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ON LONG-RANGE DEPENDENCE OF RANDOM MEASURES

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Abstract

This paper deals with long-range dependence of random measures on $\mathbb{R}^d$. By examples, it is demonstrated that one must be careful in order to define it consistently. Therefore, we define long-range dependence by a rather specific second-order condition and provide an equivalent formulation involving the asymptotic behaviour of the Bartlett spectrum near the origin. Then it is shown that the defining condition may be formulated less strictly when the additional isotropy assumption holds. Finally, we present an example of a long-range dependent random measure based on the 0-level excursion set of a Gaussian random field for which the corresponding spectral density and its asymptotics are explicitly derived.

Keywords: Long-range dependence; random measure; Bartlett spectrum

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1. Introduction

Long-range dependence of one-dimensional stochastic processes is an important concept observed in various physical, biological, geological, economic, and social systems; see, e.g. [2], [15], and references therein. One possible formulation for a second-order stationary stochastic process $\{X_n \mid n = 1, 2, \dots\}$ is based on the comparison of the variance of the sample mean $\bar{X}_n$ with $1/n$ as $n \to \infty$. Here, X_n is called long-range dependent if

$$\limsup_{n\to\infty} \frac{\operatorname{var} \bar{X}_n}{1/n} = \limsup_{n\to\infty} \frac{\operatorname{var}(\sum_{k=1}^{n} X_k)}{n} = \infty,$$

i.e. if the decrease of the variance is slower than $1/n$. This condition is strongly connected with the summability of the covariance function and also with the asymptotic behaviour of the spectral density near the origin.

Our aim in this paper is to deal with an extension of this concept to random measures on the d-dimensional Euclidean space $\mathbb{R}^d$. When $d = 1$, long-range dependence of a second-order stationary random measure ζ on $\mathbb{R}$ was defined in [7] and [8] by the condition

$$\limsup_{t\to\infty} \operatorname{var} \zeta(0, t]/t = \infty,$$

which is the straightforward generalization of the previous condition for one-dimensional stochastic processes. Again, there is a connection to the covariance and the spectrum of ζ as was shown in [7, Lemma 12.7.III].

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A possible extension of this definition to $\mathbb{R}^d$ was proposed in [7] by requiring a second-order stationary random measure ζ to satisfy

$$\limsup_{n\to\infty} \frac{\operatorname{var}\zeta(K_n)}{\nu_d(K_n)} = \infty \tag{1.1}$$

for some convex averaging sequence $(K_n)_{n\in\mathbb{N}}$, where ν_d is the d-dimensional Lebesgue measure on $\mathbb{R}^d$. Recall that a convex averaging sequence $(K_n)_{n\in\mathbb{N}}$ is a sequence of convex compact sets, nondecreasing in the sense of inclusion, such that $r(K_n) = \sup\{r \geq 0 \mid B_r(\boldsymbol{x}) \subset K_n \text{ for some } \boldsymbol{x}\} \to \infty$ as $n \to \infty$, where $B_r(\boldsymbol{x})$ is the closed ball with radius r centred at $\boldsymbol{x}$. However, as is shown in Example 3.1, this definition is inconsistent for $d \geq 2$, since the value of the limit superior may in general depend on a particular choice of the convex averaging sequence. This situation may occur when the correlation structure of ζ is directionally dependent. In [8], the criterion (1.1) was used requiring its validity for every convex averaging sequence which yields a consistent definition. However, such a condition is difficult to verify and there are no known equivalent formulations in terms of spectral properties.

In order to avoid those complications, we define long-range dependence (Definition 3.1) more strictly by

$$\limsup_{a\to\infty} \frac{\operatorname{var}\zeta(aB^d)}{\nu_d(aB^d)} = \infty, \tag{1.2}$$

where B^d is the closed unit ball. It should be noted that such a formulation was already used in [13] in connection to random closed sets, where germ–grain models with power-law grain sizes were analysed. For this definition we derive an equivalent condition (Theorem 3.1) formulated as an unboundedness of the Bartlett spectrum of ζ in the vicinity of the origin and its several consequences. After that, we deal with a question as to whether the closed ball B^d in (1.2) might be equivalently replaced by a different convex set, that is, whether one might equivalently require that

$$\limsup_{a\to\infty} \frac{\operatorname{var}\zeta(aK)}{\nu_d(aK)} = \infty \tag{1.3}$$

for some arbitrary bounded convex set $K \subset \mathbb{R}^d$ with nonempty interior. However, as is shown in Example 3.2, for $d \geq 3$ this is generally not possible and one can only show (Proposition 3.2) that long-range dependence of ζ implies (1.3) for every bounded convex set K with nonempty interior.

Then, we focus on random measures with rotation invariant reduced covariance measure and show in Proposition 3.3 that (1.3) is equivalent to long-range dependence of ζ independently of K as long as it is a bounded convex set with nonempty interior. It would be also desirable to know whether the rotation invariance is sufficient for the independence of (1.1) on the choice of the convex averaging sequence (K_n), but we have not been able to solve this fully. Hence, we only show in Proposition 3.4 that ζ is long-range dependent when (1.1) holds for some convex averaging sequence.

Finally, in Section 4 important examples having both positive and negative values of the exponent of power-law behaviour of the spectral density near the origin are introduced. We especially emphasize a random measure based on the 0-level excursion set of a Gaussian random field with Cauchy covariance function in Example 4.3 for which the spectral density is explicitly derived.

2. Random measures and the Bartlett spectrum

Here we briefly recapitulate relevant material from stochastic geometry. For more details, we refer the reader to [6], [7], and [17]. Recall that a random measure on $\mathbb{R}^d$ is a measurable mapping from some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ to a measurable space $(\mathsf{M}, \mathcal{M})$ of all locally finite Borel measures on $\mathbb{R}^d$, where $\mathcal{M}$ is the smallest σ-algebra for which all mappings $\varphi \in \mathsf{M} \mapsto \varphi(A)$, $A \subset \mathbb{R}^d$ is Borel, are measurable. A random measure ζ is second-order stationary if both its intensity measure $\Lambda = \mathbb{E}\,\zeta$ and second-order moment measure $\Lambda_2 = \mathbb{E}\,\zeta^{(2)} = \mathbb{E}(\zeta \times \zeta)$ are locally finite measures satisfying $\Lambda(A+\boldsymbol{x}) = \Lambda(A)$ and $\Lambda_2((A+\boldsymbol{x}) \times (B+\boldsymbol{x})) = \Lambda_2(A \times B)$ for all Borel $A, B \subset \mathbb{R}^d$ and all $\boldsymbol{x} \in \mathbb{R}^d$. In that case, the intensity measure Λ is absolutely continuous with respect to the Lebesgue measure ν_d on $\mathbb{R}^d$ with constant density λ called the intensity of ζ.

The covariance measure C_2 is defined by $C_2(A \times B) = \Lambda_2(A \times B) - \Lambda(A)\Lambda(B) = \operatorname{cov}(\zeta(A), \zeta(B))$ for all Borel $A, B \subset \mathbb{R}^d$, whenever both Λ and Λ_2 are locally finite. The covariance measure C_2 of a second-order stationary random measure ζ is expressible as the product of ν_d along the diagonal and the reduced covariance measure $\breve{C}_2$, see, e.g. [7, Proposition 12.6.III], i.e. in the integral form

$$\int_{\mathbb{R}^{2d}} f(\boldsymbol{x}_1, \boldsymbol{x}_2) C_2(\mathrm{d}\boldsymbol{x}_1 \times \mathrm{d}\boldsymbol{x}_2) = \int_{\mathbb{R}^d} \mathrm{d}\boldsymbol{x} \int_{\mathbb{R}^d} f(\boldsymbol{x}, \boldsymbol{x} + \boldsymbol{y}) \breve{C}_2(\mathrm{d}\boldsymbol{y}) \tag{2.1}$$

for any bounded measurable function f on $\mathbb{R}^{2d}$ of bounded support. The reduced covariance measure $\breve{C}_2$ on $\mathbb{R}^d$ (and also C_2 on $\mathbb{R}^{2d}$) is a signed measure in the sense of distributions rather than an ordinary Borel measure. This means that it is a continuous linear functional on the space $C_c(\mathbb{R}^d)$ of continuous compactly supported complex-valued functions and that it has real values on real functions from $C_c(\mathbb{R}^d)$. The upper variation $\breve{C}_2^+$ and lower variation $\breve{C}_2^-$ of $\breve{C}_2$ are (ordinary) Borel measures defined by $\breve{C}_2^+ = (|\breve{C}_2| + \breve{C}_2)/2$ and $\breve{C}_2^- = (|\breve{C}_2| - \breve{C}_2)/2$, respectively, where $|\breve{C}_2|$ is the total variation of $\breve{C}_2$ defined by $\breve{C}_2(f) = \sup_{|g| \le f,\, g \in C_c(\mathbb{R}^d)} |\breve{C}_2(g)|$ for every $f \in C_c(\mathbb{R}^d)$. It might be shown (e.g. [6, Proposition 8.1.II]) that $\breve{C}_2$ is symmetric, translation bounded, and positive semidefinite, i.e. $\breve{C}_2(A) = \breve{C}_2(-A)$ for all Borel $A \subset \mathbb{R}^d$, for each bounded Borel $B \subset \mathbb{R}^d$ there exists $K_B < \infty$ such that $|\breve{C}_2(\boldsymbol{x} + B)| \le K_B$ for all $\boldsymbol{x} \in \mathbb{R}^d$, and, for every $f \in C_c(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} (f * f^*)(\boldsymbol{x}) \breve{C}_2(\mathrm{d}\boldsymbol{x}) \ge 0,$$

where $f * f^*$ is the convolution of f and the involution f^* of f defined by $f^*(\boldsymbol{x}) = \overline{f(-\boldsymbol{x})}$ for every $\boldsymbol{x} \in \mathbb{R}^d$.

The Bartlett spectrum Γ of a second-order stationary random measure ξ is defined as the Fourier transform of $\breve{C}_2$, i.e. it is a positive Borel measure satisfying

$$\int_{\mathbb{R}^d} (f * f^*)(\boldsymbol{x}) \breve{C}_2(\mathrm{d}\boldsymbol{x}) = (2\pi)^{d/2} \int_{\mathbb{R}^d} \left| \check{f}(\boldsymbol{\xi}) \right|^2 \Gamma(\mathrm{d}\boldsymbol{\xi}) \tag{2.2}$$

for every bounded Borel f with bounded support (e.g. [6, Proposition 8.2.I]), where $\check{f}$ is the inverse Fourier transform (in the unitary sense) defined by

$$\check{f}(\boldsymbol{\xi}) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}\boldsymbol{x}\boldsymbol{\xi}} f(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}.$$

Note that the Fourier transform exists for every positive semidefinite measure on $\mathbb{R}^d$ and is determined uniquely by condition (2.2) for all $f \in C_c(\mathbb{R}^d)$, see [3, Theorem 4.7]. If $\breve{C}_2$ is

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absolutely continuous with continuous density $\operatorname{cov}(\boldsymbol{x})$ then cov is called the covariance function and it is a positive semidefinite function. Moreover, in this case, by Bochner's theorem (e.g. [14, Theorem IX.9]), Γ is bounded and

$$\operatorname{cov}(\boldsymbol{x}) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{i\boldsymbol{\xi}\boldsymbol{x}} \Gamma(d\boldsymbol{\xi}). \tag{2.3}$$

3. Long-range dependence of random measures

Let ζ be a second-order stationary random measure. Using the definition of $\breve{C}_2$ and (2.2), one may easily show that

$$\frac{\operatorname{var} \zeta(A)}{\nu_d(A)} = \int_{\mathbb{R}^d} \frac{(\mathbf{1}_A * \mathbf{1}_A^*)(\boldsymbol{x})}{\nu_d(A)} \breve{C}_2(d\boldsymbol{x}) = (2\pi)^{d/2} \int_{\mathbb{R}^d} \frac{|\check{\mathbf{1}}_A(\boldsymbol{\xi})|^2}{\nu_d(A)} \Gamma(d\boldsymbol{\xi}) \tag{3.1}$$

for every Borel $A \subset \mathbb{R}^d$ such that $0 < \nu_d(A) < \infty$, where $\mathbf{1}_A$ is the indicator function of A. This relation is the key ingredient of our further analysis.

