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RAMANUJAN SERIES UPSIDE-DOWN

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Abstract

We prove that there is a correspondence between Ramanujan-type formulas for $1/\pi$ and formulas for Dirichlet L-values. Our method also allows us to reduce certain values of the Epstein zeta function to rapidly converging hypergeometric functions. The Epstein zeta functions were previously studied by Glasser and Zucker.

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1. Introduction

Quantities such as π^2 and the Dirichlet L-values are fundamental constants which appear in many areas of mathematics and physics. It is interesting to relate them to hypergeometric functions, which are important because of their applications in number theory. For instance, Ramanujan discovered many famous hypergeometric formulas for $1/\pi$ [17]. The following example is originally due to Bauer [3], but is easily derived with Ramanujan's methods:

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{6n}} \binom{2n}{n}^3 \left(\frac{1}{2} + 2n\right). \tag{1}$$

Such results are connected to class number problems, and to the theory of complex multiplication [6], [8]. In this paper we describe identities which are closely related to Ramanujan's formulas. Our first example can be constructed by manipulating (1). Let $(1/2 + 2n) \mapsto (1/2 - 2n)$, flip the rest of the summand 'upside-down', insert a factor of $1/n^3$, and perform the summation for $n \geq 1$. Then we obtain a *companion series identity*:

$$8L_{-4}(2) = \sum_{n=1}^{\infty} \frac{(-1)^n 2^{6n}}{n^3 \binom{2n}{n}^3} \left(\frac{1}{2} - 2n\right). \tag{2}$$

As usual $L_{-4}(2) = 1 - \frac{1}{3^2} + \frac{1}{5^2} \ldots$ is Catalan's constant, $L_k(s) := \sum_{n=1}^{\infty} \chi_k(n)/n^s$ denotes the general Dirichlet L-series, and

$$\chi_k(n) = \left(\frac{k}{n}\right)$$

is the Jacobi symbol. Based on this example, we might venture a guess that the same procedure should transform each of Ramanujan's formulas into identities involving Dirichlet L-values. We prove that this guess is correct when certain technical conditions are added. It is important to note that at least nine related formulas already exist in the literature. The individual formulas were discovered piecemeal with computational techniques, and mostly proved by variations of the Wilf–Zeilberger method. We mention proofs due to Zeilberger [21], Guillera [11, 12, 14], and the Hessami Pilehroods [15]. Sun also conjectured several identities from numerical experiments [18]. We give unified proofs of all of these results and conjectures in Theorem 3. We also show how to construct vast numbers of irrational formulas (such as (65) and the examples in Table 5), which were previously unknown. We describe our results in greater detail below.

Ramanujan identified seventeen formulas for $1/\pi$ [17]. His identities all have the following form:

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} \frac{(s)_n(\frac{1}{2})_n(1-s)_n}{(1)_n^3}(a+bn)z^n, \tag{3}$$

where $(x)_n = \Gamma(x+n)/\Gamma(x)$. Each example has $s \in \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}\}$, with (a, b, z) being parameterized by modular functions [6], [8]. When $s = \frac{1}{6}$, $z = 1/j(\tau)$, where $j(\tau)$ is the j-invariant, and the expressions for a and b involve Eisenstein series. If we preserve the modular parameterizations for (a, b, z), then the general *companion series* is given by

$$\sum_{n=1}^{\infty} \frac{(1)_n^3}{(s)_n(\frac{1}{2})_n(1-s)_n} \frac{(a-bn)}{n^3} z^{-n}. \tag{4}$$

When n is large, standard asymptotics show that

$$\frac{(s)_n(\frac{1}{2})_n(1-s)_n}{(1)_n^3} \sim \frac{\sin(\pi s)}{(\pi n)^{3/2}}.$$

It follows that (3) and (4) can only converge simultaneously if $|z| = 1$ (notice that (1) and (2) occur when $s = \frac{1}{2}$ and $(a, b, z) = (\frac{1}{2}, 2, -1)$). Divergent cases make sense, as long as each divergent infinite series is replaced by an analytically-continued hypergeometric function. Once of the main goals of this work, is to transform divergent formulas for $1/\pi$, into interesting convergent formulas for Dirichlet L-values.

Suppose that $s \in \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\}$. Then Propositions 2 and 3 reduce many values of the companion series (4), to linear combinations of two Epstein zeta functions and elementary constants. In general, once we fix the modular parameterizations for (a, b, z) in (4), then Propositions 2 and 3 impose restrictions on the domain of the modular functions (see the constraints on Equations (50) and (51)). This means there are fewer *potential* companion series evaluations, compared to the number of possible

Ramanujan-type formulas from (3). Finally, if the linear combination of Epstein zeta functions reduces to Dirichlet L-values, which is not automatic, then the companion series also reduces to Dirichlet L-values. Proofs are based upon a new idea called *completing the hypergeometric function*, which we outline in Section 3. The approach fails completely when $s = \frac{1}{6}$, and we describe the rationale for this failure at the end of Section 3. The Epstein zeta functions which appear have been studied by Glasser and Zucker [10]. Following their notation, define

$$S(A, B, C; t) := \sum_{(n,m)\neq(0,0)} \frac{1}{(An^2 + Bnm + Cm^2)^t}. \tag{5}$$

We demonstrate a calculation by proving (2). Set $q = -e^{-\pi\sqrt{2}}$ in (46). Then $(a, b, z) = (\frac{1}{2}, 2, -1)$. By Equation (50), we have

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \frac{(1)_n^3}{(\frac{1}{2})_n^3}\left(\frac{1}{2} - 2n\right) = \frac{32\sqrt{2}}{\pi^2}(S(1, 0, 8; 2) - S(3, 4, 4; 2)).$$

The key to completing the proof, is to reduce $S(A, B, C; t)$ to Dirichlet L-values. It is fortunate that this is a well-known problem. Let us briefly recall that quadratic forms with fixed discriminant $D = B^2 - 4AC$, are partitioned into equivalence classes under the action of $SL_2(\mathbb{Z})$. We say that quadratic forms of discriminant $D < 0$ have *one class per genus*, when disjoint classes of forms always represent disjoint sets of integers. Glasser and Zucker conjectured that $S(A, B, C; t)$ reduces to Dirichlet L-values, if and only if $An^2 + Bnm + Cm^2$ lives in a class of quadratic forms with one class per genus. Despite the fact that Zucker and Robertson discovered a few strange counterexamples to this conjecture [23], most evidence suggests that the original conjecture is 'more-or-less' correct. Every interesting companion series boils down to two values of $S(A, B, C; 2)$, and elementary constants. The proof of (2) follows from showing

$$S(1, 0, 8; 2) = \frac{7\pi^2}{48}L_{-8}(2) + \frac{\pi^2}{8\sqrt{2}}L_{-4}(2),$$
$$S(3, 4, 4; 2) = \frac{7\pi^2}{48}L_{-8}(2) - \frac{\pi^2}{8\sqrt{2}}L_{-4}(2).$$

Notice that $S(3, 4, 4; t)$ does not correspond to a reduced quadratic form ($C \geq A \geq |B|$), but it is possible to show that $S(3, 4, 4; t) = S(3, 2, 3; t)$. This type of reasoning explains all of the previously known companion series formulas, and all of the results in Theorems 3 and 4.

There are many instances where it is probably impossible to express $S(A, B, C; t)$ in terms of Dirichlet L-values. Then our method produces nontrivial hypergeometric formulas for $S(A, B, C; 2)$. For example, set $q = -e^{-\pi/3}$ in (46). After some work we obtain

$$\frac{48}{\pi^2}S(1, 0, 36; 2) = \frac{140}{27}L_{-4}(2) + \frac{13}{\sqrt{3}}L_{-3}(2) - \sum_{n=1}^{\infty} \frac{(1)_n^3}{(\frac{1}{2})_n^3} \frac{(a - bn)}{n^3} z^{-n}, \tag{6}$$

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where

$$\begin{aligned} z &= -8(74977 + 40284r + 21644r^2 + 11629r^3), \\ a &= \tfrac{1}{18}(1038 + 558r + 300r^2 + 161r^3), \\ b &= \tfrac{1}{3}(387 + 208r + 112r^2 + 60r^3), \end{aligned}$$

and $r = \sqrt[4]{12}$. Formula (6) converges very rapidly because $z \approx -2.4 \times 10^6$. The infinite series can either be expressed as a ${}_5F_4$ function, or as a linear combination of two ${}_4F_3$'s. In either case, this partially resolves a question of Zucker[1] and McPhedran [22], who asked whether or not $S(1, 0, 36; t)$ reduces to known quantities. See Section 5 for the proof of (6), and for additional examples.

2. Review of Ramanujan's formulas

We begin with a brief review of Ramanujan's formulas. Suppose that (3) holds for certain values of (a, b, z) and s. Since we are allowing the possibility that $|z| > 1$, the identity is best interpreted as:

$$\frac{1}{\pi} = bz\frac{d}{dz}{}_3F_2\left(\begin{matrix} s, \ \frac{1}{2}, \ 1-s \\ 1, 1 \end{matrix} \,\middle|\, z\right) + a\, {}_3F_2\left(\begin{matrix} s, \ \frac{1}{2}, \ 1-s \\ 1, 1 \end{matrix} \,\middle|\, z\right). \tag{7}$$

If we generically assume that s is a fixed constant, then only two out of the three parameters (a, b, z) can be chosen independently. For brevity we use the notation

$$y_0(z) := {}_3F_2\left(\begin{matrix} s, \ \frac{1}{2}, \ 1-s \\ 1, 1 \end{matrix} \,\middle|\, z\right).$$

Let us suppose that q and z are related by the differential equation:

$$\frac{dq}{dz} = \frac{q}{z\sqrt{1-z}\, y_0(z)}, \tag{8}$$

subject to the initial condition that $z = 0$ when $q = 0$. Then we choose a and b to be given by:

$$a = \frac{1 + q\log|q|\frac{d}{dq}\log y_0(z)}{\pi\, y_0(z)}, \qquad b = -\frac{\log|q|}{\pi}\sqrt{1-z}. \tag{9}$$

Once the parametrization for b is fixed, the formula for a follows automatically from solving (7), and then simplifying the differential using (8). Thus for many choices of z, we can (in principle) calculate values of a and b which make (7) valid. Ramanujan's miraculous observation is that (a, b, z) can be algebraic *simultaneously*.

The parameters in (7) can be evaluated using the theory of modular forms. First express z in terms of q by integrating and then inverting (8). The inverse expressions

[1] Zucker's dream is to resolve $S(1, 0, 36; t)$ in terms of Dirichlet L-values with complex characters.

are related to theta functions if $s \in \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}\}$. For instance when $s = \frac{1}{2}$, we have

$$z = 4\frac{\theta_3^4(-q)}{\theta_3^4(q)}\left(1 - \frac{\theta_3^4(-q)}{\theta_3^4(q)}\right), \tag{10}$$

where

$$\theta_3(q) := \sum_{n=-\infty}^{\infty} q^{n^2}.$$

If we let $q = e^{2\pi i\tau}$, then z is a weight-zero modular function in τ. It follows from the theory of complex multiplication that z is algebraic if τ is a quadratic irrational in the upper half plane. In those instances, Equation (9) implies that b is also algebraic, because $b = -\mathrm{Im}(\tau)\sqrt{1-z}$. In order to calculate a, we require formulas such as

$$y_0(z) = \theta_3^4(q), \tag{11}$$

which are only valid if q lies in a neighborhood of zero[1]. If we use eta product expansions (in this case $\theta_3(q) = \eta^5(q^2)/\eta^2(q)\eta^2(q^4)$), and then substitute (11) into (9), we arrive at an expression involving theta functions and Eisenstein series, which can also be explicitly evaluated. It is typically quite painful to calculate z and a, however various examples of these calculations are given in [2, 6–8].

