d}/\mathbb{F}_q)$, such that

$$\sigma_0 \simeq (-1)^{d-1} R_{\mathcal{T}_0}^{GL_d(\mathbb{F}_q)}(\nu).$$

Define a character ν' of $\mathcal{G}'$ by $\nu \circ \det'$, where $\det'$ is the determinant on $\mathcal{G}'$.

Lemma 10.4. *If* $\tau' \in \hat{\mathcal{G}}'_{\text{unip}}$, *then* $\tau' = \eta_0^*((-1)^{n-n'} R_{\mathcal{G}'}^{\mathcal{G}}(\nu' \otimes \tau'))$.

Proof. Given $\alpha \in \mathcal{P}^0(n')$, let $\mathcal{P}'_\alpha$ be the associated standard parabolic subgroup of $\mathcal{G}'$. The group W' of permutation matrices in $\mathcal{G}'$ is isomorphic to the Weyl group of $\mathcal{G}'$. If $\alpha \in \mathcal{P}^0(n)$, let $W'_\alpha = W' \cap \mathcal{P}'_\alpha$. As shown in [**15**, Appendix 1], there is a unique bijection $\tau' \leftrightarrow \xi$ between $\hat{\mathcal{G}}'_{\text{unip}}$ and the set $\hat{W}'$ of irreducible representations of W', such that

$$\dim(\mathrm{Hom}_{\mathcal{G}'}(\tau', \mathrm{Ind}_{\mathcal{P}'_\alpha}^{\mathcal{G}'}(1))) = \dim(\mathrm{Hom}_{W'}(\xi, \mathrm{Ind}_{W'_\alpha}^{W'}(1))) \quad \forall \alpha \in \mathcal{P}^0(n').$$

Similarly (see [**15**, Chapter 1, § 5]), there is an analogous bijection between the set of irreducible constituents of $\mathrm{Ind}_{\mathcal{P}}^{\mathcal{G}}(\sigma)$ and $\hat{W}'$. The Hecke algebra isomorphism η_0 satisfies

$$\dim(\mathrm{Hom}_{\mathcal{G}}(\eta_0^{-1*}(\tau'), \eta_0^{-1*}(\mathrm{Ind}_{\mathcal{P}'_\alpha}^{\mathcal{G}'}(1)))) = \dim(\mathrm{Hom}_{\mathcal{G}'}(\tau', \mathrm{Ind}_{\mathcal{P}'_\alpha}^{\mathcal{G}'}(1))) \quad \forall \alpha \in \mathcal{P}^0(n'),\ \tau' \in \hat{\mathcal{G}}'_{\mathrm{unip}}, \tag{10.1 a}$$

$$\dim \eta_0^{-1*}(\tau') = \dim \sigma [\mathcal{G} : \mathcal{P}][\mathcal{G}' : \mathcal{B}']^{-1} \dim \tau' \quad \forall \tau' \in \hat{\mathcal{G}}'_{\mathrm{unip}}. \tag{10.1 b}$$

Any map from $\hat{\mathcal{G}}'_{\mathrm{unip}}$ to the set of irreducible components of $\mathrm{Ind}_{\mathcal{P}}^{\mathcal{G}} \sigma$ that satisfies (10.1 a) is a bijection. We will show that the map $\tau' \to (-1)^{n-n'} R_{\mathcal{G}'}^{\mathcal{G}}(\nu' \otimes \tau')$ takes elements of $\hat{\mathcal{G}}'_{\mathrm{unip}}$ to irreducible components of $\mathrm{Ind}_{\mathcal{P}}^{\mathcal{G}} \sigma$ and satisfies (10.1) (with η_0^{-1*} replaced by $(-1)^{n-n'} R_{\mathcal{G}'}^{\mathcal{G}}(\nu' \otimes (\cdot))$).

If H is a finite group, let $\langle \cdot, \cdot \rangle_H$ denote the usual inner product on the space of class functions on H. Given $w \in W'$, let $\mathcal{T}_w$ be the Cartan subgroup of $\mathcal{G}'$ of type w (see [**24**] for the definition). Given $\xi \in \hat{W}'$, let

$$\tau_\xi = \sum_{w \in W'} \operatorname{trace} \xi(w) R_{\mathcal{T}_w}^{\mathcal{G}'}(1).$$

By Theorem 2.2 of [**24**], $\tau_\xi \in \hat{\mathcal{G}}'_{\mathrm{unip}}$. It is well known that

$$\mathrm{Ind}_{\mathcal{P}'_\alpha}^{\mathcal{G}'}(1) = |W'_\alpha|^{-1} \sum_{w \in W'_\alpha} R_{\mathcal{T}_w}^{\mathcal{G}'}(1), \quad \alpha \in \mathcal{P}^0(n'). \tag{10.2}$$

It follows from (10.2) and the orthogonality relations for the functions $R_{\mathcal{T}_w}^{\mathcal{G}'}(1)$, $w \in W'$, that

$$\langle \tau_\xi, \mathrm{Ind}_{\mathcal{P}'_\alpha}^{\mathcal{G}'}(1) \rangle_{\mathcal{G}'} = \langle \xi, \mathrm{Ind}_{W'_\alpha}^{W'}(1) \rangle_{W'} \quad \forall \alpha \in \mathcal{P}^0(n'). \tag{10.3}$$

Let $\tau' \in \hat{\mathcal{G}}'_{\mathrm{unip}}$. By the above, if $\xi \in \hat{W}'$ is the representation corresponding to τ' via the above-mentioned canonical bijection, then $\tau_\xi = \tau'$.

Let $\nu_w = \nu'|_{\mathcal{T}_w}$, $w \in W'$. Then $\nu' \otimes R_{\mathcal{T}_w}^{\mathcal{G}'}(1) = R_{\mathcal{T}_w}^{\mathcal{G}'}(\nu_w)$, and we can apply transitivity of twisted induction and Theorem 3.2 of [**24**] to conclude that

$$(-1)^{n-n'} R_{\mathcal{G}'}^{\mathcal{G}}(\nu' \otimes \tau') = (-1)^{n-n'} \sum_{w \in W'} \operatorname{trace} \xi(w) R_{\mathcal{T}_w}^{\mathcal{G}}(\nu_w) \tag{10.4}$$

is an irreducible representation of $\mathcal{G}$. Furthermore, denoting the split Cartan in $\mathcal{B}'$ by $\mathcal{T}$, since τ' is a component of $\mathrm{Ind}_{\mathcal{B}'}^{\mathcal{G}'}(1) = R_{\mathcal{T}}^{\mathcal{G}'}(1)$, $(-1)^{n-n'} R_{\mathcal{G}'}^{\mathcal{G}}(\nu' \otimes \tau')$ is a component of

$$(-1)^{n-n'} R_{\mathcal{G}'}^{\mathcal{G}}(\nu' \otimes R_{\mathcal{T}}^{\mathcal{G}'}(1)) = (-1)^{n-n'} R_{\mathcal{G}'}^{\mathcal{G}}(R_{\mathcal{T}}^{\mathcal{G}'}(\nu'|_{\mathcal{T}})) = (-1)^{n-n'} R_{\mathcal{T}}^{\mathcal{G}}(\nu'|_{\mathcal{T}}) = \mathrm{Ind}_{\mathcal{P}}^{\mathcal{G}}(\sigma).$$

Let $\alpha \in \mathcal{P}^0(n')$. Then, applying (10.2) and transitivity of twisted induction, followed by (10.4) and the orthogonality relations for the functions $R_{\mathcal{T}_w}^{\mathcal{G}}(\nu_w)$, $w \in W'$, we obtain

$$\begin{aligned} \langle R_{\mathcal{G}'}^{\mathcal{G}}(\nu' \otimes \tau'), R_{\mathcal{G}'}^{\mathcal{G}}(\nu' \otimes \mathrm{Ind}_{\mathcal{P}'_\alpha}^{\mathcal{G}'}(1)) \rangle_{\mathcal{G}} &= |W'_\alpha|^{-1} \sum_{w \in W'_\alpha} \langle R_{\mathcal{G}'}^{\mathcal{G}}(\nu' \otimes \tau'), R_{\mathcal{T}_w}^{\mathcal{G}}(\nu_w) \rangle_{\mathcal{G}} \\ &= \langle \xi, \mathrm{Ind}_{W'_\alpha}^{W'}(1) \rangle_{W'}. \end{aligned}$$

https://doi.org/10.1017/S1474748003000124 Published online by Cambridge University Press

Comparing this with (10.3) and using $\tau' = \tau_\xi$ results in

$$\dim(\mathrm{Hom}_{\mathcal{G}}((-1)^{n-n'} R^{\mathcal{G}}_{\mathcal{G}'}(\nu' \otimes \tau'), (-1)^{n-n'} R^{\mathcal{G}}_{\mathcal{G}'}(\nu' \otimes \mathrm{Ind}^{\mathcal{G}'}_{\mathcal{P}'_\alpha}(1)))) \\ = \dim(\mathrm{Hom}_{\mathcal{G}'}(\tau', \mathrm{Ind}^{\mathcal{G}'}_{\mathcal{P}'_\alpha}(1))) \quad \forall \alpha \in \mathcal{P}^0(n'),\ \tau' \in \hat{\mathcal{G}}'_{\mathrm{unip}}. \tag{10.5}$$

We remark that (10.5) can also be obtained as a consequence of general results on twisted induction (see, for example, Theorem 13.25 of [**10**]).

Above we have shown that the map $\tau' \mapsto (-1)^{n-n'} R^{\mathcal{G}}_{\mathcal{G}'}(\nu' \otimes \tau')$ takes elements $\hat{\mathcal{G}}'_{\mathrm{unip}}$ to irreducible components of $\mathrm{Ind}^{\mathcal{G}}_{\mathcal{P}}(\sigma)$ and satisfies (10.5) (which is (10.1 a), with η_0^{-1*} replaced by $(-1)^{n-n'} R^{\mathcal{G}}_{\mathcal{G}'}(\nu' \otimes (\cdot))$). Also (see [**23**]),

$$\dim((-1)^{n-n'} R^{\mathcal{G}}_{\mathcal{G}'}(\nu' \otimes \tau')) = |\mathcal{G}|_{p'} |\mathcal{G}'|^{-1}_{p'} \dim(\nu' \otimes \tau') = |\mathcal{G}|_{p'} |\mathcal{G}'|^{-1}_{p'} \dim \tau'.$$

Here, if H is a finite group, $|H|_{p'}$ denotes the part of $|H|$ that is prime to p. Let $\mathcal{M}$ be the standard Levi component of $\mathcal{P}$. Note that $[\mathcal{G} : \mathcal{P}] = |\mathcal{G}|_{p'} |\mathcal{M}|^{-1}_{p'}$, $[\mathcal{G}' : \mathcal{B}'] = |\mathcal{G}'|_{p'} |\mathcal{T}|^{-1}$ and

$$\dim \sigma = \dim(-1)^{n-n'} R^{\mathcal{M}}_{\mathcal{T}}(\nu' | \mathcal{T}) = |\mathcal{M}|_{p'} |\mathcal{T}|^{-1},$$

so

$$\dim \sigma [\mathcal{G} : \mathcal{P}][\mathcal{G}' : \mathcal{B}']^{-1} = |\mathcal{G}|_{p'} |\mathcal{G}'|^{-1}_{p'}.$$

It follows that (10.1 b) holds if $\eta_0^{-1*}(\tau')$ is replaced by $(-1)^{n-n'} R^{\mathcal{G}}_{\mathcal{G}'}(\nu' \otimes \tau')$.

As a consequence of the above results, the map $\tau' \to \eta_0^*((-1)^{n-n'} R^{\mathcal{G}}_{\mathcal{G}'}(\nu' \otimes \tau'))$ is a bijection of $\hat{\mathcal{G}}'_{\mathrm{unip}}$ onto itself. This map induces a bijection b_q of $\hat{W}'$ onto itself. To complete the proof of the lemma, it suffices to prove that $b_q(\xi) = \xi$ for all $\xi \in \hat{W}'$.

Now we will let q vary over positive powers of p, and keep n and d fixed. For each $\xi \in \hat{W}'$, there exists a polynomial $f_\xi(t)$ in one variable t such that f_ξ is independent of q and the degree of the corresponding unipotent representation τ_ξ of $\mathcal{G}' = GL_{n'}(\mathbb{F}_{q^d})$ is equal to $f_\xi(q^d)$. This can be seen from the formulae for the degrees of the unipotent representations of finite general linear groups (see, for example, [**37**]). As shown above, the bijection b_q has the property that, for each $\xi \in \hat{W}'$, $f_{b_q(\xi)}(q^d) = f_\xi(q^d)$ for all q. The Hecke algebra isomorphism η_0 has the property that η_0 maps the characteristic function of $\mathcal{B}' w \mathcal{B}'$, $w \in W'$, to a function in $\mathcal{H}_0$ that is supported on $\mathcal{P} w \mathcal{P}$. In combination with the fact that the relation $R^{\mathcal{G}}_{\mathcal{G}'} \circ R^{\mathcal{G}'}_{\mathcal{T}_w} = R^{\mathcal{G}}_{\mathcal{T}_w}$, $w \in W'$, determines $R^{\mathcal{G}}_{\mathcal{G}'}$, this implies that the bijections b_q coincide for all positive powers of p. Denoting the bijection by b, we have, for $\xi \in \hat{W}'$, $f_\xi(q^d) = f_{b(\xi)}(q^d)$ for all q. Hence $f_\xi(t) = f_{b(\xi)}(t)$. Since $f_{\xi_1}(t) \neq f_{\xi_2}(t)$ if ξ_1 is not equivalent to ξ_2 (see [**37**]), it follows that $b(\xi) = \xi$ for all $\xi \in \hat{W}'$. □

Let m be a natural number. Recall that if $\alpha = (\alpha_1, \dots, \alpha_r) \in \mathcal{P}(m)$, then $r(\alpha) = r$. Set

$$\lambda(\alpha) = \prod_{1 \leqslant i \leqslant r} \alpha_i$$

and let $\mathrm{Stab}(\alpha)$ be the stabilizer of α in the symmetric group S_r on r letters. If $\alpha = (\alpha_1, \dots, \alpha_r) \in \mathcal{P}(n)$, fix embeddings of $\mathbb{F}^\times_{q^{\alpha_i}}$ in $GL_{\alpha_i}(\mathbb{F}_q)$, $1 \leqslant i \leqslant r$. Then the image $\mathcal{T}_\alpha$ of $\prod_{1 \leqslant i \leqslant r} \mathbb{F}^\times_{q^{\alpha_i}}$ is a Cartan subgroup of the standard Levi subgroup $\mathcal{M}_\alpha$. Also,

$\{\mathcal{T}_\alpha \mid \alpha \in \mathcal{P}^0(n)\}$ is a complete set of representatives for the conjugacy classes of Cartan subgroups of $\mathcal{G}$. Similarly, if $\alpha \in \mathcal{P}(n')$, let $\mathcal{T}'_\alpha$ be an elliptic Cartan subgroup of the standard Levi subgroup $\mathcal{M}'_\alpha$ of the parabolic subgroup $\mathcal{P}'_\alpha$ of $\mathcal{G}'$. Note that $\mathcal{T}'_\alpha \simeq \mathcal{T}_{d\alpha}$.

Given $\alpha = (\alpha_1, \dots, \alpha_r) \in \mathcal{P}(m)$, set

$$\mathcal{R}(\alpha) = \{\delta \in \mathcal{P}(m) \mid \delta \text{ is a rearrangement of } \alpha\},$$
$$\mathcal{S}(\alpha) = \{\delta \in \mathcal{P}(m) \mid \delta = \delta^{(1)} \cup \cdots \cup \delta^{(r(\alpha))},\ \delta^{(i)} \in \mathcal{P}^0(\alpha_i),\ 1 \leqslant i \leqslant r(\alpha)\}.$$

When $\alpha, \beta \in \mathcal{P}(n)$, the map $\delta \mapsto \mathcal{T}_\delta$ is a bijection between $\mathcal{R}(\alpha) \cap \mathcal{S}(\beta)$ and the set of $\mathcal{M}_\beta$-conjugacy classes of Cartan subgroups of $\mathcal{M}_\beta$ that contain a $\mathcal{G}$-conjugate of $\mathcal{T}_\alpha$. A similar statement holds when $\alpha, \beta \in \mathcal{P}(n')$. Note that if $\alpha, \beta \in \mathcal{P}(n')$, the map $\delta \mapsto d\delta$ from $\mathcal{R}(\alpha) \cap \mathcal{S}(\beta)$ to $\mathcal{R}(d\alpha) \cap \mathcal{S}(d\beta)$ is a bijection. Let $\alpha, \beta \in \mathcal{P}(n')$. Suppose that $\delta = \delta^{(1)} \cup \cdots \cup \delta^{(s)} \in \mathcal{R}(\alpha) \cap \mathcal{S}(\beta)$. Let $W_{\mathcal{M}'_\beta}(\mathcal{T}'_\delta) = \mathrm{Norm}_{\mathcal{M}'_\beta}(\mathcal{T}'_\delta)/\mathcal{T}'_\delta$ denote the Weyl group of $\mathcal{T}'_\delta$ in $\mathcal{M}'_\beta$. As $W_{\mathcal{M}'_\beta}(\mathcal{T}'_\delta)$ is the direct product of the Weyl groups of the Cartan subgroups of $GL_{\beta_i}(\mathbb{F}_{q^d})$ associated to the $\delta^{(i)}$, it follows that

$$|W_{\mathcal{M}'_\beta}(\mathcal{T}'_\delta)| = \prod_{1 \leqslant i \leqslant s} \lambda(\delta^{(i)}) |\mathrm{Stab}(\delta^i)|.$$

As $\lambda(d\delta^{(i)}) = d^{r(\delta^{(i)})} \lambda(\delta^{(i)})$, $r(d\delta^{(i)}) = r(\delta^{(i)})$ and $\mathrm{Stab}(d\delta^{(i)}) = \mathrm{Stab}(\delta^{(i)})$, it follows that

$$|W_{\mathcal{M}_{d\beta}}(\mathcal{T}_{d\delta})| = d^{\sum_{1 \leqslant i \leqslant s} r(\delta^i)} |W_{\mathcal{M}'_\beta}(\mathcal{T}'_\delta)| = d^{r(\delta)} |W_{\mathcal{M}'_\beta}(\mathcal{T}'_\delta)|, \quad \delta \in \mathcal{R}(\alpha) \cap \mathcal{S}(\beta).$$

Let $\mathcal{U}_\mathcal{G}$ be the set of unipotent elements in $\mathcal{G}$. Given $\alpha \in \mathcal{P}(n)$, the Green function Q_α corresponding to $\mathcal{T}_\alpha$ is defined by

$$Q_\alpha(x) = \begin{cases} \text{trace } R^{\mathcal{G}}_{\mathcal{T}_\alpha}(1)(x) & \text{if } x \in \mathcal{U}_\mathcal{G}, \\ 0 & \text{if otherwise.} \end{cases}$$

Similarly, attached to each $\alpha \in \mathcal{P}(n')$, we have a Green function Q'_α on $\mathcal{G}'$.

Lemma 10.5. *Let* α, $\beta \in \mathcal{P}(n')$. *Then*

$$|W_{\mathcal{M}'_\beta}(\mathcal{T}'_\delta)|^{-1} \langle Q'_\alpha, Q'_\delta \rangle_{\mathcal{G}'} = |W_{\mathcal{M}_{d\beta}}(\mathcal{T}_{d\delta})|^{-1} \langle Q_{d\alpha}, Q_{d\delta} \rangle_{\mathcal{G}}, \quad \delta \in \mathcal{R}(\alpha) \cap \mathcal{S}(\beta).$$

Proof. Let $\delta \in \mathcal{R}(\alpha) \cap \mathcal{S}(\beta)$. Applying the orthogonality relations for Green functions, using the fact that $\mathcal{T}'_\alpha$ and $\mathcal{T}'_\delta$ are $\mathcal{G}'$-conjugate, and $|\mathcal{T}'_\delta| = |\mathcal{T}_{d\delta}|$, results in

$$\begin{aligned} |W_{\mathcal{M}'_\beta}(\mathcal{T}'_\delta)|^{-1} \langle Q'_\alpha, Q'_\delta \rangle_{\mathcal{G}'} &= |W_{\mathcal{M}'_\beta}(\mathcal{T}'_\delta)|^{-1} |W_{\mathcal{G}'}(\mathcal{T}'_\delta)| |\mathcal{T}'_\delta|^{-1} \\ &= d^{r(\delta)} |W_{\mathcal{M}_{d\beta}}(\mathcal{T}_{d\delta})|^{-1} d^{-r(\delta)} |W_{\mathcal{G}}(\mathcal{T}_{d\delta})| |\mathcal{T}_{d\delta}|^{-1} \\ &= |W_{\mathcal{M}_{d\beta}}(\mathcal{T}_{d\delta})|^{-1} \langle Q_{d\alpha}, Q_{d\delta} \rangle_{\mathcal{G}}. \end{aligned}$$

□

If $\mathcal{L}$ is a finite reductive $\mathbb{F}_q$-group and p is the characteristic of $\mathbb{F}_q$, let $|\mathcal{L}|_p$ and $|\mathcal{L}|_{p'}$ be the p-part of $|\mathcal{L}|$ and the part of $|\mathcal{L}|$ prime to p, respectively.

https://doi.org/10.1017/S1474748003000124 Published online by Cambridge University Press

Lemma 10.6. *Let τ be a representation of $\mathcal{G}$ on a space $\mathcal{V}$. Then*

$$\dim(\mathcal{V}^{\mathcal{N}_\alpha}) = |\mathcal{M}_\alpha|_{p'} \sum_{\delta\in\mathcal{S}(\alpha)} (-1)^{n-r(\delta)}|W_{\mathcal{M}_\alpha}(\mathcal{T}_\delta)|^{-1}\langle\tau, Q_\delta\rangle_{\mathcal{G}}, \quad \alpha\in\mathcal{P}(n).$$

Proof. Let $1_{\mathcal{N}_\alpha}$ be the characteristic function of $\mathcal{N}_\alpha$. Then

$$\begin{aligned}\dim(\mathcal{V}^{\mathcal{N}_\alpha}) &= |\mathcal{N}_\alpha|^{-1}\sum_{x\in\mathcal{G}} \operatorname{tr}\tau(x)1_{\mathcal{N}_\alpha}(x)\\ &= |\mathcal{N}_\alpha|^{-1}|\mathcal{G}|^{-1}\sum_{x\in\mathcal{G}}\operatorname{tr}\tau(x)\biggl(\sum_{y\in\mathcal{G}}1_{\mathcal{N}_\alpha}(yxy^{-1})\biggr)\\ &= |\mathcal{N}_\alpha|^{-1}|\mathcal{G}|^{-1}\sum_{x\in\mathcal{G}}\operatorname{tr}\tau(x)|\mathcal{P}_\alpha|R^{\mathcal{G}}_{\mathcal{M}_\alpha}(e_{\mathcal{M}_\alpha})(x)\\ &= |\mathcal{M}_\alpha|\langle\tau, R^{\mathcal{G}}_{\mathcal{M}_\alpha}(e_{\mathcal{M}_\alpha})\rangle_{\mathcal{G}}.\end{aligned}$$

Here, $e_{\mathcal{M}_\alpha}$ denotes the characteristic function of the identity element in $\mathcal{M}_\alpha$. The map $\delta\mapsto\mathcal{T}_\delta$ is a bijection of $\mathcal{S}(\alpha)$ onto a set of representatives for the conjugacy classes of Cartan subgroups in $\mathcal{M}_\alpha$. Given $\delta\in\mathcal{S}(\alpha)$, let $Q^{\mathcal{M}_\alpha}_\delta$ be the Green function on $\mathcal{M}_\alpha$ attached to the Cartan subgroup $\mathcal{T}_\delta$. The Green functions $Q^{\mathcal{M}_\alpha}_\delta$, $\delta\in\mathcal{S}(\alpha)$, form a basis for the space of class functions on $\mathcal{M}_\alpha$ that are supported on the unipotent subset of $\mathcal{M}_\alpha$. It follows from the orthogonality relations for the Green functions that

$$e_{\mathcal{M}_\alpha} = |\mathcal{M}_\alpha|_p^{-1}\sum_{\delta\in\mathcal{S}(\alpha)}(-1)^{n-r(\delta)}|W_{\mathcal{M}_\alpha}(\mathcal{T}_\delta)|^{-1}Q^{\mathcal{M}_\alpha}_\delta.$$

Substituting into the above expression for $\dim(\mathcal{V}^{\mathcal{N}_\alpha})$, we obtain

$$\dim(\mathcal{V}^{\mathcal{N}_\alpha}) = |\mathcal{M}_\alpha|_{p'}\sum_{\delta\in\mathcal{S}(\alpha)}(-1)^{n-r(\delta)}|W_{\mathcal{M}_\alpha}(\mathcal{T}_\delta)|^{-1}\langle\tau, R^{\mathcal{G}}_{\mathcal{M}_\alpha}(Q^{\mathcal{M}_\alpha}_\delta)\rangle_{\mathcal{G}}.$$

If $e_{\mathcal{T}_\delta}$ is the characteristic function of the identity in $\mathcal{T}_\delta$, then it follows from the Deligne–Lusztig character formula and the definition of the Green functions that $Q^{\mathcal{M}_\alpha}_\delta = R^{\mathcal{M}_\alpha}_{\mathcal{T}_\delta}(e_{\mathcal{T}_\delta})$. Together with transitivity of twisted induction, this implies

$$Q_\delta = R^{\mathcal{G}}_{\mathcal{M}_\alpha}(Q^{\mathcal{M}_\alpha}_\delta).$$

Substituting this above gives the desired result. □

If $\beta\in\mathcal{P}(n')$, let $\mathcal{N}'_\beta$ be the unipotent radical of $\mathcal{P}'_\beta$.