We begin with an assertion that allows one to combine different covariance behaviours of random measures in different directions.

Lemma 3.1. *Let k, m be positive integers. Let ζ_1 and ζ_2 be two independent second-order stationary random measures on $\mathbb{R}^k$ and $\mathbb{R}^m$, respectively, with intensities $\lambda_1, \lambda_2 \geq \frac{1}{2}$, such that $0 \leq \zeta_1(A) \leq \nu_k(A)$ almost surely for every Borel $A \subset \mathbb{R}^k$ and $0 \leq \zeta_2(B) \leq \nu_m(B)$ almost surely for every Borel $B \subset \mathbb{R}^m$, where ν_k and ν_m are Lebesgue measures on $\mathbb{R}^k$ and $\mathbb{R}^m$, respectively. Then ζ defined by*

$$\zeta(A \times B) = \zeta_1(A)\zeta_2(B) - \zeta_1(A)(\mathbb{E}\,\zeta_2)(B) - (\mathbb{E}\,\zeta_1)(A)\zeta_2(B) + 2(\mathbb{E}\,\zeta_1)(A)(\mathbb{E}\,\zeta_2)(B)$$

for every bounded Borel $A \subset \mathbb{R}^k$, $B \subset \mathbb{R}^m$ is a second-order stationary random measure on $\mathbb{R}^{k+m}$ and its reduced covariance measure is given by $\breve{C}_2 = \breve{C}_{2;1} \times \breve{C}_{2;2}$, where $\breve{C}_{2;1}$ and $\breve{C}_{2;2}$ are covariance measures of ζ_1 and ζ_2, respectively. Moreover, $0 \leq \zeta(C) \leq 2\lambda_1\lambda_2\nu_{k+m}(C)$ almost surely for every Borel $C \subset \mathbb{R}^{k+m}$.

Proof. Let $A \subset \mathbb{R}^k$, $B \subset \mathbb{R}^m$ be bounded Borel sets. If $\nu_k(A) = 0$ or $\nu_m(B) = 0$ then $\zeta(A \times B) = 0$. Thus, assuming that $\nu_k(A) > 0$ and $\nu_m(B) > 0$, we obtain

$$\frac{\zeta(A \times B)}{\nu_k(A)\nu_m(B)} = \frac{\zeta_1(A)}{\nu_k(A)} \frac{\zeta_2(B)}{\nu_m(B)} - \lambda_2 \frac{\zeta_1(A)}{\nu_k(A)} - \lambda_1 \frac{\zeta_2(B)}{\nu_m(B)} + 2\lambda_1\lambda_2.$$

The range of possible values of $\zeta(A \times B)$ is therefore determined by the range of the function $f(x, y; \lambda_1, \lambda_2) = xy - \lambda_2 x - \lambda_1 y + 2\lambda_1\lambda_2$ on the domain $0 \leq x, y \leq 1$. Since $\lambda_1, \lambda_2 \geq \frac{1}{2}$, then $0 \leq f(x, y; \lambda_1, \lambda_2) \leq 2\lambda_1\lambda_2$ for all $0 \leq x, y \leq 1$. Hence, $\zeta(A \times B)$ is always nonnegative and therefore ζ is a random measure on $\mathbb{R}^{k+m}$. Moreover, $\Lambda(A \times B) = \mathbb{E}\,\zeta(A \times B) = (\mathbb{E}\,\zeta_1)(A)(\mathbb{E}\,\zeta_2)(B) = \lambda_1\lambda_2\nu_k(A)\nu_m(B) = \lambda_1\lambda_2\nu_{k+m}(A \times B)$. Thus, the intensity of ζ is $\lambda = \lambda_1\lambda_2$. For the covariance measure, we further obtain

$$\begin{aligned} C_2(C \times D) &= \mathbb{E}(\zeta(C) - \Lambda(C))(\zeta(D) - \Lambda(D)) \\ &= \mathbb{E}(\zeta_1(A_1) - \mathbb{E}\,\zeta_1(A_1))(\zeta_1(A_2) - \mathbb{E}\,\zeta_1(A_2)) \\ &\quad \cdot \mathbb{E}(\zeta_2(B_1) - \mathbb{E}\,\zeta_2(B_1))(\zeta_2(B_2) - \mathbb{E}\,\zeta_2(B_2)) \\ &= C_{2;1}(A_1 \times A_2)C_{2;2}(B_1 \times B_2), \end{aligned}$$

where $C = A_1 \times B_1$, $D = A_2 \times B_2$, and $A_1, A_2 \subset \mathbb{R}^k$, $B_1, B_2 \subset \mathbb{R}^m$ are Borel sets. The statement for the reduced covariance measure $\breve{C}_2$ of ζ follows from (2.1). □

Now let us present an example showing the dependence of (1.1) on a particular choice of the convex averaging sequence $(K_n)_{n\in\mathbb{N}}$.

Example 3.1. Let $d \geq 2$ and $K_{a,b} = [-a, a] \times bB^{d-1}$ for some $a, b > 0$, where B^{d-1} is the closed unit ball in $\mathbb{R}^{d-1}$. Clearly, $K_{a,b}$ is a compact convex set in $\mathbb{R}^d$ and $\nu_d(K_{a,b}) = 2ab^{d-1}c_{d-1}$, where c_{d-1} is the $(d-1)$-dimensional volume of the unit ball in $\mathbb{R}^{d-1}$. The inverse Fourier transform of its indicator function $\mathbf{1}_{K_{a,b}}$ is given by

$$\check{\mathbf{1}}_{K_{a,b}}(\boldsymbol{\xi}) = \check{\mathbf{1}}_{aB^1}(\xi_1)\check{\mathbf{1}}_{bB^{d-1}}(\boldsymbol{\xi}_2) = a\check{\mathbf{1}}_{B^1}(a\,|\xi_1|)b^{d-1}\check{\mathbf{1}}_{B^{d-1}}(b\,\|\boldsymbol{\xi}_2\|),$$

where $\boldsymbol{\xi} = (\xi_1, \boldsymbol{\xi}_2)$, and $\|\boldsymbol{\xi}_2\|$ denotes the Euclidean norm of $\boldsymbol{\xi}_2 = (\xi_2, \dots, \xi_d)$ in $\mathbb{R}^{d-1}$. Note that $\check{\mathbf{1}}_{B^n}(r) = r^{-n/2}J_{n/2}(r)$ for every $n \in \mathbb{N}$ and $r > 0$, where $J_{n/2}$ is the Bessel function of the first kind of order $n/2$; see, e.g. [18, Section 3]. Thus, from (3.1), we have

$$\frac{\operatorname{var}\zeta(K_{a,b})}{\nu_d(K_{a,b})} = \frac{(2\pi)^{d/2}}{2c_{d-1}}\int_{\mathbb{R}^d}\frac{J^2_{1/2}(a\,|\xi_1|)}{|\xi_1|}\,\frac{J^2_{(d-1)/2}(b\,\|\boldsymbol{\xi}_2\|)}{\|\boldsymbol{\xi}_2\|^{d-1}}\Gamma(\mathrm{d}\boldsymbol{\xi}).$$

Let us assume that Γ is continuous with density $f_\Gamma(\boldsymbol{\xi}) = |\xi_1|^s\,\|\boldsymbol{\xi}_2\|^{-t}\,\ell_1(|\xi_1|^{-1})\ell_2(\|\boldsymbol{\xi}_2\|^{-1})$, where $s, t \in (0, 1)$, and ℓ_1, ℓ_2 are bounded continuous positive functions on $[0, \infty)$. Note that such a ζ exists by Lemma 3.1, Example 4.1, and Example 4.3. Hence, we obtain

$$\frac{\operatorname{var}\zeta(K_{a,b})}{\nu_d(K_{a,b})} = \frac{(2\pi)^{d/2}}{2c_{d-1}}\int_{\mathbb{R}}\ell_1(|\xi_1|^{-1})\frac{J^2_{1/2}(a\,|\xi_1|)}{|\xi_1|^{1-s}}\mathrm{d}\xi_1\int_{\mathbb{R}^{d-1}}\ell_2(\|\boldsymbol{\xi}_2\|^{-1})\frac{J^2_{(d-1)/2}(b\,\|\boldsymbol{\xi}_2\|)}{\|\boldsymbol{\xi}_2\|^{d-1+t}}\mathrm{d}\boldsymbol{\xi}_2.$$

By introducing spherical coordinates, integrating over angles, and proper substitutions, we obtain

$$\frac{\operatorname{var}\zeta(K_{a,b})}{\nu_d(K_{a,b})} = (2\pi)^{d/2}(d-1)a^{-s}b^t\int_0^\infty\ell_1\left(\frac{a}{x}\right)\frac{J^2_{1/2}(x)}{x^{1-s}}\mathrm{d}x\int_0^\infty\ell_2\left(\frac{b}{y}\right)\frac{J^2_{(d-1)/2}(y)}{y^{1+t}}\mathrm{d}y.$$

By the Lebesgue dominated convergence theorem,

$$\int_0^\infty\ell_1\left(\frac{a}{x}\right)\frac{J^2_{1/2}(x)}{x^{1-s}}\mathrm{d}x \sim \ell_1(a)\int_0^\infty\frac{J^2_{1/2}(x)}{x^{1-s}}\mathrm{d}x = \ell_1(a)C_1 \quad \text{as } a \to \infty,$$

$$\int_0^\infty\ell_2\left(\frac{b}{y}\right)\frac{J^2_{(d-1)/2}(y)}{y^{1+t}}\mathrm{d}y \sim \ell_2(b)\int_0^\infty\frac{J^2_{(d-1)/2}(y)}{y^{1+t}}\mathrm{d}y = \ell_2(b)C_2 \quad \text{as } b \to \infty,$$

where $0 < C_1, C_2 < \infty$ are the values of the respective (Weber–Schafheitlin-type) integrals; see, e.g. [20, Section 13.41] for precise formulae. Let us take the convex averaging sequence $(K_n)_{n\in\mathbb{N}}$ with $K_n = K_{a_n,b_n}$ for all $n \in \mathbb{N}$, where $a_n = n^{t/2s}$ and $b_n = n$. By previous considerations,

$$\frac{\operatorname{var}\zeta(K_n)}{\nu_d(K_n)} \sim n^{t/2}(2\pi)^{d/2}(d-1)C_1C_2\ell_1(n^{t/2s})\ell_2(n) \quad \text{as } n \to \infty.$$

Since $t > 0$, it follows that $n^{t/2}\ell_1(n^{t/2s})\ell_2(n) \to \infty$ as $n \to \infty$ and, hence,

$$\limsup_{n\to\infty}\frac{\operatorname{var}\zeta(K_n)}{\nu_d(K_n)} = \infty.$$

On the other hand, taking $\tilde{K}_n = K_{a_n,b_n}$ with $a_n = n$ and $b_n = n^{s/2t}$ for all $n \in \mathbb{N}$ yields

$$\frac{\operatorname{var}\zeta(\tilde{K}_n)}{\nu_d(\tilde{K}_n)} \sim n^{-s/2}(2\pi)^{d/2}(d-1)C_1C_2\ell_1(n)\ell_2(n^{s/2t}) \quad \text{as } n \to \infty$$

and, since $n^{-s/2}\ell_1(n)\ell_2(n^{s/2t}) \to 0$ as $n \to \infty$, we obtain

$$\limsup_{n\to\infty} \frac{\operatorname{var}\zeta(\tilde{K}_n)}{\nu_d(\tilde{K}_n)} = 0.$$

Therefore, the limit superior depends on a particular choice of the convex averaging sequence.