PROPOSITION 1. *Assume that (a, b, z) and q are related by (8) and (9). Suppose that $f(z)$ is a differentiable function, and let*

$$\phi_f(q) = \frac{f(z)}{y_0(z)}.$$

Then

$$af(z) + bz\frac{df(z)}{dz} = \frac{1}{\pi}\left(\phi_f(q) - \log|q|\, q\frac{d\phi_f(q)}{dq}\right). \tag{12}$$

PROOF. From the right-hand side we have

$$\begin{aligned}
\frac{1}{\pi}\left(\phi_f(q) - \log|q|\, q\frac{d\phi_f(q)}{dq}\right) &= \frac{1}{\pi}\left(\frac{f(z)}{y_0(z)} - \log|q|\, q\frac{d}{dq}\frac{f(z)}{y_0(z)}\right)\\
&= \frac{1}{\pi}\frac{f(z)}{y_0(z)} - \log|q|\frac{q}{\pi y_0^2(z)}\left(y_0(z)\frac{df(z)}{dq} - f(z)\frac{dy_0(z)}{dq}\right)\\
&= \frac{1}{\pi}\left(\frac{1}{y_0(z)} + \frac{\log|q|}{y_0^2(z)}q\frac{dy_0(z)}{dq}\right)f(z)\\
&\quad -\left(\frac{\log|q|}{\pi y_0(z)}\frac{q}{z}\frac{dz}{dq}\right)z\frac{df(z)}{dz}\\
&= af(z) + bz\frac{df(z)}{dz}.
\end{aligned}$$

The final step follows from (9). □

[1] Equation (11) holds when q lies in a neighborhood of zero. We can analytically continue the formula along a ray from $q = 0$ until we reach a value of q for which $z \in [1, \infty)$.

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Proposition 1 allows us to insert a factor of $(a + bn)$ into a power series. For example, if $f(z) = y_0(z)$, then $\phi_f(q) = 1$. We have

$$1 = \frac{1}{y_0(z)} \sum_{n=0}^{\infty} \frac{(s)_n(\frac{1}{2})_n(1-s)_n}{(1)_n^3} z^n.$$

By Proposition 1 this becomes

$$\frac{1}{\pi}\Big(1 - \log|q|\, q\frac{d}{dq}\Big) \times 1 = \Big(a + bz\frac{d}{dz}\Big) \times \sum_{n=0}^{\infty} \frac{(s)_n(\frac{1}{2})_n(1-s)_n}{(1)_n^3} z^n,$$

hence

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} \frac{(s)_n(\frac{1}{2})_n(1-s)_n}{(1)_n^3}(a+bn)z^n.$$

We typically need to obtain a q-series expansion for $f(z)/y_0(z)$ before applying Proposition 1.

3. Completing the hypergeometric function

In this section we introduce the idea of *completing a hypergeometric function*. Hypergeometric functions are typically defined by an infinite series, and then analytically continued to a slit plane. To complete a hypergeometric function, let $n \mapsto n + x$ in the series definition, and extend the sum over $n \in \mathbb{Z}$. To fix the notation, let $y_x(z)$ denote the *extended hypergeometric series*:

$$\begin{aligned} y_x(z) &:= \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_{n+x}(s)_{n+x}(1-s)_{n+x}}{(1)_{n+x}^3} z^{n+x} \\ &= z^x \frac{(\frac{1}{2})_x(1-s)_x(s)_x}{(1)_x^3} {}_4F_3\left(\left.\begin{matrix} 1,\ \frac{1}{2}+x,\ 1-s+x,\ s+x \\ 1+x, 1+x, 1+x \end{matrix}\right| z\right). \end{aligned} \tag{13}$$

Notice that $y_x(z)$ extends $y_0(z)$ to a function of two variables. Transformations for extended hypergeometric functions often arise as byproducts when one discovers Wilf–Zeilberger pairs [12]. The completed version of $y_0(z)$ is a formal sum

$$\sum_{n\in\mathbb{Z}} \frac{(s)_{n+x}(\frac{1}{2})_{n+x}(1-s)_{n+x}}{(1)_{n+x}^3} z^{n+x}, \tag{14}$$

which involves powers of z and z^{-1}. If we interpret the positive $(n \geq 0)$ and negative $(n < 0)$ halves of the sum as hypergeometric functions, then (14) becomes a well-defined function:

$$\begin{aligned} Y_x(z) := {}& z^x \frac{(\frac{1}{2})_x(1-s)_x(s)_x}{(1)_x^3} {}_4F_3\left(\left.\begin{matrix} 1,\ \frac{1}{2}+x,\ 1-s+x,\ s+x \\ 1+x, 1+x, 1+x \end{matrix}\right| z\right) \\ & - \frac{2x^3 z^{x-1}}{s(1-s)} \frac{(-\frac{1}{2})_x(s-1)_x(-s)_x}{(1)_x^3} {}_4F_3\left(\left.\begin{matrix} 1,\ 1-x,\ 1-x,\ 1-x \\ \frac{3}{2}-x, 2-s-x, 1+s-x \end{matrix}\right| \frac{1}{z}\right). \end{aligned} \tag{15}$$

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The ${}_4\mathrm{F}_3$ functions have branch cuts on $[1, \infty)$ and $[0, 1]$, respectively, and we take the branch cut of z^x on $[0, \infty)$. Thus $Y_x(z)$ is certainly analytic for $z \in \mathbb{C} \setminus [0, \infty)$. From (14) it is obvious that $Y_x(z)$ is periodic in x:

$$Y_x(z) = Y_{x+1}(z).$$

Periodicity in x extends to (15), because ${}_4\mathrm{F}_3$ functions always obey recurrence relations in their parameters. Below we prove that $Y_x(z)$ equals a trigonometric polynomial in x, and then we use this fact to develop a q-series expansion for the companion series in Theorem 1. Before proceeding, we note that our method applies to various additional hypergeometric functions. If we apply the same procedure to

$${}_2\mathrm{F}_1\left(\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix} \middle| z \right),$$

then we can recover a q-series formula due to Duke [9, Equation (2.2)].

Lemma 1. *Suppose that $s \in (0, 1)$ and $z \notin \{0, 1\}$. There exist functions $u := u(z)$ and $v := v(z)$ which are independent of x, such that*

$$Y_x(z) = y_0(z) \frac{e^{i\pi x} \sin^2 \pi s}{\cos \pi x(\cos^2 \pi x - \cos^2 \pi s)} (-u + (u + 1) \cos 2\pi x - iv \sin 2\pi x). \tag{16}$$

Proof. Consider the Picard–Fuchs operator which annihilates $y_0(z)$. Let

$$P := \left(z\frac{d}{dz}\right)^3 - z\left(z\frac{d}{dz} + \frac{1}{2}\right)\left(z\frac{d}{dz} + s\right)\left(z\frac{d}{dz} + 1 - s\right). \tag{17}$$

If convergence issues are ignored, then it is easy to show that P also annihilates (14). This allows us to extrapolate

$$PY_x(z) = 0. \tag{18}$$

It is possible to prove (18) using standard rules for differentiating hypergeometric functions, but we leave this as an exercise. Since P annihilates $Y_x(z)$, the function has the form

$$Y_x(z) = m_0(x)y^{(0)}(z) + m_1(x)y^{(1)}(z) + m_2(x)y^{(2)}(z), \tag{19}$$

where each $y^{(i)}$ is a linearly independent solution of $Py = 0$. The linear independence property implies that $m_i(x) = m_i(x + 1)$ for all i (if the m_i's are not periodic, then $Y_x(z) - Y_{x+1}(z) = 0$ leads to a linear dependence between $y^{(i)}$'s). We derive formulas for $m_i(x)$ below. The strategy is to select a particular value of z so that we can place an upper bound on $Y_x(z)$. Since the function decomposes into independent functions of x and z, we can use Equation (19) to bound each $m_i(x)$ independently. Then we deduce that each $m_i(x)$ has a terminating Fourier series after being multiplied by suitable trigonometric functions. We then conclude that (16) holds for all $z \notin \{0, 1\}$.

Suppose that $s \in (0, 1)$, and that z is not a singular point of $Y_x(z)$ (we exclude $z = 0$ and $z = 1$). Since $Y_x(z) = Y_{x+1}(z)$, we assume without loss of generality that $\mathrm{Re}(x) \in [0, 1)$. We claim that $Y_x(z)$ is meromorphic in x, with simple poles

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at $x \in \{s, \frac{1}{2}, 1-s\}$. To prove this, first recall that ${}_4F_3(a_1, a_2, a_3, a_4; b_1, b_2, b_3; z)$ is meromorphic with respect to each b_i, provided z is not a singular point [16, page 405]. Poles occur if $b_i \in \{0, -1, -2, \ldots\}$. Since $(\mathrm{Re}(x), s) \in [0,1) \times (0,1)$, it is easy to check that the quantities $\{1+x, \frac{3}{2}-x, 2-s-x, 1+s-x\}$ are never equal to zero or negative integers. Thus the ${}_4F_3$ functions in (15) do not contribute poles. Next observe

$$\frac{(-\frac{1}{2})_x(s-1)_x(-s)_x}{(1)_x^3} = \frac{\Gamma(-\frac{1}{2}+x)\Gamma(s-1+x)\Gamma(-s+x)}{\Gamma(\frac{1}{2})\Gamma(-s)\Gamma(s-1)\Gamma^3(1+x)},$$
$$\frac{(\frac{1}{2})_x(1-s)_x(s)_x}{(1)_x^3} = \frac{\Gamma(\frac{1}{2}+x)\Gamma(1-s+x)\Gamma(s+x)}{\Gamma(\frac{1}{2})\Gamma(1-s)\Gamma(s)\Gamma^3(1+x)}.$$

The first ratio of Pochhammer symbols contributes simple poles when $x \in \{s, \frac{1}{2}, 1-s\}$, and the second ratio of Pochhammer symbols is analytic for $(\mathrm{Re}(x), s) \in [0,1) \times (0,1)$. By the linear independence argument above, we conclude that $m_i(x)$ is at worst meromorphic with simple poles when $x \in \{s, \frac{1}{2}, 1-s\}$.