Lemma 10.7. *Suppose that $\tau'\in\hat{\mathcal{G}}'_{\mathrm{unip}}$. Let $\tau = (-1)^{n-n'}R^{\mathcal{G}}_{\mathcal{G}'}(\nu'\otimes\tau')$. Let $\mathcal{V}'$ and $\mathcal{V}$ be the spaces of τ' and τ, respectively. Let $\alpha\in\mathcal{P}(n)$. Then*

$$\dim(\mathcal{V}^{\mathcal{N}_\alpha}) = \begin{cases}0 & \text{if } \alpha\notin d\mathcal{P}(n'),\\ |\mathcal{M}_{d\beta}|_{p'}|\mathcal{M}_\beta|^{-1}_{p'}\dim(\mathcal{V}'^{\mathcal{N}'_\beta}) & \text{if } \alpha = d\beta,\ \beta\in\mathcal{P}(n').\end{cases}$$

Proof. Let $\alpha \in \mathcal{P}^0(n)$. Since $\tau' \in \hat{\mathcal{G}}'_{\mathrm{unip}}$, there exist scalars a_γ, $\gamma \in \mathcal{P}^0(n')$, such that $\tau' = \sum_{\gamma\in\mathcal{P}^0(n')} a_\gamma R^{\mathcal{G}'}_{\mathcal{T}'_\gamma}(1)$. Hence

$$(\nu'\otimes\tau')|_{\mathcal{U}_{\mathcal{G}'}} = \tau'|_{\mathcal{U}_{\mathcal{G}'}} = \sum_{\gamma\in\mathcal{P}^0(n')} a_\gamma Q'_\gamma|_{\mathcal{U}_{\mathcal{G}'}}. \tag{10.6}$$

Hence

$$\tau|_{\mathcal{U}_{\mathcal{G}}} = (-1)^{n-n'} \sum_{\gamma\in\mathcal{P}^0(n')} a_\gamma R^{\mathcal{G}}_{\mathcal{T}_{d\gamma}}(1)|_{\mathcal{U}_{\mathcal{G}}} = (-1)^{n-n'} \sum_{\gamma\in\mathcal{P}^0(n')} a_\gamma Q_{d\gamma}|_{\mathcal{U}_{\mathcal{G}}}.$$

By Lemma 10.6,

$$\dim(\mathcal{V}^{\mathcal{N}_\alpha}) = |\mathcal{M}_\alpha|_{p'} \sum_{\delta\in\mathcal{S}(\alpha)} (-1)^{n'-r(\delta)} |W_{\mathcal{M}_\alpha}(\mathcal{T}_\delta)|^{-1} \sum_{\gamma\in\mathcal{P}^0(n')} a_\gamma \langle Q_{d\gamma}, Q_\delta\rangle_{\mathcal{G}}. \tag{10.7}$$

If $\langle Q_{d\gamma}, Q_\delta\rangle_{\mathcal{G}} \neq 0$, then $\mathcal{T}_\delta$ and $\mathcal{T}_{d\gamma}$ must be conjugate in $\mathcal{G}$. That is, $\delta \in \mathcal{R}(d\gamma)$. If $\delta \in \mathcal{R}(d\gamma)$, then d divides every part of δ, and, as $\delta \in \mathcal{S}(\alpha)$, d also divides every part of α. Hence (10.7) implies that $\dim(\mathcal{V}^{\mathcal{N}_\alpha}) = 0$ whenever $\alpha \notin d\mathcal{P}(n')$.

Assume that $\alpha = d\beta$ for some $\beta \in \mathcal{P}(n')$. By the above remarks, $\langle Q_{d\delta}, Q_\gamma\rangle_{\mathcal{G}} = 0$ unless $\delta \in \mathcal{R}(d\gamma)$, and, in that case, $\delta = d\omega$ for some $\omega \in \mathcal{P}(n')$. As $d\omega \in \mathcal{R}(d\gamma)\cap\mathcal{S}(d\beta)$ is equivalent to $\omega \in \mathcal{R}(\gamma)\cap\mathcal{S}(\beta)$, equation (10.7) can be rewritten as

$$\begin{aligned}
\dim(\mathcal{V}^{\mathcal{N}_{d\beta}}) &= |\mathcal{M}_{d\beta}|_{p'} \sum_{\gamma\in\mathcal{P}^0(n')} a_\gamma \sum_{\omega\in\mathcal{R}(\gamma)\cap\mathcal{S}(\beta)} (-1)^{n'-r(\omega)} |W_{\mathcal{M}_{d\beta}}(\mathcal{T}_{d\omega})|^{-1} \langle Q_{d\gamma}, Q_{d\omega}\rangle_{\mathcal{G}} \\
&= |\mathcal{M}_{d\beta}|_{p'} \sum_{\gamma\in\mathcal{P}^0(n')} a_\gamma \sum_{\omega\in\mathcal{R}(\gamma)\cap\mathcal{S}(\beta)} (-1)^{n'-r(\omega)} |W_{\mathcal{M}'_\beta}(\mathcal{T}'_\omega)|^{-1} \langle Q'_\gamma, Q'_\omega\rangle_{\mathcal{G}'} \\
&= |\mathcal{M}_{d\beta}|_{p'} \sum_{\omega\in\mathcal{S}(\beta)} (-1)^{n'-r(\omega)} |W_{\mathcal{M}'_\beta}(\mathcal{T}'_\omega)|^{-1} \sum_{\gamma\in\mathcal{P}^0(n')} a_\gamma \langle Q'_\gamma, Q'_\omega\rangle_{\mathcal{G}'} \\
&= |\mathcal{M}_{d\beta}|_{p'} \sum_{\omega\in\mathcal{S}(\beta)} (-1)^{n'-r(\omega)} |W_{\mathcal{M}'_\beta}(\mathcal{T}'_\omega)|^{-1} \langle \tau', Q'_\omega\rangle_{\mathcal{G}'} \quad \text{by (10.6)} \\
&= |\mathcal{M}_{d\beta}|_{p'} |\mathcal{M}'_\beta|^{-1}_{p'} \dim(\mathcal{V}'^{\mathcal{N}'_\beta}) \quad \text{by Lemma 10.6.}
\end{aligned}$$

To obtain the second and third equalities above, we used Lemma 10.5 and the fact that $\langle Q'_\gamma, Q'_\omega\rangle_{\mathcal{G}'} = 0$ whenever $\omega \notin \mathcal{R}(\gamma)$, respectively. □

Proposition 10.8. *Let $\pi \in \mathcal{E}(G)$ contain the pure minimal K-type $(B_{(d)^{n'}}, \sigma)$. Let $\pi' = \eta^*(\pi)$ and let $\alpha \in \mathcal{P}(n)$. Then*

$$\dim(V_\pi^{B_{\alpha,1}}) = \begin{cases} 0 & \text{if } \alpha \notin d\mathcal{P}(n'), \\ u_{d\beta}(q) u_\beta(q^d)^{-1} \dim(V_{\pi'}^{B'_{\beta,1}}) & \text{if } \alpha = d\beta,\ \beta \in \mathcal{P}(n'). \end{cases}$$

Proof. By Lemma 10.3, $\kappa(\pi', B'_{(n')}) = \eta_0^*(\kappa(\pi, B_{(n)}))$. Lemma 10.4 extends linearly to give $\kappa(\pi, B_{(n)}) = (-1)^{n-n'} R^{\mathcal{G}}_{\mathcal{G}'}(\nu' \otimes \kappa(\pi', B_{(n')}))$. Hence, if $\mathcal{V}$ and $\mathcal{V}'$ are the spaces of $\kappa(\pi, B_{(n)})$ and $\kappa(\pi', B'_{(n')})$ respectively, the conclusion of Lemma 10.7 is valid for $\mathcal{V}$

https://doi.org/10.1017/S1474748003000124 Published online by Cambridge University Press

and $\mathcal{V}'$. As $V_\pi^{B_\alpha,1} = \mathcal{V}^{\mathcal{N}_\alpha}$, $\alpha \in \mathcal{P}(n)$ and $V_{\pi'}^{B'_\beta,1} = \mathcal{V}'^{\mathcal{N}'_\beta}$, $\beta \in \mathcal{P}(n')$ (see the remarks following Lemma 10.1), the proposition follows upon observing that $|\mathcal{M}_{d\beta}|_{p'} = u_{d\beta}(q)$ and $|\mathcal{M}'_\beta|_{p'} = u_\beta(q^d)$ (notation as in § 2). □

Fix $s \in \mathfrak{o}_E^\times$ whose image in $\mathfrak{o}_E/\mathfrak{p}_E \simeq \mathbb{F}_{q^d}$ generates $\mathbb{F}_{q^d}$ over $\mathbb{F}_q$. Recall that we are assuming E is embedded in $\mathfrak{gl}_n(F)$ as in § 3. We will often identify s with its image in $\mathfrak{gl}_n(F)$. The following lemma will be used in § 13. Recall (see § 2) that Haar measure on $B_{(d)^{n'}}$ is normalized, so that $B_{(d)^{n'}}$ has volume one.

Recall that σ is the inflation to $B_{(d)^{n'}}$ of the n'-fold tensor product of σ_0, where σ_0 is an irreducible cuspidal representation of $GL_d(\mathbb{F}_q)$.

Lemma 10.9. *Suppose that $p > d$. If h is such that $d \leqslant h \leqslant p$, then*

$$\chi_\sigma\biggl(\sum_{i=1}^{h-1} \frac{X^i}{i!}\biggr) = \chi_\sigma(1) \int_{B_{(d)^{n'}}} \psi(\mathrm{tr}(s \,\mathrm{Ad}\, k(X)))\,\mathrm{d}k \quad \forall X \in \mathfrak{b}_{(d)^{n'}} \text{ such that } X^h \in \mathfrak{b}_{(d)^{n'},1}.$$

Proof. Given $X \in \mathfrak{b}_{(d)^{n'}}$, let $\bar{X}_i$, $1 \leqslant i \leqslant n'$, be the ith component of the image of X in $\mathfrak{b}_{(d)^{n'}}/\mathfrak{b}_{(d)^{n'},1} \simeq (\mathfrak{gl}_d(\mathbb{F}_q))^{n'}$. Suppose that $X^h \in \mathfrak{b}_{(d)^{n'},1}$. Then each $\bar{X}_i$ is a nilpotent element of $\mathfrak{gl}_d(\mathbb{F}_q)$. This implies that $\bar{X}_i^d = 0$ for all i, and thus $X^d \in \mathfrak{b}_{(d)^{n'},1}$. Therefore,

$$\chi_\sigma\biggl(\sum_{i=1}^{h-1} \frac{X^i}{i!}\biggr) = \prod_{i=1}^{n'} \chi_{\sigma_0}(\exp \bar{X}_i).$$

By results of [**18**], viewing the restriction of ψ to $\mathfrak{o}_F$ as a non-trivial character of $\mathbb{F}_q$, and denoting the image of s in $\mathfrak{gl}_d(\mathbb{F}_q)$ by $\bar{s}$,

$$\frac{\chi_{\sigma_0}(\exp \bar{X}_i)}{\chi_{\sigma_0}(1)} = |GL_d(\mathbb{F}_q)|^{-1} \sum_{g \in GL_d(\mathbb{F}_q)} \psi(\mathrm{tr}(\bar{s} \,\mathrm{Ad}\, g(\bar{X}_i))).$$

To complete the proof, note that

$$\int_{B_{(d)^{n'}}} \psi(\mathrm{tr}\, s \,\mathrm{Ad}\, k(X))\,\mathrm{d}k = |GL_d(\mathbb{F}_q)|^{-n'} \prod_{i=1}^{n'} \sum_{g \in GL_d(\mathbb{F}_q)} \psi(\mathrm{tr}(\bar{s} \,\mathrm{Ad}\, g(\bar{X}_i))).$$

□

11. Homogeneity of orbital integrals and germs of characters

Recall (see § 2) that if $s_0 \in \mathfrak{g}_{\mathrm{ss}}$, $\Omega_G(s_0)$ denotes the set of G-orbits in $\mathfrak{g}$ whose closure contains s_0. Given a locally integrable G-invariant function D defined on an open subset S of $\mathfrak{g}$ and an open subset S_0 of S, we say that the restriction of D to S_0 is *s_0-asymptotic*, or D is *s_0-asymptotic on* S_0, if $D|_{S_0 \cap \mathfrak{g}_{\mathrm{reg}}}$ belongs to the span of $\{\hat{\mu}_{\mathcal{O}}|_{S_0\cap\mathfrak{g}_{\mathrm{reg}}} \mid \mathcal{O} \in \Omega_G(s_0)\}$. The main results of this paper (see § 14) concern s_0-asymptotic expansions of germs of characters. An essential step in the proofs of those results involves the application of

https://doi.org/10.1017/S1474748003000124 Published online by Cambridge University Press

some important results of DeBacker and Waldspurger concerning homogeneity properties of invariant distributions and 0-asymptotic expansions of germs of characters. In the first part of this section, we summarize these homogeneity results. The original version of this paper relied upon the validity of a hypothesis concerning linear independence of the restrictions of the nilpotent orbital integrals to certain subspaces of $C_c^\infty(\mathfrak{g})$. The hypothesis is known to hold in some cases, that is, for some subspaces (see Proposition 11.6). In this newer version of the paper, in order to avoid having to assume validity of the hypothesis in cases where it has not been proved, we apply a special case of a result of [**22**]. Namely, if $\pi \in \mathcal{E}(G)$ has positive depth and contains a pure minimal K-type, then the germ of Θ_π is s_0-asymptotic on $\mathfrak{g}_{\rho(\pi)}$ for some semisimple element s_0. The precise statement is given in Theorem 11.8. Finally, the last part of the section concerns properties of the functions $\hat{\mu}_{\mathcal{O}}$, $\mathcal{O} \in \Omega_G(s_0)$, $s_0 \in \mathfrak{g}_{\mathrm{ss}}$, which will be used elsewhere in the paper.

As in § 4, given a point x in the Bruhat–Tits building $\mathcal{B}(G)$ of G, $\mathfrak{g}_{x,r}$, $r \in \mathbb{R}$, denotes the filtration of $\mathfrak{g}$ defined by Moy and Prasad [**29, 30**]. For $r \in \mathbb{R}$, set

$$\mathfrak{g}_{x,r^+} = \bigcup_{t>r} \mathfrak{g}_{x,t}, \qquad \mathfrak{g}_r = \bigcup_{x\in\mathcal{B}(G)} \mathfrak{g}_{x,r}, \qquad \mathfrak{g}_{r^+} = \bigcup_{x\in\mathcal{B}(G)} \mathfrak{g}_{x,r^+},$$
$$\mathcal{D}_r = \sum_{x\in\mathcal{B}(G)} C_{\mathrm{c}}(\mathfrak{g}/\mathfrak{g}_{x,r}).$$

Above, the sum in the definition of $\mathcal{D}_r$ should be interpreted to mean that a function in $\mathcal{D}_r$ is a sum of finitely many functions, each of which lies in $C_{\mathrm{c}}(\mathfrak{g}/\mathfrak{g}_{x,r})$ for some $x \in \mathcal{B}(G)$.

If S is a subset of $\mathfrak{g}$, let $J(S)$ be the set of G-invariant distributions on $\mathfrak{g}$ with support in the closure of $G\cdot S$. If $\mathcal{C}$ is a subspace of $C_c^\infty(\mathfrak{g})$, then $\mathrm{res}_{\mathcal{C}}\, J(S)$ denotes the restrictions of the distributions in $J(S)$ to $\mathcal{C}$. Let $\mathfrak{g}_{\mathrm{nil}}$ be the set of nilpotent elements in $\mathfrak{g}$. Versions of Theorem 11.1 and Corollary 11.4 were first proved by Waldspurger [**42**] for r integral and G belonging to a wide class of groups (including classical groups). Recently, DeBacker [**9**] proved the theorems for arbitrary r and G, subject to certain hypotheses (which hold for $G = GL_n(F)$ when p is sufficiently large). We state the theorems for $G = GL_n(F)$. The reader may refer to [**42**] and [**9**] for the general versions.

Theorem 11.1 (cf. [9, 42]). *Suppose that $p > 2n$ and $r \in \mathbb{R}$. Then $\mathrm{res}_{\mathcal{D}_r} J(\mathfrak{g}_r) = \mathrm{res}_{\mathcal{D}_r} J(\mathfrak{g}_{\mathrm{nil}})$.*

The map $f \mapsto \hat{f}$ maps $\mathcal{D}_r$ to $C_c^\infty(\mathfrak{g}_{(-r)^+})$ [**9**]. Thus Theorem 11.1 implies the following.

Corollary 11.2. *Assume that $p > 2n$. Let $r \in \mathbb{R}$. If $X \in \mathfrak{g}_r$, then $\hat{\mu}_{\mathcal{O}(X)}$ is 0-asymptotic on $\mathfrak{g}_{(-r)^+}$.*

Remark 11.3. Suppose that $s \in \mathfrak{g}_r \cap \mathfrak{g}_{\mathrm{ss}}$. Note that $\mathrm{Ad}\, g(\mathfrak{g}_{x,r}) = \mathfrak{g}_{g\cdot x,r}$, $x \in \mathcal{B}(G)$, $g \in G$, implies that $\mathfrak{g}_r$ is G-invariant. If $\mathcal{O} \in \Omega_G(s)$, after conjugation by an element in the centralizer of its semisimple part, the nilpotent part of any element of $\mathcal{O}$ can be made arbitrarily small. Hence $\mathcal{O} \subset \mathfrak{g}_r$, and thus $\hat{\mu}_{\mathcal{O}}$ is 0-asymptotic on $\mathfrak{g}_{(-r)^+}$ for all $\mathcal{O} \in \Omega_G(s)$.

Corollary 11.4. *Suppose that $s \in \mathfrak{g}_r \cap \mathfrak{g}_{\mathrm{ss}}$. Then the functions $\hat{\mu}_{\mathcal{O}}$, $\mathcal{O} \in \Omega_G(s)$, remain linearly independent upon restriction to any open neighbourhood of zero intersected with $\mathfrak{g}_{\mathrm{reg}}$ if and only if $\hat{\mu}_{\mathcal{O}}|_{\mathfrak{g}_{(-r)^+} \cap \mathfrak{g}_{\mathrm{reg}}}$, $\mathcal{O} \in \Omega_G(s)$, are linearly independent.*

Proof. This is an immediate consequence of Corollary 11.2, the above remark and the fact that the restrictions of the Fourier transforms of the nilpotent orbital integrals to any open neighbourhood of zero intersected with $\mathfrak{g}_{\mathrm{reg}}$ are linearly independent (see [**11**, Corollary 5.10]). □

Suppose that $x \in \mathcal{B}(G)$ and $X \in \mathfrak{g}_{0^+}$. Then, as $\mathfrak{g}_{x,0^+} = \mathfrak{g}_{x,r}$ for some $r > 0$, $X^h \in \mathfrak{g}_{x,rh}$ for $h \in \mathbb{N}$. Hence $\lim_{h\to\infty} X^h = 0$ for all $X \in \mathfrak{g}_{0^+}$. By Lemma 2.6 (ii) of [**42**], $G \cdot \mathfrak{b}_{(1)^n,1} = \{X \in \mathfrak{g} \mid \lim_{h\to\infty} X^h = 0\}$. This set is often referred to as the topologically nilpotent set. Since $G \cdot \mathfrak{b}_{(1)^n,1} \subset G \cdot \mathfrak{g}_{0^+} = \mathfrak{g}_{0^+}$, it follows from the above that

$$\mathfrak{g}_{0^+} = G \cdot \mathfrak{b}_{(1)^n,1} = \{X \in \mathfrak{g} \mid \lim_{h\to\infty} X^h = 0\}.$$

Given $\pi \in \mathcal{E}(G)$, by results of [**11**], we can view the character Θ_π of π as a locally integrable function on G that is locally constant on the regular subset of G. Note that if $X \in \mathfrak{g}_{0^+} \cap \mathfrak{g}_{\mathrm{reg}}$, then $1 + X$ is a regular element of G. We will refer to the restriction of the function $X \mapsto \Theta_\pi(1+X)$ to the intersection of $\mathfrak{g}_{\mathrm{reg}}$ with an open neighbourhood of zero contained in $\mathfrak{g}_{0^+}$ as the germ of Θ_π. Howe [**13**] proved that if π is an irreducible supercuspidal representation of $GL_n(F)$, then the germ of Θ_π is 0-asymptotic on some (unspecified) open neighbourhood of zero. This was later generalized to $\pi \in \mathcal{E}(G)$ and G reductive by Harish-Chandra [**11**], and Clozel [**6**], in the connected and disconnected cases, respectively. The 0-asymptotic expansion of the germ of Θ_π is sometimes referred to as the Harish-Chandra local character expansion. The following theorem (stated for $G = GL_n(F)$) says that the open neighbourhood of zero can be taken equal to $\mathfrak{g}_{\rho(\pi)^+}$, where $\rho(\pi)$ is the depth of π (see § 4).

Theorem 11.5 (cf. [9, 42]). *Let $\pi \in \mathcal{E}(G)$. Assume that $p > 2n$. Then the germ of Θ_π is 0-asymptotic on $\mathfrak{g}_{\rho(\pi)^+}$.*

If $\alpha \in \mathcal{P}^0(n)$, let $\mathfrak{b}_\alpha$ be the parahoric $\mathfrak{o}$-subalgebra of $\mathfrak{g}$ defined in § 3.

Proposition 11.6 (cf. [40]). *Let $\mathcal{F} = \mathrm{span}\{[\mathfrak{b}_\alpha] | \alpha \in \mathcal{P}^0(n)\}$. Then*

$$\dim \mathrm{res}_{\mathcal{F}}\, J(\mathfrak{g}_{\mathrm{nil}}) = |\Omega_G(0)|.$$

Suppose that n' is a positive divisor of n, j is a positive integer and E is a tamely ramified extension of F of degree n/n'. Set $e = e(E/F)$. Let $s \in \mathfrak{p}_E^{-j} - \mathfrak{p}_E^{-j+1}$ be such that $\varpi_F^j s^e$ generates $\mathfrak{o}_E/\mathfrak{p}_E$ over $\mathfrak{o}_F/\mathfrak{p}_F$. It is easy to show that such an element s is a good element in the sense of [**22**, § 2.3].

Lemma 11.7. *Suppose that $p > 2n$. Let s be as above, and let $\mathfrak{g}'$ be the Lie algebra of $C_G(s)$. If $X \in \mathfrak{g}'_{(-j)^+}$, and $\mathcal{O} = \mathcal{O}_G(s+X)$, then $\hat{\mu}_{\mathcal{O}}$ is s-asymptotic on $\mathfrak{g}_{j/e}$.*

Proof. The lemma is an immediate consequence of Lemma 3.1.5 and Theorem 3.1.7 of [**22**]. □

https://doi.org/10.1017/S1474748003000124 Published online by Cambridge University Press

Let s be as above. Define the pure minimal K-type $(Q_{n'j}, \chi_s)$ as in § 6.

Theorem 11.8. *Assume that $p > 2n$. Suppose that $\pi \in \mathcal{E}(G)$ contains $(Q_{n'j}, \chi_s)$. Then the germ of Θ_π is s-asymptotic on $\mathfrak{g}_{j/e} = \mathfrak{g}_{\rho(\pi)}$.*

Proof. According to Remark 5.3.3 of [**22**], if $p > 2n$, then Theorem 5.3.1 of [**22**] can be applied with the exponential map replaced by the map $X \mapsto 1 + X$. Thus it suffices to prove that $(Q_{n'j}, \chi_s)$ is a good minimal K-type in the sense of [**22**, § 2.3]. This is equivalent to s being a good element in the sense of [**22**, § 2.3], which, as remarked above, is easy to check. □

If $s_0 \in \mathfrak{g}$ belongs to some elliptic Cartan subalgebra of $\mathfrak{g}$, then s_0 generates a finite extension E_0 of F. Let $b = [E_0 : F]$. The centralizer G_0 of s_0 in G is isomorphic to $GL_{n/b}(E_0)$. The Jordan decomposition defines a natural bijection between $\Omega_G(s_0)$ and $\Omega_{G_0}(0)$. Given $\beta \in \mathcal{P}^0(n/b)$, let X_β be the nilpotent element in the Lie algebra $\mathfrak{g}_0 \simeq \mathfrak{gl}_{n/b}(E_0)$ of G_0 which corresponds to the element Y_β of $\mathfrak{gl}_{n/b}(E_0)$ defined at the end of § 2. Then the orbit in $\Omega_G(s_0)$ that corresponds to $\mathcal{O}_{G_0}(X_\beta)$ is $\mathcal{O}_G(s_0 + X_\beta)$.

For each $\alpha \in \mathcal{P}^0(n)$, let $\mathcal{O}_\alpha \in \Omega_G(0)$ be the corresponding nilpotent G-orbit (as discussed at the end of § 2). Given $\beta \in \mathcal{P}^0(n/b)$, let $\beta^b \in \mathcal{P}(n)$ be as defined in § 2 and let $\dot{\beta}^b$ be the unique element of $\mathcal{P}^0(n)$ that is a rearrangement of β^b.

Lemma 11.9. *Assume that $p > 2n$. Suppose that $s_0 \in \mathfrak{g}$ belongs to an elliptic Cartan subalgebra of $\mathfrak{g}$. Let $\beta \in \mathcal{P}^0(n/b)$ and let X_β be as above. For each $\alpha \in \mathcal{P}^0(n)$, let $c_\alpha(s_0, \beta)$ be the coefficient of $\hat{\mu}_{\mathcal{O}_\alpha}$ in the 0-asymptotic expansion of $\hat{\mu}_{\mathcal{O}_G(s_0+X_\beta)}$. Then the following hold.*

(1) $c_\alpha(s_0, \beta) = 0$ *unless* $\alpha \geqslant \dot{\beta}^b$.