The previous example shows that when $d \geq 2$, the fulfilment of (1.1) for some convex averaging sequence $(K_n)_{n\in\mathbb{N}}$ cannot be used for a consistent definition of long-range dependence. In order to avoid any inconsistencies and further complications we recommend the following more strict definition.

Definition 3.1. A second-order stationary random measure ζ on $\mathbb{R}^d$ is *long-range dependent* if

$$\limsup_{a\to\infty} \frac{\operatorname{var}\zeta(aB^d)}{\nu_d(aB^d)} = \infty. \tag{3.2}$$

The following proposition connects long-range dependence of ζ to its reduced covariance measure $\breve{C}_2$. For random measures on $\mathbb{R}^1$, it was shown in [7, Lemma 12.7.III].

Proposition 3.1. *Let ζ be a second-order stationary random measure and $\breve{C}_2$ be its reduced covariance measure. Further, let $\breve{C}_2^+$ and $\breve{C}_2^-$ be the upper and lower variations, respectively, of $\breve{C}_2$. If $\breve{C}_2^-$ is totally finite then ζ is long-range dependent if and only if $\breve{C}_2^+$ is not totally finite.*

Proof. Let $f_a(\boldsymbol{x}) = (\mathbf{1}_{aB^d} * \mathbf{1}^*_{aB^d})(\boldsymbol{x})/a^d\nu_d(B^d)$. Since

$$(\mathbf{1}_{aB^d} * \mathbf{1}^*_{aB^d})(\boldsymbol{x}) = a^d(\mathbf{1}_{B^d} * \mathbf{1}^*_{B^d})(\boldsymbol{x}/a) \quad \text{and} \quad (\mathbf{1}_{B^d} * \mathbf{1}^*_{B^d})(\mathbf{0}) = \nu_d(B^d),$$

$f_a(\mathbf{0}) = 1$, and $f_a(\boldsymbol{x})$ converges pointwise monotonically to 1 as $a \to \infty$. Hence,

$$\int_{\mathbb{R}^d} f_a(\boldsymbol{x})\breve{C}_2^+(\mathrm{d}\boldsymbol{x}) \to \breve{C}_2^+(\mathbb{R}^d) \quad \text{and} \quad \int_{\mathbb{R}^d} f_a(\boldsymbol{x})\breve{C}_2^-(\mathrm{d}\boldsymbol{x}) \to \breve{C}_2^-(\mathbb{R}^d) \quad \text{as } a \to \infty.$$

Thus, if $\breve{C}_2^-(\mathbb{R}^d) < \infty$ then $\lim_{a\to\infty} \operatorname{var}\zeta(aB^d)/\nu_d(aB^d) = \breve{C}_2^+(\mathbb{R}^d) - \breve{C}_2^-(\mathbb{R}^d)$ and the statement follows. □

A very useful characterization of long-range dependence can be formulated using the Bartlett spectrum.

Theorem 3.1. *Let ζ be a second-order stationary random measure with Bartlett spectrum Γ and $K \subset \mathbb{R}^d$ be an arbitrary bounded convex set containing the origin $\mathbf{0}$ in its interior. Then ζ is long-range dependent if and only if*

$$\limsup_{b\to 0_+} \frac{\Gamma(bK)}{\nu_d(bK)} = \infty. \tag{3.3}$$

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Proof. First we show the equivalence when $K = B^d$. Let us assume that ζ is a long-range dependent random measure. By (3.1),

$$\frac{\operatorname{var}\zeta(aB^d)}{\nu_d(aB^d)} = (2\pi)^{d/2}\int_{\mathbb{R}^d} f_{aB^d}(\boldsymbol{\xi})\Gamma(\mathrm{d}\boldsymbol{\xi})$$

with notation $f_{aB^d}(\boldsymbol{\xi}) = |\check{\mathbf{1}}_{aB^d}(\boldsymbol{\xi})|^2/\nu_d(aB^d)$. Since $\check{\mathbf{1}}_{aB^d}(\boldsymbol{\xi}) = a^d\check{\mathbf{1}}_{B^d}(a\boldsymbol{\xi})$, we have

$$f_{aB^d}(\boldsymbol{\xi}) = a^d f_{B^d}(a\boldsymbol{\xi}) \quad \text{and} \quad f_{aB^d}(\mathbf{0}) = (2\pi)^{-d}a^d c_d,$$

where $c_d = \nu_d(B^d)$. It is easy to show (see, e.g. [18, Section 3]) that $\check{\mathbf{1}}_{B^d}(\boldsymbol{\xi}) = \check{\mathbf{1}}_{B^d}(\|\boldsymbol{\xi}\|) = \|\boldsymbol{\xi}\|^{-d/2}J_{d/2}(\|\boldsymbol{\xi}\|)$ for all $\boldsymbol{\xi} \neq \mathbf{0}$, where $J_{d/2}$ is the Bessel function of the first kind of order $d/2$. Thus, $f_{aB^d}(\boldsymbol{\xi}) = (a^d/c_d)(a\|\boldsymbol{\xi}\|)^{-d}J_{d/2}^2(a\|\boldsymbol{\xi}\|)$. The Bessel function may be bounded (see [12]) as $J_{d/2}^2(x) \le C/x$ for all $x \ge \sqrt{\mu}$, where $\mu = (d+1)(d+3)$ and $C = 16\sqrt{\mu}/\pi\sqrt{3}^3(\sqrt{\mu}-1)$. Hence, $f_{aB^d}(\boldsymbol{\xi}) \le (C/c_d)(1/a)\|\boldsymbol{\xi}\|^{-(d+1)}$ whenever $a\|\boldsymbol{\xi}\| \ge \sqrt{\mu}$. Furthermore, if $\|\boldsymbol{\xi}\| = \sqrt{\mu}a^{-1/(d+1)} \ge \sqrt{\mu}a^{-1}$, we obtain $f_{aB^d}(\boldsymbol{\xi}) \le (C/c_d)(1/\sqrt{\mu})$. The integration of f_{aB^d} over $\mathbb{R}^d$ may be therefore separated into three parts, $\|\boldsymbol{\xi}\| > \sqrt{\mu}a^{-1/(d+1)}$, $\sqrt{\mu}a^{-1} < \|\boldsymbol{\xi}\| \le \sqrt{\mu}a^{-1/(d+1)}$, and $\|\boldsymbol{\xi}\| \le \sqrt{\mu}a^{-1}$. That is,

$$\int_{\mathbb{R}^d} f_{aB^d}(\boldsymbol{\xi})\Gamma(\mathrm{d}\boldsymbol{\xi}) = I_1 + I_2 + I_3,$$

where

$$\begin{aligned}
I_1 &= \int_{\|\boldsymbol{\xi}\|>\sqrt{\mu}/a^{1/(d+1)}} f_{aB^d}(\boldsymbol{\xi})\Gamma(\mathrm{d}\boldsymbol{\xi}) \le \frac{C}{c_d}\int_{\|\boldsymbol{\xi}\|>\sqrt{\mu}/a^{1/(d+1)}} \frac{1}{a\|\boldsymbol{\xi}\|^{d+1}}\Gamma(\mathrm{d}\boldsymbol{\xi}),\\
I_2 &= \int_{\sqrt{\mu}/a<\|\boldsymbol{\xi}\|\le\sqrt{\mu}/a^{1/(d+1)}} f_{aB^d}(\boldsymbol{\xi})\Gamma(\mathrm{d}\boldsymbol{\xi}) \le \frac{C}{c_d}\int_{\sqrt{\mu}/a<\|\boldsymbol{\xi}\|\le\sqrt{\mu}/a^{1/(d+1)}} \frac{1}{a\|\boldsymbol{\xi}\|^{d+1}}\Gamma(\mathrm{d}\boldsymbol{\xi}),\\
I_3 &= \int_{\|\boldsymbol{\xi}\|\le\sqrt{\mu}/a^{-1}} f_{aB^d}(\boldsymbol{\xi})\Gamma(\mathrm{d}\boldsymbol{\xi}) \le (2\pi)^{-d}a^d c_d\Gamma(\sqrt{\mu}a^{-1}B^d).
\end{aligned}$$

The integrand of I_1 is always bounded by $C/(c_d\sqrt{\mu})$ and for every $\boldsymbol{\xi}$ it decreases as a increases. Let $g_a(\boldsymbol{\xi}) = \min\{a^{-1}\|\boldsymbol{\xi}\|^{-(d+1)}, \mu^{(d+1)/2}\}$ for every $a > 0$, $\boldsymbol{\xi} \in \mathbb{R}^d$. The translation boundedness (see, e.g. [3, Proposition 4.9]) of Γ implies that $g_a \in L^1(\mathbb{R}^d, \Gamma)$ for all $a > 0$. Moreover, since $g_a(\boldsymbol{\xi}) \le g_1(\boldsymbol{\xi})$ for all $a > 1$, there exists $K_1 < \infty$ such that

$$I_1 \le \frac{C}{c_d}\int_{\|\boldsymbol{\xi}\|>\sqrt{\mu}a^{-1/(d+1)}} g_a(\boldsymbol{\xi})\Gamma(\mathrm{d}\boldsymbol{\xi}) \le \frac{C}{c_d}\int_{\mathbb{R}^d} g_a(\boldsymbol{\xi})\Gamma(\mathrm{d}\boldsymbol{\xi}) \le K_1 \quad \text{for all } a > 1.$$

Arguing by contradiction, let us assume that there exists $M < \infty$ such that $\Gamma(bB^d) \le M\nu_d(bB^d)$ for all $b < 1$, which yields $I_3 \le (c_d^2\mu^{d/2}/(2\pi)^d)M$. For I_2, let us assume that $a > 1$ and for arbitrary $n \in \mathbb{N}$ introduce $y_0 < \cdots < y_n$ by $y_0 = \sqrt{\mu}/a^{1/(d+1)}$, $y_n = \sqrt{\mu}/a$, and $y_i = y_0(y_n/y_0)^i = \sqrt{\mu}/a^{(1/(d+1))(1+id/n)}$ for every $i = 1, \dots, n-1$. We obtain

$$\begin{aligned}
\int_{y_n<\|\boldsymbol{\xi}\|\le y_0} & \frac{1}{a\,\|\boldsymbol{\xi}\|^{d+1}}\Gamma(\mathrm{d}\boldsymbol{\xi}) \\
&= \sum_{i=1}^{n} \frac{1}{a y_i^{d+1}}(\Gamma(y_{i-1}B^d) - \Gamma(y_i B^d)) \\
&\le \frac{1}{a y_1^{d+1}}\Gamma(y_0 B^d) + \sum_{i=1}^{n}\bigg(\frac{1}{a y_{i+1}^{d+1}} - \frac{1}{a y_i^{d+1}}\bigg)\Gamma(y_i B^d) \\
&\le \frac{M c_d}{\sqrt{\mu}}\bigg(\frac{1}{a^{(d/(d+1))(1-(d+1)/n)}} + \bigg(1 - \frac{1}{a^{d/(d+1)}}\bigg)\frac{a^{d/n(d+1)}(a^{d/n}-1)}{a^{d/n(d+1)}-1}\bigg).
\end{aligned}$$

Taking $n = [a]$, the integer part of a, yields that the first term converges to 0 as $a \to \infty$. The second term converges to $d+1$. Thus, I_2 is also bounded, which is a contradiction because $\limsup_{a\to\infty} I_1 + I_2 + I_3 = \infty$ by the long-range dependence of ζ.