Now we show that $m_i(x) = O(|\mathrm{Im}(x)|^{-3/2} e^{-\pi\,\mathrm{Im}(x)})$ when $|\mathrm{Im}(x)|$ is sufficiently large. Let z be a negative real number in a compact subinterval of $(-1, 0)$. Then $z = \rho e^{\pi i}$ for some $\rho \in (0,1)$. Thus $|z^x| = |\rho^x e^{\pi i x}| = \rho^{\mathrm{Re}(x)} e^{-\pi\,\mathrm{Im}(x)} < e^{-\pi\,\mathrm{Im}(x)}$. Formula (15) becomes

$$|Y_x(z)| < e^{-\pi\,\mathrm{Im}(x)} \left| \frac{(\frac{1}{2})_x(1-s)_x(s)_x}{(1)_x^3} {}_4F_3\left(\begin{matrix} 1,\ \frac{1}{2}+x,\ 1-s+x,\ s+x \\ 1+x, 1+x, 1+x \end{matrix} \middle| z\right) \right.$$
$$\left. - \frac{2x^3 z^{-1}}{s(1-s)} \frac{(-\frac{1}{2})_x(s-1)_x(-s)_x}{(1)_x^3} {}_4F_3\left(\begin{matrix} 1,\ 1-x,\ 1-x,\ 1-x \\ \frac{3}{2}-x, 2-s-x, 1+s-x \end{matrix} \middle| \frac{1}{z}\right) \right|.$$

The terms inside the absolute value vanish when $|\mathrm{Im}(x)| \mapsto \infty$. To see this, use the estimates

$${}_4F_3\left(\begin{matrix} 1,\ \frac{1}{2}+x,\ 1-s+x,\ s+x \\ 1+x, 1+x, 1+x \end{matrix} \middle| z\right) \approx {}_1F_0\left(\begin{matrix} 1 \\ \ \end{matrix} \middle| z\right) = \frac{1}{1-z}$$
$${}_4F_3\left(\begin{matrix} 1,\ 1-x,\ 1-x,\ 1-x \\ \frac{3}{2}-x, 2-s-x, 1+s-x \end{matrix} \middle| \frac{1}{z}\right) \approx {}_1F_0\left(\begin{matrix} 1 \\ \ \end{matrix} \middle| \frac{1}{z}\right) = \frac{z}{z-1},$$
$$\frac{(1-s)_x(\frac{1}{2})_x(s)_x}{(1)_x^3} \approx \frac{\sin \pi s}{(\pi i\,\mathrm{Im}(x))^{3/2}},$$
$$\frac{2x^3}{s(1-s)} \frac{(-\frac{1}{2})_x(s-1)_x(-s)_x}{(1)_x^3} \approx -\frac{\sin \pi s}{(\pi i\,\mathrm{Im}(x))^{3/2}},$$

which are valid when $|\mathrm{Im}(x)|$ is large, and when $z \notin [0, \infty)$. Thus if $|\mathrm{Im}(x)|$ is sufficiently large (which rules out the possibility of x lying in a neighborhood of the poles $\{s, \frac{1}{2}, 1-s\}$), then $Y_x(z) = O(|\mathrm{Im}(x)|^{-3/2} e^{-\pi\,\mathrm{Im}(x)})$. The estimate holds uniformly if z lies in a compact subinterval of $(-1, 0)$, so a linear independence argument suffices to show that $m_i(x) = O(|\mathrm{Im}(x)|^{-3/2} e^{-\pi\,\mathrm{Im}(x)})$ for each i.

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We have proved that $m_i(x)$ is periodic and meromorphic, with possible simple poles if $x \in \{s, \frac{1}{2}, 1-s\}$. We conclude that

$$e^{-i\pi x} \cos \pi x(\cos^2 \pi x - \cos^2 \pi s) m_i(x) \tag{20}$$

is analytic for $\mathrm{Re}(x) \in [0, 1)$. This new function has period one, so it is also analytic on $\mathbb{C}$. If $|\mathrm{Im}(x)|$ is sufficiently large, then $m_i(x) = O(|\mathrm{Im}(x)|^{-3/2} e^{-\pi \,\mathrm{Im}(x)})$. Thus by elementary properties of the trigonometric functions, (20) becomes

$$\begin{aligned} e^{-i\pi x} \cos \pi x(\cos^2 \pi x - \cos^2 \pi s) m_i(x) &= O(e^{\pi \,\mathrm{Im}(x)} e^{3\pi|\mathrm{Im}(x)|} |m_i(x)|) \\ &= O(|\mathrm{Im}(x)|^{-3/2} e^{3\pi|\mathrm{Im}(x)|}). \end{aligned}$$

Therefore the function has a Fourier series which terminates:

$$e^{-i\pi x} \cos \pi x(\cos^2 \pi x - \cos^2 \pi s) m_i(x) = a_i^{(0)} + a_i^{(1)} \cos(2\pi x) + a_i^{(2)} \sin(2\pi x).$$

After collecting constants in (19), and noting that $Y_0(z) = y_0(z)$, we conclude that $Y_x(z)$ has the form given in (16). □

Since $y_x(z)$ is analytic in a neighborhood of $x = 0$, we have a Maclaurin series of the form

$$\frac{y_x(z)}{y_0(z)} = 1 + \phi_1(q)x + \phi_2(q)x^2 + \phi_3(q)x^3 + O(x^4), \tag{21}$$

where z and q are related by (8). Since $y_x(z)/y_0(z)$ is nonholomorphic in z, we expect each $\phi_i(q)$ to be nonholomorphic in q.

Theorem 1. *Assume that $s \in (0, 1)$, $z \notin \{0, 1\}$ and let $\phi_i(q)$ be as in (21). Then*

$$\begin{aligned} &\frac{1}{\pi y_0(z)} \sum_{n=1}^{\infty} \frac{(1)_n^3}{(s)_n(\frac{1}{2})_n(1-s)_n} \frac{z^{-n}}{n^3} \\ &\quad = \pi^2 i \csc^2(\pi s) - \frac{\pi}{3}(1 + 3\csc^2(\pi s))\phi_1(q) - i\phi_2(q) + \frac{1}{\pi}\phi_3(q). \end{aligned} \tag{22}$$

By Proposition 1, we also have

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{(1)_n^3}{(s)_n(\frac{1}{2})_n(1-s)_n} \frac{(a-bn)}{n^3} z^{-n} \\ &\quad = \pi^2 i \csc^2(\pi s) - \frac{\pi}{3}(1 + 3\csc^2(\pi s))\Big(\phi_1(q) - q \log|q| \frac{d\phi_1(q)}{dq}\Big) \\ &\qquad - i\Big(\phi_2(q) - q \log|q| \frac{d\phi_2(q)}{dq}\Big) + \frac{1}{\pi}\Big(\phi_3(q) - q \log|q| \frac{d\phi_3(q)}{dq}\Big). \end{aligned} \tag{23}$$

The sums in (22) and (23) diverge if $|z| < 1$, however the identities remain valid when ${}_4\mathrm{F}_3$ and ${}_5\mathrm{F}_4$ functions are substituted.

Proof. From (15) and (13) we see that

$$Y_x(z) = y_x(z) + O(x^3).$$

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This is sufficient to determine u and v in (16). From (13) we find

$$\frac{y_x(z)}{y_0(z)} = 1 + \phi_1(q)x + \phi_2(q)x^2 + \phi_3(q)x^3 + O(x^4).$$

By (16) we also have

$$\begin{aligned}\frac{Y_x(z)}{y_0(z)} = 1 &+ i\pi(1-2v)x + \pi^2(-2-2u+2v+\csc^2(\pi s))x^2 \\ &- \frac{i\pi^3}{3}(5+6u-4v+(-3+6v)\csc^2(\pi s))x^3 + O(x^4),\end{aligned} \tag{24}$$

where s and z satisfy the appropriate restrictions. The Taylor coefficients of $Y_x(z)$ and $y_x(z)$ agree up to order x^2. This leads to a pair of equations

$$\begin{aligned}\phi_1(q) &= i\pi(1-2v)\\ \phi_2(q) &= \pi^2(-2-2u+2v+\csc^2(\pi s)),\end{aligned}$$

from which it is easy to solve for u and v.

The companion series arises from the x^3 coefficient of $Y_x(z)$. By (15) and (13) we have

$$\begin{aligned}\frac{1}{y_0(z)}\sum_{n=1}^{\infty}\frac{(1)_n^3}{(s)_n(\frac12)_n(1-s)_n}\frac{z^{-n}}{n^3} &= \frac{1}{y_0(z)}\frac{2z^{-1}}{s(1-s)}{}_4\mathrm{F}_3\left(\left.\begin{matrix}1,\ 1,\ 1,\ 1\\ \frac32, 2-s, 1+s\end{matrix}\ \right|\ \frac{1}{z}\right)\\ &= \lim_{x\to 0}\left(\frac{y_x(z)-Y_x(z)}{y_0(z)\ x^3}\right)\\ &= \phi_3(q) + \frac{i\pi^3}{3}(5+6u-4v+(-3+6v)\csc^2(\pi s)).\end{aligned}$$

We recover (22) by eliminating u and v. □

Despite the fact that (22) and (23) hold for many values of s, it is probably only possible to evaluate $\phi_i(q)$ if $s \in \{\frac12, \frac13, \frac14\}$. We prove formulas for $\phi_i(q)$ below.

THEOREM 2. *Suppose that q lies in a neighborhood of zero. When $s = \frac12$,*

$$\phi_1(q) = \log q, \tag{25}$$

$$\phi_2(q) = \frac12\log^2 q + \frac{\pi^2}{2}, \tag{26}$$

$$\phi_3(q) = \frac16\log^3 q + \frac{\pi^2}{2}\log q - 6\zeta(3) - 16\sum_{n=1}^{\infty}\frac{\sigma_3(n)}{n^3}q^n + 4\sum_{n=1}^{\infty}\frac{\sigma_3(n)}{n^3}q^{4n}. \tag{27}$$

When $s = \frac13$,

$$\phi_1(q) = \log q, \tag{28}$$

$$\phi_2(q) = \frac12\log^2 q + \frac{2\pi^2}{3}, \tag{29}$$

$$\phi_3(q) = \frac16\log^3 q + \frac{2\pi^2}{3}\log q - 10\zeta(3) - 30\sum_{n=1}^{\infty}\frac{\sigma_3(n)}{n^3}q^n + 10\sum_{n=1}^{\infty}\frac{\sigma_3(n)}{n^3}q^{3n}. \tag{30}$$

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When $s = \frac{1}{4}$,

$$\phi_1(q) = \log q, \tag{31}$$

$$\phi_2(q) = \frac{1}{2}\log^2 q + \pi^2, \tag{32}$$

$$\phi_3(q) = \frac{1}{6}\log^3 q + \pi^2 \log q - 20\zeta(3) - 80\sum_{n=1}^{\infty}\frac{\sigma_3(n)}{n^3}q^n + 40\sum_{n=1}^{\infty}\frac{\sigma_3(n)}{n^3}q^{2n}. \tag{33}$$

Proof. The essential idea is to apply the Picard–Fuchs operator which annihilates $y_0(z)$. Recall that P is defined in (17). It was proved in [13, Proposition 2.2], that

$$Py_x(z) = \frac{(1-s)_x(\frac{1}{2})_x(s)_x}{(1)_x^3}z^x x^3 = x^3 + O(x^4). \tag{34}$$

When $x = 0$, we immediately obtain the homogeneous differential equation $Py_0(z) = 0$. If $y_x(z)$ is expanded in a Maclaurin series with respect to x, then by (21) we have $P(y_0(z)\phi_1(q)) = 0$ and $P(y_0(z)\phi_2(q)) = 0$. Appealing to [19, Lemma 1], we see that

$$\left(q\frac{d}{dq}\right)^3\phi_1(q) = 0, \quad \left(q\frac{d}{dq}\right)^3\phi_2(q) = 0, \tag{35}$$

and integrating gives

$$\phi_1(q) = \alpha_0 + \alpha_1 \log q + \alpha_2 \log^2 q, \tag{36}$$

$$\phi_2(q) = \beta_0 + \beta_1 \log q + \beta_2 \log^2 q, \tag{37}$$

where the α_i's and β_i's are undetermined constants. Examining the x^3 coefficient of $y_x(z)$ leads to the inhomogeneous differential equation $P[y_0(z)\phi_3(q)] = 1$. By [19, Lemma 1] and [13, Equation (2.33)], we find that

$$\left(q\frac{d}{dq}\right)^3\phi_3(q) = \sqrt{1-z}\,y_0^2(z). \tag{38}$$

In order to solve (38), and to determine the constants in (36) and (37), it is necessary to specify the value of s.