(2) $c_{\dot{\beta}^b}(s_0, \beta) > 0$.

Proof. Let $i \geqslant 1$. There exists a $k \geqslant 0$ such that $\varpi^k s_o \in \mathfrak{b}_{(n),i}$. Let $\beta \in \mathcal{P}^0(n/b)$. By the remark following Corollary 11.2, $\mathcal{O}_G(\varpi^k(s_0 + X_\beta)) = \mathcal{O}_G(\varpi^k s_0 + X_\beta) \subset \mathfrak{g}_i$. By Corollary 11.2, $\hat{\mu}_{\mathcal{O}_G(\varpi^k s_0 + X_\beta)}$ is 0-asymptotic on $\mathfrak{g}_{(-i)^+}$. Let $c_\alpha(\varpi^k s_0, \beta)$ be the coefficient of $\hat{\mu}_{\mathcal{O}_\alpha}$ in the corresponding 0-asymptotic expansion. As $\mathfrak{b}_{(n),-i+1} \subset \mathfrak{g}_{(-i)^+}$, and $f \in C_{\mathrm{c}}(\mathfrak{g}/\mathfrak{b}_{(n),i})$ if and only if $\hat{f}$ is supported on $\mathfrak{b}_{(n),-i+1}$, we have

$$\mu_{\mathcal{O}_G(\varpi^k s_0 + X_\beta)}(f) = \sum_{\alpha \in \mathcal{P}^0(n)} c_\alpha(\varpi^k s_0, \beta)\mu_{\mathcal{O}_\alpha}(f), \quad f \in C_{\mathrm{c}}(\mathfrak{g}/\mathfrak{b}_{(n),i}).$$

If $\alpha \in \mathcal{P}^0(n)$, let $Y_\alpha \in \mathcal{O}_\alpha$ be the nilpotent element defined at the end of § 2. Note that if α, $\gamma \in \mathcal{P}^0(n)$, $(Y_\alpha + \mathfrak{b}_{(n),i}) \cap \mathcal{O}_\gamma \neq \emptyset$ implies $\gamma \leqslant \alpha$. Hence, given $\alpha \in \mathcal{P}^0(n)$, there exists a function $f_\alpha \in C_{\mathrm{c}}^\infty(\mathfrak{g})$ that is a linear combination of the characteristic functions $[Y_\gamma + \mathfrak{b}_{(n),i}]$, $\gamma \in \mathcal{P}^0(n)$ such that $\gamma \leqslant \alpha$, and is such that $\mu_{\mathcal{O}_\alpha}(f_\alpha) = 1$ and $\mu_{\mathcal{O}_\gamma}(f_\alpha) = 0$ whenever $\gamma \neq \alpha$. Suppose that $\alpha \in \mathcal{P}^0(n)$, $\beta \in \mathcal{P}^0(n/b)$, and $c_\alpha(\varpi^k s_0, \beta) \neq 0$. Then $\mu_{\mathcal{O}_G(\varpi^k s_0 + X_\beta)}(f_\alpha) \neq 0$, which implies that $\mathcal{O}_G(\varpi^k s_0 + X_\beta) \cap (Y_\gamma + \mathfrak{b}_{(n),i}) \neq \emptyset$ for some $\gamma \leqslant \alpha$. By the Lie algebra version of Proposition 6.8 (1) of [**31**], there exists $i \geqslant 1$ such that

$$\mathcal{O}_G(\varpi^k s_0 + X_\beta) \cap (Y_\gamma + \mathfrak{b}_{(n),i}) = \emptyset \quad \text{unless } \dot{\beta}^b \leqslant \gamma.$$

Hence $\dot{\beta}^b \leqslant \gamma$ for some $\gamma \leqslant \alpha$. This implies that $\dot{\beta}^b \leqslant \alpha$, and part (1) holds with $\varpi^k s_0$ replacing s_0.

By the Lie algebra version of Proposition 6.8 (2) of [**31**], if k is sufficiently large, then

$$\mathcal{O}_G(\varpi^k s_0 + X_\beta) \cap (Y_{\dot{\beta}^b} + \mathfrak{b}_{(n),i}) \neq \emptyset.$$

It follows from the above that

$$\mu_{\mathcal{O}_G(\varpi^k s_0 + X_\beta)}([Y_{\dot{\beta}^b} + \mathfrak{b}_{(n),i}]) = c_{\dot{\beta}^b}(\varpi^k s_0, \beta) > 0.$$

To show that the lemma holds for s_0, note that

$$\hat{\mu}_{\mathcal{O}_G(\varpi^k(s_0 + X_\beta))}(X) = \hat{\mu}_{\mathcal{O}_G(s_0 + X_\beta)}(\varpi^k X), \quad X \in \mathfrak{g}_{\mathrm{reg}}.$$

As $\hat{\mu}_{\mathcal{O}_\alpha}(\varpi^k X)$ is a positive multiple of $\hat{\mu}_{\mathcal{O}_\alpha}(X)$ for $X \in \mathfrak{g}_{\mathrm{reg}}$, $c_\alpha(\varpi^k s_0, \beta)$ is a positive multiple of $c_\alpha(s_0, \beta)$ for all $\alpha \in \mathcal{P}^0(n)$ and $\beta \in \mathcal{P}^0(n/b)$. □

Remark 11.10. The results of [**31**] give a more precise range of α for which $c_\alpha(s_0, \beta)$ can be non-zero, but Lemma 11.9 (1) suffices for our purposes.

Corollary 11.11. *Suppose that $s_0 \in \mathfrak{g}$ belongs to an elliptic Cartan subalgebra of $\mathfrak{g}$. Then the functions $\hat{\mu}_{\mathcal{O}}$, $\mathcal{O} \in \Omega_G(s_0)$, remain linearly independent upon restriction to any open neighbourhood of zero in $\mathfrak{g}$.*

Proof. Let the notation be as in Lemma 11.9. As the restrictions of $\hat{\mu}_{\mathcal{O}_\alpha}$, $\alpha \in \mathcal{P}^0(n)$, are linearly independent upon restriction to any open neighbourhood of zero (see [**11**, Corollary 5.10]), it suffices to show that the matrix $(a_{\alpha,\beta})_{\beta,\alpha \in \mathcal{P}^0(n/b)}$, $a_{\alpha,\beta} = c_{\dot{\beta}^b}(s_0, \alpha)$ is invertible. This is immediate from Lemma 11.9, as $a_{\alpha,\alpha} \neq 0$, and $a_{\alpha,\beta} \neq 0$ implies $\dot{\beta}^b \geqslant \dot{\alpha}^b$, which implies $\beta \geqslant \alpha$. □

We remark that if $s_0 \in \mathfrak{g}_{\mathrm{ss}}$ does not belong to an elliptic Cartan subalgebra of $\mathfrak{g}$, then the Fourier transforms $\hat{\mu}_{\mathcal{O}}$, $\mathcal{O} \in \Omega_G(s_0)$, might become linearly dependent on sufficiently small neighbourhoods of zero. For example, let $s_0 \in \mathfrak{gl}_4(F)$ be a diagonal matrix with two diagonal entries equal to 1 and the other two equal to -1. Then there are two nilpotent orbits in the Lie algebra of $C_G(s_0)$ that are neither trivial nor regular. Let Y_1 and Y_2 be representatives for these orbits. It is easy to check (see Lemma 17.1) that

$$\hat{\mu}_{\mathcal{O}(s_0+Y_1)}|_{\mathfrak{g}_{0+} \cap \mathfrak{g}_{\mathrm{reg}}} = \hat{\mu}_{\mathcal{O}(s_0+Y_2)}|_{\mathfrak{g}_{0+} \cap \mathfrak{g}_{\mathrm{reg}}} = \hat{\mu}_{\mathcal{O}_{(3,1)}}|_{\mathfrak{g}_{0+} \cap \mathfrak{g}_{\mathrm{reg}}}.$$

12. Asymptotic expansions of germs of characters: the pure case

In this section, we study the relation between asymptotic expansions of the germ of Θ_π and asymptotic expansions of the germ of $\Theta_{\pi'}$ in the case where π contains a pure minimal K-type, and π' is a representation corresponding to π via the associated Hecke algebra isomorphism. The proofs involve applying several results. Namely, results of DeBacker and Waldspurger concerning homogeneity properties of germs of characters and of Fourier transforms of orbital integrals (as discussed in § 11), results comparing multiplicities of

https://doi.org/10.1017/S1474748003000124 Published online by Cambridge University Press

certain K-types contained in π and π' (Propositions 8.6 and 10.8), descent properties of orbital integrals (Proposition 5.6) and a result from [**22**] (see Theorem 11.6). The results of this section are applied later in the paper (see the proofs of Theorem 14.1, Corollary 14.3 and Theorem 14.5).

When comparing germs of characters, it is convenient to impose some compatibility conditions on invariant measures on orbits. If $s_0 \in \mathfrak{g}_{\mathrm{ss}}$ and $H = C_G(s_0)$, there is a natural bijection $\mathcal{O}_H \mapsto G \cdot \mathcal{O}_H$ between $\Omega_H(0)$ and $\Omega_G(s_0)$. Fix Haar measures on G and H. For each $\mathcal{O}_H \in \Omega_H(0)$, fix a nilpotent element $Y \in \mathcal{O}_H$ and choose a left Haar measure on the centralizer $C_H(Y)$. These Haar measures induce G-invariant measures on the orbits $\mathcal{O}_G(s_0 + Y) \simeq G/C_H(Y)$ and $\mathcal{O}_H(Y) \simeq H/C_H(Y)$. Throughout this paper, we assume that the measures on the orbits in $\Omega_G(s_0)$ and $\Omega_H(0)$ are compatible in this sense. Furthermore, the map $\mathcal{O}_H(Y) \mapsto s_0 + \mathcal{O}_H(Y)$ induces a bijection between $\Omega_H(0)$ and $\Omega_H(s_0)$, and $\mathcal{O}_H(s_0 + Y) \simeq H/C_H(Y)$. Thus the H-invariant measure on the orbit $\mathcal{O}_H(s_0 + Y)$ will be taken to be the one induced by the above H-invariant measure on $H/C_H(Y)$. Note that since we can choose left Haar measures on H and on each $C_H(Y)$ freely, we may use any normalizations of measures on the orbits in $\Omega_H(0)$ (or $\Omega_H(s_0)$). However, having made these choices, the G-invariant measures on the orbits in $\Omega_G(s_0)$ are all determined up to the same positive constant depending on the choice of Haar measure on G. Conversely, we may take any normalization of measures on the orbits in $\Omega_G(s_0)$, and this then determines the measures on orbits in $\Omega_H(0)$ and $\Omega_H(s_0)$ up to a positive constant depending on choice of Haar measure on H.

Let $\pi \in \mathcal{E}(G)$. Suppose that π contains a pure minimal K-type. As usual, $\rho(\pi)$ denotes the depth of π (see § 4). If $\rho(\pi) > 0$, then $\rho(\pi) = j/e$ for some divisor e of n and a positive integer j which is prime to e, and π contains some $(Q_{n'j}, \chi_s)$, where s is as in § 6. We continue to use the notation of §§ 3–8. If $\rho(\pi) = 0$, then π contains $(B_{(d)^{n'}}, \sigma)$, where σ and s are as in § 10, $E = F(s)$, $d = [E : F]$, $n' = n/d$ and $e = e(E/F) = 1$. In this case, setting $j = 0$, we also have $\rho(\pi) = j/e$.

Suppose that $s' \in \mathfrak{g}'_{\mathrm{ss}} \cap \mathfrak{g}'_{(-j)^+}$. As $\mathfrak{g}'_{0+} = G' \cdot \mathfrak{b}'_{(1)^{n'},1}$ (see § 11), $s' \in G' \cdot \mathfrak{b}'_{(1)^{n'},-n'j+1}$, so Lemma 5.5 implies $C_G(s + s') = C_{G'}(s')$. We will be comparing $(s + s')$-asymptotic expansions on $\mathfrak{g}$ and $s + s'$ or s'-asymptotic expansions on $\mathfrak{g}'$, so we require compatibility conditions on the measures on the orbits in $\Omega_G(s + s')$ and $\Omega_{G'}(s')$ or $\Omega_{G'}(0)$. Consider the bijection between $\Omega_G(s + s')$ and $\Omega_{G'}(s')$ arising from the above-mentioned bijections of $\Omega_G(s + s')$ with $\Omega_{C_{G'}(s')}(0)$ and $\Omega_{G'}(s')$ with $\Omega_{C_{G'}(s')}(0)$. Fix Haar measures on G and G'. Choosing a representative Y for an orbit in $\Omega_{C_{G'}(s')}(0)$, fix a left Haar measure on $C_{C_{G'}(s')}(Y)$. This results in a G-invariant measure on $\mathcal{O}_G(s + s' + Y) \simeq G/C_{C_{G'}(s')}(Y)$ an a G'-invariant measure on the corresponding orbit $\mathcal{O}_{G'}(s' + Y) \simeq G'/C_{C_{G'}(s')}(Y)$. We will assume that measures on the orbits in $\Omega_G(s+s')$ and $\Omega_{G'}(s')$ are chosen so as to be compatible in this sense. And the measures on the orbits in $\Omega_{G'}(s + s')$ will be chosen to correspond to those on the orbits in $\Omega_{G'}(s')$ in the obvious way.

Let $\eta : \mathcal{H}' \to \mathcal{H}$ be the Hecke algebra isomorphism of Howe and Moy associated to the above pure minimal K-type (see Theorems 7.1 and 10.2). Recall that, if $\alpha \in \mathcal{P}^0(n)$ and $i \in \mathbb{Z}$, $[\mathfrak{b}_{\alpha,i}]$ is the characteristic function of $\mathfrak{b}_{\alpha,i}$. Let $u_\alpha(q)$ be defined as in § 2. Set $\mathfrak{e}_1(X) = 1 + X$ for $X \in \mathfrak{g}_{0+}$. Let $\mathfrak{e}'_1$ be the restriction of $\mathfrak{e}_1$ to $\mathfrak{g}'_{0+}$.

Proposition 12.1. *Let $\pi' = \eta^*(\pi)$. Suppose that there exists an $s' \in \mathfrak{g}'_{\mathrm{ss}} \cap \mathfrak{g}'_{(-j)+}$ such that $\Theta_{\pi'} \circ \mathfrak{e}'_1$ is s'-asymptotic on $\mathfrak{g}'_{j+}$. Let $c_{\mathcal{O}'}(\pi')$, $\mathcal{O}' \in \Omega_{G'}(s')$, be the coefficients in some s'-asymptotic expansion of $\Theta_{\pi'} \circ \mathfrak{e}'_1$. Given $\mathcal{O} \in \Omega_G(s+s')$, let $\mathcal{O}' \in \Omega_{G'}(s')$ be such that $\mathcal{O} = G \cdot (s + \mathcal{O}')$, and set*

$$\lambda_{\mathcal{O}} = \upsilon_G(B_{(n)})^{-1}\upsilon_{G'}(B'_{(n')})q^{fn'((n-n')j-e+1)/2}u_n(q)u_{n'}(q^f)^{-1}c_{\mathcal{O}'}(\pi').$$

Define $D = \sum_{\mathcal{O}\in\Omega_G(s+s')} \lambda_{\mathcal{O}}\hat{\mu}_{\mathcal{O}}$. Then $(\Theta_\pi \circ \mathfrak{e}_1 - D)([\mathfrak{b}_{\alpha,\lfloor jr(\alpha)/e\rfloor+1}]) = 0$ for every $\alpha \in \mathcal{P}(n)$.

Proof. First we note that, as $\mathfrak{g}'_{0+} = G' \cdot \mathfrak{b}'_{(1)^{n'},1}$, there is no loss of generality in assuming that $s' \in \varpi_E^{-j}\mathfrak{b}'_{(1)^{n'},1} = \mathfrak{q}'_{-n'j+1}$.

As $\mathfrak{e}_1(\mathfrak{b}_{\alpha,\lfloor jr(\alpha)/e\rfloor+1}) = B_{\alpha,\lfloor jr(\alpha)/e\rfloor+1}$,

$$(\Theta_\pi \circ \mathfrak{e}_1)([\mathfrak{b}_{\alpha,\lfloor jr(\alpha)/e\rfloor+1}]) = \upsilon_{\mathfrak{g}}(\mathfrak{b}_{\alpha,\lfloor jr(\alpha)/e\rfloor+1})\dim(V_\pi^{B_{\alpha,\lfloor jr(\alpha)/e\rfloor+1}}), \quad \alpha \in \mathcal{P}(n). \quad (12.1)$$

By Propositions 8.6 ($j > 0$) and 10.8 ($j = 0$), the right-hand side of (12.1) is zero unless $\alpha = (f\beta)^e$ for some $\beta \in \mathcal{P}(n')$. Note that the Fourier transform of $[\mathfrak{b}_{\alpha,\lfloor jr(\alpha)/e\rfloor+1}]$ is equal to $\upsilon_{\mathfrak{g}}(\mathfrak{b}_{\alpha,\lfloor jr(\alpha)/e\rfloor+1})[\mathfrak{b}_{\alpha,-\lfloor jr(\alpha)/e\rfloor}]$. Hence, by Proposition 5.6 (1), $D([\mathfrak{b}_{\alpha,\lfloor jr(\alpha)/e\rfloor+1}]) = 0$ unless $\alpha = (f\beta)^e$ for some $\beta \in \mathcal{P}(n')$. Thus it suffices to consider $\alpha = (f\beta)^e$, $\beta \in \mathcal{P}(n')$.

By Propositions 8.6 and 10.8, if $\alpha = (f\beta)^e$, $\beta \in \mathcal{P}(n')$, equation (12.1) can be rewritten as

$$\begin{aligned}\upsilon_{\mathfrak{g}}(\mathfrak{b}_{(f\beta)^e,jr(\beta)+1})^{-1}&(\Theta_\pi \circ \mathfrak{e}_1)([\mathfrak{b}_{(f\beta)^e,jr(\beta)+1}])\\ &= q^{fn'((n-n')j-e+1)/2}u_{(f\beta)^e}(q)u_\beta(q^f)^{-1}\dim(V_{\pi'}^{B'_{\beta,jr(\beta)+1}}),\end{aligned}$$

which, by the analogue of (12.1) for π', is equal to

$$q^{fn'((n-n')j-e+1)/2}u_{(f\beta)^e}(q)u_\beta(q^f)^{-1}\upsilon_{\mathfrak{g}'}(\mathfrak{b}'_{\beta,jr(\beta)+1})^{-1}(\Theta_{\pi'} \circ \mathfrak{e}'_1)([\mathfrak{b}'_{\beta,jr(\beta)+1}]),$$

which, since $\Theta_{\pi'} \circ \mathfrak{e}'_1$ is s'-asymptotic on $\mathfrak{g}'_{j+}$, $\mathfrak{b}'_{jr(\beta)+1} = \varpi_E^j\mathfrak{b}'_{\beta,1} \subset \varpi_E^j\mathfrak{b}'_{(1)^{n'},1} \subset \mathfrak{g}'_{j+}$ and $[\mathfrak{b}'_{\beta,jr(\beta)+1}]\hat{} = \upsilon_{\mathfrak{g}'}(\mathfrak{b}'_{\beta,jr(\beta)+1})[\mathfrak{b}'_{\beta,-jr(\beta)}]$, equals

$$q^{fn'((n-n')j-e+1)/2}u_{(f\beta)^e}(q)u_\beta(q^f)^{-1}\sum_{\mathcal{O}'\in\Omega_{G'}(s')} c_{\mathcal{O}'}(\pi')\mu_{\mathcal{O}'}([\mathfrak{b}'_{\beta,-jr(\beta)}]),$$

which, by Proposition 5.6 and the definition of $\lambda_{\mathcal{O}}$, $\mathcal{O} \in \Omega_G(s+s')$, and D, can be rewritten as

$$\begin{aligned}\upsilon_G(B_{(n)})^{-1}&\upsilon_{G'}(B'_{(n')})q^{fn'((n-n')j-e+1)/2}u_n(q)u_{n'}(q^f)^{-1}\\ &\times \sum_{\mathcal{O}'\in\Omega_{G'}(s')} c_{\mathcal{O}'}(\pi')\mu_{G\cdot(s+\mathcal{O}')}([\mathfrak{b}_{(f\beta)^e,-jr(\beta)}])\\ &= \sum_{\mathcal{O}\in\Omega_G(s+s')} \lambda_{\mathcal{O}}\mu_{\mathcal{O}}([\mathfrak{b}_{(f\beta)^e,-jr(\beta)}])\\ &= \upsilon_{\mathfrak{g}}(\mathfrak{b}_{(f\beta)^e,jr(\beta)+1})^{-1}D([\mathfrak{b}_{(f\beta)^e,jr(\beta)+1}]).\end{aligned}$$

□

https://doi.org/10.1017/S1474748003000124 Published online by Cambridge University Press

Remark 12.2. Suppose that $j > 0$ and there exists s' as in Proposition 12.1 such that $\Theta_{\pi'}$ is $(s+s')$-asymptotic on $\mathfrak{g}'_j$. Then the $(s+s')$-asymptotic expansion restricts to an s'-asymptotic expansion on $\mathfrak{g}'_{j+}$, and if $\mathcal{O}' \in \Omega_{G'}(s')$, the coefficient $c_{s+\mathcal{O}'}(\pi')$ of $\hat{\mu}_{s+\mathcal{O}'}$ in the $(s+s')$-asymptotic expansion equals the coefficient $c_{\mathcal{O}'}(\pi')$ in the s'-asymptotic expansion of $\Theta_{\pi'} \circ \mathfrak{e}'_1$.

Theorem 12.3. *Assume that $p > 2n$. Let $\pi' = \eta^*(\pi)$.*

(1) *Suppose that $j = 0$. Then $\Theta_\pi \circ \mathfrak{e}_1$ is s-asymptotic on $\mathfrak{g}_{0+}$. Furthermore, given a 0-asymptotic expansion of $\Theta_{\pi'} \circ \mathfrak{e}'_1$ with coefficients $c_{\mathcal{O}'}(\pi')$, $\mathcal{O}' \in \Omega_{G'}(0)$, there exists a unique s-asymptotic expansion of $\Theta_\pi \circ \mathfrak{e}_1$ with coefficients given by*

$$c_{G\cdot(s+\mathcal{O}')}(\pi) = \upsilon_G(B_{(n)})^{-1}\upsilon_{G'}(B'_{(n')})u_n(q)u_{n'}(q^{n/n'})^{-1}c_{\mathcal{O}'}(\pi'), \quad \mathcal{O}' \in \Omega_{G'}(0').$$

(2) *Suppose that $j > 0$ and that $\Theta_{\pi'} \circ \mathfrak{e}'_1$ is $(s+s')$-asymptotic on $\mathfrak{g}'_j$, for some $s' \in \mathfrak{g}'_{\mathrm{ss}} \cap \mathfrak{g}'_{(-j)^+}$. Then $\Theta_\pi \circ \mathfrak{e}_1$ is $(s+s')$-asymptotic on $\mathfrak{g}_{j/e}$. Furthermore, given an $(s+s')$-asymptotic expansion of $\Theta_{\pi'} \circ \mathfrak{e}'_1$ with coefficients $c_{\mathcal{O}'}(\pi')$, $\mathcal{O}' \in \Omega_{G'}(s+s')$, there exists an $(s+s')$-asymptotic expansion of $\Theta_\pi \circ \mathfrak{e}_1$ with coefficients given by*

$$c_{G\cdot\mathcal{O}'}(\pi) = \upsilon_G(B_{(n)})^{-1}\upsilon_{G'}(B'_{(n')})q^{fn'((n-n')j-e+1)/2}u_n(q)u_{n'}(q^f)^{-1}c_{\mathcal{O}'}(\pi'),$$
$$\mathcal{O}' \in \Omega_{G'}(s+s').$$

Proof. Let D be as in Proposition 12.1. We must show that $\Theta_\pi \circ \mathfrak{e}_1 - D$ vanishes on $\mathfrak{g}_{0+} \cap \mathfrak{g}_{\mathrm{reg}}$ (respectively, $\mathfrak{g}_{j/e} \cap \mathfrak{g}_{\mathrm{reg}}$) if $j = 0$ (respectively, $j > 0$) (see the remark above).

First, suppose that $j = 0$. By Corollary 11.2 and Theorem 11.5, $\Theta_\pi \circ \mathfrak{e}_1 - D$ is 0-asymptotic on $\mathfrak{g}_{0+}$. As $\mathfrak{b}_{\alpha,1} \subset \mathfrak{g}_{0+}$, Propositions 11.6 and 12.1 imply that $(\Theta_\pi \circ \mathfrak{e}_1 - D) \mid \mathfrak{g}_{0+} \cap \mathfrak{g}_{\mathrm{reg}} \equiv 0$. Hence, by definition of D, $\Theta_\pi \circ \mathfrak{e}_1$ is s-asymptotic on $\mathfrak{g}_{0+}$, and the coefficients $c_{\mathcal{O}}(\pi)$, $\mathcal{O} \in \Omega_G(s)$, can be chosen as given in (1). Furthermore, the linear independence of the restrictions of the Fourier transforms $\hat{\mu}_{\mathcal{O}}$, $\mathcal{O} \in \Omega_G(s)$, to $\mathfrak{g}_{0+} \cap \mathfrak{g}_{\mathrm{reg}}$ (see Corollary 11.11) guarantee uniqueness of the coefficients $c_{\mathcal{O}}(\pi)$.