For the opposite implication let us assume that (3.3) holds for $K = B^d$. Since f_{B^d} is continuous at the origin with $f_{B^d}(\mathbf{0}) = (2\pi)^{-d}c_d$, there exists $r_0 > 0$ such that $f_{aB^d}(\boldsymbol{\xi}) = a^d f_{B^d}(a\boldsymbol{\xi}) \ge a^d(2\pi)^{-d}c_d/2$ for all $\boldsymbol{\xi} \in (r_0/a)B^d$. Therefore,

$$\int_{\mathbb{R}^d} f_{aB^d}(\boldsymbol{\xi})\Gamma(\mathrm{d}\boldsymbol{\xi}) \ge \int_{\|\boldsymbol{\xi}\|\le r_0/a} f_{aB^d}(\boldsymbol{\xi})\Gamma(\mathrm{d}\boldsymbol{\xi}) \ge \frac{a^d c_d}{2(2\pi)^d}\Gamma\bigg(\frac{r_0}{a}B^d\bigg) = \frac{r_0^d c_d^2}{2(2\pi)^d}\frac{\Gamma((r_0/a)B^d)}{\nu_d((r_0/a)B^d)}$$

and the statement follows.

Finally, let $K_1, K_2 \subset \mathbb{R}^d$ be two bounded convex sets containing the origin in their interior. Indeed, there exist $c > 0$ such that $cK_1 \subset K_2$ and, thus,

$$\frac{\Gamma(bK_2)}{\nu_d(bK_2)} \ge \frac{\Gamma(bcK_1)}{\nu_d(bK_2)} = \frac{\Gamma(bcK_1)}{\nu_d(bcK_1)}\frac{\nu_d(cK_1)}{\nu_d(K_2)}.$$

Therefore, (3.3) for $K = K_1$ implies (3.3) for $K = K_2$. Since K_1, K_2 were arbitrary, the value of the limit superior is independent of K, which completes the proof. □

Remark 3.1. By choosing $K = U^d$, the open ball in $\mathbb{R}^d$, the condition in the previous theorem may be formulated as: the upper derivative of Γ with respect to the Lebesgue measure ν_d at the origin is infinite.

Remark 3.2. It is interesting to note that the dual condition (3.3) for long-range dependence is formulated in a less restrictive way than condition (3.2). As follows from the proof of Theorem 3.1, the key ingredient is the estimate $f_{B^d}(\boldsymbol{\xi}) \le C/\|\boldsymbol{\xi}\|^{d+1}$, whenever $\|\boldsymbol{\xi}\|$ is greater than some constant. Hence, any convex set satisfying this estimate may be equivalently used in (3.2) instead of B^d. We refer the reader to [11] for a deeper discussion showing that such a convex set should be bounded with sufficiently smooth boundary having everywhere nonvanishing Gaussian curvature.

There are several obvious consequences of the previous theorem.

Corollary 3.1. *If* $\Gamma(\{\mathbf{0}\}) > 0$ *then* ζ *is long-range dependent.*

Corollary 3.2. *Let* Γ *be absolutely continuous with spectral density* f_Γ.

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(i) *If f_Γ is ν_d almost everywhere bounded in some neighbourhood of* $\mathbf{0}$ *then ζ is not long-range dependent.*

(ii) *If, for every $c > 0$, there exists a neighbourhood U_c of* $\mathbf{0}$ *such that $f_\Gamma(\boldsymbol{x}) \geq c$ for ν_d almost all $\boldsymbol{x} \in U_c$ then ζ is long-range dependent.*

Let us continue with Remark 3.2 and inspect whether the closed ball B^d in (3.2) may be equivalently replaced by other bounded convex sets with nonempty interior and without any further restrictions, which yields condition (1.3). As our next example shows, the general replacement is not possible in dimension greater than two.

Example 3.2. Let $d \geq 3$ and let ζ be a second-order stationary random measure on $\mathbb{R}^d$ with Bartlett spectrum Γ given by

$$\Gamma = \frac{(2\pi)^{d/2}}{2^d} \sum_{\boldsymbol{n}\in\mathbb{N}^d,\, i_1,\dots,i_d\in\{-1,1\}} c_{\boldsymbol{n}} \delta_{(n_1 i_1,\dots,n_d i_d)}.$$

For the construction of ζ, see Example 4.2. Let $\theta \in \mathrm{SO}_d$ be a rotation in $\mathbb{R}^d$ such that $\sqrt{d}\theta(1, 0, 0, \dots) = (1, \dots, 1)$. Let C be the cube $[-1, 1]^d$ rotated by θ, i.e. $C = \theta([-1, 1]^d)$. For the inverse Fourier transform $\check{\mathbf{1}}_C(\boldsymbol{\xi})$ of the indicator of C, we obtain $\check{\mathbf{1}}_C(\boldsymbol{\xi}) = \check{\mathbf{1}}_{[-1,1]^d}(\theta^{-1}\boldsymbol{\xi})$. Since $\check{\mathbf{1}}_{[-1,1]^d}(\boldsymbol{\omega}) = 2^d(2\pi)^{-d/2}\prod_{i=1}^d \sin(|\omega_i|)/|\omega_i|$, we have

$$\frac{|\check{\mathbf{1}}_{aC}(\boldsymbol{\xi})|^2}{(2a)^d} = \frac{a^d}{\pi^d}\prod_{i=1}^d \frac{\sin^2(a(\theta^{-1}\boldsymbol{\xi})_i)}{a^2\left|(\theta^{-1}\boldsymbol{\xi})_i\right|^2}.$$

Hence, we obtain

$$\begin{aligned}
(2\pi)^{d/2}\int_{\mathbb{R}^d} \frac{|\check{\mathbf{1}}_{aC}(\boldsymbol{\xi})|^2}{\nu_d(aC)}\Gamma(\mathrm{d}\boldsymbol{\xi}) &= a^d \sum_{\boldsymbol{n}\in\mathbb{N}^d,\, i_1,\dots,i_d\in\{-1,1\}} c_{\boldsymbol{n}} \prod_{k=1}^d \frac{\sin^2(a(\theta^{-1}(n_1 i_1,\dots,n_d i_d))_k)}{a^2\left|(\theta^{-1}(n_1 i_1,\dots,n_d i_d))_k\right|^2} \\
&\geq a^d \prod_{k=1}^d \frac{\sin^2(a(\theta^{-1}(1,\dots,1))_k)}{a^2\left|(\theta^{-1}(1,\dots,1))_k\right|^2} \\
&= a^{d-2}\frac{\sin^2(a\sqrt{d})}{d}.
\end{aligned}$$

By (3.1), this yields

$$\limsup_{a\to\infty} \frac{\operatorname{var}\zeta(aC)}{\nu_d(aC)} = \infty.$$

On the other side, taking $[-1, 1]^d$ leads directly to

$$\begin{aligned}
(2\pi)^{d/2}\int_{\mathbb{R}^d} \frac{|\check{\mathbf{1}}_{a[-1,1]^d}(\boldsymbol{\xi})|^2}{\nu_d(a[-1,1]^d)}\Gamma(\mathrm{d}\boldsymbol{\xi}) &= a^d \sum_{\boldsymbol{n}\in\mathbb{N}^d,\, i_1,\dots,i_d\in\{-1,1\}} c_{\boldsymbol{n}} \prod_{k=1}^d \frac{\sin^2(a\,|n_k i_k|)}{a^2\,|n_k i_k|^2} \\
&= (2a)^d \sum_{\boldsymbol{n}\in\mathbb{N}^d} c_{\boldsymbol{n}} \prod_{k=1}^d \frac{\sin^2(a n_k)}{a^2 n_k^2} \\
&= \frac{2^d}{a^d}\sum_{\boldsymbol{n}\in\mathbb{N}^d} c_{\boldsymbol{n}} \prod_{k=1}^d \frac{\sin^2(a n_k)}{n_k^2}.
\end{aligned}$$

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Hence,

$$\lim_{a\to\infty} \frac{\operatorname{var}\zeta([-a,a]^d)}{\nu_d([-a,a]^d)} = 0.$$

Thus, the limit superior depends on the particular choice of K.

Remark 3.3. The random measure ζ from Example 3.2 is not long-range dependent as easily follows from Theorem 3.1 since $\Gamma(\frac{1}{2}B^d) = 0$.

Even though the closed ball B^d cannot be replaced by a general convex set in Definition 3.1, it is possible to show the following implication.

Proposition 3.2. *Let ζ be a long-range dependent second-order stationary random measure. Then*

$$\limsup_{a\to\infty} \frac{\operatorname{var}\zeta(aK)}{\nu_d(aK)} = \infty$$

for every bounded convex set $K \subset \mathbb{R}^d$ with nonempty interior.

Proof. Let $f_{aK}(\boldsymbol{\xi}) = |\check{\mathbf{1}}_{aK}(\boldsymbol{\xi})|^2/\nu_d(aK)$. Since $\check{\mathbf{1}}_{aK}(\boldsymbol{\xi}) = a^d\check{\mathbf{1}}_K(a\boldsymbol{\xi})$, we obtain $f_{aK}(\boldsymbol{\xi}) = a^d f_K(a\boldsymbol{\xi})$. Hence, as in the proof of Theorem 3.1, there exists $r_0 > 0$ such that $f_{aK}(\boldsymbol{\xi}) = a^d f_K(a\boldsymbol{\xi}) \geq a^d(2\pi)^{-d}\nu_d(K)/2$ for all $\boldsymbol{\xi} \in (r_0/a)B^d$. Therefore, by (3.1),

$$\begin{aligned}\frac{\operatorname{var}\zeta(aK)}{(2\pi)^{d/2}\nu_d(aK)} &= \int_{\mathbb{R}^d} f_{aK}(\boldsymbol{\xi})\Gamma(\mathrm{d}\boldsymbol{\xi}) \\ &\geq \int_{\|\boldsymbol{\xi}\|\leq r_0/a} f_{aK}(\boldsymbol{\xi})\Gamma(\mathrm{d}\boldsymbol{\xi}) \\ &= \frac{r_0^d\nu_d(K)\nu_d(B^d)}{2(2\pi)^d}\frac{\Gamma((r_0/a)B^d)}{\nu_d((r_0/a)B^d)}\end{aligned}$$

and the statement follows by Theorem 3.1. □

In what follows we focus on the case when ζ has rotation invariant reduced covariance measure $\breve{C}_2$, i.e. $\breve{C}_2(\theta A) = \breve{C}_2(A)$ for any bounded Borel $A \subset \mathbb{R}^d$ and any rotation $\theta \in \mathrm{SO}_d$. We start by showing the equivalence with the rotation invariance of the Bartlett spectrum Γ of ζ, which then has a specific form.

Lemma 3.2. *A second-order stationary random measure ζ has rotation invariant reduced covariance measure $\breve{C}_2$ if and only if its Bartlett spectrum Γ is rotation invariant. Moreover, in that case, Γ can be factorized such that*

$$\int_{\mathbb{R}^d} f(\boldsymbol{\xi})\Gamma(\mathrm{d}\boldsymbol{\xi}) = f(\mathbf{0})\Gamma(\{\mathbf{0}\}) + \int_0^\infty \tilde{\Gamma}(\mathrm{d}\varrho)\int_{S^{d-1}} f(\varrho\boldsymbol{\omega})\sigma_{d-1}(\mathrm{d}\boldsymbol{\omega})$$

for any bounded measurable function f of bounded support, where σ_{d-1} is the spherical measure ($d-1$ Hausdorff measure) on the unit sphere S^{d-1} in $\mathbb{R}^d$ and $\tilde{\Gamma}$ is a positive measure on $(0,\infty)$ satisfying the following boundedness condition:

$$\sup_{s>1}\int_{B+s} \varrho^{-d+1}\tilde{\Gamma}(\mathrm{d}\varrho) < \infty \tag{3.4}$$

for any bounded Borel set $B \subset (0,\infty)$.