Suppose that q lies in a neighborhood of zero. When $s = \frac{1}{2}$ we have $\sqrt{1-z} = 1 - 2\lambda(q)$, where $\lambda(q) = \theta_2^4(q)/\theta_3^4(q)$ is the elliptic lambda function [13, Section 2.5]. By standard theta function inversion formulas, we also have

$$y_0(z) = \theta_3^4(q). \tag{39}$$

Identity (39) *does not* hold for $|q| < 1$. For instance, if q is close to one we have to replace (39) with $y_0(z) = (\log^2(q)/\pi^2)\theta_3^4(q)$. For $|q|$ sufficiently small

$$\begin{aligned} y_0^2(z)\sqrt{1-z} &= \theta_3^8(q) - 2\theta_3^4(q)\theta_2^4(q) \\ &= 1 - 16\sum_{n=1}^{\infty}\sigma_3(n)q^n + 16^2\sum_{n=1}^{\infty}\sigma_3(n)q^{4n}, \end{aligned}$$

where the second equality follows from [4, page 126, Entry 13]. Integrating (38) gives

$$\begin{aligned}\phi_3(q) &= \gamma_0 + \gamma_1 \log q + \gamma_2 \log^2 q + \frac{1}{6}\log^3 q \\ &\quad - 16\sum_{n=1}^{\infty} \sigma_3(n)\frac{q^n}{n^3} + 4\sum_{n=1}^{\infty} \sigma_3(n)\frac{q^{4n}}{n^3},\end{aligned} \tag{40}$$

where the γ_is are constants.

There are nine constants left to calculate. Let q tend to zero in (21). Since z has a q-series of the form $z = 64q + O(q^2)$, it follows that $z \approx 64q$ when q approaches zero. In a similar manner we find that $y_0(z) \approx 1$. By (21) we have

$$\begin{aligned} q^{-x} y_x(z) &= q^{-x} y_0(z)(1 + \phi_1(q)x + \phi_2(q)x^2 + \phi_3(q)x^3 + O(x^4)) \\ &\approx q^{-x}(1 + \phi_1(q)x + \phi_2(q)x^2 + \phi_3(q)x^3 + O(x^4)).\end{aligned} \tag{41}$$

From the definition of $q^{-x}y_x(z)$, we calculate

$$\begin{aligned} q^{-x} y_x(z) &= q^{-x} z^x \frac{(\frac{1}{2})_x^3}{(1)_x^3}\Big(1 + \sum_{n=1}^{\infty} z^n \frac{(\frac{1}{2}+x)_n^3}{(1+x)_n^3}\Big) \\ &\approx 64^x \frac{(\frac{1}{2})_x^3}{(1)_x^3}(1+0).\end{aligned} \tag{42}$$

Compare the Maclaurin series coefficients of (41) and (42) in x, x^2 and x^3. Since (42) is holomorphic at $x = 0$, it follows that (41) is holomorphic at $x = 0$ as well. Since q tends to zero, this implies that the powers of $\log(q)$ must drop out of the series obtained from (41). Comparing coefficients then provides sufficiently many relations to determine the values of α_i, β_i and γ_i explicitly. The cases when $s = \frac{1}{3}$ and $s = \frac{1}{4}$ require analogous arguments, using appropriate theta functions from [5]. □

The method fails when $s = \frac{1}{6}$, because of our inability to solve (38). The calculation is difficult because Ramanujan's theory of signature-6 modular equations is incomplete, and as a result it seems to be impossible to find a nice q-series expansion for

$$\sqrt{1-z}\, y_0^2(z) = \sqrt{1-z}\; {}_3F_2\left(\begin{matrix}\frac{1}{6}, \frac{1}{2}, \frac{5}{6} \\ 1, 1\end{matrix}\,\middle|\, z\right)^2.$$

Notice that (38) is equivalent to

$$\Big(q\frac{d}{dq}\Big)^3 \phi_3(q) = \frac{1 - 504\sum_{n=1}^{\infty}\frac{n^5 q^n}{1-q^n}}{\sqrt{1 + 240\sum_{n=1}^{\infty}\frac{n^3 q^n}{1-q^n}}}. \tag{43}$$

If we could obtain a reasonable expression for $\phi_3(q)$, then it might be possible to evaluate a companion series with $s = \frac{1}{6}$. Experimental searches failed to turn up any interesting identities, and this suggests that the task is impossible.

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4. Explicit formulas

Now we prove companion series evaluations. Proposition 2 reduces every companion series to elementary constants and values of the following special function:

$$F(q) := -\frac{\log^3|q|}{3\pi} + \frac{120}{\pi}\zeta(3) + \frac{240}{\pi}\sum_{j=1}^{\infty}\mathrm{Li}_3(q^j) - \log|q^j|\,\mathrm{Li}_2(q^j). \tag{44}$$

Notice that $F(q)$ is closely related to the elliptic trilogarithm [20]. Set $q = e^{2\pi i\tau}$, with $\tau = x + iy$, and $y > 0$. In Proposition 3 we prove

$$\mathrm{Re}(F(q)) = \frac{120y^3}{\pi^2}S(1, 2x, x^2 + y^2; 2). \tag{45}$$

It is easy to see that $F(q)$ is real-valued if $q \in (-1, 1)$, so (45) becomes a formula for $F(q)$ whenever $x \in \mathbb{Z}/2$. Glasser and Zucker proved that $S(A, B, C; t)$ reduces to Dirichlet L-values quite often. Their formulas lead to precisely 65 evaluations of $F(q)$, when $x = 0$ and $y^2 \in \mathbb{N}$. For instance, when $(x, y) = (0, \sqrt{7})$, we have

$$F(e^{-2\pi\sqrt{7}}) = 175\sqrt{7}L_{-7}(2).$$

Various additional values of $F(q)$ are provided in Table 1. The formulas in Theorems 3 and 4 are proved by evaluating linear combinations of $F(q)$s.

PROPOSITION 2. *Suppose that q lies in a neighborhood of zero, and that (a, b, z) and q are related by (8) and (9). When $s = \frac{1}{2}$,*

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(1)_n^3}{(\frac{1}{2})_n^3}\frac{(a-bn)}{n^3}z^{-n} = &-\frac{1}{15}F(q) + \frac{1}{60}F(q^4)\\ &+ \frac{\log^3(q)}{6\pi} - \frac{\log^2(q)\log|q|}{2\pi} + \frac{\log^3|q|}{3\pi}\\ &- \frac{i}{2}\log^2(q) + i\log(q)\log|q|\\ &- \frac{5}{6}\pi\log(q) + \frac{5}{6}\pi\log|q| + \frac{i\pi^2}{2}.\end{aligned} \tag{46}$$

When $s = \frac{1}{3}$,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(1)_n^3}{(\frac{1}{3})_n(\frac{1}{2})_n(\frac{2}{3})_n}\frac{(a-bn)}{n^3}z^{-n} = &-\frac{1}{8}F(q) + \frac{1}{24}F(q^3)\\ &+ \frac{\log^3(q)}{6\pi} - \frac{\log^2(q)\log|q|}{2\pi} + \frac{\log^3|q|}{3\pi}\\ &- \frac{i}{2}\log^2(q) + i\log(q)\log|q|\\ &- \pi\log(q) + \pi\log|q| + \frac{2i\pi^2}{3}.\end{aligned} \tag{47}$$

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TABLE 1. Selected values of $F(q)$.

q	$F(q)$
$e^{-2\pi}$	$80L_{-4}(2)$
$e^{-2\pi\sqrt{2}}$	$80\sqrt{2}L_{-8}(2)$
$e^{-2\pi\sqrt{3}}$	$135\sqrt{3}L_{-3}(2)$
$e^{-2\pi\sqrt{4}}$	$280L_{-4}(2)$
$e^{-2\pi\sqrt{5}}$	$100\sqrt{5}L_{-20}(2)+96L_{-4}(2)$
$e^{-2\pi\sqrt{6}}$	$120\sqrt{6}L_{-24}(2)+90\sqrt{3}L_{-3}(2)$
$e^{-2\pi\sqrt{7}}$	$175\sqrt{7}L_{-7}(2)$
$e^{-2\pi\sqrt{8}}$	$280\sqrt{2}L_{-8}(2)+240L_{-4}(2)$
$e^{-2\pi\sqrt{9}}$	$560L_{-4}(2)+180\sqrt{3}L_{-3}(2)$
$e^{-2\pi\sqrt{10}}$	$200\sqrt{10}L_{-40}(2)+192\sqrt{2}L_{-8}(2)$
$e^{-2\pi\sqrt{12}}$	$480L_{-4}(2)+\frac{1035}{2}\sqrt{3}L_{-3}(2)$
$e^{-2\pi\sqrt{13}}$	$260\sqrt{13}L_{-52}(2)+480L_{-4}(2)$
$e^{-2\pi\sqrt{15}}$	$\frac{375}{2}\sqrt{15}L_{-15}(2)+468\sqrt{3}L_{-3}(2)$
$e^{-2\pi\sqrt{16}}$	$480\sqrt{2}L_{-8}(2)+1100L_{-4}(2)$
$e^{-2\pi\sqrt{18}}$	$880\sqrt{2}L_{-8}(2)+540\sqrt{3}L_{-3}(2)$
$e^{-2\pi\sqrt{21}}$	$210\sqrt{21}L_{-84}(2)+210\sqrt{7}L_{-7}(2)+480L_{-4}(2)+360\sqrt{3}L_{-3}(2)$
$e^{-2\pi\sqrt{22}}$	$440\sqrt{22}L_{-88}(2)+330\sqrt{11}L_{-11}(2)$
$e^{-2\pi\sqrt{24}}$	$420\sqrt{6}L_{-24}(2)+480\sqrt{2}L_{-8}(2)+720L_{-4}(2)+495\sqrt{3}L_{-3}(2)$
$e^{-2\pi\sqrt{25}}$	$480\sqrt{5}L_{-20}(2)+2320L_{-4}(2)$
$e^{-2\pi\sqrt{28}}$	$\frac{1435}{2}\sqrt{7}L_{-7}(2)+1920L_{-4}(2)$
$e^{-2\pi\sqrt{30}}$	$300\sqrt{30}L_{-120}(2)+288\sqrt{6}L_{-24}(2)+225\sqrt{15}L_{-15}(2)+630\sqrt{3}L_{-3}(2)$
$e^{-2\pi\sqrt{33}}$	$330\sqrt{33}L_{-132}(2)+330\sqrt{11}L_{-11}(2)+1440L_{-4}(2)+630\sqrt{3}L_{-3}(2)$

When $s=\frac{1}{4}$,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(1)_n^3}{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}\frac{(a-bn)}{n^3}z^{-n} &= -\frac{1}{3}F(q)+\frac{1}{6}F(q^2)\\ &\quad+\frac{\log^3(q)}{6\pi}-\frac{\log^2(q)\log|q|}{2\pi}+\frac{\log^3|q|}{3\pi}\\ &\quad-\frac{1}{2}i\log^2(q)+i\log(q)\log|q|\\ &\quad-\frac{4}{3}\pi\log(q)+\frac{4}{3}\pi\log|q|+i\pi^2.\end{aligned} \tag{48}$$

PROOF. Proofs follow from combining Theorems 1 and 2. In particular, we obtain formulas (46) through (48) by substituting the results of Theorem 2 into (23). □

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Proposition 3. *Let $q=e^{2\pi i\tau}$, with $\tau=x+iy$, and $y>0$. Then*

$$F(q)=\frac{120y^3}{\pi^2}S(1,2x,x^2+y^2;2)+\frac{60i}{\pi^2}\sum_{\substack{n,k\\n\neq 0}}\frac{(k+nx)((k+nx)^2+3n^2y^2)}{n^3((k+nx)^2+n^2y^2)^2}. \tag{49}$$

If $x\in\mathbb{Z}/2$ and $y>0$, then

$$F(q)=\frac{120y^3}{\pi^2}S(1,2x,x^2+y^2;2). \tag{50}$$

If $2x/(x^2+y^2)\in\mathbb{Z}$ and $y>0$, then

$$F(q)=\frac{120y^3}{\pi^2}S(1,2x,x^2+y^2;2)+\frac{4i\pi^2}{3}x\Big(\frac{x^2+3y^2}{(x^2+y^2)^2}+x^2+3y^2-5\Big). \tag{51}$$