Next, suppose that $j > 0$. Since π contains $(Q_{n'j}, \chi_s)$, we may apply Theorem 11.8 to conclude that $\Theta_\pi \circ \mathfrak{e}_1$ is s-asymptotic on $\mathfrak{g}_{j/e}$. Then, combining this with Lemma 11.7, $\Theta_\pi \circ \mathfrak{e}_1 - D$ is s-asymptotic on $\mathfrak{g}_{j/e}$. By Corollary 11.11, $\{\hat{\mu}_\xi \mid \xi \in \Omega_G(s)\}$ remain linearly independent upon restriction to any open neighbourhood of zero in $\mathfrak{g}$. Therefore, to prove that $\Theta_\pi \circ \mathfrak{e}_1 - D$ vanishes on $\mathfrak{g}_{j/e} \cap \mathfrak{g}_{\mathrm{reg}}$, it suffices to find a set of functions supported in $\mathfrak{g}_{j/e}$ having the properties that on the span of this set of functions $\Theta_\pi \circ \mathfrak{e}_1 - D$ vanishes, and the restrictions of the $\hat{\mu}_\xi$, $\xi \in \Omega_G(s)$, to this span are linearly independent. By Proposition 12.1, $(\Theta_\pi \circ \mathfrak{e}_1 - D)([\mathfrak{b}_{(f\beta)^e,-jr(\beta)+1}]) = 0$ for all $\beta \in \mathcal{P}(n')$. By Proposition 5.6 (2), Proposition 11.6 and homogeneity properties of nilpotent orbital integrals, the restrictions of $\hat{\mu}_\xi$, $\xi \in \Omega_G(s)$, are linearly independent on the span of the functions $[\mathfrak{b}_{(f\beta)^e,-jr(\beta)+1}]$, $\beta \in \mathcal{P}(n')$. □

Remark 12.4. If π and π' are as above and $j > 0$, then an $(s+s')$-asymptotic expansion of $\Theta_{\pi'} \circ \mathfrak{e}'_1$ might not be unique (see comments at the end of § 11). Clearly, in that case, an $(s+s')$-asymptotic expansion of $\Theta_\pi \circ \mathfrak{e}_1$ will not be unique. However, if $s' = 0$, then

https://doi.org/10.1017/S1474748003000124 Published online by Cambridge University Press

because of the linear independence of the restrictions of $\hat{\mu}_{\mathcal{O}}$, $\mathcal{O} \in \Omega_G(s)$, to any open neighbourhood of zero in $\mathfrak{g}$ (Corollary 11.11), the s-asymptotic expansion of $\Theta_\pi \circ \mathfrak{e}_1$ is unique.

13. Characters of refined minimal K-types

We assume throughout this section that $p > n$. In [**17**], Howe and Moy show that there exist families of irreducible representations of parahoric subgroups, called refined minimal K-types. Via a Hecke algebra isomorphism attached to some refined minimal K-type, each $\pi \in \mathcal{E}(G)$ corresponds to a unipotent representation of a direct product of general linear groups over finite extensions of F. (Recall that an irreducible admissible representation of a general linear group is unipotent if it has non-zero Iwahori-invariant vectors).

Theorem 13.1 (cf. Theorem 5.6 of [17]).

(1) *Every $\pi \in \mathcal{E}(G)$ contains a refined minimal K-type.*

(2) *Let (B, τ) be a refined minimal K-type. Then there exist extensions E_i/F and $n_i \in \mathbb{N}$, $1 \leqslant i \leqslant u$, such that $n = \sum_{1 \leqslant i \leqslant u} n_i [E_i : F]$, $B'' = B \cap G''$ is an Iwahori subgroup of $G'' = \prod_{i=1}^{u} GL_{n_i}(E_i)$, and there exists a Hecke algebra isomorphism*

$$\iota : \mathcal{H}'' = \mathcal{H}(G'' /\!\!/ B'') \to \mathcal{H}(\tau) = \mathcal{H}(G /\!\!/ B, \tilde{\tau})$$

such that

$$\operatorname{supp}(\iota(f)) = B \operatorname{supp}(f) B \quad \text{and} \quad \operatorname{supp}(\iota(f)) \cap G'' = \operatorname{supp}(f), \quad f \in \mathcal{H}''.$$

Furthermore, ι is an L^2-isometry for the natural L^2-structures on $\mathcal{H}(\tau)$ and $\mathcal{H}''$.

We will say that a refined minimal K-type τ is *totally pure* if $u = 1$, that is, if there exists an extension L/F of degree dividing n such that $G'' \simeq GL_a(L)$, where $a = n/[L : F]$. In this case, letting $e_o = e(L/F)$ and $f_o = f(L/F)$, B is conjugate to $B_{(f_o)^{ae_o}} = B_{(f_o)^{n/f_o}}$. The totally pure refined minimal K-types are described in the proof of Theorem 5.4 of [**17**] (see below for a summary). We are using the terminology totally pure because it is possible to show that each such K-type is attached to a finite sequence of pure unrefined minimal K-types for general linear groups arising as centralizers of elements of L. The totally pure K-types are the refined minimal K-types that are contained in the discrete series representations. If τ is trivial, then $G = G''$ and τ is the trivial representation of the standard Iwahori subgroup $B = B_{(1)^n}$. For the remainder of this section, we fix a totally pure refined minimal K-type (B, τ). Let t be the smallest non-negative integer such that the space of τ contains non-zero B_{t+1}-fixed vectors. Set $\rho(\tau) = t/ae_o = f_o t/n$. Recall that if $\pi \in \mathcal{E}(G)$, then $\rho(\pi)$ denotes the depth of π (see § 11). We are using the notation $\rho(\tau)$ because $\rho(\pi) = \rho(\tau)$ for all $\pi \in \mathcal{E}(G)$ that contain (B, τ).

Let h be an integer such that $n \leqslant h < p$. Set

$$\mathfrak{e}_h(X) = \sum_{0 \leqslant u \leqslant h-1} \frac{X^u}{u!}, \quad X \in \mathfrak{g}_{0+}. \tag{13.1}$$

This section is devoted to proving that the character χ_τ of a totally pure refined minimal K-type satisfies a Kirillov-type character formula (Theorem 13.2). For each h as above, there exists $s_{\tau,h} \in L$ such that, on an open subset of B, $\chi_\tau \circ \mathfrak{e}_h$ coincides with a multiple of the $\operatorname{Ad} B$-orbit of the linear functional $\psi(\operatorname{tr}(s_{\tau,h}\cdot))$.

Theorem 13.2. *Let (B,τ) be a totally pure refined minimal K-type. Then, for each h as above, there exists $s_{\tau,h} \in L$ such that the following hold.*

(1) $\nu_L(s_{\tau,h}) = -e_o\rho(\tau)$ *and the centralizer of $s_{\tau,h}$ in G is equal to G''.*

(2)
$$\chi_\tau(\mathfrak{e}_h(X)) = \dim\tau \int_B \psi(\operatorname{tr}(s_{\tau,h} \operatorname{Ad} k(X)))\,\mathrm{d}k$$
for all $X \in \mathfrak{b}$ such that $X^h \in \mathfrak{b}_{(n\rho(\tau)/f_o)+1} = \varpi_L^{e_o\rho(\tau)}\mathfrak{b}_1$.

The proof of the theorem will be given later in the section. Next we outline the construction (given in the proof of Theorem 5.4 of [**17**]) of the non-trivial totally pure minimal K-types.

If $\rho(\tau) = 0$, then there exists a divisor d of n such that $B = B_{(d)^{n'}}$ and $\tau = \sigma$ is as in § 10. In this case, $L = E$ is the unramified extension of F of degree d and $G'' \simeq G' \simeq GL_{n'}(E)$.

Suppose that $\rho(\tau) > 0$. Let π be a discrete series representation of G that contains (B,τ). As shown in [**17**], there exists a pure unrefined minimal K-type $(Q_{n'j},\chi_s)$ (notation as in § 6) contained in π. Let E, d, n', e, f, G', etc., be as in § 6. Then L is an extension of E and τ contains $(B_{nj/ef_o},\chi_s)$. By definition of $\rho(\tau)$, we have $\rho(\tau) = j/e$. In § 6, we saw that $\rho(\pi) = j/e$. Hence

$$\rho(\tau) = \rho(\pi) = \frac{j}{e} = -\frac{\nu_E(s)}{e} = -\frac{\nu_L(s)}{e_o}.$$

As $E \subset L$, f divides f_o and f_o divides n/e, so we can define parahoric filtrations $\dot{\mathfrak{q}}_i$, $i \in \mathbb{Z}$, and $\dot{Q}_i$, $i \geqslant 0$, as in § 9. Let $\dot\sigma$ be the irreducible representation of the compact open subgroup $\dot{J}$ that appears in § 9. The parahoric $\dot{Q}$ has the property that $B = \dot{Q}$ and $B'' = \dot{Q} \cap G''$. Setting $\mathfrak{b}' = \dot{\mathfrak{q}}'$ and $B' = \dot{Q}'$, we have $\mathfrak{b}'_i = \mathfrak{b}_i \cap \mathfrak{g}'$ for all $i \in \mathbb{Z}$. The K-type (B,τ) is defined inductively, using a totally pure refined minimal K-type (B',τ') for G' and an extension of the representation $\dot\sigma$ to $B'\dot{J}$. Let $\dot\eta : \mathcal{H}(G' /\!\!/ B'_{nj/ef_o},\chi'_s) \simeq \mathcal{H}(G /\!\!/ \dot{J},\dot\sigma)$ be the Hecke algebra isomorphism attached to the unrefined minimal K-type $(B_{nj/ef_o},\chi_s)$ (as in Theorem 9.1). Fix a quasi-character θ of $E^\times$ such that (denoting the determinant on $G' \simeq GL_{n'}(E)$ by $\det'$) $\theta \circ \det'$ is an extension of the character χ'_s of $B'_{n\rho(\tau)/f_o}$. Set $\dot\pi = \dot\eta^*(\pi)$. As $\dot\pi|_{B'_{n\rho(\tau)/f_o}}$ contains χ'_s and $\dot\pi$ belongs to the discrete series, the representation $(\theta^{-1} \circ \det')\dot\pi$ is a discrete series representation

https://doi.org/10.1017/S1474748003000124 Published online by Cambridge University Press

of G' such that $\rho((\theta^{-1}\circ\det')\dot\pi)<\rho(\dot\pi)=j$. There exists a totally pure refined minimal K-type (B',τ') contained in $(\theta^{-1}\circ\det')\dot\pi$, and $\rho(\tau')=\rho((\theta^{-1}\circ\det')\dot\pi)$. Hence

$$\rho(\tau')/e<\rho(\tau). \tag{13.2}$$

This inequality is essential for the induction step in the proof of Theorem 13.2.

There exists a uniquely defined extension $\dot\sigma_{\mathrm{ext}}$ of $\dot\sigma$ from $\dot J$ to $B'\dot J$ which is obtained from $\dot\sigma$ using the character $\theta\circ\det'|_{B'}$, and possibly also a Weil representation over $\mathbb{F}_p$ (depending on the parity of nj/ef_o). This extension $\dot\sigma_{\mathrm{ext}}$ is described in more detail in the proof of Proposition 13.7. The representation τ' extends trivially across $\dot J$ to give a representation of $B'\dot J$, also denoted by τ'. Also (see [**17**, p. 422]),

$$\tau=\mathrm{Ind}_{B'\dot J}^{B}(\tau'\otimes\dot\sigma_{\mathrm{ext}}). \tag{13.3}$$

Fix an integer h such that $n\leqslant h\leqslant p$. Define $s_h\in E$ as follows. First, in the case $j=0$, set $s_h=s$. Otherwise, let θ be as above. As θ is trivial on $1+\mathfrak{p}_E^{j+1}$ and non-trivial on $1+\mathfrak{p}_E^{j}$, the map $x\mapsto\theta(\sum_{i=1}^{h-1}x^i/i!)$ is a character of $\mathfrak{p}_E^{\lfloor j/h\rfloor+1}$. Thus there exists $s_h\in E$ such that

$$\theta\biggl(\sum_{i=0}^{h-1}\frac{x^i}{i!}\biggr)=\psi(\mathrm{tr}_{E/F}(s_hx)),\quad x\in\mathfrak{p}_E^{\lfloor j/h\rfloor+1}. \tag{13.4}$$

Note that if $x\in\mathfrak{p}_E^{j}$, then

$$\psi(\mathrm{tr}_{E/F}(s_hx))=\theta\biggl(\sum_{i=0}^{h-1}\frac{x^i}{i!}\biggr)=\theta(1+x)=\chi_s'(1+x)=\psi(\mathrm{tr}_{E/F}(sx)).$$

This implies $s_h-s\in\mathfrak{p}_E^{-j+1}$. In fact, s is the unique element of the subgroup of $E^\times$ generated by ϖ_E and the roots of unity in $E^\times$ of order prime to p such that $|s_h-s|_E<|s_h|$. Hence (see the remarks following Lemma 8 of [**14**]), $F(s)\subset F(s_h)$. But $E=F(s)$, so $E=F(s_h)$.

Given $v\in\mathbb{Z}$, let $\mathfrak{b}_v'^{\perp}=\mathfrak{b}_v\cap\mathfrak{g}'^{\perp}$, where $\mathfrak{g}'^{\perp}$ is the orthogonal complement of $\mathfrak{g}'$ relative to trace. Recall that $\mathfrak{b}_v'=\mathfrak{b}_v\cap\mathfrak{g}'$ and, if $v\geqslant 0$, $B_v'=B_v\cap G'$.

Lemma 13.3.

(1) $\mathfrak{b}_v=\mathfrak{b}_v'\oplus\mathfrak{b}_v'^{\perp}$, $v\in\mathbb{Z}$.

(2) $B'(1+\mathfrak{b}_v'^{\perp})=B'B_v$ *if* $v\geqslant 1$.

(3) $[s,\mathfrak{b}_v'^{\perp}-\mathfrak{b}_{v+1}'^{\perp}]=\mathfrak{b}_{v-nj/ef_o}'^{\perp}-\mathfrak{b}_{v-nj/ef_o+1}'^{\perp}$.

(4) *If* $v\geqslant 1$ *and* $X\in\mathfrak{b}$ *is such that* $X^h\in\mathfrak{b}_v$, *then*

$$\mathfrak{e}_h(X)\in B'B_v\quad\Leftrightarrow\quad X\in\mathfrak{b}'+\mathfrak{b}_v.$$

(5) *If* $X=X'+X^{\perp}$ *where* $X'\in\mathfrak{b}'$, $X^{\perp}\in\mathfrak{b}_v$ *and* $X^h\in\mathfrak{b}_{v+r}$ *for some* r *such that* $0\leqslant r\leqslant v$, *then* $\mathfrak{e}_h(X)\in\mathfrak{e}_h(X')(1+\mathfrak{b}_v'^{\perp}+\mathfrak{b}_{2v})$ *and* $X'^h\in\mathfrak{b}_{v+r}'$.

https://doi.org/10.1017/S1474748003000124 Published online by Cambridge University Press

(6) *Let X', v and r be as in (5). Then*

$$(\mathrm{tr}'\, X')^h \in \mathfrak{p}_E^{\lfloor (v+r-1)ef_o/n\rfloor+1}$$

and

$$\det{}'(\mathfrak{e}_h(X')) \in \biggl(\sum_{i=0}^{h-1} \frac{(\mathrm{tr}'\, X')^i}{i!}\biggr)(1+\mathfrak{p}_E^{\lfloor (v+r-1)ef_o/n\rfloor+1}).$$

Proof. Parts (1)–(3) are restatements of parts (1)–(3) of Lemma 5.1.

For (4), let $X \in \mathfrak{b}$ be such that $X^h \in \mathfrak{b}_v$. Then the image $\bar{X}$ of X in $\mathfrak{b}/\mathfrak{b}_1$ is nilpotent. This implies that $\overline{\mathfrak{e}_h(X)} = \exp(\bar{X})$ is unipotent. Hence $\mathfrak{e}_h(X) \in \mathfrak{b}^\times = B$. Suppose that $X \in \mathfrak{b}' + \mathfrak{b}_v$. Then $X^i/i! \in \mathfrak{b}' + \mathfrak{b}_v$ for $i \leqslant h-1$, so $\mathfrak{e}_h(X) \in (\mathfrak{b}' + \mathfrak{b}_v) \cap B = B'B_v$. Conversely, if $\mathfrak{e}_h(X) \in B'B_v$, then $\mathfrak{e}_h(X) - 1 \in \mathfrak{b}' + \mathfrak{b}_v$. Hence, since $X^h \in \mathfrak{b}_v$,

$$X \in \sum_{i=1}^{h-1} \frac{(-1)^i(\mathfrak{e}_h(X)-1)^i}{i} + \mathfrak{b}_v \subset \mathfrak{b}' + \mathfrak{b}_v.$$

For (5), note that $\mathfrak{g}'^{\perp}$ is stable under left and right multiplication by elements of $\mathfrak{g}'$. Fixing $k \leqslant h$, any monomial of the form $X'^i X^{\perp} X'^{k-i}$, $0 \leqslant i \leqslant k-1$, belongs to $\mathfrak{g}'^{\perp} \cap \mathfrak{b}_v = \mathfrak{b}_v'^{\perp}$. If $X^{\perp}$ occurs at least twice in a monomial involving X' and $X^{\perp}$, then, since $X' \in \mathfrak{b}'$ and $X^{\perp} \in \mathfrak{b}_v$, that monomial lies in $\mathfrak{b}_{2v}$. Writing X^k as a sum of momomials involving products of powers of X' and $X^{\perp}$, and decomposing the monomials into sums of elements in $\mathfrak{g}'$ and $\mathfrak{g}'^{\perp}$, we see that the $\mathfrak{g}'$-component of X^k belongs to $X'^k + \mathfrak{b}'_{2v}$.

Since $X^h \in \mathfrak{b}_{v+r}$, part (1) implies that the $\mathfrak{g}'$-component of X^h belongs to $\mathfrak{b}'_{v+r}$. Thus, in view of the above, and $\mathfrak{b}_{2v} \subset \mathfrak{b}_{v+r}$, $X'^h \in \mathfrak{b}_{r+v}$.

Since $h-1 < p$,

$$\frac{X^k}{k!} \in \frac{X'^k}{k!} + \mathfrak{b}_v'^{\perp} + \mathfrak{b}_{2v}, \quad 1 \leqslant k \leqslant h-1.$$

Therefore, $x = \mathfrak{e}_h(X) \in \mathfrak{e}_h(X') + \mathfrak{b}_v'^{\perp} + \mathfrak{b}_{2v}$. By arguing as in the proof of (4), $X'^h \in \mathfrak{b}'_1$ implies $\mathfrak{e}_h(X') \in (\mathfrak{b}')^\times = B'$. Thus, as $\mathfrak{b}_v'^{\perp}$ and $\mathfrak{b}_{2v}$ are both left B'-stable, $x \in \mathfrak{e}_h(X')(1 + \mathfrak{b}_v'^{\perp} + \mathfrak{b}_{2v})$.

For (6), E' be a finite extension of E containing the eigenvalues λ_i, $1 \leqslant i \leqslant n'$, of X'. Let $|\cdot|$ be an extension of $|\cdot|_E$ from E to E'. By (5), $X'^{nh/ef_o} \in \mathfrak{b}_{(v+r)n/ef_o}$. As $\varpi_E^k \mathfrak{b} = \mathfrak{b}_{kn/ef_o}$, $k \in \mathbb{Z}$, we have $\varpi_E^{-(v+r)} X'^h \in \mathfrak{b}'$. This implies $|\varpi_E^{-(v+r)}\lambda_i|^{nh/ef_o} \leqslant 1$ for all i. Thus $|\,\mathrm{tr}'\, X'|_E^h \leqslant q_E^{-(v+r)ef_o/n}$. Equivalently, $(\mathrm{tr}'\, X')^h \in \mathfrak{p}_E^{\lfloor (v+r-1)ef_o/n\rfloor+1}$. Any multinomial occurring in

$$\det{}'(\mathfrak{e}_h(X')) - \sum_{k=0}^{h-1} \frac{(\mathrm{tr}\, X')^k}{k!} = \prod_{i=1}^{n'}\biggl(\sum_{k=0}^{h-1} \frac{\lambda_i^k}{k!}\biggr) - \sum_{k=0}^{h-1}\biggl(\sum_{i=1}^{n'} \lambda_i\biggr)^k / k!$$

is of the form $\prod_{i=1}^{n'} \lambda_i^{n'_i}$, where $h \leqslant \sum_{i=1}^{n'} n'_i$. And

$$\biggl|\prod_{i=1}^{n'} \lambda_i^{n'_i}\biggr| \leqslant q_E^{-(v+r)ef_o/n}.$$

https://doi.org/10.1017/S1474748003000124 Published online by Cambridge University Press

Thus

$$\det{}'(\mathfrak{e}_h(X')) \in \biggl(\sum_{k=0}^{h-1} \frac{(\operatorname{tr}' X')^k}{k!}\biggr)(1 + \mathfrak{p}_E^{\lfloor (v+r-1)ef_o/n \rfloor + 1}).$$

□

Recall from § 9 that $\ell_o = \lfloor \frac{1}{2}((nj/ef_o) + 1) \rfloor$ and $m_o = \lfloor nj/2ef_o \rfloor + 1$.

Lemma 13.4. *Suppose that $\rho(\tau) = j/e > 0$. Fix $t \in \{\ell_o, m_o\}$. Assume that $X \in \mathfrak{b}' + \mathfrak{b}_t$ is such that $X^h \in \mathfrak{b}_{(n\rho(\tau)/f_o)+1} = \mathfrak{b}_{(nj/ef_o)+1}$. Write $X = Y + Z$, with $Y \in \mathfrak{b}'$ and $Z \in \mathfrak{b}_t$. Set $x = \mathfrak{e}_h(X)$, $y = \mathfrak{e}_h(Y)$, and $z = y^{-1}x$. Then we have the following.*

(1) $Y^h \in \mathfrak{b}_{\ell_o + t}$.

(2) *If $t = m_o$, then $\theta(\det' y) = \psi(\operatorname{tr}(s_h Y))$ and $\psi(\operatorname{tr}(s_h(z-1))) = 1$.*

Proof. Note that $\ell_o + m_o = (nj/ef_o) + 1$. For (1), if $t = \ell_o$ (respectively, $t = m_o$) apply Lemma 13.3 (5) with $v = r = \ell_o$ (respectively, $v = m_o$) and $r = \ell_o$. For (2), applying Lemma 13.3 (6) with $v = t = m_o$ and $r = \ell_o$, we have

$$(\operatorname{tr}' Y)^h \in \mathfrak{p}_E^{j+1} \quad \text{and} \quad \theta(\det{}' y) = \theta\biggl(\sum_{i=1}^{h-1} \frac{(\operatorname{tr}' Y)^i}{i!}\biggr),$$

which, by (13.4), implies

$$\theta(\det{}' y) = \psi(\operatorname{tr}_{E/F} s_h \operatorname{tr}' Y) = \psi(\operatorname{tr}_{E/F} \operatorname{tr}'(s_h Y)) = \psi(\operatorname{tr}(s_h Y)).$$

By Lemma 13.3 (5), $z \in 1 + \mathfrak{b}'^{\perp}_{m_o} + \mathfrak{b}_{2m_o}$, so, since $s_h \in \mathfrak{b}'_{-nj/ef_o}$,

$$\operatorname{tr}(s_h(z-1)) \in \operatorname{tr}(\mathfrak{b}_{-(nj/ef_o)+2m_o}) \subset \operatorname{tr}(\mathfrak{b}_1) \subset \mathfrak{p}_F.$$

Thus $\psi(\operatorname{tr}(s_h(z-1))) = 1$. □

Lemma 13.5. *Suppose that $\rho(\tau) > 0$. Let s_h be as above. Let $X \in \mathfrak{b}$ and $s' \in \varpi_E^{-j+1}\mathfrak{b}'$. Then the following hold.*

(1) *We have*

$$\int_{B_{m_o}} \psi(\operatorname{tr}((s_h + s') \operatorname{Ad} k(X)))\, \mathrm{d}k = 0 \quad \textit{if } X \notin \mathfrak{b}' + \mathfrak{b}_{\ell_o}.$$

(2) *If $X \in \mathfrak{b}_1$, then*

$$\int_{B_{\ell_o}} \psi(\operatorname{tr}(s_h \operatorname{Ad} k(X)))\, \mathrm{d}k = 0 \quad \textit{if } X \notin \mathfrak{b}' + \mathfrak{b}_{m_o}.$$

Proof. Since $s_h + s' = s + (s_h - s) + s' \in s + \mathfrak{p}_E^{-j+1} + \varpi_E^{-j+1} \in s + \varpi_E^{-j+1}\mathfrak{b}'$, it suffices to prove the lemma with s_h replaced by s.

https://doi.org/10.1017/S1474748003000124 Published online by Cambridge University Press

For (1), given $k \in B_{m_o}$, let $Y = k - 1 \in \mathfrak{b}_{m_o}$. Then

$$\operatorname{Ad} k(X) \in X + [Y, X] + \mathfrak{b}_{2m_o} \subset X + [Y, X] + \mathfrak{b}_{(nj/ef_o)+1}.$$

Combining this with $s + s' \in \varpi_E^{-j}\mathfrak{b}' \subset \mathfrak{b}_{-nj/ef_o} = \mathfrak{b}^*_{(nj/ef_o)+1}$, we have

$$\begin{aligned}\int_{B_{m_o}} \psi(\operatorname{tr}(s + s') \operatorname{Ad} k(X))\, \mathrm{d}k &= \psi(\operatorname{tr}(s + s')X) \int_{\mathfrak{b}_{m_o}} \psi(\operatorname{tr}(s + s')[Y, X])\, \mathrm{d}Y \\ &= \psi(\operatorname{tr}(s + s')X) \int_{\mathfrak{b}_{m_o}} \psi(\operatorname{tr}([X, s + s']Y))\, \mathrm{d}Y.\end{aligned}$$

Assume that the above integral is non-zero. Then $[X, s + s'] \in \mathfrak{b}^*_{m_o} = \mathfrak{b}_{-m_o+1}$. There exists a unique $i \geqslant 0$ such that $X \in \mathfrak{b}_i - \mathfrak{b}_{i+1}$. By Lemma 13.3, $X = X' + X^{\perp}$ for some $X' \in \mathfrak{b}'_i$ and $X^{\perp} \in \mathfrak{b}'^{\perp}_i$. As $s + s' \in \mathfrak{g}'$, and both $\mathfrak{g}'$ and $\mathfrak{g}'^{\perp}$ are $\operatorname{ad}\mathfrak{g}'$-stable, it follows from Lemma 13.3 that

$$[s + s', X] \in \mathfrak{b}_{-m_o+1} \quad \Rightarrow \quad [s + s', X'] \in \mathfrak{b}'_{-m_o+1} \quad \text{and} \quad [s + s', X^{\perp}] \in \mathfrak{b}'^{\perp}_{-m_o+1}.$$

Let v be such that $X^{\perp} \in \mathfrak{b}'^{\perp}_v - \mathfrak{b}'^{\perp}_{v+1}$. By Lemma 13.3,

$$[s, X^{\perp}] \in \mathfrak{b}'^{\perp}_{-(nj/ef_o)+v} - \mathfrak{b}'^{\perp}_{-(nj/ef_o)+v+1}.$$

As

$$s' \in \varpi_E^{-j+1}\mathfrak{b}' = \mathfrak{b}'_{n(-j+1)/ef_o}, \qquad [s', X^{\perp}] \in \mathfrak{b}'^{\perp}_{(n(-j+1)/ef_o)+v}.$$

Therefore,

$$[s + s', X^{\perp}] \in \mathfrak{b}'^{\perp}_{-(nj/ef_o)+v} - \mathfrak{b}'^{\perp}_{-(nj/ef_o)+v+1}.$$

Hence, if the above integral is non-vanishing, $-(nj/ef_o) + v \geqslant -m_o + 1$. That is,

$$v \geqslant (nj/ef_o) + 1 - m_o = \ell_o + m_o - m_o = \ell_o.$$

This implies $X = X' + X^{\perp} \in \mathfrak{b}' + \mathfrak{b}'^{\perp}_{\ell_o} = \mathfrak{b}' + \mathfrak{b}_{\ell_o}$.