Proof. To prove the equivalence of rotation invariances let $\theta \in \mathrm{SO}_d$ and $f \in C_c(\mathbb{R}^d)$ be arbitrary. Let $f_\theta \in C_c(\mathbb{R}^d)$ be defined by $f_\theta(\boldsymbol{x}) = f(\theta^{-1}\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}^d$. It is easy to show that $(f_\theta * f_\theta^*) = (f * f^*)_\theta$ and $\check{f}_\theta = (\check{f})_\theta$. Hence,

$$\begin{aligned}
(2\pi)^{d/2} \int_{\mathbb{R}^d} \left|\check{f}(\boldsymbol{\xi})\right|^2 \Gamma(\mathrm{d}\boldsymbol{\xi}) &= \int_{\mathbb{R}^d} (f * f^*)(\boldsymbol{x}) \check{C}_2(\mathrm{d}\boldsymbol{x}) \\
&= \int_{\mathbb{R}^d} (f * f^*)(\boldsymbol{x}) \check{C}_2(\theta \mathrm{d}\boldsymbol{x}) \\
&= \int_{\mathbb{R}^d} (f * f^*)_\theta(\boldsymbol{y}) \check{C}_2(\mathrm{d}\boldsymbol{y}) \\
&= \int_{\mathbb{R}^d} (f_\theta * f_\theta^*)(\boldsymbol{y}) \check{C}_2(\mathrm{d}\boldsymbol{y}) \\
&= (2\pi)^{d/2} \int_{\mathbb{R}^d} \left|\check{f}_\theta(\boldsymbol{\omega})\right|^2 \Gamma(\mathrm{d}\boldsymbol{\omega}) \\
&= (2\pi)^{d/2} \int_{\mathbb{R}^d} \left|\check{f}(\theta^{-1}\boldsymbol{\omega})\right|^2 \Gamma(\mathrm{d}\boldsymbol{\omega}) \\
&= (2\pi)^{d/2} \int_{\mathbb{R}^d} \left|\check{f}(\boldsymbol{\xi})\right|^2 \Gamma(\theta \mathrm{d}\boldsymbol{\xi})
\end{aligned}$$

and the invariance follows from the uniqueness of Γ, see [3, Theorem 4.7]. Now the factorization lemma (see [6, Lemma A2.7.II]) implies that $\Gamma = \Gamma(\{\mathbf{0}\})\delta_{\mathbf{0}} + \tilde{\Gamma} \times \sigma_{d-1}$, where $\delta_{\mathbf{0}}$ is the Dirac measure at the origin, $\tilde{\Gamma}$ is some positive measure on $(0, \infty)$, and σ_{d-1} is the spherical measure ($d-1$ Hausdorff measure) on the unit sphere S^{d-1} in $\mathbb{R}^d$, i.e. in the integral form

$$\int_{\mathbb{R}^d} f(\boldsymbol{\xi})\Gamma(\mathrm{d}\boldsymbol{\xi}) = f(\mathbf{0})\Gamma(\{\mathbf{0}\}) + \int_0^\infty \tilde{\Gamma}(\mathrm{d}\varrho) \int_{S^{d-1}} f(\varrho\boldsymbol{\omega})\sigma_{d-1}(\mathrm{d}\boldsymbol{\omega})$$

for any bounded measurable function f of bounded support, see [6, Proposition A2.7.III] and also [6, Example A2.7(b)]. The translation boundedness of Γ (see, e.g. [3, Proposition 4.9]) implies the boundedness (3.4) of $\tilde{\Gamma}$. For the proof take an arbitrary point $\boldsymbol{\omega}_0 \in S^{d-1}$ and for any $\varrho \le \pi$ define the spherical cap $P_\varrho = \{\boldsymbol{\omega} \in S^{d-1} \mid d_{S^{d-1}}(\boldsymbol{\omega}, \boldsymbol{\omega}_0) \le \varrho\}$, where $d_{S^{d-1}}$ is the distance function on the sphere S^{d-1}. Clearly, $\sigma_{d-1}(P_\varrho) = \varrho^{d-1}\sigma_{d-1}(S^{d-1}) = \varrho^{d-1} d c_d$. Let $B \subset (0, \infty)$ be a bounded Borel set. For $s > 1$, we define $A_{B+s} = \{\varrho\boldsymbol{\omega} \in \mathbb{R}^d \mid \varrho - s \in B,\ \boldsymbol{\omega} \in P_{1/\varrho}\}$. By construction of P_ϱ, there exists a bounded Borel set $A \subset \mathbb{R}^d$ such that $A_{B+s} \subset A + s\boldsymbol{\omega}_0$ for every $s > 1$. Thus, we obtain

$$\begin{aligned}
\int_{B+s} \varrho^{-d+1}\tilde{\Gamma}(\mathrm{d}\varrho) &= \frac{1}{d c_d} \int_{B+s} \sigma_{d-1}(P_{1/\varrho})\tilde{\Gamma}(\mathrm{d}\varrho) \\
&= \frac{1}{d c_d} \int_0^\infty \tilde{\Gamma}(\mathrm{d}\varrho) \int_{S^{d-1}} \mathbf{1}_{A_{B+s}} f(\varrho\boldsymbol{\omega})\sigma_{d-1}(\mathrm{d}\boldsymbol{\omega}) \\
&= \frac{1}{d c_d} \Gamma(A_{B+s}) \\
&\le \frac{1}{d c_d} \Gamma(A + s\boldsymbol{\omega}_0)
\end{aligned}$$

and the translation boundedness of Γ yields the result. □

Remark 3.4. The conditions of the previous lemma are satisfied by any second-order stationary isotropic random measure.

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In the following assertion we show that the implication in Proposition 3.2 turns into an equivalence for rotation invariant covariance measures.

Proposition 3.3. *Let ζ be a second-order stationary random measure with rotation invariant reduced covariance measure $\breve{C}_2$ and $K \subset \mathbb{R}^d$ be an arbitrary bounded convex set with nonempty interior. Then ζ is long-range dependent if and only if*

$$\limsup_{a\to\infty} \frac{\operatorname{var}\zeta(aK)}{\nu_d(aK)} = \infty.$$

Proof. If ζ is long-range dependent then the statement follows from Proposition 3.2. For the remaining implication let $K \subset \mathbb{R}^d$ satisfy the assumptions. By Lemma 3.2,

$$\frac{\operatorname{var}\zeta(aK)}{\nu_d(aK)} = (2\pi)^{d/2} f_{aK}(\mathbf{0})\Gamma(\{\mathbf{0}\}) + (2\pi)^{d/2}\int_0^\infty \tilde{\Gamma}(\mathrm{d}\varrho)\int_{S^{d-1}} f_{aK}(\varrho\boldsymbol{\omega})\sigma_{d-1}(\mathrm{d}\boldsymbol{\omega}),$$

where $f_{aK}(\boldsymbol{\xi}) = |\check{\mathbf{1}}_{aK}(\boldsymbol{\xi})|^2/\nu_d(aK)$. If $\Gamma(\{\mathbf{0}\}) > 0$ then ζ is long-range dependent by Corollary 3.1. Thus, let us assume that $\Gamma(\{\mathbf{0}\}) = 0$. The $L^2(S^{d-1}, \sigma_{d-1})$-norm of $\check{\mathbf{1}}_K(\varrho\boldsymbol{\omega})$ decreases with increasing ϱ as $\varrho^{-(d+1)}$, see [4, Theorem 1.1], i.e. there exists $C > 0$ such that

$$\int_{S^{d-1}} \left|\check{\mathbf{1}}_K(\varrho\boldsymbol{\omega})\right|^2 \sigma_{d-1}(\mathrm{d}\boldsymbol{\omega}) \le \frac{C}{\varrho^{d+1}} \quad \text{for every } \varrho > 0.$$

For $f_{aK}(\boldsymbol{\xi}) = a^d f_K(a\boldsymbol{\xi})$ this yields

$$\int_{S^{d-1}} f_{aK}(\varrho\boldsymbol{\omega})\sigma_{d-1}(\mathrm{d}\boldsymbol{\omega}) \le \frac{C}{a\varrho^{d+1}} \quad \text{for every } \varrho > 0,\ a > 0.$$

Thus, similarly as in the proof of Theorem 3.1, we may split the integration in the radial coordinate ϱ into three parts, $\varrho > a^{-1/(d+1)}$, $a^{-1} < \varrho \le a^{-1/(d+1)}$, and $0 < \varrho \le a^{-1}$; that is,

$$\int_0^\infty \tilde{\Gamma}(\mathrm{d}\varrho)\int_{S^{d-1}} f_{aK}(r\boldsymbol{\omega})\sigma_{d-1}(\mathrm{d}\boldsymbol{\omega}) = I_1 + I_2 + I_3,$$

where

$$I_1 = \int_{1/a^{1/(d+1)}}^\infty \tilde{\Gamma}(\mathrm{d}\varrho)\int_{S^{d-1}} f_{aK}(\varrho\boldsymbol{\omega})\sigma_{d-1}(\mathrm{d}\boldsymbol{\omega}) \le C\int_{1/a^{1/(d+1)}}^\infty \frac{1}{a\varrho^{d+1}}\tilde{\Gamma}(\mathrm{d}\varrho),$$
$$I_2 = \int_{1/a}^{1/a^{1/(d+1)}} \tilde{\Gamma}(\mathrm{d}\varrho)\int_{S^{d-1}} f_{aK}(\varrho\boldsymbol{\omega})\sigma_{d-1}(\mathrm{d}\boldsymbol{\omega}) \le C\int_{1/a}^{1/a^{1/(d+1)}} \frac{1}{a\varrho^{d+1}}\tilde{\Gamma}(\mathrm{d}\varrho),$$
$$I_3 = \int_0^{1/a} \tilde{\Gamma}(\mathrm{d}\varrho)\int_{S^{d-1}} f_{aK}(\varrho\boldsymbol{\omega})\sigma_{d-1}(\mathrm{d}\boldsymbol{\omega}) \le \frac{a^d\nu_d(K)}{dc_d(2\pi)^d}\Gamma(a^{-1}B^d),$$

since $\int_0^r \tilde{\Gamma}(\mathrm{d}\varrho) = (1/dc_d)\Gamma(rB^d)$ for all $r > 0$ as follows from the previous lemma, from $\Gamma(\{\mathbf{0}\}) = 0$, and from $\sigma_{d-1}(S^{d-1}) = dc_d$. The integrand of I_1 is always bounded by C and for every ϱ it decreases as a increases. Let $g_a(\varrho) = \min\{a^{-1}\varrho^{-(d+1)}, 1\}$ for every $a > 0$, $\varrho > 0$. The translation boundedness (3.4) of $\tilde{\Gamma}$ implies that $g_a \in L^1(\mathbb{R}^+, \tilde{\Gamma})$ for every $a > 0$. Hence, there exists $K_1 < \infty$ such that

$$I_1 \le C\int_{1/a^{1/(d+1)}}^\infty \frac{1}{a\varrho^{d+1}}\tilde{\Gamma}(\mathrm{d}\varrho) \le C\int_0^\infty g_a(\varrho)\tilde{\Gamma}(\mathrm{d}\varrho) \le K_1 \quad \text{for all } a > 1.$$

Now let us argue by contradiction and assume that ζ is not long-range dependent. By Theorem 3.1, there exists $M < \infty$ such that $\Gamma(bB^d) \le M\nu_d(bB^d)$ for all $b < 1$, which yields $I_3 \le (\nu_d(K)/d(2\pi)^d)M$. Similarly as in the proof of Theorem 3.1,

$$\int_{1/a}^{1/a^{1/(d+1)}} \frac{1}{a\varrho^{d+1}} \tilde{\Gamma}(\mathrm{d}\varrho) \le \frac{M}{d}\left(\frac{1}{a^{d/(d+1)(1-(d+1)/n)}} + \left(1 - \frac{1}{a^{d/(d+1)}}\right)\frac{a^{d/n(d+1)}(a^{d/n}-1)}{a^{d/n(d+1)}-1}\right)$$

for every $n \in \mathbb{N}$. Taking $n = [a]$ yields the boundedness of I_2 which is a contradiction. □

Finally, we show that the rotation invariance of the reduced covariance measure enables us to relate long-range dependence with condition (1.1).