Proof. The proof below is slightly technical, but we have included it for the sake of completeness. It is important to note that the statement of this proposition, and more, is contained in [20, Part II, Section 7]. By (44) we obtain

$$\begin{aligned}F(q)&=\frac{8\pi^2}{3}(\operatorname{Im}\tau)^3+\frac{120}{\pi}\sum_{n=1}^{\infty}\Big(\frac{1}{n^3}+\frac{2}{n^3}\sum_{j=1}^{\infty}q^{jn}+\frac{4\pi\operatorname{Im}(\tau)}{n^2}\sum_{j=1}^{\infty}jq^{jn}\Big)\\&=\frac{8\pi^2}{3}(\operatorname{Im}\tau)^3+\frac{120}{\pi}\sum_{n=1}^{\infty}\Big(\frac{1}{n^3}\frac{1+q^n}{1-q^n}+\frac{4\pi\operatorname{Im}(\tau)}{n^2}\frac{q^n}{(1-q^n)^2}\Big)\\&=\frac{8\pi^2}{3}(\operatorname{Im}\tau)^3+\frac{60}{\pi}\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}\Big(\frac{i\cot(\pi n\tau)}{n^3}-\frac{\pi\operatorname{Im}(\tau)\csc^2(\pi n\tau)}{n^2}\Big).\end{aligned}$$

Substitute the partial fractions decompositions:

$$\cot(\pi n\tau)=\frac{1}{\pi}\sum_{k=-\infty}^{\infty}\frac{1}{k+\tau n},\qquad \pi\csc^2(\pi n\tau)=\frac{1}{\pi}\sum_{k=-\infty}^{\infty}\frac{1}{(k+\tau n)^2},$$

to obtain

$$F(q)=\frac{8\pi^2}{3}(\operatorname{Im}\tau)^3+\frac{60}{\pi^2}\sum_{\substack{n,k=-\infty\\n\neq 0}}^{\infty}\frac{i}{n^3(k+n\tau)}-\frac{\operatorname{Im}(\tau)}{n^2(k+n\tau)^2}. \tag{52}$$

Formula (49) follows from setting $\tau=x+iy$, and then isolating the real and imaginary parts of the function. We complete the proof of (50) by noting that $F(q)$ is real valued whenever $x\in\mathbb{Z}/2$.

To complete the proof of (51) we need to evaluate the following sum:

$$T(x,y):=\sum_{\substack{n,k\\n\neq 0}}\frac{(k+nx)((k+nx)^2+3n^2y^2)}{n^3((k+nx)^2+n^2y^2)^2}.$$

Extract the $k=0$ term to obtain

$$T(x,y)=\frac{\pi^4}{45}\frac{x(x^2+3y^2)}{(x^2+y^2)^2}+\sum_{\substack{k\\k\neq 0}}\sum_{\substack{n\\n\neq 0}}\frac{(k+nx)((k+nx)^2+3n^2y^2)}{n^3((k+nx)^2+n^2y^2)^2}.$$

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When $k \neq 0$ the inner sum can be evaluated by the residues method. `Mathematica` produces the following formula:

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(k+nx)((k+nx)^2+3n^2y^2)}{n^3((k+nx)^2+n^2y^2)^2}$$
$$= -\frac{x(\pi^2k^2-9y^2-3x^2)}{3k^4}$$
$$-\pi \sin\Big(\frac{2\pi kx}{x^2+y^2}\Big)\frac{(x^2+y^2)(\cosh^2\frac{\pi ky}{x^2+y^2}-\cos^2\frac{\pi kx}{x^2+y^2})+k\pi y\sinh\frac{2k\pi y}{x^2+y^2}}{2k^3(\cosh^2\frac{\pi ky}{x^2+y^2}-\cos^2\frac{\pi kx}{x^2+y^2})^2}.$$

If $2x/(x^2+y^2) \in \mathbb{Z}$, then the second term vanishes. Thus we are left with

$$T(x,y) = \frac{\pi^4}{45}\frac{x(x^2+3y^2)}{(x^2+y^2)^2} - \sum_{\substack{k \\ k\neq 0}} \frac{x(\pi^2k^2-9y^2-3x^2)}{3k^4}$$
$$= \frac{\pi^4}{45}x\Big(\frac{x^2+3y^2}{(x^2+y^2)^2}+x^2+3y^2-5\Big),$$

and (51) follows. □

4.1. Convergent rational formulas. Now we prove rational, convergent, companion series formulas. Virtually all of these results have appeared in the literature before, although we believe this is their first unified treatment. Equation (55) was proved by Zeilberger [21, Theorem 8]. Formulas (53), (54), (56) are due to Guillera [11, 12]. Equations (57) through (61) were conjectured by Sun [18]. Formula (58) was subsequently proved by Guillera [14], and the Hessami Pilehroods proved (59) [15]. Our strategy is to express each companion series in terms of $F(q)$'s, and then to evaluate $F(q)$ using properties of Epstein zeta functions. The hypergeometric-side of the formula also requires values of (a, b, z). We will refrain from rigorously calculating (a, b, z) in this paper; however, all three quantities can be calculated in a reasonably straightforward manner using techniques outlined in [2] or [7]. We summarize the values of (a, b, z) and the corresponding q's in Table 2.

THEOREM 3. *The following formulas are true:*

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(1)_n^3}{(\frac{1}{2})_n^3}\frac{(4n-1)}{n^3} = 16L_{-4}(2), \tag{53}$$

$$\sum_{n=1}^{\infty}\frac{(1)_n^3}{(\frac{1}{2})_n^3}\frac{(3n-1)}{n^3}\frac{1}{2^{2n}} = \frac{\pi^2}{2}, \tag{54}$$

$$\sum_{n=1}^{\infty}\frac{(1)_n^3}{(\frac{1}{2})_n^3}\frac{(21n-8)}{n^3}\frac{1}{2^{6n}} = \frac{\pi^2}{6}, \tag{55}$$

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$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(1)_n^3}{(\frac{1}{2})_n^3}\frac{(3n-1)}{n^3}\frac{1}{2^{3n}}=2L_{-4}(2), \tag{56}$$

$$\sum_{n=1}^{\infty}\frac{(1)_n^3}{(\frac{1}{2})_n(\frac{1}{3})_n(\frac{2}{3})_n}\frac{(10n-3)}{n^3}\left(\frac{2}{27}\right)^{2n}=\frac{\pi^2}{2}, \tag{57}$$

$$\sum_{n=1}^{\infty}\frac{(1)_n^3}{(\frac{1}{2})_n(\frac{1}{3})_n(\frac{2}{3})_n}\frac{(11n-3)}{n^3}\left(\frac{16}{27}\right)^{n}=8\pi^2, \tag{58}$$

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(1)_n^3}{(\frac{1}{2})_n(\frac{1}{3})_n(\frac{2}{3})_n}\frac{(15n-4)}{n^3}\frac{1}{4^n}=27L_{-3}(2), \tag{59}$$

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(1)_n^3}{(\frac{1}{2})_n(\frac{1}{4})_n(\frac{3}{4})_n}\frac{(5n-1)}{n^3}\left(\frac{3}{4}\right)^{2n}=\frac{45}{2}L_{-3}(2), \tag{60}$$

$$\sum_{n=1}^{\infty}\frac{(1)_n^3}{(\frac{1}{2})_n(\frac{1}{4})_n(\frac{3}{4})_n}\frac{(35n-8)}{n^3}\left(\frac{3}{4}\right)^{4n}=12\pi^2. \tag{61}$$

Proof. We begin by proving (53). Set $q=-e^{-\pi\sqrt{2}}$ in (46). We have $(a,b,z)=(\frac{1}{2},2,-1)$.

Table 2. Values of (a,b,z) in Theorem 3.

s	q	a	b	z
$\frac{1}{2}$	$-e^{-\pi\sqrt{2}}$	$\frac{1}{2}$	2	-1
$\frac{1}{2}$	$ie^{-\pi\sqrt{3}/2}$	$-\frac{i}{2}$	$-\frac{3i}{2}$	4
$\frac{1}{2}$	$e^{3\pi i/4}e^{-\pi\sqrt{7}/4}$	$-2i$	$-\frac{21i}{4}$	64
$\frac{1}{2}$	$-e^{-\pi}$	1	3	-8
$\frac{1}{3}$	$e^{2\pi i/3}e^{-2\pi\sqrt{2}/3}$	$-i$	$-\frac{10i}{3}$	$\frac{27}{2}$
$\frac{1}{3}$	$e^{\pi i/3}e^{-\pi\sqrt{11}/3}$	$-\frac{i}{4}$	$-\frac{11i}{12}$	$\frac{27}{16}$
$\frac{1}{3}$	$-e^{-\pi\sqrt{15}/3}$	$\frac{4}{3\sqrt{3}}$	$\frac{5}{\sqrt{3}}$	-4
$\frac{1}{4}$	$-e^{-\pi\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$\frac{5}{\sqrt{3}}$	$-\frac{16}{9}$
$\frac{1}{4}$	$ie^{-\pi\sqrt{7}/2}$	$-\frac{4i}{9}$	$-\frac{35i}{18}$	$\frac{256}{81}$

The formula reduces to

$$\frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(1)_n^3}{(\frac{1}{2})_n^3}\frac{(4n-1)}{n^3} = -\frac{1}{15}F(-e^{-\pi\sqrt{2}}) + \frac{1}{60}F(e^{-4\pi\sqrt{2}}).$$

Apply (50) to reduce the equation to

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(1)_n^3}{(\frac{1}{2})_n^3}\frac{(4n-1)}{n^3} &= \frac{64\sqrt{2}}{\pi^2}S(1,0,8;2) - \frac{4\sqrt{2}}{\pi^2}S\left(1,1,\frac{3}{4};2\right)\\ &= \frac{64\sqrt{2}}{\pi^2}(S(1,0,8;2) - S(3,4,4;2)).\end{aligned}$$

Glasser and Zucker have evaluated $S(1,0,8;t)$ for all t [10]. Their method also applies to $S(3,4,4;t) = S(3,2,3;t)$. When $t = 2$, the formulas become

$$\begin{aligned}S(1,0,8;2) &= \frac{7\pi^2}{48}L_{-8}(2) + \frac{\pi^2}{8\sqrt{2}}L_{-4}(2),\\ S(3,4,4;2) &= \frac{7\pi^2}{48}L_{-8}(2) - \frac{\pi^2}{8\sqrt{2}}L_{-4}(2),\end{aligned}$$

and the result follows.