The proof of part (2) is omitted as it is very similar to that of (1). □

If S is a subset of $\mathfrak{g}$, let $[S]$ be the characteristic function of S.

Corollary 13.6. *If $\rho(\tau) > 0$, $s' \in \mathfrak{p}_E^{-j+1}\mathfrak{b}'$ and $X \in \mathfrak{b}$, then*

$$\int_B \psi(\operatorname{tr}((s_h + s') \operatorname{Ad} k(X)))\, \mathrm{d}k = \int_B [\mathfrak{b}' + \mathfrak{b}_{\ell_o}](\operatorname{Ad} k(X)) \psi(\operatorname{tr}((s_h + s') \operatorname{Ad} k(X)))\, \mathrm{d}k.$$

Proposition 13.7. *Suppose that $\rho(\tau) > 0$. Let s_h be as in (13.4). Then*

$$\chi_{\dot{\sigma}_{\mathrm{ext}}}(\mathfrak{e}_h(X)) = (\dim \dot{\sigma}_{\mathrm{ext}}) \int_{B_{\ell_o}} \psi(\operatorname{tr}(s_h \operatorname{Ad} k(X)))\, \mathrm{d}k$$

for all $X \in \mathfrak{b}' + \mathfrak{b}_{\ell_o}$ such that $X^h \in \mathfrak{b}_{(n\rho(\tau)/f_o)+1} = \mathfrak{b}_{(nj/ef_o)+1}$.

https://doi.org/10.1017/S1474748003000124 Published online by Cambridge University Press

Proof. Given X as in the statement of the proposition, let $Y \in \mathfrak{b}'$, $Z \in \mathfrak{b}_{\ell_o}$ be such that $X = Y + Z$. Let $x = \mathfrak{e}_h(X)$, $y = \mathfrak{e}_h(Y)$, and $z = x^{-1}y$. Note that, by Lemma 13.3 (2), $B'\dot{J} = B'B_{\ell_o}$ and $B'\dot{J}_+ = B'B_{m_o}$. Define a character φ of $B'\dot{J}_+$ by

$$\varphi|_{B'} = \theta \circ \det{}'|_{B'}, \quad \varphi(w) = \psi(\operatorname{tr}(s_h(w-1))) = \psi(\operatorname{tr}(s(w-1))), \quad w \in \dot{J}_+.$$

If $\ell_o = m_o$, then $\dot{\sigma}_{\text{ext}} = \varphi$. As shown in Lemma 13.4 (2),

$$\varphi(x) = \varphi(y)\varphi(z) = \theta(\det{}'y)\psi(\operatorname{tr}(s_h(z-1))) = \psi(\operatorname{tr}(s_hY)) = \psi(\operatorname{tr}(s_hX)).$$

As $\ell_o = m_o$, the function $X \mapsto \psi(\operatorname{tr}(s_hX))$ is $\operatorname{Ad} B_{\ell_o}$-invariant. The proposition now follows.

For the remainder of the proof, assume that $m_o = \ell_o + 1$. The representation $\dot{\sigma}_{\text{ext}}$ can be described as follows. Let $N = 1 + \dot{\mathfrak{q}}_{(nj/ef_o)+1} + \dot{\mathfrak{q}}'^{\perp}_{\mathfrak{m}}$. Then $\dot{J}/N$ is isomorphic to the direct product of finitely many Heisenberg groups H_i. (This is discussed in [**17**, pp. 413, 414] in the description of σ—the idea is the same for $\dot{\sigma}$.) Let φ_i be the restriction of φ to the centre of H_i. Attached to each H_i, there is a natural finite symplectic vector space $\mathcal{V}_i$. Let $Sp(\mathcal{V}_i)$ be the corresponding symplectic group. The Weil representation ω_{φ_i} is a uniquely defined irreducible representation of $Sp(\mathcal{V}_i) \ltimes H_i$. One of its properties is that $\omega_{\varphi_i}|_{H_i}$ is the Heisenberg representation of H_i with central character φ_i. The conjugation action of B' on $\dot{J}$ induces a homomorphism $B' \ltimes \dot{J} \to \prod_i Sp(\mathcal{V}_i) \ltimes H_i$. We can view the restriction of $\omega_\varphi = \bigotimes_i \omega_{\varphi_i}$ to the image of $B' \ltimes \dot{J}/N$ as a representation of $B' \ltimes \dot{J}$, also denoted by ω_φ. Let $inf(\varphi)$ be the representation of $B' \ltimes \dot{J}$ which is equal to φ on B' and trivial on $\dot{J}$. Then $\omega_\varphi \otimes inf(\varphi)$ factors through the natural map $B' \ltimes \dot{J} \to B'\dot{J}$. The resulting representation of $B'\dot{J} = B'B_{\ell_o}$ is $\dot{\sigma}_{\text{ext}}$.

Note that $\dot{\sigma}_{\text{ext}}|_{\dot{J}} = \omega_\varphi|_{\dot{J}}$ is the inflation of the tensor products of the Heisenberg representations of the H_i with central characters φ_i. This implies that $\dot{\sigma}_{\text{ext}}|_{\dot{J}}$ is the unique irreducible component of $\operatorname{Ind}_{\dot{J}_+}^{\dot{J}}(\varphi|_{\dot{J}_+})$, which is, by definition, $\dot{\sigma}$.

Case 1. Suppose that $X \in \mathfrak{b}'_1 + \mathfrak{b}_{\ell_o}$. Then $x \in B'_1B_{\ell_o}$. Because $\omega_\varphi|_{B'_1}$ is a multiple of the trivial representation, it follows that $\dot{\sigma}_{\text{ext}}|_{B'_1}$ is a multiple of φ. As remarked above, $\dot{\sigma}_{\text{ext}}|\dot{J} = \dot{\sigma}$. It follows that $\dot{\sigma}_{\text{ext}}|_{B'_1B_{\ell_o}}$ is the unique irreducible component of $\operatorname{Ind}_{B'_1\dot{J}_+}^{B'_1\dot{J}} \varphi = \operatorname{Ind}_{B'_1B_{m_o}}^{B'_1B_{\ell_o}} \varphi$. As $B'_1B_{m_o}$ is normal in $B'_1B_{\ell_o}$,

$$\frac{\chi_{\dot{\sigma}_{\text{ext}}}(x)}{\dim \dot{\sigma}_{\text{ext}}} = \begin{cases} \varphi(x) & \text{if } x \in B'_1B_{m_o}, \\ 0 & \text{if } x \notin B'_1B_{m_o}. \end{cases}$$

If $Z \notin \mathfrak{b}_{m_o}$, then $X \notin \mathfrak{b}'_1 + \mathfrak{b}_{m_o}$, so $x \notin B'_1B_{m_o}$ (Lemma 13.3 (4)) and hence $\chi_{\dot{\sigma}_{\text{ext}}}(x) = 0$. Also, by Lemma 13.5 (2), $\int_{B_{\ell_o}} \psi(\operatorname{tr}(s_h \operatorname{Ad} k(X)))\,\mathrm{d}k = 0$. If $Z \in \mathfrak{b}_{m_o}$ and $k \in B_{\ell_o}$, then, by Lemma 13.4 (2),

$$\psi(\operatorname{tr}(s_h(\operatorname{Ad} k(X)))) = \psi(\operatorname{tr}(s_hX)) = \psi(\operatorname{tr}(s_hY)) = \theta(\det{}'y)\psi(\operatorname{tr}(s_h(z-1))) = \varphi(x).$$

Hence the right-hand side of the above-displayed formula is equal to

$$\int_{B_{\ell_o}} \psi(\operatorname{tr}(s_h \operatorname{Ad} k(X)))\,\mathrm{d}k \quad \text{for all } X \in \mathfrak{b}'_1 + \mathfrak{b}_{\ell_o}.$$

That is, the proposition holds when $X \in \mathfrak{b}'_1 + \mathfrak{b}_{\ell_o}$.

Case 2. Suppose that $X \in (\mathfrak{b}' + \mathfrak{b}_{\ell_o}) - (\mathfrak{b}'_1 + \mathfrak{b}_{\ell_o})$. Then $Y \in \mathfrak{b}' - \mathfrak{b}'_1$ and $x \notin B'_1 B_{\ell_o}$. Let U be the subgroup of B' generated by y and B'_1. The image of U in the direct products of the symplectic groups $Sp(\mathcal{V}_i)$ (see the comments preceding the discussion of Case 1) is contained in the direct product of the cyclic group generated by the image of the element y. Remarks in [**12**, p. 295] imply that

$$\dot{\sigma}_{\mathrm{ext}}|_{UB_{\ell_o}} = \mathrm{Ind}_{UK}^{UB_{\ell_o}}(\bar{\varphi}),$$

where K is a subgroup of B_{ℓ_o} containing B_{m_o} having the property that the image of $B'_1 K$ in $B'_1 B_{\ell_o} / \ker(\varphi|_{B'_1 B_{m_o}})$ is a maximal abelian subgroup fixed under the action induced by conjugation by y. Here, $\bar{\varphi}$ is any character of UB_{ℓ_o} that coincides with φ on UB_{m_o}. We shall take $\bar{\varphi}$ given by

$$\bar{\varphi}|_U = \theta \circ \det{}' \quad \text{and} \quad \bar{\varphi}(1 + W) = \psi(\mathrm{tr}(s_h(W - \tfrac{1}{2}W^2))), \quad W \in K - 1.$$

It is a simple matter to check that $\bar{\varphi}$ is a well-defined representation of UK. (The argument is the same as that in [**33**, p. 441].)

Let $\mathcal{F}(x) = \{w \in UB_{\ell_o} \mid w^{-1}xw \in UK\}$. Then it follows from the Frobenius formula for characters of induced representations that

$$\frac{\chi_{\dot{\sigma}_{\mathrm{ext}}}(x)}{\dim \dot{\sigma}_{\mathrm{ext}}} = \int_{\mathcal{F}(x)} \bar{\varphi}(w^{-1}xw)\, \mathrm{d}w.$$

Conjugation by U fixes $\bar{\varphi}$, and U normalizes B_{ℓ_o}, so $\mathcal{F}(x)$ may be replaced by $\mathcal{F}_1(x) = \mathcal{F}(x) \cap B_{\ell_o}$. Since $\chi_{\dot{\sigma}_{\mathrm{ext}}}$ is supported on the set of conjugacy classes in UB_{ℓ_o} that intersect UB_{m_o}, after replacing x by $w^{-1}xw$, $w \in \mathcal{F}_1(x)$, if necessary, we assume that $x \in UB_{m_o}$. As will be shown in Lemma 13.9 (1),

$$\bar{\varphi}(w^{-1}xw) = \bar{\varphi}(x)\psi(\mathrm{tr}(s_h(\mathrm{Ad}\, w^{-1}(X) - X))), \quad w \in \mathcal{F}_1(x).$$

It follows that

$$\frac{\chi_{\dot{\sigma}_{\mathrm{ext}}}(x)}{\dim \dot{\sigma}_{\mathrm{ext}}} = \bar{\varphi}(x)\psi(-\mathrm{tr}(s_h X)) \int_{\mathcal{F}_1(x)} \psi(\mathrm{tr}(s_h\, \mathrm{Ad}\, w^{-1}(X)))\, \mathrm{d}w.$$

Recall that $\bar{\varphi}|_{UB_{m_o}} = \varphi|_{UB_{m_o}}$. As $x \in UB_{m_o}$ and $U \in B'$, we have $X \in \mathfrak{b}' + \mathfrak{b}_{m_o}$. By Lemma 13.4 (2) and the definition of φ, $\bar{\varphi}(x) = \varphi(x) = \psi(\mathrm{tr}(s_h X))$. Hence

$$\frac{\chi_{\dot{\sigma}_{\mathrm{ext}}}(x)}{\dim \dot{\sigma}_{\mathrm{ext}}} = \int_{\mathcal{F}_1(x)} \psi(\mathrm{tr}(s_h\, \mathrm{Ad}\, w^{-1}(X)))\, \mathrm{d}w.$$

Together with Lemma 13.9 (2), this completes the proof. □

The proof of the following lemma is omitted as it is very similar to that of Lemma 3.17 of [**33**].

https://doi.org/10.1017/S1474748003000124 Published online by Cambridge University Press

Lemma 13.8. *Let X, Z, x and z be as in Lemma 13.4. Then the following hold.*

(1) $\operatorname{Ad} x^{-1}(T) \in \sum_{i=0}^{h-1} \frac{(-1)^i}{i!}(\operatorname{ad} X)^i(T) + \mathfrak{b}_{m_o}$, $T \in \mathfrak{b}_{\ell_o}$.

(2) $[X,T] \in \sum_{i=1}^{h-1} \frac{(-1)^i}{i}(\operatorname{Ad} x^{-1} - 1)^i(T) + \mathfrak{b}_{m_o}$, $T \in \mathfrak{b}_{\ell_o}$.

(3) $z \in 1 + \sum_{i=0}^{h-2} \frac{(-1)^i}{(i+1)!}(\operatorname{ad} X)^i(Z) + \mathfrak{b}_{m_o}$.

(4) $Z \in \sum_{i=0}^{h-2} \frac{(-1)^i}{i+1}(\operatorname{Ad} x^{-1} - 1)^i(z-1) + \mathfrak{b}_{m_o}$.

Lemma 13.9. *Suppose that $\rho(\tau) > 0$, $X \in (\mathfrak{b}' + \mathfrak{b}_{\ell_o}) - (\mathfrak{b}'_1 + \mathfrak{b}_{\ell_o})$ and $X^h \in \mathfrak{b}_{(nj/ef_o)+1}$. Let the notation be as in the proof of Proposition 13.7. Assume that $x = \mathfrak{e}_h(X) \in UK$. Then the following hold.*

(1) $\bar{\varphi}(w^{-1}xw) = \bar{\varphi}(x)\psi(\operatorname{tr}(s_h(\operatorname{Ad} w^{-1}(X) - X)))$, $w \in \mathcal{F}_1(x)$.

(2) $\int_{B_{\ell_o}} \psi(\operatorname{tr}(s_h \operatorname{Ad} w^{-1}(X)))\, \mathrm{d}X = \int_{\mathcal{F}_1(x)} \psi(\operatorname{tr}(s_h \operatorname{Ad} w^{-1}(X)))\, \mathrm{d}w$.

Proof. As $B'_1 \subset U \subset B'$ and $K \subset B_{\ell_o}$, it follows that $(UK) \cap B_{\ell_o} = B'_{\ell_o}K$. Since y normalizes $B'_{\ell_o}K$, $x \in yB'_1B_{\ell_o}$, x normalizes $B'_{\ell_o}K$. This implies that

$$\operatorname{Ad} x^{-1}(\mathfrak{b}'_{\ell_o} + K - 1) \subset \mathfrak{b}'_{\ell_o} + K - 1. \tag{13.5}$$

Combining this with Lemma 13.8 (2) results in

$$[X, \mathfrak{b}'_{\ell_o} + K - 1] \subset \mathfrak{b}'_{\ell_o} + K - 1. \tag{13.6}$$

Let $w \in \mathcal{F}_1(x)$. As $x \in UK$, $x^{-1}w^{-1}xw \in (UK) \cap B_{\ell_o} = B'_{\ell_o}K$. This implies, setting $W = w - 1$, that $W - \operatorname{Ad} x^{-1}(W) \in \mathfrak{b}'_{\ell_o} + K - 1$. Hence, by (13.5) and Lemma 13.8 (2),

$$[X, W] \in \mathfrak{b}'_{\ell_o} + K - 1, \quad W \in \mathcal{F}_1(x) - 1.$$

Using (13.6) and the fact that, as B'_1K is abelian modulo $\ker(\varphi|_{B'_1B_{m_o}})$, $\psi(\operatorname{tr}(s_h(\cdot)))$ is trivial on the commutator of $\mathfrak{b}'_{\ell_o} + K - 1$, it is possible to show that (for details, see the proof of (8.12) in [**34**, p. 91]),

$$\psi(\operatorname{tr}(s_h([(\operatorname{ad} X)^i(T), W]))) = 1, \quad i \geqslant 1, \quad T \in \mathfrak{b}'_{\ell_o} + K - 1, \quad W \in \mathcal{F}_1(x) - 1. \tag{13.7}$$

From $z = y^{-1}x \in (UK) \cap B_{\ell_o} = B'_{\ell_o}K$, it follows that $z - 1 \in \mathfrak{b}'_{\ell_o} + K - 1$. By (13.5) and Lemma 13.8 (4),

$$Z \in \mathfrak{b}'_{\ell_o} + K - 1. \tag{13.8}$$

Let $w = 1 + W \in \mathcal{F}_1(x)$. As $y \in G'$, y commutes with s_h, $\mathrm{tr}(s_h(W - \mathrm{Ad}\, x^{-1}(W))) = \mathrm{tr}(s_h(\mathrm{Ad}\, z(W) - W))$. Furthermore, $\mathrm{Ad}\, z(W) - W \in [z-1, W] + \mathfrak{b}_{(nj/ef_o)+1}$. Using Lemma 13.8 (3) and $[\mathfrak{b}_{m_o}, W] \in \mathfrak{b}_{(nj/ef_o)+1}$, we can show that

$$\psi(\mathrm{tr}(s_h(W - \mathrm{Ad}\, x^{-1}(W)))) = \prod_{i=0}^{h-2} \psi\biggl(\mathrm{tr}\biggl(s_h\biggl[\frac{(-1)^i (\mathrm{ad}\, X)^i (Z)}{(i+1)!}, W\biggr]\biggr)\biggr).$$

By (13.8), we may apply (13.7) with $T = (-1)^i Z/(i+1)!$ and $i \geqslant 1$. This results in

$$\psi(\mathrm{tr}(s_h(W - \mathrm{Ad}\, x^{-1}(W)))) = \psi(\mathrm{tr}(s_h[Z, W])) = \psi(\mathrm{tr}(s_h[X, W])). \tag{13.9}$$

By Lemma 13.8 (1),

$$\begin{aligned}\psi(\mathrm{tr}(s_h[\mathrm{Ad}\, x^{-1}(W), W])) &= \psi(\mathrm{tr}(s_h[\mathrm{Ad}\, x^{-1}(W) - W, W])) \\ &= \prod_{i=1}^{h-1} \psi\biggl(\mathrm{tr}\biggl(s_h\biggl[\frac{(-1)^i (\mathrm{ad}\, X)^i (W)}{i!}, W\biggr]\biggr)\biggr).\end{aligned}$$

By (13.6), $T = (\mathrm{ad}\, X)^i (W)/i! \in \mathfrak{b}'_{\ell_o} + K - 1$. Applying (13.7) with $i-1$ instead of i, we see that the terms corresponding to $2 \leqslant i \leqslant h-1$ in the above product are equal to one. Thus

$$\psi(\mathrm{tr}(s_h[\mathrm{Ad}\, x^{-1}(W), W])) = \psi(\mathrm{tr}(s_h[W, [X, W]])). \tag{13.10}$$

Proof of (1). A straightforward argument shows that

$$\begin{aligned}\bar{\varphi}(x^{-1} w^{-1} x w) &= \psi(\mathrm{tr}(s_h(W - \mathrm{Ad}\, x^{-1}(W) - \tfrac{1}{2}[\mathrm{Ad}\, x^{-1}(W), W]))) \\ &= \psi(\mathrm{tr}(s_h([X, W] - \tfrac{1}{2}[W, [X, W]]))) \quad \text{by (13.9) and (13.10).}\end{aligned}$$

It is easy to see that $\psi(\mathrm{tr}(s_h(\mathrm{Ad}\, w^{-1}(X)))) = \psi(\mathrm{tr}(s_h(X + [X, W] - \frac{1}{2}[W, [X, W]])))$, so part (1) follows. □

Proof of (2). Introducing an extra integration over K, we have

$$\int_{B_{\ell_o}} \psi(\mathrm{tr}(s_h\, \mathrm{Ad}\, k(X)))\, \mathrm{d}k = \int_{B_{\ell_o}} \biggl(\int_K \psi(\mathrm{tr}(s_h\, \mathrm{Ad}\, wk(X)))\, \mathrm{d}w\biggr)\, \mathrm{d}k.$$

Fix $k \in B_{\ell_o}$. Set $\tilde{X} = \mathrm{Ad}\, k(X)$. To prove (2), it suffices to prove that if the inner integral above, which we will denote by $I(\tilde{X})$, is non-zero, then $k \in \mathcal{F}_1(x)$.

Note that $I(\tilde{X})$ can be rewritten as

$$I(\tilde{X}) = \psi(\mathrm{tr}(s_h \tilde{X})) \int_{K-1} \psi(\mathrm{tr}(s_h([\tilde{X}, W] + \tfrac{1}{2}[W, [W, \tilde{X}]])))\, \mathrm{d}W.$$

As $\tilde{X} \in X + \mathfrak{b}_{\ell_o}$, it follows from (13.6) and the fact that $\psi(\mathrm{tr}(s_h \cdot))$ is trivial on the commutator of $\mathfrak{b}'_{\ell_o} + K - 1$ that

$$\psi(\mathrm{tr}(s_h[W, [W, \tilde{X}]])) = \psi(\mathrm{tr}(s_h[W, [W, X]])) = 1 \quad \forall W \in K - 1.$$

https://doi.org/10.1017/S1474748003000124 Published online by Cambridge University Press

Applying Lemma 13.4, write $\tilde{X} = \tilde{Y} + \tilde{Z}$, $\tilde{x} = \mathfrak{e}_h(\tilde{X})$, $\tilde{y} = \mathfrak{e}_h(\tilde{Y})$, $\tilde{z} = \tilde{y}^{-1}\tilde{x}$. Then

$$I(\tilde{X}) = \psi(\mathrm{tr}(s_h\tilde{X}))\int_{K-1}\psi(\mathrm{tr}(s_h[\tilde{Z},W]))\,\mathrm{d}W.$$

Suppose that $I(\tilde{X}) \neq 0$. Then $\psi(\mathrm{tr}(s_h[\tilde{Z},W])) = 1$ for all $W \in K-1$. However, we can easily check that $\varphi((1+\tilde{Z})^{-1}w^{-1}(1+\tilde{Z})w) = \psi(\mathrm{tr}(s_h[\tilde{Z},W]))$. As $B_1'K$ is a maximal abelian subgroup modulo the kernel of φ, it follows that $1+\tilde{Z} \in (B_1'K)\cap B_{\ell_o} = B_{\ell_o}'K$. Equivalently, $\tilde{Z} \in \mathfrak{b}_1' + K - 1$. By (13.6) and Lemma 13.8 (3), it follows that $\tilde{z} \in B_{\ell_o}'K$. From

$$\tilde{y} = \tilde{x}\tilde{z}^{-1} = y(y^{-1}kyk^{-1})(kzk^{-1})\tilde{z}^{-1} \in yB_{\ell_o}$$

and $y, \tilde{y} \in G'$, we get $\tilde{y} \in yB_{\ell_o}' \subset U$. Thus, if $I(\tilde{X}) \neq 0$, then $kxk^{-1} = \tilde{x} = \tilde{y}\tilde{z} \in UK$. That is, $k \in \mathcal{F}_1(x)$. □

Proof of Theorem 13.2. First suppose that $\rho(\tau) = 0$. In this case, $\tau = \sigma$, where σ is as in § 10, and Theorem 13.2 is equivalent to Lemma 10.9. (We remark that if τ is trivial, then $s_{\tau,h} = 0$.)