Proposition 3.4. *Let ζ be a second-order stationary random measure with rotation invariant reduced covariance measure $\check{C}_2$. If*

$$\limsup_{n\to\infty} \frac{\operatorname{var}\zeta(K_n)}{\nu_d(K_n)} = \infty$$

for some convex averaging sequence $(K_n)_{n\in\mathbb{N}}$ then ζ is long-range dependent.

Proof. Assume that $\Gamma(\{\mathbf{0}\}) = 0$ since, otherwise, ζ is long-range dependent by Corollary 3.1. By the same arguments as in the proof of Proposition 3.3, for every $n \in \mathbb{N}$, there exist $C_n > 0$ such that

$$\int_{S^{d-1}} \left|\check{\mathbf{1}}_{K_n}(\varrho\boldsymbol{\omega})\right|^2 \sigma_{d-1}(\mathrm{d}\boldsymbol{\omega}) \le \frac{C_n}{\varrho^{d+1}} \quad \text{for all } \varrho > 0.$$

Moreover, following the proof of [4, Theorem 1.1], one may show that $C_n \le C(S(K_n))^3$, where $S(K_n)$ is the surface area ($(d-1)$-dimensional Hausdorff measure) of K_n and C is independent of K_n. Let $(a_n)_{n\in\mathbb{N}}$ be the sequence of nonnegative real numbers such that $S(K_n) = a_n^{d-1}$, i.e. $S(a_n^{-1}K_n) = 1$. Since $K_n \subset K_{n+1}$ for all $n \in \mathbb{N}$, it follows that $a_n \le a_{n+1}$ for all $n \in \mathbb{N}$. Moreover, $a_n \to \infty$ as $n \to \infty$ since $r(K_n) = \sup\{r \ge 0 \mid B_r(\boldsymbol{x}) \subset K_n \text{ for some } \boldsymbol{x}\} \to \infty$ as $n \to \infty$ by the convex averaging property of (K_n). For $f_{K_n}(\boldsymbol{\xi})$, we thus obtain $f_{K_n}(\boldsymbol{\xi}) = f_{a_nK_n/a_n}(\boldsymbol{\xi}) = a_n^d f_{K_n/a_n}(a_n\boldsymbol{\xi})$ and, therefore,

$$\int_{S^{d-1}} f_{K_n}(\varrho\boldsymbol{\omega})\sigma_{d-1}(\mathrm{d}\boldsymbol{\omega}) \le \frac{C}{a_n\varrho^{d+1}} \quad \text{for every } n \in \mathbb{N},\ \varrho > 0.$$

Again, we may write

$$\int_0^\infty \tilde{\Gamma}(\mathrm{d}\varrho)\int_{S^{d-1}} f_{K_n}(r\boldsymbol{\omega})\sigma_{d-1}(\mathrm{d}\boldsymbol{\omega}) = I_1 + I_2 + I_3,$$

where

$$I_1 = \int_{1/a_n^{1/(d+1)}}^\infty \tilde{\Gamma}(\mathrm{d}\varrho)\int_{S^{d-1}} f_{K_n}(\varrho\boldsymbol{\omega})\sigma_{d-1}(\mathrm{d}\boldsymbol{\omega}) \le C\int_{1/a_n^{1/(d+1)}}^\infty \frac{1}{a_n\varrho^{d+1}}\tilde{\Gamma}(\mathrm{d}\varrho),$$

$$I_2 = \int_{1/a_n}^{1/a_n^{1/(d+1)}} \tilde{\Gamma}(\mathrm{d}\varrho)\int_{S^{d-1}} f_{K_n}(\varrho\boldsymbol{\omega})\sigma_{d-1}(\mathrm{d}\boldsymbol{\omega}) \le C\int_{1/a_n}^{1/a_n^{1/(d+1)}} \frac{1}{a_n\varrho^{d+1}}\tilde{\Gamma}(\mathrm{d}\varrho),$$

$$I_3 = \int_0^{1/a_n} \tilde{\Gamma}(\mathrm{d}\varrho)\int_{S^{d-1}} f_{K_n}(\varrho\boldsymbol{\omega})\sigma_{d-1}(\mathrm{d}\boldsymbol{\omega}) \le \frac{\nu_d(K_n)}{dc_d(2\pi)^d}\Gamma(a_n^{-1}B^d).$$

Let $g_n(\varrho) = \min\{a_n^{-1}\varrho^{-(d+1)}, 1\}$ for every $n \in \mathbb{N}$, $\varrho > 0$. The translation boundedness (3.4) of $\tilde{\Gamma}$ implies that $g_n \in L^1(\mathbb{R}^+, \tilde{\Gamma})$ for every $n \in \mathbb{N}$. Let $n_0 \in \mathbb{N}$ be such that $a_n > a_1$ for $n > n_0$. Then, there exists $C_1 < \infty$ such that

$$I_1 \leq C \int_{1/a_n^{1/(d+1)}}^{\infty} \frac{1}{a_n \varrho^{d+1}} \tilde{\Gamma}(\mathrm{d}\varrho) \leq C \int_0^{\infty} g_n(\varrho) \tilde{\Gamma}(\mathrm{d}\varrho) \leq C_1 \quad \text{for all } n > n_0.$$

We again argue by contradiction and assume that ζ is not long-range dependent. Hence, by Theorem 3.1, there exists $M < \infty$ such that $\Gamma(bB^d) \leq M\nu_d(bB^d)$ for all $b < 1$, which yields

$$\begin{aligned} I_3 &\leq \frac{\nu_d(K_n)}{dc_d(2\pi)^d} \Gamma(a_n^{-1}B^d) \\ &\leq \frac{M}{d(2\pi)^d} \frac{\nu_d(K_n)}{a_n^d} \\ &= \frac{M}{d(2\pi)^d} \frac{\nu_d(K_n)}{a_n S(K_n)} \\ &\leq \frac{M}{d(2\pi)^d} \frac{1}{d^{d/(d-1)} c_d^{1/(d-1)}} \quad \text{for all } n \in \mathbb{N} \end{aligned}$$

since a set that maximizes its volume for the fixed surface area $S(K_n)$ is the ball of radius $\sqrt[d-1]{S(K_n)/(dc_d)}$ as follows from the isoperimetric inequality; see, e.g. [16, Section 7.1]. Therefore, I_3 is bounded. To deal with I_2 let for any n such that $a_n > 1$ and any $k \in \mathbb{N}$ introduce $y_0 < \cdots < y_k$ by $y_0 = 1/a_n^{1/(d+1)}$, $y_k = 1/a_n$, and $y_i = y_0(y_k/y_0)^i = a_n^{-(1/(d+1))(1+id/k)}$ for every $i = 1, \dots, k-1$. Similar arguments as in the proof of Theorem 3.1 yield

$$\int_{y_k}^{y_0} \frac{1}{a_n \varrho^{d+1}} \tilde{\Gamma}(\mathrm{d}\varrho) \leq \frac{M}{d} \left(\frac{1}{a_n^{(d/(d+1))(1-(d+1)/k)}} + \left(1 - \frac{1}{a_n^{d/(d+1)}}\right) \frac{a_n^{d/k(d+1)}(a_n^{d/k} - 1)}{a_n^{d/k(d+1)} - 1} \right).$$

Setting $k = [a_n]$, the integer part of a_n, implies the boundedness of I_2 as $n \to \infty$ which is a contradiction. This completes the proof. □

Remark 3.5. It is an open question as to whether the implication from the previous proposition may be reversed, i.e. if ζ is long-range dependent second-order stationary random measure with rotation invariant reduced covariance measure, whether (1.1) is valid for every convex averaging sequence. However, for this situation we were not able to either prove the statement or to find a counterexample.

4. Several examples

In this section we construct several specific random measures that were needed in the previous examples.

Example 4.1. Suppose that $\alpha > 0$. We construct a random measure on $\mathbb{R}$ with spectral density $f_\Gamma(\omega)$ that behaves like $|\omega|^\alpha$ as $\omega \to 0$. Let A be a nonnegative continuous random variable with bounded probability density function f_α and let U_a be for every $a > 0$ a random variable uniformly distributed on $[0, 2\pi/a]$, i.e. $U_a \sim \mathrm{Uniform}[0, 2\pi/a]$. Let X_a be a stationary random closed set (see, e.g. [17, Chapter 2] or [5, Chapter 6]) given as the union of equidistant closed intervals shifted by U_a, i.e.

$$X_a = \bigcup_{n \in \mathbb{Z}} \left[(4n-1)\frac{\pi}{2a}, (4n+1)\frac{\pi}{2a} \right] + U_a.$$

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The volume fraction $\mathbb{P}(x \in X_a)$ is equal to $\frac{1}{2}$ for every $a > 0$. The covariance function $\mathrm{cov}_a(x) = \mathbb{P}(0, x \in X_a) - \frac{1}{4}$ is a triangle function, $\mathrm{cov}_a(x) = (1/2\pi)\arcsin(\cos(ax))$ for every $x \in \mathbb{R}$, $a > 0$. As an odd function it may be expanded into the following cosine series:

$$\mathrm{cov}_a(x) = \sum_{k=0}^{\infty} \frac{2}{(2k+1)^2\pi^2} \cos((2k+1)ax).$$

Let X be a doubly stochastic random closed set obtained form A and $\{X_a\}_{a>0}$ as $X = X_A$. The volume fraction of X is $\mathbb{P}(x \in X) = \frac{1}{2}$ and the covariance of X is $\mathrm{cov}(x) = \mathbb{E}_A \mathrm{cov}_A(x)$, where $\mathbb{E}_A$ is the expectation with respect to A. Hence,

$$\mathrm{cov}(x) = \sum_{k=0}^{\infty} \frac{2}{(2k+1)^2\pi^2} \mathbb{E}_A \cos((2k+1)Ax).$$

A random measure ζ on $\mathbb{R}$ with desired properties is given by the so-called volume measure of X, i.e. a random measure defined by $\zeta(B) = \nu_1(X \cap B)$ for every Borel set $B \subset \mathbb{R}$, where ν_1 is the Lebesgue measure on $\mathbb{R}$. By construction, ζ is second-order stationary with intensity $\lambda = \frac{1}{2}$ and covariance function $\mathrm{cov}(x)$. Let us now determine the Bartlett spectrum Γ of ζ. The expectation of cosines may be interpreted as the inverse Fourier transform of g_α, where $g_\alpha(a) = f_\alpha(|a|)$ for all $a \in \mathbb{R}$,

$$\mathbb{E}_A \cos(nAx) = \int_0^{\infty} \cos(nxa) f_\alpha(a)\mathrm{d}a = \frac{\sqrt{\pi}}{\sqrt{2}} \check{g}_\alpha(nx).$$

Since $\mathrm{cov}(x)$ is integrable, Γ is absolutely continuous with spectral density f_Γ given by the Fourier transform of the covariance function,

$$f_\Gamma(\omega) = \widehat{\mathrm{cov}}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}x\omega} \mathrm{cov}(x)\mathrm{d}x.$$

By previous considerations, the Fourier transform of $\mathbb{E}_A \cos(nAr)$ is

$$\mathbb{E}_A \widehat{\cos(nAr)}(\omega) = \frac{\sqrt{\pi}}{n\sqrt{2}} f_\alpha\left(\frac{|\omega|}{n}\right).$$

Thus,

$$f_\Gamma(\omega) = \sum_{k=0}^{\infty} \frac{\sqrt{2\pi}}{(2k+1)^3\pi^2} f_\alpha\left(\frac{|\omega|}{2k+1}\right),$$

which is a uniformly convergent series. Taking $f_\alpha(x) = (1/\Gamma(1+\alpha))x^\alpha \mathrm{e}^{-x}$ for $x > 0$, i.e. the gamma distribution determined by shape parameter $1+\alpha$ and scale parameter 1, we finally obtain

$$f_\Gamma(\omega) = |\omega|^\alpha \frac{\sqrt{2\pi}}{\pi^2\Gamma(1+\alpha)} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{3+\alpha}} \mathrm{e}^{-|\omega|/(2k+1)}.$$

Hence, the spectral density behaves as $|\omega|^\alpha$ as $\omega \to 0$. Note that by Corollary 3.2 (a), ζ is not long-range dependent.