Next consider (54). Set $q = ie^{-\pi\sqrt{3}/2}$ in (46). We have $(a,b,z) = (-i/2, -3i/2, 4)$. The formula reduces to

$$\frac{i}{2}\sum_{n=1}^{\infty}\frac{(1)_n^3}{(\frac{1}{2})_n^3}\frac{(3n-1)}{n^3}\frac{1}{2^{2n}} = \frac{3i\pi^2}{8} - \frac{1}{15}F(ie^{-\pi\sqrt{3}/2}) + \frac{1}{60}F(e^{-2\pi\sqrt{3}}).$$

Equate the imaginary parts, and apply (51). The equation reduces to

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(1)_n^3}{(\frac{1}{2})_n^3}\frac{(3n-1)}{n^3}\frac{1}{2^{2n}} &= \frac{3\pi^2}{4} - \frac{2}{15}\,\mathrm{Im}\,F(ie^{-\pi\sqrt{3}/2})\\ &= \frac{\pi^2}{2}.\end{aligned}$$

Next we prove (55). Set $q = e^{3\pi i/4}e^{-\pi\sqrt{7}/4}$ in (46). We have $(a,b,z) = (-2i, -21i/4, 64)$. The formula reduces to

$$\frac{i}{4}\sum_{n=1}^{\infty}\frac{(1)_n^3}{(\frac{1}{2})_n^3}\frac{(21n-8)}{n^3}\frac{1}{2^{6n}} = \frac{9\pi^2 i}{64} - \frac{1}{15}F(e^{3\pi i/4}e^{-\pi\sqrt{7}/4}) + \frac{1}{60}F(-e^{-\pi\sqrt{7}}).$$

Equate the imaginary parts, then apply (51). We obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(1)_n^3}{(\frac{1}{2})_n^3}\frac{(21n-8)}{n^3}\frac{1}{2^{6n}} &= \frac{9\pi^2}{16} - \frac{4}{15}\,\mathrm{Im}\,F(e^{3\pi i/4}e^{-\pi\sqrt{7}/4})\\ &= \frac{\pi^2}{6}.\end{aligned}$$

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Next consider (56). Set $q = -e^{-\pi}$ in (46). We have $(a, b, z) = (1, 3, -8)$. The formula reduces to

$$\sum_{n=1}^{\infty} \frac{(1)_n^3}{(\frac{1}{2})_n^3} \frac{(3n-1)}{n^3} \frac{(-1)^{n+1}}{2^{3n}} = -\frac{1}{15} F(-e^{-\pi}) + \frac{1}{60} F(e^{-4\pi}).$$

Apply (50) to obtain

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(1)_n^3}{(\frac{1}{2})_n^3} \frac{(3n-1)}{n^3} \frac{(-1)^{n+1}}{2^{3n}} &= -\frac{1}{\pi^2} S(1, 1, \tfrac{1}{2}; 2) + \frac{16}{\pi^2} S(1, 0, 4; 2) \\ &= 2L_{-4}(2).\end{aligned}$$

In the final step we use $S(1, 0, 4; 2) = (7\pi^2/24)L_{-4}(2)$, and $S(1, 1, \frac{1}{2}; 2) = 4S(2, 2, 1; 2) = 4S(1, 0, 1; 2) = (8\pi^2/3)L_{-4}(2)$. Both of these evaluations follow from the results of Glasser and Zucker [10].

Now consider (57). Set $q = e^{2\pi i/3} e^{-2\pi\sqrt{2}/3}$ in (47). We have $(a, b, z) = (-i, -10i/3, 27/2)$. The formula reduces to

$$\frac{i}{3} \sum_{n=1}^{\infty} \frac{(1)_n^3}{(\frac{1}{3})_n (\frac{1}{2})_n (\frac{2}{3})_n} \frac{(10n-3)}{n^3} \left(\frac{2}{27}\right)^n = \frac{26\pi^2 i}{81} - \frac{1}{8} F(e^{2\pi i/3} e^{-2\pi\sqrt{2}/3}) + \frac{1}{24} F(e^{-2\pi\sqrt{2}}).$$

Take the imaginary parts, then apply (51). We obtain

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(1)_n^3}{(\frac{1}{3})_n (\frac{1}{2})_n (\frac{2}{3})_n} \frac{(10n-3)}{n^3} \left(\frac{2}{27}\right)^n &= \frac{26\pi^2}{27} - \frac{3}{8} \operatorname{Im} F(e^{2\pi i/3} e^{-2\pi\sqrt{2}/3}) \\ &= \frac{\pi^2}{2}.\end{aligned}$$

Next we prove (58). Set $q = e^{\pi i/3} e^{-\pi\sqrt{11}/3}$ in (47). We have $(a, b, z) = (-i/4, -11i/12, 27/16)$. The formula reduces to

$$\frac{i}{12} \sum_{n=1}^{\infty} \frac{(1)_n^3}{(\frac{1}{3})_n (\frac{1}{2})_n (\frac{2}{3})_n} \frac{(11n-3)}{n^3} \left(\frac{16}{27}\right)^n = \frac{64\pi^2 i}{81} - \frac{1}{8} F(e^{\pi i/3} e^{-\pi\sqrt{11}/3}) + \frac{1}{24} F(-e^{-\pi\sqrt{11}}).$$

Take the imaginary parts, then apply (51). We have

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(1)_n^3}{(\frac{1}{3})_n (\frac{1}{2})_n (\frac{2}{3})_n} \frac{(11n-3)}{n^3} \left(\frac{16}{27}\right)^n &= \frac{256\pi^2}{27} - \frac{3}{2} \operatorname{Im} F(e^{\pi i/3} e^{-\pi\sqrt{11}/3}) \\ &= 8\pi^2.\end{aligned}$$

Now prove (59). Set $q = -e^{-\pi\sqrt{15}/3}$ in (47). We have $(a, b, z) = (4/3\sqrt{3}, 5/\sqrt{3}, -4)$. The formula reduces to

$$\frac{1}{3\sqrt{3}} \sum_{n=1}^{\infty} \frac{(1)_n^3}{(\frac{1}{3})_n (\frac{1}{2})_n (\frac{2}{3})_n} \frac{(15n-4)}{n^3} \frac{(-1)^{n+1}}{4^n} = -\frac{1}{8} F(-e^{-\pi\sqrt{15}/3}) + \frac{1}{24} F(-e^{-\pi\sqrt{15}}).$$

Apply (50) to obtain

$$\sum_{n=1}^{\infty}\frac{(1)_n^3}{(\frac{1}{3})_n(\frac{1}{2})_n(\frac{2}{3})_n}\frac{(15n-4)}{n^3}\frac{(-1)^{n+1}}{4^n}=-\frac{75\sqrt{5}}{8\pi^2}S\left(1,1,\frac{2}{3};2\right)+\frac{675\sqrt{5}}{8\pi^2}S(1,1,4;2)$$
$$=\frac{675\sqrt{5}}{8\pi^2}(S(1,1,4;2)-S(2,3,3;2)).$$

Glasser and Zucker have calculated $S(1,1,4;t)$ for all t [10]. Their method also applies to $S(2,3,3;t)=S(2,1,2;t)$. When $t=2$ the formulas reduce to

$$S(1,1,4;2)=\frac{\pi^2}{6}L_{-15}(2)+\frac{4\pi^2}{25\sqrt{5}}L_{-3}(2),$$
$$S(2,3,3;2)=\frac{\pi^2}{6}L_{-15}(2)-\frac{4\pi^2}{25\sqrt{5}}L_{-3}(2),$$

and (59) follows.

Next we prove (60). Set $q=-e^{-\pi\sqrt{3}}$ in (48). We have $(a,b,z)=(1/\sqrt{3},5/\sqrt{3},-16/9)$. The formula reduces to

$$\frac{1}{\sqrt{3}}\sum_{n=1}^{\infty}\frac{(1)_n^3}{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}\frac{(5n-1)}{n^3}(-1)^{n+1}\left(\frac{3}{4}\right)^{2n}=-\frac{1}{3}F(-e^{-\pi\sqrt{3}})+\frac{1}{6}F(e^{-2\pi\sqrt{3}}).$$

By (50), we have

$$\sum_{n=1}^{\infty}\frac{(1)_n^3}{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}\frac{(5n-1)}{n^3}(-1)^{n+1}\left(\frac{3}{4}\right)^{2n}=-\frac{45}{\pi^2}S(1,1,1;2)+\frac{180}{\pi^2}S(1,0,3;2)$$
$$=\frac{45}{2}L_{-3}(2).$$

Glasser and Zucker proved that $S(1,0,3;2)=(3\pi^2/8)L_{-3}(2)$, and $S(1,1,1;2)=\pi^2L_{-3}(2)$ [10].

Finally, we prove (61). Set $q=ie^{-\pi\sqrt{7}/2}$ in (48). We have $(a,b,z)=(-4i/9,-35i/18,256/81)$. The formula reduces to

$$\frac{i}{18}\sum_{n=1}^{\infty}\frac{(1)_n^3}{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}\frac{(35n-8)}{n^3}\left(\frac{3}{4}\right)^{4n}=\frac{7\pi^2 i}{8}-\frac{1}{3}F(ie^{-\pi\sqrt{7}/2})+\frac{1}{6}F(-e^{-\pi\sqrt{7}}).$$

Take the imaginary part, then apply (51). We obtain

$$\sum_{n=1}^{\infty}\frac{(1)_n^3}{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}\frac{(35n-8)}{n^3}\left(\frac{3}{4}\right)^{4n}=\frac{63\pi^2}{4}-6\,\mathrm{Im}\,F(ie^{-\pi\sqrt{7}/2})$$
$$=12\pi^2. \qquad \square$$

Table 2 summarizes the values of (a,b,z) and q in Theorem 3. These values also lead to divergent formulas for $1/\pi$. For instance, when $s=\frac{1}{3}$ and $(a,b,z)=$

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$(4/3\sqrt{3}, 5/\sqrt{3}, -4)$, we obtain (59), and

$$\frac{1}{\pi} = \frac{4}{3\sqrt{3}}\,{}_4F_3\left(\begin{matrix}\frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{19}{15}\\ 1, 1, \frac{4}{15}\end{matrix}\,\middle|\, -4\right).$$

The above formula is rigorously proved, but it is worth noting that `Mathematica` returns a numerical value for the the right-hand side, which agrees perfectly with $1/\pi = 0.318\,309\,8\ldots$.

4.2. Divergent rational formulas. Next we examine divergent hypergeometric formulas for Dirichlet L-values. These are companions to the convergent formulas for $1/\pi$. Since the identities have $|z| < 1$, we have substituted a ${}_5F_4$ function for the divergent companion series:

$$\sum_{n=1}^{\infty} \frac{(1)_n^3}{(s)_n(\frac{1}{2})_n(1-s)_n}\frac{(a-bn)}{n^3}z^{-n} = \frac{2(a-b)}{s(1-s)z}\,{}_5F_4\left(\begin{matrix}1, 1, 1, 1, 2-\frac{a}{b}\\ \frac{3}{2}, 1+s, 2-s, 1-\frac{a}{b}\end{matrix}\,\middle|\, z^{-1}\right). \tag{62}$$

The ${}_5F_4$ function has a branch cut on the interval $[1, \infty)$ [16, page 405]. When z^{-1} lies on the branch cut, the function takes a complex value. The real part of the function is uniquely defined, but the sign of the imaginary part depends on the direction from which we approach the branch cut. We use the same computational method as `Mathematica 8`, if $z^{-1} \in [1, \infty)$ then we define ${}_5F_4(\cdots|z^{-1}) = \lim_{\delta\mapsto 0^+} {}_5F_4(\cdots|z^{-1} - i\delta)$. We note that the values of (a, b, z) and q in Tables 3 and 4 were extracted directly from the tables of Chan and Cooper [7].

THEOREM 4. *The following identity holds:*

$$\frac{2(a-b)}{s(1-s)z}\,{}_5F_4\left(\begin{matrix}1, 1, 1, 1, 2-\frac{a}{b}\\ \frac{3}{2}, 1+s, 2-s, 1-\frac{a}{b}\end{matrix}\,\middle|\, z^{-1}\right) = L(2), \tag{63}$$

for the values of s, (a, b, z) and $L(2)$ in Tables 3 and 4.