Now suppose that $\rho(\tau) > 0$. Assume that the theorem holds for those totally pure refined minimal K-types (B^*, τ^*) of groups $G^* \simeq GL_{n/[E^*:F]}(E^*)$, where E^*/F is an extension of degree dividing n, having the property that $\rho(\tau^*)/e(E^*/F) < \rho(\tau)$. Let $\dot{\chi}_{\tau'}$ (respectively, $\dot{\chi}_{\dot{\sigma}_{\mathrm{ext}}}$) be the function on B given by $\chi_{\tau'}$ (respectively, $\chi_{\dot{\sigma}_{\mathrm{ext}}}$) on $B'\dot{J}$, and zero elsewhere. We have

$$\begin{aligned}
\frac{\chi_\tau(\mathfrak{e}_h(X))}{\dim\tau} &= (\dim\tau')^{-1}(\dim\dot{\sigma}_{\mathrm{ext}})^{-1}\int_B \dot{\chi}_{\tau'}(k\mathfrak{e}_h(X)k^{-1})\dot{\chi}_{\dot{\sigma}_{\mathrm{ext}}}(k\mathfrak{e}_h(X)k^{-1})\,\mathrm{d}k \\
&= (\dim\tau')^{-1}(\dim\dot{\sigma}_{\mathrm{ext}})^{-1}\int_B \dot{\chi}_{\tau'}(\mathfrak{e}_h(\mathrm{Ad}\,k(X)))\dot{\chi}_{\dot{\sigma}_{\mathrm{ext}}}(\mathfrak{e}_h(\mathrm{Ad}\,k(X)))\,\mathrm{d}k \\
&= (\dim\tau')^{-1}[\mathfrak{b}' + \mathfrak{b}_{\ell_o}](\mathrm{Ad}\,k(X)) \\
&\qquad\times\int_B \dot{\chi}_{\tau'}(\mathfrak{e}_h(\mathrm{Ad}\,k(X)))\left(\int_{B_{\ell_o}}\psi(\mathrm{tr}(s_h\,\mathrm{Ad}(yk)(X)))\,\mathrm{d}y\right)\mathrm{d}k \\
&= (\dim\tau')^{-1}\int_B\int_{B_{\ell_o}}\dot{\chi}_{\tau'}(\mathfrak{e}_h(\mathrm{Ad}\,yk(X)))[\mathfrak{b}' + \mathfrak{b}_{\ell_o}] \\
&\qquad\qquad\times(\mathrm{Ad}\,yk(X))\psi(\mathrm{tr}(s_h\,\mathrm{Ad}\,yk(X)))\,\mathrm{d}y\,\mathrm{d}k \\
&= (\dim\tau')^{-1}\int_B \dot{\chi}_{\tau'}(\mathfrak{e}_h(\mathrm{Ad}\,k(X)))[\mathfrak{b}' + \mathfrak{b}_{\ell_o}](\mathrm{Ad}\,k(X))\psi(\mathrm{tr}(s_h\,\mathrm{Ad}\,k(X)))\,\mathrm{d}k.
\end{aligned} \tag{13.11}$$

The first equality above follows from (13.3), and the second follows from $k\mathfrak{e}_h(X)k^{-1} = \mathfrak{e}_h(\mathrm{Ad}\,k(X))$. By parts (2) and (4) of Lemma 13.3,

$$\mathfrak{e}_h(\mathrm{Ad}\,k(X)) \in B'\dot{J} = B'B_{\ell_o} \quad\Leftrightarrow\quad \mathrm{Ad}\,k(X) \in \mathfrak{b}' + \mathfrak{b}_{\ell_o}.$$

Combining this with Proposition 13.7 results in the third equality. For the fourth equality, note that the function $[\mathfrak{b}' + \mathfrak{b}_{\ell_o}]$ is $\mathrm{Ad}\,B_{\ell_o}$-invariant, and, as τ' is a representation of $B'B_l$,

https://doi.org/10.1017/S1474748003000124 Published online by Cambridge University Press

the function $\dot\chi_{\tau'}$ is invariant under conjugation by B_{ℓ_o}. Finally, the order of integration is reversed and the B_{ℓ_o}-integral disappears.

If τ' is the trivial representation of $B'\dot{J}$, then (13.11) can be rewritten as

$$\begin{aligned}\frac{\chi_\tau(\mathfrak{e}_h(X))}{\dim\tau} &= \int_B [\mathfrak{b}' + \mathfrak{b}_{\ell_o}](\operatorname{Ad} k(X))\psi(\operatorname{tr}(s_h \operatorname{Ad} k(X)))\,\mathrm{d}k \\ &= \int_B \psi(\operatorname{tr}(s_h \operatorname{Ad} k(X)))\,\mathrm{d}k \quad \text{by Corollary 13.6 (with } s' = 0).\end{aligned}$$

When τ' is trivial, we have $G' = G''$ and $L = E$. We know that s_h generates E over F, so its centralizer in G' is G'. Hence Theorem 13.2 holds with $s_{\tau,h} = s_h$.

Next, assume that τ' is non-trivial. Suppose that $\mathfrak{e}_h(\operatorname{Ad} k(X)) \in B'\dot{J}$. Then, by Lemma 13.4 (applied to $\operatorname{Ad} k(X)$), $\operatorname{Ad} k(X) = Y + Z$ with $Y \in \mathfrak{b}'$, $Z \in \mathfrak{b}'^{\perp}_{\ell_o}$, and $\mathfrak{e}_h(\operatorname{Ad} k(X)) \in \mathfrak{e}_h(Y')\dot{J}$. As τ' is trivial on $\dot{J}$, we have $\chi_{\tau'}(\mathfrak{e}_h(\operatorname{Ad} k(X))) = \chi_{\tau'}(\mathfrak{e}_h(Y))$. By Lemma 13.4 (1) and (13.2),

$$Y^h \in \mathfrak{b}'_{2\ell_o} \subset \mathfrak{b}'_{n\rho(\tau)/f_o} \subset \mathfrak{b}'_{(nj\rho(\tau')/(ef_o))+1}.$$

By (13.2) and our inductive assumption, there exists $s_{\tau',h} \in L$ such that $\nu_L(s_{\tau',h}) = -e(L/E)\rho(\tau') = -(e_0/e)\rho(\tau')$, the centralizer of $s_{\tau',h}$ in G' is equal to G'', and

$$\frac{\chi_{\tau'}(\mathfrak{e}_h(\operatorname{Ad} k(X)))}{\dim\tau'} = \frac{\chi_{\tau'}(\mathfrak{e}_h(Y))}{\dim\tau'} = \int_{B'} \psi(\operatorname{tr}(s_{\tau',h} \operatorname{Ad} k'(Y)))\,\mathrm{d}k'. \tag{13.12}$$

Note that we have taken $\psi \circ \operatorname{tr}_{E/F}$ for our non-trivial character ψ_E of E, so that $\psi_E \circ \operatorname{tr}' = \psi \circ \operatorname{tr}$. As $\operatorname{Ad} B'(\mathfrak{b}'^{\perp}_{\ell_o}) = \mathfrak{b}'^{\perp}_{\ell_o}$ and $s_{\tau',h} \in E \subset \mathfrak{g}'$, we have $\operatorname{tr}(s_{\tau',h} \operatorname{Ad} k'(Z)) = 0$. This allows us to replace Y by $Y + Z = \operatorname{Ad} k(X)$ in the right-hand side of (13.12). Combining (13.12) with (13.11) results in

$$\begin{aligned}\frac{\chi_\tau(\mathfrak{e}_h(X))}{\dim\tau} &= \int_B [\mathfrak{b}' + \mathfrak{b}_{\ell_o}](\operatorname{Ad} k(X)) \\ &\qquad \times \left(\int_{B'} \psi(\operatorname{tr}(s_{\tau',h} \operatorname{Ad} k'k(X)))\,\mathrm{d}k'\right)\psi(\operatorname{tr}(s_h \operatorname{Ad} k(X)))\,\mathrm{d}k \\ &= \int_B \int_{B'} [\mathfrak{b}' + \mathfrak{b}_{\ell_o}](\operatorname{Ad} k'k(X))\psi(\operatorname{tr}(s_{\tau',h} \operatorname{Ad} k'k(X)))\psi(s_h \operatorname{Ad} k'k(X))\,\mathrm{d}k'\,\mathrm{d}k \\ &= \int_B [\mathfrak{b}' + \mathfrak{b}_{\ell_o}](\operatorname{Ad} k(X))\psi(\operatorname{tr}(s_h + s_{\tau',h}) \operatorname{Ad} k(X))\,\mathrm{d}k \\ &= \int_B \psi(\operatorname{tr}(s_h + s_{\tau',h}) \operatorname{Ad} k(X))\,\mathrm{d}k.\end{aligned}$$

For the second equality above note that $[\mathfrak{b}' + \mathfrak{b}_{\ell_o}]$ is $\operatorname{Ad} B'$-invariant, and $s_{\tau',h} \in E$ so belongs to centre of $\mathfrak{g}'$ and

$$\operatorname{tr}(s_{\tau',h} \operatorname{Ad} k'k(X)) = \operatorname{tr}(\operatorname{Ad} k'^{-1}(s_{\tau',h}) \operatorname{Ad} k(X)) = \operatorname{tr}(s_{\tau',h} \operatorname{Ad} k(X)).$$

The same statement holds if $s_{\tau',h}$ is replaced by s_h. For the third equality, reverse the order of integration and then absorb the B_{ℓ_o}-integration. The final equality follows after an application of Corollary 13.6 with $s' = s_h + s_{\tau',h}$.

https://doi.org/10.1017/S1474748003000124 Published online by Cambridge University Press

Set $s_{\tau,h} = s_h + s_{\tau',h}$. To complete the proof, it remains to show that $s_{\tau,h}$ generates L/F. Since $s_{\tau',h} \in \varpi_E^{-j+1}\mathfrak{b}'$, by Lemma 5.5, the centralizer of $s_{\tau,h}$ in G is contained in G'. As s_h is in the centre of G', and $s_{\tau',h}$ generates L over E, the centralizer of $s_{\tau,h}$ in G' is equal to G''. Hence the centralizer of $s_{\tau,h}$ in G is G''. Since $s_{\tau,h} \in L$, it follows that this element generates L over F. □

The following result will be needed for the proof of Theorem 14.1.

Lemma 13.10. *Let (B,τ) be as in Theorem 13.2.*

(1) *If $\rho(\tau) = 0$, then $\dim\tau = u_d(q)^{n'} u_1(q^d)^{-n'}$.*

(2) *If $\rho(\tau) > 0$, then $\dim\tau = (\dim\tau')u_{f_o}(q)^{n/f_o} u_{f_o/f}(q^f)^{-n/ef_o} q^{(n/2e)((n-n')j-e+1)}$.*

Proof. For (1), recall that $\tau = \sigma$, where σ is as in § 10. Thus $\dim\tau = (\dim\sigma_0)^{n'}$, where σ_0 is the inflation to $GL_d(\mathfrak{o}_F)$ of an irreducible cuspidal representation of $GL_d(\mathbb{F}_q)$. As shown in Appendix 3 of [**17**], $\dim\sigma_0 = u_d(q)u_1(q^d)^{-1}$.

Suppose that $\rho(\tau) > 0$. By (13.3),

$$\dim\tau = [B : B'\dot{J}](\dim\tau')(\dim\dot{\sigma}_{\mathrm{ext}}) = (\dim\tau')[B : B'B_{\ell_o}](\dim\dot{\sigma}).$$

Arguing as in the proof of Proposition 8.4, except with $\dot{\sigma}$, B_{ℓ_o}, $\dot{J}$ in place of σ, Q_ℓ, J, etc., we see that

$$[B_1 : B_{\ell_o}][B_1' : B_{\ell_o}']^{-1}\dim\sigma = q^{(f_o/2)(n-n/(ef))((nj/ef_o)-1)}.$$

Hence

$$\begin{aligned}[B : B'B_{\ell_o}]\dim\dot{\sigma} &= [B : B_1][B' : B_1']^{-1} q^{(f_o/2)(n-n/(ef))((nj/ef_o)-1)} \\ &= u_{f_o}(q)^{n/f_o} u_{f_o/f}(q^f)^{-n/ef_o} q^{(n/2e)((n-n')j-e+1)}.\end{aligned}$$

Here we have used (3.1) to compute various group indices. □

14. Reduction to germs of unipotent characters

Throughout this section, in order to be able to apply Theorems 11.5 and 12.3, we assume that $p > 2n$. Proofs of the results stated in this section appear in §§ 15–17.

One of our main results (Theorem 14.1) concerns the germs of characters of those representations $\pi \in \mathcal{E}(G)$ that contain totally pure refined minimal K-types. As discussed in § 13 (see Theorem 13.1), Howe and Moy [**17**] showed that there exist families of K-types, called refined minimal K-types, having the property that every $\pi \in \mathcal{E}(G)$ contains some refined minimal K-type. Furthermore, the Hecke algebra attached to a refined minimal K-type is naturally isomorphic to the Iwahori Hecke algebra of a direct product of general linear groups over finite extensions of F. Recall that we have defined a pure refined minimal K-type (B,τ) to be totally pure if the Hecke algebra isomorphism of Howe and Moy is of the form

$$\iota : \mathcal{H}'' = \mathcal{H}(G''/\!\!/B'') \to \mathcal{H}(\tau) = \mathcal{H}(G/\!\!/B, \tilde{\tau}),$$

https://doi.org/10.1017/S1474748003000124 Published online by Cambridge University Press

where $G'' \simeq GL_a(L)$ for some positive divisor a of n and some extension L/F such that $a[L:F]=n$, and $B''=B\cap G$ is an Iwahori subgroup of G''. The totally pure refined minimal K-types are the ones that are contained in the essentially square integrable representations (although they are also contained in many other representations). If (B,τ) is a totally pure refined minimal K-type, let $\rho(\tau)$ be the rational number defined in § 13. Recall that the depth $\rho(\pi)$ of any $\pi \in \mathcal{E}(G)$ that contains (B,τ) is equal to $\rho(\tau)$.

Before stating Theorem 14.1, we remind the reader of some of our notation. Given $r\in\mathbb{R}$, let $\mathfrak{g}_{r+}$ be as in § 11. If s_0 is a semisimple element in $\mathfrak{g}$, $\Omega_G(s_0)$ denotes the set of G-orbits in $\mathfrak{g}$ whose closures contain s_0. If $\mathcal{O}$ is a G-orbit in $\mathfrak{g}$, $\hat{\mu}_{\mathcal{O}}$ is the locally integrable function on $\mathfrak{g}$ that represents the Fourier transform of the distribution $\mu_{\mathcal{O}}$ given by integration over the orbit $\mathcal{O}$ relative to a G-invariant measure on $\mathcal{O}$. We say that a locally integrable G-invariant function D defined on an open subset S of $\mathfrak{g}$ is s_0-asymptotic on an open subset S_0 of S if $D|_{S_0\cap\mathfrak{g}_{\mathrm{reg}}}$ belongs to the span of $\{\hat{\mu}_{\mathcal{O}}|_{S_0\cap\mathfrak{g}_{\mathrm{reg}}} \mid \mathcal{O}\in\Omega_G(s_0)\}$. The resulting linear combination of the functions $\hat{\mu}_{\mathcal{O}}$, $\mathcal{O}\in\Omega_G(s_0)$, is referred to as an s_0-asymptotic expansion of D (on S_0). Recall (Corollary 11.11) that if s_0 belongs to an elliptic Cartan subalgebra of $\mathfrak{g}$, then the functions $\hat{\mu}_{\mathcal{O}}$, $\mathcal{O}\in\Omega_G(s_0)$, remain linearly independent upon restriction to any open neighbourhood of zero intersected with $\mathfrak{g}_{\mathrm{reg}}$. If $\pi\in\mathcal{E}(G)$, and Θ_π is the character of π, the function $X\mapsto\Theta_\pi(1+X)$, which we refer to as the germ of Θ_π, is locally integrable on $\mathfrak{g}_{0+}$. If $s_0\in\mathfrak{g}$ is semisimple and if the germ of Θ_π is s_0-asymptotic on some open neighbourhood of zero, then, for each $\mathcal{O}\in\Omega_G(s_0)$, we denote the coefficient of $\hat{\mu}_{\mathcal{O}}$ in the s_0-asymptotic expansion by $c_{\mathcal{O}}(\pi)$.

If B is a parahoric subgroup of G, let $\mathfrak{b}_i$, $i\in\mathbb{Z}$, be the filtration of $\mathfrak{g}$ given by powers of the nilradical of the associated parahoric $\mathfrak{o}_F$-subalgebra of $\mathfrak{g}$ (see § 3). Given $S\subset G$, $\upsilon_G(S)$ denotes the measure of S with respect to some fixed Haar measure on G.

Theorem 14.1. *Let (B,τ) be a totally pure refined minimal K-type. Let L be as above. Set $f_o=f(L/F)$. Suppose that $n\leqslant h\leqslant p$. Let $\mathfrak{e}_h$ be the truncated exponential of (13.1). Then the following hold.*

(1) *There exists $s_{\tau,h}\in L$ such that the centralizer of $s_{\tau,h}$ in G equals G'' and*

$$\chi_\tau(\mathfrak{e}_h(X)) = \dim\tau\int_B \psi(\mathrm{tr}(s_{\tau,h}\,\mathrm{Ad}\,k(X)))\,\mathrm{d}k \qquad \forall X\in\mathfrak{b} \text{ such that } X^h\in\mathfrak{b}_{(n\rho(\tau)/f_o)+1}.$$

(2) *There exist coefficients $c_{\mathcal{O}}(\pi)$, one for each $\mathcal{O}\in\Omega_G(s_{\tau,h})$, such that*

$$\Theta_\pi(1+X) = \sum_{\mathcal{O}\in\Omega_G(s_{\tau,h})} c_{\mathcal{O}}(\pi)\hat{\mu}_{\mathcal{O}}(X), \quad X\in\begin{cases}\mathfrak{g}_{\rho(\tau)}\cap\mathfrak{g}_{\mathrm{reg}} & \text{if } \rho(\tau)>0,\\ \mathfrak{g}_{0+}\cap\mathfrak{g}_{\mathrm{reg}} & \text{otherwise.}\end{cases}$$

(3) *Assume that the measures on the orbits in $\Omega_G(s_{\tau,h})$ and $\Omega_{G''}(0)$ are compatible in the sense described in § 12. Let π'' be the irreducible unipotent representation of G'' that corresponds to π via the Hecke algebra isomorphism ι. Given $\mathcal{O}''\in\Omega_{G''}(0)$, let $\mathcal{O}=G\cdot(s_{\tau,h}+\mathcal{O}'')$. Then*

$$c_{\mathcal{O}}(\pi)=\upsilon_G(B)^{-1}\upsilon_{G''}(B'')(\dim\tau)c_{\mathcal{O}''}(\pi'').$$

https://doi.org/10.1017/S1474748003000124 Published online by Cambridge University Press

Remarks 14.2.

(1) If $\rho(\tau) = 0$, then $s_{\tau,h}$ is independent of the choice of h such that $n \leqslant h \leqslant p$ (see the proof of Theorem 14.1). Suppose that $\rho(\tau) > 0$ and $n \leqslant h,\ \ell \leqslant p$. Given $\mathcal{O}_h \in \Omega_G(s_{\tau,h})$, if $\mathcal{O}_h = G \cdot (s_{\tau,h} + \mathcal{O}'')$, then $\mathcal{O}_h \cap \mathfrak{g}'' = s_{\tau,h} + \mathcal{O}''$. The map $\mathcal{O}_h \to \mathcal{O}_\ell = G \cdot (s_{\tau,\ell} + \mathcal{O}'')$ is a bijection from $\Omega_G(s_{\tau,h})$ to $\Omega_G(s_{\tau,\ell})$. By (3), $c_{\mathcal{O}_h}(\pi) = c_{\mathcal{O}_\ell}(\pi)$. It can be shown by a straightforward inductive argument using descent properties of orbital integrals (see § 5) that $\hat{\mu}_{\mathcal{O}_h}$ and $\hat{\mu}_{\mathcal{O}_\ell}$ coincide on $\mathfrak{g}_{\rho(\tau)} \cap \mathfrak{g}_{\mathrm{reg}}$. Hence the expression in Theorem 14.1 (2) for the germ of Θ_π is independent of the choice of h. (We allow h to vary because it is convenient for the induction step in the proof of Theorem 14.1 (1) (Theorem 13.2).)

(2) Suppose that π is essentially square integrable and contains (B, τ). Let $d(\pi)$ be the formal degree of π, and let St_G be the Steinberg representation of G. It is easy to show, using the fact that $\iota(\pi)$ is a twist of $St_{G''}$, together with the relation between $d(\pi)$ and $d(\iota(\pi))$ arising via ι (see [**17**, § 5] and the discussion in [**7**, § 2]) that

$$\frac{d(\pi)}{d(St_G)} = f_o(q^n - 1)(q^{f_o} - 1)^a (q^{af_o} - 1)^{-1} u_{f_o}(q)^{-n/a} \dim \tau.$$

Here, we have divided by $d(St_G)$ on the left-hand side in order to remove dependence on measures. In Theorem 2.4.7 of [**7**], $d(\pi)/d(St_G)$ is expressed in terms of data involved in the supercuspidal support of π.

Suppose that $\pi \in \mathcal{E}(G)$ is supercuspidal. Then π contains a totally pure refined minimal K-type such that $[L : F] = n$ and $G'' = L^\times$. In this case, an analogue of Theorem 14.1 was proved in [**33**]. There exists a compact open mod centre subgroup H of $L^\times B$ and a representation κ of H such that $\pi = \mathrm{Ind}_H^G \kappa$. Let $\tau_0 = \mathrm{Ind}_H^{L^\times B} \kappa$. Then $\pi = \mathrm{Ind}_{L^\times B}^G \tau_0$ and $\tau = \tau_0 \mid B$ is a refined minimal K-type contained in π. Proposition 3.10 of [**33**] is essentially Theorem 14.1 (1) for $h = n$. This implies that the s_π of [**33**] is the $s_{\tau,n}$ of Theorem 13.2. Taking $h = n$, parts (2) and (3) of Theorem 14.1 are analogues of Theorem 4.3 of [**33**], with one important difference. In [**33**], we do not specify the neighbourhood on which the s_π-asymptotic expansion holds. Recently, Adler and DeBacker [**2**] have shown, by refinements of the methods of [**33**], that the s_π-asymptotic expansion holds on $\mathfrak{g}_{\rho(\pi)^+}$. In [**33**], the Kirillov-type character formula of Proposition 3.10 was proved via methods similar to those used here to prove Theorem 13.2. But the analogues of (2) and (3) were proved differently in [**33**], via a comparison of Harish-Chandra's integral formulae for $\Theta_\pi \circ \exp$ and the Fourier transform $\hat{\mu}_{\mathcal{O}(s_\pi)}$.

Given $\pi \in \mathcal{E}(G)$, the wavefront set $WF(\pi)$ of π is

$$WF(\pi) = \bigcup_{\{\mathcal{O} \in \Omega_G(0) \mid c_{\mathcal{O}}(\pi) \neq 0\}} \bar{\mathcal{O}}.$$

As shown in Proposition II.2 of [**25**], there exists a unique $\mathcal{O} \in \Omega_G(0)$ such that $WF(\pi) = \bar{\mathcal{O}}$. If $\alpha \in \mathcal{P}^0(n/a)$, let $\alpha^{n/a} \in \mathcal{P}(n)$ be as defined in § 2. Let $\dot{\alpha}^{n/a}$ be the unique element of $\mathcal{P}^0(n)$ that is a rearrangement of $\alpha^{n/a}$.

Corollary 14.3. *Let $\pi \in \mathcal{E}(G)$ and $\pi'' \in \mathcal{E}(G'')$ be as in Theorem 14.1. If $\beta \in \mathcal{P}^0(a)$ is such that $WF(\pi'') = \bar{\mathcal{O}}_\beta$, then $WF(\pi) = \bar{\mathcal{O}}_{\dot{\beta}^{n/a}}$.*

Recall that $\pi \in \mathcal{E}(G)$ is said to be essentially square integrable if some twist of π by a linear character of G belongs to the discrete series. Next, we give the coefficients in the $s_{\tau,h}$-asymptotic expansion of the germ of Θ_π for essentially square integrable π. Let $\alpha \mapsto \mathcal{O}''_\alpha$ be the bijection between $\mathcal{P}^0(a)$ and $\Omega_{G''}(0)$ discussed in § 2. As shown by Howe [**13**], given $\alpha \in \mathcal{P}^0(a)$, there exists a parabolic subgroup P_α of G'' such that the germ of $\Theta_{\mathrm{Ind}_{P_\alpha}^{G''} 1}$ coincides with $\lambda_\alpha \hat{\mu}_{\mathcal{O}''_\alpha}$ for some positive constant λ_α depending on the normalization of measure on $\mathcal{O}''_\alpha$. For the next result, we will assume that the measure on $\mathcal{O}''_\alpha$ has been normalized so that $\lambda_\alpha = 1$. If $r = r(\alpha)$ is the length of α, let $|\mathrm{Stab}\,\alpha|$ be the cardinality of the stabilizer of α in the symmetric group on r letters.