Example 4.2. Here, we introduce a random measure with purely discrete spectrum separated by a positive distance form the origin $\mathbf{0}$. It is based on a multidimensional extension of the construction from the previous example. Let p be such that $\frac{1}{2} \le p < 1$ and $U \sim \mathrm{Uniform}[0, 2\pi]$. We define a stationary random closed set X in $\mathbb{R}$ by

$$X = \bigcup_{n \in \mathbb{Z}} [-p\pi + 2n\pi, p\pi + 2n\pi] + U.$$

Its volume fraction is clearly $\mathbb{P}(x \in X) = p$ and the covariance function is an even continuous piecewise linear 2π periodic function given on $[0, 2\pi]$ by

$$\mathrm{cov}(x) = \begin{cases} p - p^2 - \dfrac{r}{2\pi} & \text{for } x \in [0, 2\pi(1-p)], \\ 2p - 1 - p^2 & \text{for } x \in (2\pi(1-p), 2p\pi], \\ p - 1 - p^2 + \dfrac{r}{2\pi} & \text{for } x \in (2p\pi, 2\pi]. \end{cases}$$

It can be expanded into the Fourier cosine series,

$$\mathrm{cov}(x) = \sum_{n=1}^{\infty} \frac{2\sin^2(np\pi)}{n^2\pi^2} \cos(nx),$$

where the absolute term is 0 since $\int_0^{\pi} \mathrm{cov}(x)\mathrm{d}x = 0$. A straightforward multidimensional extension of this construction to a random closed set X in $\mathbb{R}^d$ is given by $X = X_1 \times \cdots \times X_d$, where $X_1, \ldots, X_d$ are independent and identically distributed random closed sets of the previous type. The covariance function of X is then

$$\mathrm{cov}(\boldsymbol{x}) = \prod_{i=1}^{d} \mathrm{cov}(x_i) = \prod_{i=1}^{d} \sum_{n_i=1}^{\infty} \frac{2\sin^2(n_i p\pi)}{n_i^2\pi^2} \cos(n_i x_i) = \sum_{\boldsymbol{n} \in \mathbb{N}^d} c_{\boldsymbol{n}} \prod_{i=1}^{d} \cos(n_i x_i),$$

where $\boldsymbol{n} = (n_1, \ldots, n_d)$ and $c_{\boldsymbol{n}} = (2^d/\pi^{2d}) \prod_{i=1}^{d} \sin^2(n_i p\pi)/n_i^2$. Let ζ be a random measure on $\mathbb{R}^d$ defined as the volume measure of X, $\zeta(B) = \nu_d(X \cap B)$ for every Borel set $B \subset \mathbb{R}^d$. Clearly, its intensity is $\lambda = p$ and covariance function is $\mathrm{cov}(\boldsymbol{x})$. The Bartlett spectrum Γ of ζ can be determined from (2.3) as

$$\Gamma = \frac{(2\pi)^{d/2}}{2^d} \sum_{\boldsymbol{n} \in \mathbb{N}^d,\, i_1, \ldots, i_d \in \{-1, 1\}} c_{\boldsymbol{n}} \delta_{(n_1 i_1, \ldots, n_d i_d)}.$$

Note that ζ is not long-range dependent as follows from Theorem 3.1 since $\Gamma(\frac{1}{2}B^d) = 0$.

Example 4.3. Finally, we construct a random measure determined by a parameter $\alpha \in (0, d)$ which has absolutely continuous Bartlett spectrum with density $f_\Gamma(\boldsymbol{\xi})$ that behaves like $\|\boldsymbol{\xi}\|^{d-\alpha}$ as $\boldsymbol{\xi} \to \mathbf{0}$.

Let us start with the 0-level excursion set X of a stationary isotropic Gaussian random field Z in $\mathbb{R}^d$ determined by the mean $\mathbb{E}\, Z = 0$ and covariance function

$$\mathrm{cov}_Z(r) = (1 + |r|^2)^{-\alpha/2},$$

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where $\alpha > 0$ is a parameter; see, e.g. [1]. A stationary isotropic random field Z with such a covariance function is known as the Cauchy model and it is a part of the so-called Cauchy class; see [10]. The 0-level excursion set X of Z is defined by $X = \{\boldsymbol{x} \in \mathbb{R}^d \mid Z(\boldsymbol{x}) \geq 0\}$ and it is a stationary random closed set; see, e.g. [5, Section 6.6.3]. It is easy to check (e.g. with the help of [5, Equations (6.157) and (6.159)]) that $\mathbb{P}(\boldsymbol{x} \in X) = \frac{1}{2}$ and $\operatorname{cov}(\boldsymbol{x}) = \mathbb{P}(\boldsymbol{0}, \boldsymbol{x} \in X) - \frac{1}{4} = (1/2\pi)\arcsin(\operatorname{cov}_Z(\|\boldsymbol{x}\|))$.

A random measure ζ with the desired properties is given by the volume measure of X, $\zeta(B) = \nu_d(X \cap B)$ for every Borel set $B \subset \mathbb{R}^d$. It follows immediately that ζ is isotropic and second-order stationary with intensity $\lambda = \frac{1}{2}$ and covariance function

$$\operatorname{cov}(\boldsymbol{x}) = \frac{1}{2\pi} \arcsin \frac{1}{(1 + \|\boldsymbol{x}\|^2)^{\alpha/2}}.$$

First, we inspect the connection between α and the long-range dependence of ζ. By (3.1),

$$\frac{\operatorname{var} \zeta(aB^d)}{\nu_d(aB^d)} = \int_{\mathbb{R}^d} \frac{(\mathbf{1}_{aB^d} * \mathbf{1}^*_{aB^d})(\boldsymbol{x})}{a^d c_d} \operatorname{cov}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}.$$

A simple calculation shows that $(\mathbf{1}_{aB^d} * \mathbf{1}^*_{aB^d})(\boldsymbol{x}) = a^d \nu_d(B^d \cap (B^d - \boldsymbol{x}/a))$ and

$$\nu_d(B^d \cap (B^d - \boldsymbol{y})) = c_d I_{1-\|y\|^2/4}\bigg(\frac{d+1}{2}, \frac{1}{2}\bigg)$$

whenever $\|\boldsymbol{y}\| \leq 2$, and $\nu_d(B^d \cap (B^d - \boldsymbol{y})) = 0$ otherwise, where

$$c_d = \nu_d(B^d) = \frac{\pi^{d/2}}{\Gamma(d/2+1)}$$

and $I_x(p, q)$, $p, q > 0$, is the regularized incomplete beta function; see, e.g. [19, Section 11.3] for its definition and basic properties. Therefore, we may use spherical coordinates and integrate over angles to obtain

$$\frac{\operatorname{var} \zeta(aB^d)}{\nu_d(aB^d)} = \frac{dc_d}{2\pi} \int_0^{2a} r^{d-1} I_{1-r^2/4a^2}\bigg(\frac{d+1}{2}, \frac{1}{2}\bigg) \arcsin \frac{1}{(1+r^2)^{\alpha/2}} \mathrm{d}r.$$

For $\alpha > d$, the integral is bounded since $I_x(p, q) \in [0, 1]$, $\arcsin((1 + r^2)^{-\alpha/2}) \sim r^{-\alpha}$ as $r \to \infty$, and $r^{d-1-\alpha}$ is integrable at ∞. Therefore, ζ is not long-range dependent. Regarding $\alpha \leq d$, it is useful to substitute $u = r/(2a)$ in order to obtain

$$\frac{\operatorname{var} \zeta(aB^d)}{\nu_d(aB^d)} = \frac{dc_d}{2\pi}(2a)^d \int_0^1 u^{d-1} I_{1-u^2}\bigg(\frac{d+1}{2}, \frac{1}{2}\bigg) \arcsin \frac{1}{(1+4a^2u^2)^{\alpha/2}} \mathrm{d}u.$$

Since $\arcsin x \sim x$ as $x \to 0_+$ and $(2a)^\alpha(1 + 4a^2u^2)^{-\alpha/2}$ increases to $u^{-\alpha}$ as $a \to \infty$, Lebesgue dominated and monotone convergence theorems imply that

$$\frac{\operatorname{var} \zeta(aB^d)}{\nu_d(aB^d)} \sim \frac{dc_d}{2\pi}(2a)^{d-\alpha} \int_0^1 u^{d-1-\alpha} I_{1-u^2}\bigg(\frac{d+1}{2}, \frac{1}{2}\bigg) \mathrm{d}u \quad \text{as } a \to \infty.$$

For $\alpha \in (0, d)$, the integral is convergent to some positive value and $(2a)^{d-\alpha} \to \infty$ as $a \to \infty$. Hence, ζ is long-range dependent. For $\alpha = d$, we may bound $\arcsin x$ from below by x and

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use the similar monotone convergence argument to conclude that the limit is again ∞ since the integral is now divergent. Therefore, ζ is long-range dependent.