PROOF. The proofs are the same as for Theorem 3, so we only consider one example in detail. Set $q = e^{-\pi\sqrt{7}}$ in (46). By Table 4, we have $s = \frac{1}{2}$ and $(a, b, z) = (5/16, 21/8, 1/64)$. Applying (50) and then (62), reduces the formula to

$$\begin{aligned}-1184\,{}_5F_4\left(\begin{matrix}1, 1, 1, 1, \frac{79}{42}\\ \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{37}{42}\end{matrix}\,\middle|\, 64\right) &= 4i\pi^2 - \frac{1}{15}F(e^{-\pi\sqrt{7}}) + \frac{1}{60}F(e^{-4\pi\sqrt{7}})\\ &= 4i\pi^2 - \frac{112\sqrt{7}}{\pi^2}(S(4, 0, 7; 2) - S(1, 0, 28; 2)).\end{aligned}$$

By the results of Glasser and Zucker [10], we obtain

$$S(1, 0, 28; 2) = \frac{41\pi^2}{384}L_{-7}(2) + \frac{2\pi^2}{7\sqrt{7}}L_{-4}(2),$$
$$S(4, 0, 7; 2) = \frac{41\pi^2}{384}L_{-7}(2) - \frac{2\pi^2}{7\sqrt{7}}L_{-4}(2),$$

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TABLE 3. Values of (a, b, z) with $z < 0$ in Theorem 4.

s	q	a	b	$z < 0$	$L(2)$
$\frac{1}{2}$	$-e^{-\pi\sqrt{2}}$	$\frac{1}{2}$	$\frac{4}{2}$	-1	$8L_{-4}(2)$
$\frac{1}{2}$	$-e^{-\pi\sqrt{4}}$	$\frac{1}{2\sqrt{2}}$	$\frac{6}{2\sqrt{2}}$	$-\frac{1}{8}$	$16\sqrt{2}L_{-8}(2)$
$\frac{1}{3}$	$-e^{-\pi\sqrt{9/3}}$	$\frac{\sqrt{3}}{4}$	$\frac{5\sqrt{3}}{4}$	$-\frac{9}{16}$	$10\sqrt{3}L_{-3}(2)$
$\frac{1}{3}$	$-e^{-\pi\sqrt{17/3}}$	$\frac{7}{12\sqrt{3}}$	$\frac{51}{12\sqrt{3}}$	$-\frac{1}{16}$	$30\sqrt{3}L_{-3}(2)$
$\frac{1}{3}$	$-e^{-\pi\sqrt{25/3}}$	$\frac{\sqrt{15}}{12}$	$\frac{9\sqrt{15}}{12}$	$-\frac{1}{80}$	$15\sqrt{15}L_{-15}(2)$
$\frac{1}{3}$	$-e^{-\pi\sqrt{41/3}}$	$\frac{106}{192\sqrt{3}}$	$\frac{1230}{192\sqrt{3}}$	$-\frac{1}{2^{10}}$	$120\sqrt{3}L_{-3}(2)$
$\frac{1}{3}$	$-e^{-\pi\sqrt{49/3}}$	$\frac{26\sqrt{7}}{216}$	$\frac{330\sqrt{7}}{216}$	$-\frac{1}{3024}$	$70\sqrt{7}L_{-7}(2)$
$\frac{1}{3}$	$-e^{-\pi\sqrt{89/3}}$	$\frac{827}{1500\sqrt{3}}$	$\frac{14\,151}{1500\sqrt{3}}$	$-\frac{1}{500^2}$	$390\sqrt{3}L_{-3}(2)$
$\frac{1}{4}$	$-e^{-\pi\sqrt{5}}$	$\frac{3}{8}$	$\frac{20}{8}$	$-\frac{1}{4}$	$32L_{-4}(2)$
$\frac{1}{4}$	$-e^{-\pi\sqrt{7}}$	$\frac{8}{9\sqrt{7}}$	$\frac{65}{9\sqrt{7}}$	$-\frac{16^2}{63^2}$	$\frac{35}{2}\sqrt{7}L_{-7}(2)$
$\frac{1}{4}$	$-e^{-\pi\sqrt{9}}$	$\frac{3\sqrt{3}}{16}$	$\frac{28\sqrt{3}}{16}$	$-\frac{1}{48}$	$60\sqrt{3}L_{-3}(2)$
$\frac{1}{4}$	$-e^{-\pi\sqrt{13}}$	$\frac{23}{72}$	$\frac{260}{72}$	$-\frac{1}{18^2}$	$160L_{-4}(2)$
$\frac{1}{4}$	$-e^{-\pi\sqrt{25}}$	$\frac{41\sqrt{5}}{288}$	$\frac{644\sqrt{5}}{288}$	$-\frac{1}{5\cdot 72^2}$	$160\sqrt{5}L_{-20}(2)$
$\frac{1}{4}$	$-e^{-\pi\sqrt{37}}$	$\frac{1123}{3528}$	$\frac{21\,460}{3528}$	$-\frac{1}{882^2}$	$800L_{-4}(2)$

and we recover the value of $L(2)$ in Table 4. After simplifying, we find that

$$ {}_5F_4\left(\begin{matrix} 1, & 1, & 1, & 1, & \frac{79}{42} \\ \frac{3}{2}, & \frac{3}{2}, & \frac{3}{2}, & \frac{37}{42} & \end{matrix}\,\middle|\, 64\right) = -\frac{2}{37}L_{-4}(2) - \frac{1}{296}\pi^2 i. $$

All of the formulas in Tables 3 and 4 follow from analogous arguments. □

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TABLE 4. Values of (a, b, z) with $z > 0$ in Theorem 4.

s	q	a	b	$z > 0$	$L(2)$
$\frac{1}{2}$	$e^{-\pi\sqrt{3}}$	$\frac{1}{4}$	$\frac{6}{4}$	$\frac{1}{4}$	$16L_{-4}(2) + 2\pi^2 i$
$\frac{1}{2}$	$e^{-\pi\sqrt{7}}$	$\frac{5}{16}$	$\frac{42}{16}$	$\frac{1}{64}$	$64L_{-4}(2) + 4\pi^2 i$
$\frac{1}{3}$	$e^{-\pi\sqrt{8/3}}$	$\frac{1}{3\sqrt{3}}$	$\frac{6}{3\sqrt{3}}$	$\frac{1}{2}$	$\frac{15}{2}\sqrt{3}L_{-3}(2) + 2\pi^2 i$
$\frac{1}{3}$	$e^{-\pi\sqrt{16/3}}$	$\frac{8}{27}$	$\frac{60}{27}$	$\frac{2}{27}$	$40L_{-4}(2) + \frac{10}{3}\pi^2 i$
$\frac{1}{3}$	$e^{-\pi\sqrt{20/3}}$	$\frac{8}{15\sqrt{3}}$	$\frac{66}{15\sqrt{3}}$	$\frac{4}{125}$	$39\sqrt{3}L_{-3}(2) + 4\pi^2 i$
$\frac{1}{4}$	$e^{-2\pi}$	$\frac{2}{9}$	$\frac{14}{9}$	$\frac{32}{81}$	$20L_{-4}(2) + 3\pi^2 i$
$\frac{1}{4}$	$e^{-\pi\sqrt{6}}$	$\frac{1}{2\sqrt{3}}$	$\frac{8}{2\sqrt{3}}$	$\frac{1}{9}$	$30\sqrt{3}L_{-3}(2) + 4\pi^2 i$
$\frac{1}{4}$	$e^{-\pi\sqrt{10}}$	$\frac{4}{9\sqrt{2}}$	$\frac{40}{9\sqrt{2}}$	$\frac{1}{81}$	$64\sqrt{2}L_{-8}(2) + 6\pi^2 i$
$\frac{1}{4}$	$e^{-\pi\sqrt{18}}$	$\frac{27}{49\sqrt{3}}$	$\frac{360}{49\sqrt{3}}$	$\frac{1}{7^4}$	$180\sqrt{3}L_{-3}(2) + 10\pi^2 i$
$\frac{1}{4}$	$e^{-\pi\sqrt{22}}$	$\frac{19}{18\sqrt{11}}$	$\frac{280}{18\sqrt{11}}$	$\frac{1}{99^2}$	$110\sqrt{11}L_{-11}(2) + 12\pi^2 i$
$\frac{1}{4}$	$e^{-\pi\sqrt{58}}$	$\frac{4412}{9801\sqrt{2}}$	$\frac{105\,560}{9801\sqrt{2}}$	$\frac{1}{99^4}$	$960\sqrt{2}L_{-8}(2) + 30\pi^2 i$

4.3. Irrational formulas. We emphasize that the *vast majority* of companion series formulas involve irrational values of (a, b, z). Consider the narrow class of formulas which arises from setting $q = e^{-2\pi\sqrt{v}}$ in (48). The companion series with $s = \frac{1}{4}$ reduces to a linear combination of $S(1, 0, v; 2)$, $S(1, 0, 4v; 2)$ and elementary constants. There are 24 values of $v \in \mathbb{N}$, for which both sums reduce to Dirichlet L-values [10]. The $v = 1$ case produces a rational, albeit divergent, companion series (Theorem 4 with $s = \frac{1}{4}$ and $(a, b, z) = (2/9, 14/9, 32/81)$). The other 23 choices lead to formulas with complicated algebraic values of (a, b, z). While it is possible to determine those numbers from modular equations, it is usually much easier to use a computer. Formulas (8) and (9) are rather unwieldy for computational purposes, so we find it convenient to use theta functions. Suppose that $s = \frac{1}{2}$, and that q lies in a neighborhood of zero. Then substituting (11) directly into (9) gives

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TABLE 5. Selected convergent irrational companion series evaluations.

s	q	a	b	$\lvert z\rvert > 1$	Value of Equation (4)
$\frac{1}{2}$	$-e^{-\pi\frac{\sqrt{2}}{2}}$	$\frac{3+2\sqrt{2}}{2}$	$\frac{8+5\sqrt{2}}{2}$	$\frac{-8}{(\sqrt{2}-1)^3}$	$2L_{-4}(2)-\sqrt{2}L_{-8}(2)$
$\frac{1}{2}$	$-e^{-\frac{\pi}{2}}$	$\frac{14+10\sqrt{2}}{2}$	$\frac{33+24\sqrt{2}}{2}$	$\frac{-16\sqrt{2}}{(\sqrt{2}-1)^6}$	$-\frac{13}{4}L_{-4}(2)+2\sqrt{2}L_{-8}(2)$
$\frac{1}{2}$	$-e^{-\pi\frac{\sqrt{2}}{3}}$	$\frac{59+24\sqrt{6}}{6}$	$\frac{140+56\sqrt{6}}{6}$	$\frac{-1}{(5-2\sqrt{6})^4}$	$\frac{136}{9}L_{-4}(2)-\frac{16}{3}\sqrt{6}L_{-24}(2)$
$\frac{1}{2}$	$-e^{-\pi\frac{2\sqrt{3}}{3}}$	$\frac{3\sqrt{6}+7\sqrt{2}}{24}$	$\frac{6\sqrt{6}+30\sqrt{2}}{24}$	$\frac{-1}{2(\sqrt{3}-1)^6}$	$16\sqrt{2}L_{-8}(2)-8\sqrt{6}L_{-24}(2)$
$\frac{1}{2}$	$-e^{-\pi\frac{\sqrt{6}}{3}}$	$\frac{5+4\sqrt{2}}{6}$	$\frac{12+12\sqrt{2}}{6}$	$\frac{-1}{(\sqrt{2}-1)^4}$	$-8L_{-4}(2)+\frac{16}{3}\sqrt{2}L_{-8}(2)$
$\frac{1}{2}$	$-e^{-\pi\frac{\sqrt{10}}{5}}$	$\frac{23+10\sqrt{5}}{10}$	$\frac{60+24\sqrt{5}}{10}$	$\frac{-1}{(\sqrt{5}-2)^4}$	$\frac{56}{5}L_{-4}(2)-4\sqrt{5}L_{-20}(2)$
$\frac{1}{2}$	$e^{\frac{9\pi i}{8}}e^{-\pi\frac{\sqrt{15}}{8}}$	$\frac{4(11+5\sqrt{5})}{8}i$	$\frac{3(35+16\sqrt{5})}{8}i$	$\frac{2^{14}}{(\sqrt{5}-1)^8}$	$-\frac{1}{240}\pi^2 i$
$\frac{1}{3}$	$-e^{-\pi\frac{\sqrt{21}}{3}}$	$\frac{10+7\sqrt{7}}{54}$	$\frac{21+39\sqrt{7}}{54}$	$\frac{-1}{26\sqrt{7}-68}$	$-20L_{-4}(2)+\frac{35}{4}\sqrt{7}L_{-7}(2)$
$\frac{1}{4}$	$-e^{-\pi\frac{\sqrt{21}}{3}}$	$\frac{27+20\sqrt{3}}{72}$	$\frac{84+112\sqrt{3}}{72}$	$\frac{-1}{(42-24\sqrt{3})^2}$	$-\frac{160}{3}L_{-4}(2)+40\sqrt{3}L_{-3}(2)$
$\frac{1}{4}$	$-e^{-\frac{3\pi\sqrt{5}}{5}}$	$\frac{3987+2124\sqrt{3}}{4840}$	$\frac{19\,380+7440\sqrt{3}}{4840}$	$\frac{-1}{(680\sqrt{3}-1178)^2}$	$\frac{544}{5}L_{-4}(2)-72\sqrt{3}L_{-3}(2)$

$$
\begin{aligned}
z &= 4\frac{\theta_3^4(-q)}{\theta_3^4(q)}\left(1-\frac{\theta_3^4(-q)}{\theta_3^4(q)}\right),\\
a &= \frac{1}{\pi\theta_3^4(q)}\left(1+\frac{8\log|q|}{\theta_3(q)}\sum_{n=1}^{\infty}n^2q^{n^2}\right),\\
b &= \frac{\log|q|}{\pi}\left(1-2\frac{\theta_3^4(-q)}{\theta_3^4(q)}\right),
\end{aligned}
\tag{64}
$$

where

$$
\theta_3(q) = 1+2\sum_{n=1}^{\infty}q^{n^2}.
$$

More complicated formulas are required if $s \in \{\frac{1}{3}, \frac{1}{4}\}$.