Theorem 14.4. *Suppose that $\pi \in \mathcal{E}(G)$ is essentially square integrable. Then the following hold.*

(1) *π contains a totally pure refined minimal K-type τ.*

(2) *Let $s_{\tau,h}$, G'' and π'' be as in Theorem 14.1. Assume that the measures on the orbits in $\Omega_G(s_{\tau,h})$ and $\Omega_{G''}(0)$ are compatible in the sense described in § 12. Let $\alpha \in \mathcal{P}^0(a)$. If $\mathcal{O} = G \cdot (s_{\tau,h} + \mathcal{O}''_\alpha)$, then*
$$\begin{aligned} c_{\mathcal{O}}(\pi) &= \upsilon_G(B)^{-1}\upsilon_{G''}(B'')(\dim\tau) c_{\mathcal{O}''_\alpha}(St_{G''}) \\ &= \upsilon_G(B)^{-1}\upsilon_{G''}(B'')(\dim\tau)\frac{(-1)^{a-r(\alpha)}r(\alpha)!}{|\mathrm{Stab}\,\alpha|}. \end{aligned}$$

It follows from Theorem 14.4 that if π is essentially square integrable, in order to compute the coefficients in the 0-asymptotic expansion of the germ of Θ_π, it suffices to compute the coefficients in the 0-asymptotic expansions of the functions $\hat{\mu}_{\mathcal{O}}$, $\mathcal{O} \in \Omega_G(s_{\tau,h})$. It is not known how to do this in general. In a separate paper [**36**], the coefficients in the 0-asymptotic expansions of the germs of characters of certain discrete series representations are computed, via a slightly different approach, using some results of this paper.

If $\pi \in \mathcal{E}(G)$ contains a refined minimal K-type τ that is not totally pure, then Howe and Moy do not give an explicit construction of τ, so we do not have an analogue of Theorem 14.1 (1) (Theorem 13.2). However, we can relate the germ of Θ_π to the germ of the character of a unipotent representation of the centralizer of some semisimple element.

Theorem 14.5. *Let $\pi \in \mathcal{E}(G)$. Then there exists $s_\pi \in \mathfrak{g}_{\mathrm{ss}} \cap \mathfrak{g}_{-\rho(\pi)}$ and an irreducible unipotent representation π_u of $H = C_G(s_\pi)$ such that the following hold.*

(1) *The germ of Θ_π is s_π-asymptotic on $\mathfrak{g}_{\rho(\pi)}$ (respectively, $\mathfrak{g}_{0+}$) if $\rho(\pi) > 0$ (respectively, $\rho(\pi) = 0$).*

(2) *Assume that the measures on the orbits in $\Omega_G(s_\pi)$ and $\Omega_H(0)$ are compatible in the sense described in § 12. Then there exists an s_π-asymptotic expansion of the germ of Θ_π such that the coefficients are given by*
$$c_{G\cdot(s_\pi+\mathcal{O}_H)}(\pi) = \lambda c_{\mathcal{O}_H}(\pi_u), \quad \mathcal{O}_H \in \Omega_H(0),$$
for some positive constant λ.

The constant λ has the property that $\upsilon_G(K)^{-1}\upsilon_H(K_H)\lambda$ is independent of measures for any open compact sugroups K and K_H of G and H, respectively. In order to relate s_π to the character of a refined minimal K-type contained in π (as was done for those π containing a totally pure refined minimal K-type), it would be necessary to derive a precise relation between the characters of the refined minimal K-types contained in representations which are related via parabolic induction.

15. Proof of Theorem 14.1

Assume that $p > 2n$. Let (B, τ) be a totally pure refined minimal K-type. Let $\iota : \mathcal{H}'' \to \mathcal{H}(\tau)$ be the associated Hecke algebra isomorphism (see Theorem 13.1). Given $\pi \in \mathcal{E}(G)$, which contains (B, τ), let $\iota^*(\pi) \in \mathcal{E}(G'')$ denote the representation corresponding to π via ι. Let L be the extension of F such that $G'' \simeq GL_a(L)$, and let $f_o = f(L/F)$.

Proof of Theorem 14.1. Note that part (1) is Theorem 13.2. Hence we need only prove parts (2) and (3). Suppose that $\pi \in \mathcal{E}(G)$ contains (B, τ). Let h and $s_{\tau,h}$ be as in part (1). Let $\mathfrak{e}_1(X) = 1 + X$, $X \in \mathfrak{g}_{0+}$.

We start with the case $\rho(\tau) = 0$. Let $E = L$, $d = f_o$ and $n' = n/d$. In this case, $B = B_{(d)^{n'}}$ and τ is the representation σ of B defined in § 10. Define s, $\mathcal{H}$, $\mathcal{H}'$, G', η, etc, as in § 10. Then $G'' = G'$, $\mathcal{H}'' = \mathcal{H}'$ and $\iota = \eta$, so $\iota^*(\pi) = \eta^*(\pi) = \pi'$. Note that $e = e(E/F) = 1$, $f = f(E/F) = d$ and $j = e\rho(\tau) = e\rho(\pi) = 0$.

By Lemma 10.9, the element $s_{\tau,h}$ of part (1) is equal to s. By Theorem 12.3 (1), $\Theta_\pi \circ \mathfrak{e}_1$ is s-asymptotic on $\mathfrak{g}_{0+}$ and, given $\mathcal{O}' \in \Omega_{G'}(0)$,

$$c_{G\cdot(s+\mathcal{O}')}(\pi) = \upsilon_G(B_{(n)})^{-1}\upsilon_{G'}(B'_{(n')})u_n(q)u_{n'}(q^d)^{-1}c_{\mathcal{O}'}(\iota^*(\pi)).$$

After expressing $\upsilon_G(B_{(n)})$ and $\upsilon_{G'}(B'_{(n')})$ in terms of $\upsilon_G(B)$ and $\upsilon_G(B')$ $(B' = B'_{(1)^{n'}})$ with the help of (3.1), we obtain

$$\begin{aligned} c_{G\cdot(s+\mathcal{O}')}(\pi) &= \upsilon_G(B)^{-1}\upsilon_{G'}(B')u_d(q)^{n'}u_1(q^d)^{-n'}c_{\mathcal{O}'}(\iota^*(\pi)) \\ &= \upsilon_G(B)^{-1}\upsilon_{G'}(B')(\dim\tau)c_{\mathcal{O}'}(\iota^*(\pi)) \end{aligned}$$

for $\mathcal{O}' \in \Omega_{G'}(0)$. Above, we have applied Lemma 13.10 to get the second equality. Hence Theorem 14.1 holds in the case $\rho(\tau) = 0$.

For the rest of the proof, we assume that $\rho(\tau) > 0$. As discussed in § 13 (following the statement of Theorem 13.2), there exists a pure unrefined minimal K-type $(Q_{n'j}, \chi_s)$ such that τ contains $(B_{nj/ef_o}, \chi_s)$ and $\rho(\tau) = j/e$. Here, s is as in § 5 and $Q_{n'j}$ is as defined in § 6. Recall that $E = F(s)$ is an extension of F of degree $d = n/n'$, $e = e(E/F)$ and $s \in \mathfrak{p}_E^{-j} - \mathfrak{p}_E^{-j+1}$. The extension E is embedded in $\mathfrak{g}$ as in § 3, $G' = C_G(s) \simeq GL_{n'}(E)$, and $\mathfrak{g}'$ denotes the Lie algebra of $\mathfrak{g}$.

Let η be the Hecke algebra isomorphism attached to the K-type $(Q_{n'j}, \chi_s)$ (see Theorem 7.1). Set $\pi' = \eta^*(\pi)$. Let θ and s_h be as in § 13 (see (13.4)). Recall that $\theta \mid 1 + \mathfrak{p}_E^j = \chi_s \mid 1 + \mathfrak{p}_E^j$.

For a description of the construction of τ, the reader may refer to the discussion following the statement of Theorem 13.2. The field E is a subfield of L.

First we consider the case $E = L$. In this case, $G'' = G'$, $B'' = B' = Q' = Q \cap G'$ (see § 6) and $\iota^*(\pi)$ is a twist of π' by some linear character of G'. Replacing θ by another quasi-character of $E^\times$ that agrees with χ_s on $1 + \mathfrak{p}_E^j$, if necessary, we may assume that

$$\iota^*(\pi) = (\theta^{-1} \circ \det')\pi' = (\theta^{-1} \circ \det')\eta^*(\pi).$$

Let $\mathfrak{e}'_1 = \mathfrak{e}_1|_{\mathfrak{g}'_{0+}}$. Note that if $X \in \mathfrak{g}'_j = 1 + \varpi_E^j \mathfrak{g}'_0$, then $\theta(\mathfrak{e}'_1(X)) = \psi(\mathrm{tr}'(s_h X))$. Thus, because π' contains the standard unrefined minimal K-type $(Q'_{n'j}, \chi'_s) = (Q'_{n'j}, \chi'_{s_h})$, $\iota^*(\pi)$ must contain the trivial representation of $Q'_{n'j}$. Hence $\rho(\iota^*(\pi)) < j$ and, by Theorem 11.5, $\Theta_{\iota^*(\pi)} \circ \mathfrak{e}'_1$ is 0-asymptotic on $\mathfrak{g}'_{\rho(\iota^*(\pi))+} \supset \mathfrak{g}'_j$. It follows from the above relation between $\iota^*(\pi)$ and π' that $\Theta_{\iota^*(\pi)}(\mathfrak{e}'_1(X)) = \psi(\mathrm{tr}'(-s_h X))\Theta_{\pi'}(\mathfrak{e}'_1(X))$ for all $X \in \mathfrak{g}'_j \cap \mathfrak{g}'_{\mathrm{reg}}$. Also, for each $\mathcal{O}' \in \Omega_{G'}(0)$,

$$\hat{\mu}_{(s_h + \mathcal{O}')}(X) = \psi(\mathrm{tr}'(s_h X))\hat{\mu}_{\mathcal{O}'}(X), \quad X \in \mathfrak{g}'_j \cap \mathfrak{g}'_{\mathrm{reg}}.$$

Thus $\Theta_{\pi'} \circ \mathfrak{e}'_1$ is s_h-asymptotic on $\mathfrak{g}'_j$, with

$$c_{s_h + \mathcal{O}'}(\pi') = c_{\mathcal{O}'}(\iota^*(\pi)), \qquad \mathcal{O}' \in \Omega_{G'}(0).$$

Applying Theorem 12.3 (2) with $s' = s - s_h$, we see that $\Theta_\pi \circ \mathfrak{e}_1$ is s_h-asymptotic on $\mathfrak{g}_{j/e}$. After expressing $\upsilon_G(B_{(n)})$ and $\upsilon_{G'}(B_{(n')})$ in terms of $\upsilon_G(B)$ and $\upsilon_{G'}(B')$ (recall that B is conjugate to $B_{(f)^{n/f}}$ and B' is conjugate to $B_{(1)^{n'}}$) with the help of (3.1), the relation between $c_{G\cdot(s_h + \mathcal{O}')}(\pi)$ and $c_{s_h + \mathcal{O}'}(\pi') = c_{\mathcal{O}'}(\iota^*(\pi))$ given in Theorem 12.3 (2) can be rewritten in the form

$$\begin{aligned} c_{G\cdot(s_h+\mathcal{O}')}(\pi) &= \upsilon_G(B)^{-1}\upsilon_{G'}(B')u_f(q)^{n/f}u_1(q^f)^{-n'}q^{fn'((n-n')j-e+1)/2}c_{\mathcal{O}'}(\iota^*(\pi)) \\ &= \upsilon_G(B)^{-1}\upsilon_{G'}(B')(\dim\tau)c_{\mathcal{O}'}(\iota^*(\pi)), \quad \mathcal{O}' \in \Omega_{G'}(0). \end{aligned}$$

Here we have used Lemma 13.10 for the second equality. This completes the proof in the case $E = L$.

Now suppose that $E \subsetneq L$. Then G'' is a proper subgroup of G'. Assume that the theorem holds for those totally pure refined minimal K-types (B^*, τ^*) of groups $G^* \simeq GL_{n/[E^*:F]}(E^*)$, where E^*/F is an extension of degree dividing n, having the property that $\rho(\tau^*)/e(E^*/F) < \rho(\tau)$.

Let $\dot{\sigma}$ be the irreducible representation of the compact open subgroup $\dot{J}$ defined in § 9, and let $\dot{\eta} : \dot{\mathcal{H}}' \to \dot{\mathcal{H}}$ be the associated Hecke algebra isomorphism. As discussed in § 13, there exists a totally pure unrefined minimal K-type (B', τ') of G' such that (13.2) $\rho(\tau')/e < \rho(\tau)$ and (13.3) $\tau = \mathrm{Ind}_{B'\dot{J}}^{B}(\tau' \otimes \dot{\sigma}_{\mathrm{ext}})$, where $\dot{\sigma}_{\mathrm{ext}}$ is a uniquely defined extension of $\dot{\sigma}$ to $B'\dot{J}$. Let $\mathcal{H}(\tau') = \mathcal{H}(G' /\!/ B', \tilde{\tau}')$. As follows from remarks in the proof of Theorem 5.4 of [**17**], the Hecke algebra isomorphism attached to (B', τ') is an isomorphism $\iota' : \mathcal{H}'' \to \mathcal{H}(\tau')$. Let ι'_θ be the composition of ι' and the map which to each $f \in \mathcal{H}(\tau')$ attaches the function $(\theta^{-1} \circ \det')f$. Then, setting $\tau'_\theta = (\theta \circ \det')\tau'$, $\iota'_\theta : \mathcal{H}'' \to \mathcal{H}(\tau'_\theta)$ is an isomorphism of Hecke algebras. Also (see [**17**]), $\mathcal{H}(\tau'_\theta)$ is a subalgebra of $\dot{\mathcal{H}}'$ whose image under the isomorphism $\dot{\eta} : \dot{\mathcal{H}}' \to \dot{\mathcal{H}}$ is equal to $\mathcal{H}(\tau)$, and $\iota = \dot{\eta} \circ \iota'_\theta$. Hence, setting $\dot{\pi} = \dot{\eta}^*(\pi)$ and $\dot{\pi}'_\theta = (\theta^{-1} \circ \det')\dot{\pi}'$,

$$\iota^*(\pi) = \iota'^*((\theta^{-1} \circ \det')\dot{\eta}^*(\pi)) = \iota'^*((\theta^{-1} \circ \det')\dot{\pi}') = \iota'^*(\dot{\pi}'_\theta). \tag{15.1}$$

https://doi.org/10.1017/S1474748003000124 Published online by Cambridge University Press

As $\dot{\pi}'_\theta$ contains (B',τ') and $\rho(\tau')/e < \rho(\tau)$, the theorem holds for $\dot{\pi}'_\theta$, with $s_{\tau',h} \in L$ as in Theorem 13.2 (applied to (B',τ')). Note that $\mathfrak{g}'_j = \mathfrak{g}'_{\rho(\tau)e} \subset \mathfrak{g}'_{\rho(\tau')+}$. Thus $\Theta_{\dot{\pi}'_\theta} \circ \mathfrak{e}'_1$ is $s_{\tau',h}$-asymptotic on $\mathfrak{g}'_j$. If $\mathcal{O}'' \in \Omega_{G''}(0)$ and $\mathcal{O}' = G' \cdot (s_{\tau',h} + \mathcal{O}'')$, then

$$c_{\mathcal{O}'}(\dot{\pi}'_\theta) = \upsilon_{G'}(B')^{-1}\upsilon_{G''}(B'')(\dim\tau')c_{\mathcal{O}''}(\iota'^*(\dot{\pi}'_\theta)). \tag{15.2}$$

In addition, arguing as in the case $E = L$, we can see that, for $X \in \mathfrak{g}'_j \cap \mathfrak{g}'_{\mathrm{reg}}$,

$$\Theta_{\dot{\pi}'}(\mathfrak{e}'_1(X)) = \psi(\mathrm{tr}'(s_h X))\Theta_{\dot{\pi}'_\theta}(\mathfrak{e}'_1(X)),$$
$$\hat{\mu}_{(s_h+\mathcal{O}')}(X) = \psi(\mathrm{tr}'(s_h X))\hat{\mu}_{\mathcal{O}'}(X), \quad \mathcal{O}' \in \Omega_{G'}(s_{\tau',h}).$$

Hence $\Theta_{\dot{\pi}'} \circ \mathfrak{e}'_1$ is $(s_h + s_{\tau',h})$-asymptotic on $\mathfrak{g}'_j$, with

$$c_{s_h+\mathcal{O}'}(\dot{\pi}') = c_{\mathcal{O}'}(\dot{\pi}'_\theta), \quad \mathcal{O}' \in \Omega_{G'}(s_{\tau',h}). \tag{15.3}$$

Recall from Proposition 9.2 that $\pi' = \eta^*(\pi) \simeq \dot{\pi}$. By Theorem 12.3 (2), applied with $s' = s_h - s + s_{\tau',h}$, $\Theta_\pi \circ \mathfrak{e}_1$ is $(s_h + s_{\tau',h})$-asymptotic on $\mathfrak{g}_{j/e}$ and, if $\mathcal{O}' \in \Omega_{G'}(s_h + s_{\tau',h})$,

$$\begin{aligned} c_{G\cdot\mathcal{O}'}(\pi) &= \upsilon_G(B_{(n)})^{-1}\upsilon_{G'}(B'_{(n')})u_n(q)u_{n'}(q^f)^{-1}q^{fn'((n-n')j-e+1)/2}c_{\mathcal{O}'}(\dot{\pi}') \\ &= \upsilon_G(B)^{-1}\upsilon_{G'}(B')u_{f_o}(q)^{n/f_o}u_{f_o/f}(q^f)^{-n'f/f_o}q^{fn'((n-n')j-e+1)/2}c_{\mathcal{O}'}(\dot{\pi}'). \end{aligned} \tag{15.4}$$

Note that the second equality follows from (3.1) and the fact that B is conjugate to $B_{(f_o)^{n/f_o}}$ and B' is conjugate to $B'_{(f_o/f)^{n'f_o/f}}$.

As shown in the proof of Theorem 13.2, Theorem 14.1 (1) holds with $s_{\tau,h} = s_h + s_{\tau',h}$. It follows from (15.1)–(15.4) and Lemma 13.10 that

$$\begin{aligned} &c_{G\cdot(s_{\tau,h}+\mathcal{O}'')}(\pi) \\ &\quad = \upsilon_G(B)^{-1}\upsilon_{G'}(B')u_{f_o}(q)^{n/f_o}u_{f_o/f}(q^f)^{-n'f/f_o}q^{fn'((n-n')j-e+1)/2}(\dim\tau')c_{\mathcal{O}''}(\iota^*(\pi)) \\ &\quad = \upsilon_G(B)^{-1}\upsilon_{G'}(B')(\dim\tau)c_{\mathcal{O}''}(\iota^*(\pi)), \quad \mathcal{O}'' \in \Omega_{G''}(0). \end{aligned}$$

□

16. Proofs of Corollary 14.3 and Theorem 14.4

Proof of Corollary 14.3. Let π and π'' be as in Theorem 14.1. Let notation be as in § 14 and Lemma 11.9 (with $s_0 = s_{\tau,h}$).

Let $\alpha \in \mathcal{P}^0(n)$. By Theorem 14.1 (2), the coefficient $c_{\mathcal{O}_\alpha}(\pi)$ of $\hat{\mu}_{\mathcal{O}_\alpha}$ in the 0-asymptotic expansion of $\Theta_\pi \circ \mathfrak{e}_1$ on $\mathfrak{g}_{\rho(\pi)+}$ is equal to the coefficient of $\hat{\mu}_{\mathcal{O}_\alpha}$ in the 0-asymptotic expansion of

$$\sum_{\gamma\in\mathcal{P}^0(a)} c_{G\cdot(s_{\tau,h}+\mathcal{O}_\gamma)}(\pi)\hat{\mu}_{G\cdot(s_{\tau,h}+\mathcal{O}_\gamma)}.$$

There exists a unique $\beta \in \mathcal{P}^0(a)$ such that $WF(\pi'') = \bar{\mathcal{O}}_\beta$. By Theorem 14.1 (3) and the definition of $WF(\pi'')$,

$$c_{\mathcal{O}_\alpha}(\pi) = \upsilon_G(B)^{-1}\upsilon_{G''}(B'')(\dim\tau)\sum_{\{\gamma\in\mathcal{P}^0(a)\mid\gamma\geqslant\beta\}} c_{\mathcal{O}_\gamma}(\pi'')c_\alpha(s_{\tau,h},\gamma),$$

https://doi.org/10.1017/S1474748003000124 Published online by Cambridge University Press

where $c_\alpha(s_{\tau,h},\gamma)$ is the coefficient of $\hat\mu_{\mathcal{O}_\alpha}$ in the 0-asymptotic expansion of $\hat\mu_{\mathcal{O}_G(s_{\tau,h}+\mathcal{O}_\gamma)}$. By Lemma 11.9 (1), $c_\alpha(s_{\tau,h},\gamma)\neq 0$ implies $\alpha\geqslant\dot\gamma^{n/a}$. Note that $\gamma\geqslant\beta$ implies $\dot\gamma^{n/a}\geqslant\dot\beta^{n/a}$. Hence

$$c_{\mathcal{O}_\alpha}(\pi)=\upsilon_G(B)^{-1}\upsilon_{G''}(B'')(\dim\tau)\sum_{\{\gamma\in\mathcal{P}^0(a)\mid\alpha\geqslant\dot\gamma^{n/a}\geqslant\dot\beta^{n/a}\}}c_{\mathcal{O}_\gamma}(\pi'')c_\alpha(s_{\tau,h},\gamma).$$

Thus $c_{\mathcal{O}_\alpha}(\pi)\neq 0$ implies $\alpha\geqslant\dot\beta^{n/a}$. To complete the proof, it suffices to show that $c_{\mathcal{O}_{\dot\beta^{n/a}}}(\pi)\neq 0$. By the above,

$$c_{\mathcal{O}_{\dot\beta^{n/a}}}(\pi)=\upsilon_G(B)^{-1}\upsilon_{G''}(B'')(\dim\tau)c_{\mathcal{O}_\beta}(\pi'')c_{\dot\beta^{n/a}}(s_{\tau,h},\beta).$$

As $WF(\pi'')=\bar{\mathcal{O}}_\beta$, $c_{\mathcal{O}_\beta}(\pi'')\neq 0$. By Lemma 11.9 (2), $c_{\dot\beta^{n/a}}(s_{\tau,h},\beta)>0$. □

Proof of Theorem 14.4. Part (1) is Theorem 5.4 of [**17**]. For (2), by Theorem 5.4 of [**17**], π'' is essentially square integrable and unipotent, hence is a twist of the Steinberg representation $St_{G''}$ of G''. Thus $c_{\mathcal{O}''}(\pi'')=c_{\mathcal{O}''}(St_{G''})$ for all $\mathcal{O}''\in\Omega_{G''}(0)$. The character of $St_{G''}$ is expressed in terms of characters of representations parabolically induced from one-dimensional representations of Levi subgroups (see [**5**]). Using Howe's result concerning $\hat\mu_{\mathcal{O}''_\alpha}$ (see above), it is straightforward to show that (if the measure on $\mathcal{O}''_\alpha$ is normalized as in § 14) $c_{\mathcal{O}''_\alpha}(St_{G''})=(-1)^{a-r(\alpha)}r(\alpha)!/|\mathrm{Stab}\,\alpha|$ for $\alpha\in\mathcal{P}^0(a)$. □

17. Proof of Theorem 14.5

Before proving the theorem, we need a result relating germs of characters and parabolic induction, as well as a refinement of Proposition 4.3.

Let P be a parabolic subgroup of G, with Levi component M and unipotent radical N. Let $\mathfrak{m}$ and $\mathfrak{n}$ be the Lie algebras of M and N, respectively. Normalize Haar measures on $K=B_{(n)}$ and $\mathfrak{n}$ so that $\upsilon_K(K)=v_{\mathfrak{n}}(\mathfrak{n}\cap\mathfrak{b}_{(n)})=1$. Given $f\in C_c^\infty(\mathfrak{g})$, let $f_P\in C_c^\infty(\mathfrak{m})$ be defined by

$$f_P(X)=\int_K\int_{\mathfrak{n}}f(\mathrm{Ad}\,k(X+Z))\,\mathrm{d}k\,\mathrm{d}Z,\quad X\in\mathfrak{m}.$$

If $s_0\in\mathfrak{m}_{\mathrm{ss}}$ is such that $C_M(s_0)=C_G(s_0)$, then the map $\mathcal{O}_M\mapsto G\cdot\mathcal{O}_M$ is a bijection from $\Omega_M(s_0)$ to $\Omega_G(s_0)$. If $Y\in\mathfrak{m}_{\mathrm{nil}}$ and $\mathcal{O}_M=\mathcal{O}_M(s_0+Y)\simeq M/C_M(s_0+Y)$, then left Haar measures on M on $C_M(s_0+Y)$ determine an M-invariant measure on $\mathcal{O}_M$, and we will take the G-invariant measure on $G\cdot\mathcal{O}_M\simeq G/C_M(s_0+Y)$ determined by the above Haar measure on $C_M(s_0+Y)$ and Haar measure on G.