Now, let us focus on the behaviour of the Bartlett spectrum Γ. We proceed similarly as in [9, Section 56]. The covariance function $\mathrm{cov}(\boldsymbol{x})$ can be regarded as a tempered distribution, i.e. the continuous liner functional on the Schwartz space $\mathcal{S}$ of rapidly decreasing functions on $\mathbb{R}^d$, for all complex values of α. Moreover, it may be regarded analytic in α when $\operatorname{Re}\alpha > 0$, which means that for every test function $\psi \in \mathcal{S}$, the complex-valued function $\alpha \mapsto \mathrm{cov}(\psi)$ is analytic in α when $\operatorname{Re}\alpha > 0$. This analytic quality extends the usual concept in a natural way including the analytic continuation; see [9, Section 24]. The Fourier transform of the covariance in the distributional sense is equal to the Bartlett spectrum Γ of ζ and it is again a tempered distribution analytic in α when $\operatorname{Re}\alpha > 0$. We may determine its explicit form by analytic continuation from larger values of α, where the covariance is integrable and its Fourier transform is given by the ordinary integral formula. For $\operatorname{Re}\alpha > d$, the covariance function is integrable and, hence, Γ is absolutely continuous with continuous density f_Γ given by

$$f_\Gamma(\boldsymbol{\xi}) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}\boldsymbol{x}\boldsymbol{\xi}} \,\mathrm{cov}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \frac{1}{(2\pi)^{d/2+1}} \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}\boldsymbol{x}\boldsymbol{\xi}} \arcsin \frac{1}{(1+\|\boldsymbol{x}\|^2)^{\alpha/2}} \mathrm{d}\boldsymbol{x}.$$

Expressing arcsin by its Taylor series and interchanging the sum and integral due to the Lebesgue monotone convergence theorem yields

$$f_\Gamma(\boldsymbol{\xi}) = \frac{1}{(2\pi)^{d/2+1}} \sum_{k=0}^{\infty} C_k \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}\boldsymbol{x}\boldsymbol{\xi}} \frac{1}{(1+\|\boldsymbol{x}\|^2)^{\alpha(2k+1)/2}} \mathrm{d}\boldsymbol{x},$$

where $C_k = \binom{2k}{k}/4^k(2k+1)$ for $k = 0, 1, \dots$ are coefficients of the Taylor expansion of arcsin. Then we use

$$\frac{1}{(1+s^2)^\beta} = \frac{1}{\Gamma(\beta)} \int_0^\infty \mathrm{e}^{-s^2 u} \mathrm{e}^{-u} u^{\beta-1}\,\mathrm{d}u,$$

where $\Gamma(\cdot)$ denotes the gamma function, valid for any real s and any β with positive real part, in order to obtain

$$f_\Gamma(\boldsymbol{\xi}) = \frac{1}{(2\pi)^{d/2+1}} \sum_{k=0}^{\infty} \frac{C_k}{\Gamma(\alpha(2k+1)/2)} \int_{\mathbb{R}^d} \int_0^\infty \mathrm{e}^{-\mathrm{i}\boldsymbol{x}\boldsymbol{\xi}} \mathrm{e}^{-|\boldsymbol{x}|^2 u} \mathrm{e}^{-u} u^{\alpha(2k+1)/2-1} \mathrm{d}u \mathrm{d}\boldsymbol{x}.$$

By interchanging the order of integration, the Fourier transform of each term may be computed explicitly since it is just the transform of the Gaussian. Thus, finally,

$$f_\Gamma(\boldsymbol{\xi}) = \frac{1}{(2\pi)2^{d/2}} \sum_{k=0}^{\infty} \frac{C_k}{\Gamma(\alpha(2k+1)/2)} \int_0^\infty \mathrm{e}^{-\|\boldsymbol{\xi}\|^2/4u} \mathrm{e}^{-u} u^{\alpha(2k+1)/2-d/2-1} \mathrm{d}u.$$

For any $\|\boldsymbol{\xi}\| > 0$ and any $\operatorname{Re}\alpha > 0$, the right-hand side is given by the convergent series and it is an analytic function of α. Hence, the analytic continuation of the Bartlett spectrum Γ in the distributional sense is absolutely continuous with density f_Γ given by the previous expression. To see the convergence and analyticity it is useful to split the sum into two parts determined by k_0 chosen such that $\operatorname{Re}\alpha(2k_0+1)/2 - d/2 \le 0$ and $\operatorname{Re}\alpha(2k+1)/2 - d/2 > 0$ for all $k > k_0$. The first part then contains a finite number of convergent integrals and the second part may be bounded by a convergent series,

$$\sum_{k=k_0+1}^{\infty} \frac{C_k}{\Gamma(\alpha(2k+1)/2)} \int_0^\infty \mathrm{e}^{-\|\boldsymbol{\xi}\|^2/4u} \mathrm{e}^{-u} u^{\alpha(2k+1)/2-d/2-1} \mathrm{d}u \\ \leq \sum_{k=k_0+1}^{\infty} C_k \frac{\Gamma(\alpha(2k+1)/2-d/2)}{\Gamma(\alpha(2k+1)/2)}$$

since

$$\int_0^\infty \mathrm{e}^{-\|\boldsymbol{\xi}\|^2/4u} \mathrm{e}^{-u} u^{\alpha(2k+1)/2-d/2-1} \mathrm{d}u \leq \int_0^\infty \mathrm{e}^{-u} u^{\alpha(2k+1)/2-d/2-1} \mathrm{d}u = \Gamma\bigg(\frac{\alpha}{2}(2k+1) - \frac{d}{2}\bigg).$$

Let us now focus on the asymptotic behaviour of $f_\Gamma(\boldsymbol{\xi})$ as $\boldsymbol{\xi} \to 0$. For $0 < \alpha < d$, we again split the sum into two parts determined by the index k_0 chosen such that $\alpha(2k_0+1)/2 - d/2 \leq 0$ and $\alpha(2k+1)/2 - d/2 > 0$ for all $k > k_0$. The first part is, after the substitution $u = \|\boldsymbol{\xi}\|^2/t$, given by

$$\frac{1}{(2\pi)2^{d/2}} \sum_{k=0}^{k_0} \frac{C_k}{\Gamma(\alpha(2k+1)/2)} \int_0^\infty \mathrm{e}^{-\|\boldsymbol{\xi}\|^2/4u} \mathrm{e}^{-u} u^{\alpha(2k+1)/2-d/2-1} \mathrm{d}u \\ = \|\boldsymbol{\xi}\|^{\alpha-d} \frac{1}{(2\pi)2^{d/2}} \sum_{k=0}^{k_0} \frac{C_k}{\Gamma(\alpha(2k+1)/2)} \|\boldsymbol{\xi}\|^{\alpha 2k} \int_0^\infty \mathrm{e}^{-t/4} \mathrm{e}^{-\|\boldsymbol{\xi}\|^2/t} t^{-\alpha(2k+1)/2+d/2-1} \mathrm{d}t.$$

By the additional assumption $\alpha(2k_0+1)/2 - d/2 < 0$, the integrals converge as a consequence of the Lebesgue monotone convergence theorem to

$$\int_0^\infty \mathrm{e}^{-t/4} t^{-\alpha(2k+1)/2+d/2-1} \mathrm{d}u = 2^{d-\alpha(2k+1)} \Gamma\bigg(\frac{d}{2} - \frac{\alpha}{2}(2k+1)\bigg).$$

Only the first term $k = 0$ is not diminished by the $\|\boldsymbol{\xi}\|^{\alpha 2k}$ factor and we obtain

$$\frac{1}{(2\pi)2^{d/2}} \sum_{k=0}^{k_0} \frac{C_k}{\Gamma(\alpha(2k+1)/2)} \int_0^\infty \mathrm{e}^{-\|\boldsymbol{\xi}\|^2/4u} \mathrm{e}^{-u} u^{\alpha(2k+1)/2-d/2-1} \mathrm{d}u \\ \sim \|\boldsymbol{\xi}\|^{\alpha-d} \frac{\Gamma((d-\alpha)/2)}{(2\pi)2^{\alpha-d/2}\Gamma(\alpha/2)} \quad \text{as } \boldsymbol{\xi} \to 0.$$

The equality case $\alpha(2k_0+1)/2 - d/2 = 0$ leads to the same result but one must show the boundedness of the last term in the sum differently by the same method as is used later for $\alpha = d$. For the second part, one can use the Lebesgue monotone convergence theorem directly and obtain

$$\frac{1}{(2\pi)2^{d/2}} \sum_{k=k_0+1}^{\infty} \frac{C_k}{\Gamma(\alpha(2k+1)/2)} \int_0^\infty \mathrm{e}^{-\|\boldsymbol{\xi}\|^2/4u} \mathrm{e}^{-u} u^{\alpha(2k+1)/2-d/2-1} \mathrm{d}u \\ \to \frac{1}{(2\pi)2^{d/2}} \sum_{k=k_0+1}^{\infty} C_k \frac{\Gamma(\alpha(2k+1)/2-d/2)}{\Gamma(\alpha(2k+1)/2)} \\ < \infty \quad \text{as } \boldsymbol{\xi} \to \mathbf{0}.$$

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Hence, in total, for $0 < \alpha < d$, we obtain

$$f_\Gamma(\boldsymbol{\xi}) = \|\boldsymbol{\xi}\|^{\alpha-d} \frac{\Gamma((d-\alpha)/2)}{(2\pi)2^{\alpha-d/2}\Gamma(\alpha/2)}(1+h(\|\boldsymbol{\xi}\|)),$$

where $h(\|\boldsymbol{\xi}\|)$ is a continuous function of $\boldsymbol{\xi}$ that converges to 0 with $\|\boldsymbol{\xi}\|$. As a consequence of Corollary 3.2(ii), ζ is long-range dependent, which we already know from the previous analysis of the covariance function.

When $\alpha = d$, we take separately only the first term of the sum in f_Γ. Here, after the substitution $u = \|\boldsymbol{\xi}\|^2/t$, we obtain

$$\frac{1}{(2\pi)2^{d/2}\Gamma(d/2)} \int_0^\infty \mathrm{e}^{-t/4}\mathrm{e}^{-\|\boldsymbol{\xi}\|^2/t}t^{-1}\,\mathrm{d}t.$$

For $\|\boldsymbol{\xi}\| < 1$, the integral may be split into three parts $\int_0^{\|\boldsymbol{\xi}\|^2} + \int_{\|\boldsymbol{\xi}\|^2}^1 + \int_1^\infty$. It is easy to see that the first and last integrals are bounded independently on $\|\boldsymbol{\xi}\|$. For the middle integral, using the mean value theorem we obtain

$$\int_{\|\boldsymbol{\xi}\|^2}^1 \mathrm{e}^{-t/4}\mathrm{e}^{-\|\boldsymbol{\xi}\|^2/t}t^{-1}\,\mathrm{d}t = \mathrm{e}^{-u_{\boldsymbol{\xi}}/4-\|\boldsymbol{\xi}\|^2/u_{\boldsymbol{\xi}}} \int_{\|\boldsymbol{\xi}\|^2}^1 t^{-1}\,\mathrm{d}t = \mathrm{e}^{-u_{\boldsymbol{\xi}}/4-\|\boldsymbol{\xi}\|^2/u_{\boldsymbol{\xi}}} \log(\|\boldsymbol{\xi}\|^{-2}),$$

where $u_{\boldsymbol{\xi}} \in [\|\boldsymbol{\xi}\|, 1]$ depends on $\|\boldsymbol{\xi}\|$. The factor $\mathrm{e}^{-u_{\boldsymbol{\xi}}/4-\|\boldsymbol{\xi}\|^2/u_{\boldsymbol{\xi}}}$ is always in $[\mathrm{e}^{-1-\|\boldsymbol{\xi}\|^2/4}, \mathrm{e}^{-\|\boldsymbol{\xi}\|}]$, which converges to $[\mathrm{e}^{-1}, 1]$ as $\boldsymbol{\xi} \to \mathbf{0}$. The remaining terms of the sum in f_Γ again converge, now to the constant

$$\frac{1}{(2\pi)2^{d/2}} \sum_{k=1}^\infty C_k \frac{\Gamma(dk)}{\Gamma(dk+d/2)}.$$

Therefore, for $\alpha = d$,

$$f_\Gamma(\boldsymbol{\xi}) \sim \log(\|\boldsymbol{\xi}\|^{-2}) \frac{1}{(2\pi)2^{d/2}\Gamma(d/2)}(1-h(\|\boldsymbol{\xi}\|)) \quad \text{as } \boldsymbol{\xi} \to \mathbf{0},$$

where $h(\|\boldsymbol{\xi}\|)$ is asymptotically contained in the interval $[0, 1-\mathrm{e}^{-1}]$. Again, Corollary 3.2(ii) consistently with previous considerations yields long-range dependence of ζ in this case.

For $\alpha > d$, one uses the Lebesgue monotone convergence theorem for all terms in order to obtain

$$f_\Gamma(\boldsymbol{\xi}) \to \frac{1}{(2\pi)2^{d/2}} \sum_{k=0}^\infty C_k \frac{\Gamma(\alpha(2k+1)/2 - d/2)}{\Gamma(\alpha(2k+1)/2)} < \infty \quad \text{as } \boldsymbol{\xi} \to \mathbf{0}.$$

Thus, the density of the Bartlett spectrum is bounded in some neighbourhood of $\mathbf{0}$ and ζ is not long-range dependent by Corollary 3.2(i).

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