Lemma 17.1. *Let P be as above. Suppose that $s_0\in\mathfrak{m}_{\mathrm{ss}}$ is such that $C_M(s_0)=C_G(s_0)$. Let $\mathcal{O}_M\in\Omega_M(s_0)$. Then*

$$\mu_{G\cdot\mathcal{O}_M}(f)=\upsilon_G(K)^{-1}\upsilon_M(K\cap M)|\det(\mathrm{ad}\,s_0)_{\mathfrak{n}}|^{-1}\mu_{\mathcal{O}_M}(f_P),\quad f\in C_c^\infty(\mathfrak{g}).$$

Proof. Note that $C_M(s_0)=C_G(s_0)$ guarantees that $(\mathrm{ad}\,s_0)_{\mathfrak{n}}$ is invertible. If $Y\in\mathfrak{m}_{\mathrm{nil}}$, then $(\mathrm{ad}\,s_0)_{\mathfrak{n}}$ and $(\mathrm{ad}\,Y)_{\mathfrak{n}}$ are the semisimple and nilpotent parts of $(\mathrm{ad}(s_0+Y))_{\mathfrak{n}}$. Hence

https://doi.org/10.1017/S1474748003000124 Published online by Cambridge University Press

$\det(\operatorname{ad} s_0)_{\mathfrak{n}} = \det(\operatorname{ad}(s_0+Y))_{\mathfrak{n}}$. If $\mathcal{O}_M = \mathcal{O}_M(s_0+Y)$, arguing as in [**11**, § 1.2], but without discriminants, results in

$$\mu_{G\cdot\mathcal{O}_M}(f) = \upsilon_G(K)^{-1}\upsilon_M(K\cap M)|\det(\operatorname{ad}(s_0+Y))_{\mathfrak{n}}|^{-1}\mu_{\mathcal{O}_M}(f_P), \quad f\in C_{\mathrm{c}}^{\infty}(\mathfrak{g}).$$

□

Lemma 17.2. *Let $\pi\in\mathcal{E}(G)$. Suppose that $\pi = \operatorname{Ind}_P^G \pi_M$ for some $\pi_M\in\mathcal{E}(M)$ and the germ of Θ_{π_M} is s_0-asymptotic on $\mathfrak{m}_{\rho(\pi_M)}\cap\mathfrak{m}_{0+}$ for some $s_0\in\mathfrak{m}_{\mathrm{ss}}$ such that $C_M(s_0) = C_G(s_0)$. Let $K = B_{(n)}$ and let $\mathfrak{n}$ be the Lie algebra of the unipotent radical of a parabolic subgroup with Levi component M. Let $c_{\mathcal{O}_M}(\pi_M)$, $\mathcal{O}_M\in\Omega_M(s_0)$, be the coefficients in some s_0-asymptotic expansion of the germ of Θ_{π_M}. Then there exists an s_0-asymptotic expansion of the germ of Θ_π on $\mathfrak{g}_{\rho(\pi)}\cap\mathfrak{g}_{0+}$ for which the coefficients $c_{G\cdot\mathcal{O}_M}(\pi)$ of the Fourier transforms $\hat\mu_{G\cdot\mathcal{O}_M}$ are given by*

$$c_{G\cdot\mathcal{O}_M}(\pi) = \upsilon_G(K)^{-1}\upsilon_M(K\cap M)|\det(\operatorname{ad} s_0)_{\mathfrak{n}}|^{-1}c_{\mathcal{O}_M}(\pi_M), \quad \mathcal{O}_M\in\Omega_M(s_0).$$

Proof. If $f\in C^{\infty}(\mathfrak{g}_{0+})$, define $\tilde f\in C^{\infty}(G_{0+})$ by $\tilde f(x) = f(x-1)$, $x\in G_{0+}$. Suppose that $f\in C_{\mathrm{c}}^{\infty}(\mathfrak{g})$ is supported on $\mathfrak{g}_{\rho(\pi)}\cap\mathfrak{g}_{0+}$. Then, if P is a parabolic subgroup with Levi component M and $f_P\in C_{\mathrm{c}}^{\infty}(\mathfrak{m})$ is as defined in § 11, f_P is supported on $\mathfrak{m}_{\rho(\pi)}\cap\mathfrak{m}_{0+}$ (see Remark 4.2.10 of [**1**]). By Theorem 5.2 of [**30**], $\rho(\pi_M) = \rho(\pi)$. Hence, if f is supported on $\mathfrak{g}_{\rho(\pi)}\cap\mathfrak{g}_{0+}$,

$$\begin{aligned}\Theta_{\pi_M}((f_P)\tilde{}) &= \sum_{\mathcal{O}_M\in\Omega_M(s_0)} c_{\mathcal{O}_M}(\pi_M)\hat\mu_{\mathcal{O}_M}(f_P)\\ &= \upsilon_G(K)^{-1}\upsilon_M(K\cap M)|\det(\operatorname{ad} s_0)_{\mathfrak{n}}|^{-1}\sum_{\mathcal{O}_M\in\Omega_G(s_0)} c_{\mathcal{O}_M}(\pi_M)\hat\mu_{G\cdot\mathcal{O}_M}(f).\end{aligned}$$

To obtain the second equality, we have used Lemma 17.1 and $(f_P)\hat{} = (\hat f)_P$ (see Lemma 1.7 of [**11**]). It follows from [**38**, § 5] that $\Theta_\pi(\tilde f) = \Theta_{\pi_M}((f_P)\tilde{})$ for $f\in C^{\infty}(\mathfrak{g}_{0+})$.

Hence the function

$$\upsilon_G(K)^{-1}\upsilon_M(K\cap M)|\det(\operatorname{ad} s_0)_{\mathfrak{n}}|^{-1}\sum_{\mathcal{O}_M\in\Omega_M(s_0)} c_{\mathcal{O}_M}(\pi_M)\hat\mu_{G\cdot\mathcal{O}_M}$$

and the germ of Θ_π agree on $\mathfrak{g}_{\rho(\pi)}\cap\mathfrak{g}_{0+}\cap\mathfrak{g}_{\mathrm{reg}}$. □

If M is the Levi component of a parabolic subgroup P of G, there exists a partition $(n_1,\dots,n_r)$ of n such that $M\simeq\prod_{i=1}^r GL_{n_i}(F)$. Set $G^{(i)} = GL_{n_i}(F)$. Let $\pi_M = \bigotimes_{i=1}^r \pi_M^{(i)}$, where $\pi_M^{(i)}\in\mathcal{E}(G^{(i)})$, $1\leqslant i\leqslant r$. Suppose that $\pi_M^{(i)}$ contains a pure minimal K-type. If $\rho(\pi_M^{(i)})>0$, then, by Lemma 6.1, $\pi_M^{(i)}$ contains a pure minimal K-type $(B^{(i)}_{n_i'j_i},\chi_{s_i})$, where s_i is a semisimple element in $\mathfrak{g}^{(i)}$ that generates an extension E_i of F, $s_i\in\mathfrak{p}_{E_i}^{-j_i}-\mathfrak{p}_{E_i}^{-j_i+1}$, $n_i' = n_i/[E_i:F]$, $f_i = f(E_i/F)$, and $B_m^{(i)} = B^{(i)}_{(f_i)^{n_i/f_i},m}$, $m\geqslant 0$, where $B^{(i)}_{(f_i)^{n_i/f_i}}$ is the parahoric subgroup of $G^{(i)}$ corresponding to the partition $(f_i)^{n_i/f_i}$ of n_i, as defined in § 3. The parahoric filtration that appears in the statement of Lemma 6.1 is actually a conjugate of $B_m^{(i)}$, but we are replacing it by $B_m^{(i)}$ as it is convenient for the proof of the next result.

If $\rho(\pi_M^{(i)}) = 0$, then (see § 10) $\pi_M^{(i)}$ contains

$$\Big(B^{(i)}_{(d_i)^{n_i'}}, \sigma^{(i)}\Big),$$

where d_i is a positive divisor of n_i, $n_i' = n_i/d_i$, and $\sigma^{(i)}$ is the inflation to $B^{(i)}_{(d_i)^{n_i'}}$ of the n_i'-fold tensor product of an irreducible cuspidal representation of $GL_{d_i}(\mathbb{F}_q)$. In this case, define s_i relative to $\sigma^{(i)}$ as s was defined relative to the representation σ of § 10. Let $E_i = F(s_i)$.

Proposition 17.3. *Let $\pi \in \mathcal{E}(G)$. Then there exists a parabolic subgroup $P = MN$ of G and $\pi_M \in \mathcal{E}(M)$ such that, with notation as above, the following hold.*

(1) $\pi = \operatorname{Ind}_P^G \pi_M$.

(2) *Each $\pi_M^{(i)}$, $1 \leqslant i \leqslant r$, contains a pure minimal K-type.*

(3) *If $s = (s_1, \dots, s_r)$, where $s_i \in E_i$ is as above, then $C_M(s) = C_G(s)$.*

Proof. After repeated applications of Proposition 4.3 and transitivity of induction, we may assume that there exists $P = MN$ and $\pi_M \in \mathcal{E}(M)$ such that (1) and (2) are satisfied. As there is nothing to show otherwise, we assume that $P \neq G$.

First, consider the case where $\rho(\pi_M^{(i)}) > 0$ for $1 \leqslant i \leqslant r$. Suppose that $C_M(s) \neq C_G(s)$. After renumbering the n_i if necessary, we can assume that

$$C_{GL_{n_1+n_2}(F)}(s_1, s_2) \supsetneq C_{G^{(1)}}(s_1) \times C_{G^{(2)}}(s_2) \simeq GL_{n_1'}(E_1) \times GL_{n_2'}(E_2).$$

It follows that $E_1 \simeq E_2$, $j_1 = j_2$, and $C_{GL_{n_1+n_2}(F)}(s_1, s_2) \simeq GL_{n_1'+n_2'}(E_1)$. After replacing s_2 by a conjugate, we may (and will) assume that $E_1 = E_2$.

Let $B_m^* = B_{(f_1)^{(n_1+n_2)/f_1}, m}$, $m \geqslant 0$, be the parahoric filtration associated to the partition $(f_1)^{(n_1+n_2)/f_1}$. Set $s^* = (s_1, s_2)$ in $\mathfrak{g}^* = \mathfrak{gl}_{n_1+n_2}(F)$. Then $(B^*_{(n_1'+n_2')j_1}, \chi_{s^*})$ is a pure minimal K-type. Let $P^* = M^*N^*$ be an upper triangular parabolic subgroup of $G^* = GL_{n_1+n_2}(F)$ with standard Levi component $M^* = GL_{n_1}(F) \times GL_{n_2}(F)$. If m is a positive integer and N^{*-} is the unipotent radical of the parabolic subgroup opposite to P^*, then

$$B_m^* = (B_m^* \cap N^{*-})(B_m^* \cap M^*)(B_m^* \cap N^*).$$

Furthermore, $B^*_{(n_1'+n_2')j_1} \cap M^* = B^{(1)}_{n_1'j_1} \times B^{(2)}_{n_2'j_1}$. Observe that the character χ_{s^*} is equal to $\chi_{s_1} \otimes \chi_{s_2}$ extended trivially across $B^*_{(n_1'+n_2')j_1} \cap N^{*-}$ and $B^*_{(n_1'+n_2')j_1} \cap N^*$. Hence, since $(B^{(1)}_{n_1'j_1} \times B^{(2)}_{n_2'j_1}, \chi_{s_1} \otimes \chi_{s_2})$ is an unrefined minimal K-type contained in $\pi_M^{(1)} \otimes \pi_M^{(2)}$, it follows from Theorem 4.5 of [**30**] that $(B^*_{(n_1'+n_2')j_1}, \chi_{s^*})$ is contained in $\pi_M^* = \operatorname{Ind}_{P^*}^{G^*}(\pi_M^{(1)} \otimes \pi_M^{(2)})$. Note that G^*P is a parabolic subgroup of G with Levi component $G^* \times \prod_{i=3}^r G^{(i)}$. By transitivity of induction, $\pi \simeq \operatorname{Ind}_{G^*P}^G(\pi_M^* \otimes \bigotimes_{i=3}^r \pi_M^i)$, so

https://doi.org/10.1017/S1474748003000124 Published online by Cambridge University Press

we may replace M by $G^* \times \prod_{i=3}^r G^{(i)}$ and π_M by $\pi_M^* \otimes \bigotimes_{i=3}^r \pi_M^{(i)}$. Continuing in this manner, after a finite number of steps, part (3) is satisfied.

Next, we turn to the case where $\rho(\pi_M^{(i)}) = 0$ for $1 \leqslant i \leqslant r$. Recall that $\sigma^{(i)}$ is the inflation of the n_i'-fold tensor product of $\sigma_0^{(i)}$, where $\sigma_0^{(i)}$ is an irreducible cuspidal representation of $GL_{d_i}(\mathbb{F}_q)$, $1 \leqslant i \leqslant r$.

Suppose that $r \geqslant 2$, $d_1 = d_2$, and $\sigma_0^{(1)} \simeq \sigma_0^{(2)}$. As shown in § 10, s_i can be any element of $\mathfrak{o}_{E_i}^\times$ whose image $\bar{s}_i$ in $\mathfrak{o}_{E_i}/\mathfrak{p}_{E_i} \simeq \mathbb{F}_{q^{d_i}}$ generates $\mathbb{F}_{q^{d_i}}$ over $\mathbb{F}_q$. So we can (and will) take $s_1 = s_2$. Set $d = d_1$. Let G^*, P^* and M^* be as above. It is easy to see from the definitions of the parahorics that

$$B_{(d)^{n_1'+n_2'}} \cap M^* = B_{(d)^{n_1'}} \times B_{(d)^{n_2'}},$$
$$(B_{(d)^{n_1'}} \times B_{(d)^{n_2'}})/(B_{(d)^{n_1'},1} \times B_{(d)^{n_2'},1}) \simeq B_{(d)^{n_1'+n_2'}}/B_{(d)^{n_1'+n_2'},1}.$$

Let σ^* be the lift of $\sigma^{(1)} \otimes \sigma^{(2)}$ to $B_{(d)^{n_1'+n_2'}}$. As $\sigma_0^{(1)} \simeq \sigma_0^{(2)}$, $(B_{(d)^{n_1'+n_2'}}, \sigma^*)$ is a pure minimal K-type. By Theorem 5.2 (2) of [**30**], $(B_{(d)^{n_1'+n_2'}}, \sigma^*)$ is contained in $\pi_M^* = \operatorname{Ind}_{P^*}^{G^*}(\pi_M^{(1)} \otimes \pi_M^{(2)})$. Arguing as in the positive depth case using transitivity of induction, after a finite number of steps we are reduced to considering the case where $\sigma_0^{(i)} \not\simeq \sigma_0^{(j)}$ whenever $d_i = d_j$ and $i \neq j$.

Let d be a positive integer. There is a bijection between the set of equivalence classes of irreducible cuspidal representations of $GL_d(\mathbb{F}_q)$ and the set of $\mathrm{Gal}(\mathbb{F}_{q^d}/\mathbb{F}_q)$-orbits of characters of $\mathbb{F}_{q^d}^\times$ that are not fixed by any non-trivial element of $\mathrm{Gal}(\mathbb{F}_{q^d}/\mathbb{F}_q)$. This second set is in bijection with the $\mathrm{Gal}(\mathbb{F}_{q^d}/\mathbb{F}_q)$-orbits of elements of $\mathbb{F}_{q^d}$ that generate $\mathbb{F}_{q^d}$ over $\mathbb{F}_q$. Hence the number of distinct possible choices for $\sigma_0^{(i)}$, up to equivalence, is equal to the number of distinct possible choices for $\bar{s}_i$, up to $\mathrm{Gal}(\mathbb{F}_{q^d}/\mathbb{F}_q)$-conjugacy. Thus, assuming that $\sigma_0^{(i)} \not\simeq \sigma_0^{(j)}$ whenever $d_i = d_j$ and $i \neq j$, we can (and will) arrange for the s_i to be chosen so that whenever $d_i = d_j$, and $i \neq j$, $\bar{s}_i$ and $\bar{s}_j$ belong to distinct $\mathrm{Gal}(\mathbb{F}_{q^{d_i}}/\mathbb{F}_q)$ orbits. This guarantees that $C_M(s) = C_G(s)$.

The general case can be dealt with by writing $\pi_M = \pi_M^0 \otimes \pi_M^+$, where π_M^0 is the tensor product of those $\pi_M^{(i)}$ that have depth zero, and π_M^+ is the tensor product of the $\pi_M^{(i)}$ that have positive depth, and treating π_M^0 and π_M^+ as above. □

Proof of Theorem 14.5. Let $\pi \in \mathcal{E}(G)$. Let $P = MN$ and $\pi_M = \bigotimes_{i=1}^r \pi_M^{(i)}$, $\pi_M^{(i)} \in \mathcal{E}(G^{(i)})$, $s_i \in E_i$, $1 \leqslant i \leqslant r$, be as in Proposition 17.3. Let $G^{(i)\prime} = C_{G^{(i)}}(s_i)$. Suppose that the following holds for $1 \leqslant i \leqslant r$.

$$\text{There exists } s_i' \in \mathfrak{g}^{(i)\prime} \cap \mathfrak{g}^{(i)}_{-\rho(\pi_M^{(i)})} \text{ such that the theorem holds for } \pi_M^{(i)}, \text{ with } s_{\pi_M^{(i)}} = s_i + s_i'. \quad (17.1)$$

https://doi.org/10.1017/S1474748003000124 Published online by Cambridge University Press

Let $\pi_u^{(i)}$ be the corresponding irreducible unipotent representation of $H^{(i)} = C_{G^{(i)}}(s_i + s_i')$. Set

$$s^{(i)} = s_i + s_i', \quad s_\pi = (s^{(1)}, \dots, s^{(r)}), \quad H = \prod_{i=1}^{r} H^{(i)} \quad \text{and} \quad \pi_u = \bigotimes_{i=1}^{r} \pi_u^{(i)}.$$

Note that $s_i' \subset \mathfrak{g}^{(i)\prime}_{(-j_i)+} = G^{(i)\prime} \cdot \mathfrak{b}_{(1)^{n_i'}, -n_i' j_i + 1}$, so $C_{G^{(i)}}(s^{(i)}) = C_{G^{(i)\prime}}(s_i')$ by Lemma 5.5. This combines with $C_M(s) = C_G(s)$ $(s = (s_1, \dots, s_r))$ to give $C_M(s_\pi) = C_G(s_\pi)$. Since $\rho(\pi_M)$ is the maximum of the $\rho(\pi_M^{(i)})$, and $\rho(\pi) = \rho(\pi_M)$ (see Theorem 5.2 of [**30**]), $s_\pi \in \mathfrak{g}_{-\rho(\pi)}$. Assuming that (17.1) holds for each i, the theorem now follows from Lemma 17.2.

It remains to show (17.1). Without loss of generality, we may assume that $M = G$. First, if $\rho(\pi) = 0$, equation (17.1) holds with $s' = 0$, by Theorem 14.1. Next, assume that $\rho(\pi) > 0$. By induction, we may assume that the theorem holds for $\pi^* \in \mathcal{E}(G^*)$, where $G^* \simeq GL_m(E^*)$, $m \leqslant n$, E^*/F is a finite extension, and $\rho(\pi^*)/e(E^*/F) < \rho(\pi)$. By Lemma 6.1, π contains a pure minimal K-type $(Q_{n'j}, \chi_s)$. Let η be the associated Hecke algebra isomorphism (see § 7). Set $G' = C_G(s)$, $\pi' = \eta^*(\pi)$ and $\pi_\theta' = (\theta^{-1} \circ \det')\pi'$, where θ is as in § 13. By the same argument used in the proof of Theorem 14.1, $\rho(\pi_\theta') < \rho(\pi')$. As $\rho(\pi') = \rho(\pi)/e(F(s)/F)$ (see Lemma 4.2 (2)), by induction, the theorem holds for π_θ'. Let $H = C_{G'}(s_{\pi_\theta'})$ and $s_\pi = s + s_{\pi_\theta'}$. By Lemma 5.5, as $s_{\pi_\theta'} \in \mathfrak{g}'_{(-j)+}$, $C_G(s_\pi) = C_{G'}(s_{\pi_\theta'}) = H$. An application of Theorem 12.3 (2) yields (17.1). □

Acknowledgements. We thank J. D. Adler, S. DeBacker and A. Moy for many helpful conversations during the course of this work.

This research was supported in part by the NSERC.

References

1. J. D. Adler and S. DeBacker, Some applications of Bruhat–Tits theory to harmonic analysis on the Lie algebra of a reductive p-adic group, *Michigan Math. J.* **50** (2002), 263–286.
2. J. D. Adler and S. DeBacker, Murnaghan–Kirillov theory for supercuspidal representations of tame general linear groups, preprint.
3. C. J. Bushnell, Hereditary orders, Gauss sums and supercuspidal representations of $GL(N)$, *J. Reine Angew. Math.* **375** (1987), 184–210.
4. C. J. Bushnell and P. C. Kutzko, *The admissible dual of $GL(N)$ via compact open subgroups*, Annals of Mathematical Studies, vol. 129 (Princeton University Press, Princeton, NJ, 1993).
5. W. Casselman, *The Steinberg character as a true character*, Proceedings of Symposia of Pure Mathematics, vol. 26, pp. 414–417 (American Mathematical Society, Providence, RI, 1973).
6. L. Clozel, Characters of non-connected, reductive p-adic groups, *Can. J. Math.* **39** (1987), 149–167.
7. L. Corwin, A. Moy and P. Sally, Degrees and formal degrees for division algebras and GL_n over a p-adic field, *Pac. J. Math.* **141** (1990), 21–45.
8. S. DeBacker, On supercuspidal characters of GL_ℓ, ℓ a prime, PhD thesis, University of Chicago (1997).

9. S. DeBacker, Homogeneity results for invariant distributions of a reductive p-adic group, *Annls Sci. Ec. Norm. Super.* **35** (2002), 391–422.
10. F. Digne and J. Michel, *Representations of finite groups of Lie type*, London Mathematical Society Student Texts, vol. 21 (Cambridge University Press, 1991).
11. Harish-Chandra, *Admissible distributions on reductive p-adic groups*, Preface and notes by Stephen DeBacker and Paul J. Sally Jr, University Lecture Series, vol. 16 (American Mathematical Society, Providence, RI, 1999).
12. R. Howe, On the character of Weil's representation, *Trans. Am. Math. Soc.* **117** (1973), 287–298.
13. R. Howe, The Fourier transform and germs of characters (case of $\boldsymbol{GL}_n$ over a p-adic field), *Math. Ann.* **208** (1974), 305–322.
14. R. Howe, Tamely ramified supercuspidal representations of GL_n, *Pac. J. Math.* **73** (1977), 437–460.
15. R. Howe and A. Moy, *Harish-Chandra homomorphisms for p-adic groups*, CBMS Regional Conf. Series, no. 59 (American Mathematical Society, Providence, RI, 1985).
16. R. Howe and A. Moy, Minimal K-types for GL_n over a p-adic field, *Astérisque* **171** (1989), 257–271.
17. R. Howe and A. Moy, Hecke algebra isomorphisms for $GL(n)$ over a p-adic field, *J. Alg.* **131** (1990), 388–424.
18. D. Kazhdan, Proof of Springer's hypothesis, *Isr. J. Math.* **28** (1977), 272–286.
19. J.-L. Kim, Hecke algebras of classical groups over p-adic fields and supercuspidal representations, *Am. J. Math.* **121** (1999), 967–1029.
20. J.-L. Kim, Hecke algebras of classical groups over p-adic fields, II, *Comput. Math.*, in press.
21. J.-L. Kim, Hecke algebras of classical groups over p-adic fields, III, When ρ_Σ is a Heisenberg representation, preprint.
22. J.-L. Kim and F. Murnaghan, Character expansions and unrefined minimal K-types (2002), preprint.
23. G. Lusztig, On the finiteness of the number of unipotent classes, *Invent. Math.* **34** (1976), 201–213.
24. G. Lusztig and B. Srinivasan, The characters of the finite unitary groups, *J. Alg.* **49** (1977), 167–171.
25. C. Moeglin and J.-L. Waldspurger, Modèles de Whittaker dégénérés pour des groupes p-adiques, *Math. Z.* **196** (1987), 427–452.
26. A. Moy, Representations of $GSp(4)$ over a p-adic field, Part 1, *Comput. Math.* **66** (1988), 237–284.
27. A. Moy, Representations of $GSp(F)$ over a p-adic field, Part 2, *Comput. Math.* **66** (1988), 285–328.
28. A. Moy, A conjecture on minimal K-types for GL_n over a p-adic field, representation theory and number theory in connection with the local Langlands conjecture, *Contemp. Math.* **86** (1989), 249–254.
29. A. Moy and G. Prasad, Unrefined minimal K-types for p-adic groups, *Invent. Math.* **116** (1994), 393–408.
30. A. Moy and G. Prasad, Jacquet functors and unrefined minimal K-types, *Comment. Math. Helv.* **71** (1996), 98–121.
31. F. Murnaghan, Invariant meromorphic distributions on p-adic GL_n, *Am. J. Math.* **111** (1989), 143–196.
32. F. Murnaghan, Characters of supercuspidal representations of $SL(n)$, *Pac. J. Math.* **170** (1995), 217–235.
33. F. Murnaghan, Local character expansions and Shalika germs for $\boldsymbol{GL}(n)$, *Math. Ann.* **304** (1996), 423–455.

34. F. Murnaghan, Characters of supercuspidal representations of classical groups, *Annls Sci. Ec. Norm. Super.* **29** (1996), 49–105.
35. F. Murnaghan, Germs of characters of admissible representations, in *The mathematical legacy of Harish-Chandra: a celebration of representation theory and harmonic analysis*, Proceedings of Symposia of Pure Mathematics, vol. 68, pp. 501–515 (American Mathematical Society, Providence, RI, 2000).
36. F. Murnaghan, Local character expansions of admissible representations of p-adic general linear groups, *J. Reine Angew. Math.* **554** (2003), 139–155.
37. R. Steinberg, A geometric approach to the representations of the full general linear group over a Galois field, *Trans. Am. Math. Soc.* **71** (1951), 274–282.
38. G. van Dijk, Computation of certain induced characters of p-adic groups, *Math. Ann.* **199** (1972), 229–240.
39. J.-L. Waldspurger, Algèbres de Hecke et induites de représentations cuspidales pour $GL(N)$, *J. Reine Angew. Math.* **370** (1986), 127–191.
40. J.-L. Waldspurger, Sur les germes de Shalika pour les groupes linéaires, *Math. Ann.* **284** (1989), 199–221.
41. J.-L. Waldspurger, Sur les intégrales orbitales tordues pour les groupes linéaires: un lemme fondamental, *Can. J. Math.* **43** (1991), 852–896.
42. J.-L. Waldspurger, Homogénéité de certaines distributions sur les groupes p-adiques, *Publ. Math. IHES* **81** (1995), 25–72.
43. A. V. Zelevinski, Induced representations of reductive p-adic groups, II, *Annls Sci. Ec. Norm. Super.* **13** (1980), 165–210.