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To give an example of an irrational formula, set $q = e^{9\pi i/8}e^{-\pi\sqrt{15}/8}$ in (46). We calculate $(a, b, z) \approx (11.09i, 26.54i, 3006.63)$. The PSLQ algorithm returns the following polynomials:

$$
\begin{aligned}
0 &= 1 - 11ia + a^2,\\
0 &= 495 - 1680ib + 64b^2,\\
0 &= 4096 - 3008z + z^2.
\end{aligned}
$$

Therefore $(a, b, z) = (\frac{1}{2}i(11 + 5\sqrt{5}), \frac{3}{8}i(35 + 16\sqrt{5}), \frac{1}{4}(1 + \sqrt{5})^8)$. After simplifying with (51), we arrive at the following identity:

$$\frac{\pi^2}{30} = \sum_{n=1}^{\infty} \frac{3(35 + 16\sqrt{5})n - 4(11 + 5\sqrt{5})}{n^3\binom{2n}{n}^3}\left(\frac{\sqrt{5}-1}{2}\right)^{8n}. \tag{65}$$

This should be compared to Ramanujan's irrational formula for $1/\pi$, since both formulas involve powers of the golden ratio [17]. Table 5 contains many additional irrational formulas.

5. Irreducible values of $S(A, B, C; 2)$

Irreducible values of $S(A, B, C; 2)$ occur when the quadratic form $An^2 + Bnm + Cm^2$ fails the one-class-per-genus test. Apart from a few oddball cases, it is probably impossible to reduce these sums to Dirichlet L-functions [23]. In this section, we prove that it is still possible to express some irreducible values of $S(A, B, C; 2)$ in terms of hypergeometric functions. Propositions 2 and 3 reduce every interesting companion series to two values of $S(A, B, C; 2)$. Sometimes it is possible to select q such that one sum reduces to Dirichlet L-values and one sum does not. Sometimes both values of $S(A, B, C; 2)$ are irreducible, but one of them can be eliminated by finding a multi-term linear dependence with Dirichlet L-functions.

To make a first attempt at finding a formula, set $q = e^{-3\pi}$ in (46). Then $s = \frac{1}{2}$ and $(a, b, z) = (\frac{1}{4}(18r - 5r^3), 12r - 3r^3, (7 + 4\sqrt{3})^{-2})$, where $r = \sqrt[4]{12}$. By (50), the companion series equals a linear combination of $S(1, 0, 36; 2)$, $S(4, 0, 9; 2)$ and elementary constants. We eliminate $S(4, 0, 9; 2)$ with a result from [22]:

$$
\begin{aligned}
S(1, 0, 36; t) + S(4, 0, 9; t) = {}& (1 - 2^{-t} + 2^{1-2t})(1 + 3^{1-2t})L_1(t)L_{-4}(t)\\
&+ (1 + 2^{-t} + 2^{1-2t})L_{-3}(t)L_{12}(t).
\end{aligned} \tag{66}
$$

After noting that $L_1(2) = \pi^2/6$ and $L_{12}(2) = \pi^2/6\sqrt{3}$, we obtain a divergent formula:

$$
\begin{aligned}
\frac{2}{\pi^2}S(1, 0, 36; 2) = {}& \frac{49}{18^2}L_{-4}(2) + \frac{11}{48\sqrt{3}}L_{-3}(2)\\
&- \left(\frac{161 + 93\sqrt{3}}{18\sqrt[4]{12}}\right)\mathrm{Re}\left[{}_5F_4\left(\begin{matrix}1,\ 1,\ 1,\ 1,\ \frac{21+\sqrt{3}}{12}\\ \frac{3}{2},\ \frac{3}{2},\ \frac{3}{2},\ \frac{9+\sqrt{3}}{12}\end{matrix}\ \middle|\ (7 + 4\sqrt{3})^2\right)\right].
\end{aligned}
$$

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Many additional divergent formulas exist, but these formulas are virtually useless from a computational perspective. Rapidly converging formulas are somewhat more exciting.

Consider the restriction on q imposed in Proposition 2. To obtain an $s = \frac{1}{2}$ companion series from (46), we must select q to lie in a neighborhood of zero. Unwinding the proof of Theorem 2 shows that we can only select values of q for which

$$\theta_3^4(q) = {}_3\mathrm{F}_2\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1\end{matrix} \,\middle|\, 4\frac{\theta_3^4(-q)}{\theta_3^4(q)}\left(1 - \frac{\theta_3^4(-q)}{\theta_3^4(q)}\right)\right)$$

holds (similar restrictions exist when $s = \frac{1}{3}$ and $s = \frac{1}{4}$). This constraint implies that the allowable values on the real axis are $q \in (-1, e^{-\pi})$. If $q \in (-e^{-\pi\sqrt{2}}, e^{-\pi})$ then $|z| < 1$, and the companion series diverges. On the other hand, if $q \in (-1, -e^{-\pi\sqrt{2}})$ then $|z| > 1$, and we obtain convergent formulas. Suppose that $q = e^{2\pi i(\frac{1}{2}+iy)}$, so that q lives on the negative real axis. Then by (50) we find

$$\begin{aligned} F(q) = &F(-e^{-2\pi y}) = \frac{120y^3}{\pi^2} S(1, 1, \tfrac{1}{4} + y^2; 2), \\ F(q^4) = &F(e^{-8\pi y}) = \frac{120(4y)^3}{\pi^2} S(1, 0, 16y^2; 2). \end{aligned} \tag{67}$$

Elementary manipulations suffice to prove

$$S(1, 1, \tfrac{1}{4} + y^2; t) = -S(1, 0, y^2; t) + 18S(1, 0, 4y^2; t) - 16S(1, 0, 16y^2; t). \tag{68}$$

Now we prove the formula for $S(1, 0, 36; 2)$ quoted in the introduction (Equation (6), Section 1). Set $q = -e^{-\pi/3}$ in (46). Using the results above (with $y = \frac{1}{6}$), we conclude

$$\begin{aligned} F(-e^{-\pi/3}) &= \frac{90}{\pi^2}(9S(1, 0, 9; 2) - 8S(1, 0, 36; 2) - 8S(4, 0, 9; 2)) \\ F(e^{-4\pi/3}) &= \frac{2880}{\pi^2} S(4, 0, 9; 2). \end{aligned}$$

We can eliminate $S(4, 0, 9; 2)$ with (66), and $S(1, 0, 9; 2)$ simplifies via

$$S(1, 0, 9; t) = (1 + 3^{1-2t})L_1(t)L_{-4}(t) + L_{-3}(t)L_{12}(t).$$

Putting everything together in (46), and simplifying (a, b, z) with (64), produces the desired formula for $S(1, 0, 36; 2)$.

Next consider (46) when $q = -e^{-\pi/\sqrt{5}}$. Applying (67) and (68) with $y = 1/\sqrt{20}$ reduces the companion series to a linear combination of $S(1, 0, 20; 2)$, $S(4, 0, 5; 2)$ and $S(1, 0, 5; 2)$. We can eliminate the latter two sums with

$$\begin{aligned} S(4, 0, 5; t) + S(1, 0, 20; t) &= (1 - 2^{-t} + 2^{1-2t})L_1(t)L_{-20}(t) + (1 + 2^{-t} + 2^{1-2t})L_{-4}(t)L_5(t) \\ S(1, 0, 5; t) &= L_1(t)L_{-20}(t) + L_{-4}(t)L_5(t). \end{aligned}$$

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Zucker proved the first identity, and the second appears in [10]. Thus we arrive at

$$\frac{16\sqrt{5}}{\pi^2}S(1,0,20;2) = \frac{5\sqrt{5}}{3}L_{-20}(2) + \frac{104}{25}L_{-4}(2) - \sum_{n=1}^{\infty}\frac{(1)_n^3}{(\frac{1}{2})_n^3}\frac{(a-bn)}{n^3}z^{-n}, \tag{69}$$

where

$$\begin{aligned} z &= -8\Big(617 + 276\sqrt{5} + 2\sqrt{5(38\,078 + 17\,029\sqrt{5})}\Big), \\ a &= \frac{34}{5} + 3\sqrt{5} + \frac{1}{2}\sqrt{\frac{9032}{25} + \frac{808}{\sqrt{5}}}, \\ b &= 16 + 7\sqrt{5} + \frac{1}{2}\sqrt{\frac{9728}{5} + \frac{4352}{\sqrt{5}}}. \end{aligned}$$

This formula also converges rapidly, because $z \approx -1.9 \times 10^4$.

We conclude the paper with one final example. To obtain a formula for $S(1,0,52;2)$, set $q = -e^{-\pi/\sqrt{13}}$ in (46). Applying (67) and (68) with $y = 1/\sqrt{52}$ reduces the companion series to an expression involving $S(1,0,52;2)$, $S(4,0,13;2)$ and $S(1,0,13;2)$. The latter two sums can be eliminated with

$$\begin{aligned} S(1,0,52;t) + S(4,0,13;t) &= (1 - 2^{-t} + 2^{1-2t})L_1(t)L_{-52}(t) + (1 + 2^{-t} + 2^{1-2t})L_{-4}(t)L_{13}(t) \\ S(1,0,13;t) &= L_1(t)L_{-52}(t) + L_{-4}(t)L_{13}(t). \end{aligned}$$

Zucker proved the first formula, and the second appears in [10]. Therefore, we obtain

$$\frac{16\sqrt{13}}{\pi^2}S(1,0,52;2) = \frac{5\sqrt{13}}{3}L_{-52}(2) + 8L_{-4}(2) - \sum_{n=1}^{\infty}\frac{(1)_n^3}{(\frac{1}{2})_n^3}\frac{(a-bn)}{n^3}z^{-n}, \tag{70}$$

where

$$\begin{aligned} z &= -8\Big(3\,367\,657 + 934\,020\sqrt{13} + 90\sqrt{2\,800\,274\,982 + 776\,656\,541\sqrt{13}}\Big), \\ a &= \frac{4266}{13} + 91\sqrt{13} + \frac{1}{13}\sqrt{2(18\,194\,697 + 5\,046\,301\sqrt{13})}, \\ b &= 720 + \frac{2595}{\sqrt{13}} + \frac{48}{26}\sqrt{13(23\,382 + 6485\sqrt{13})}. \end{aligned}$$

Notice that $z \approx -1.07 \times 10^8$, so the formula converges rapidly.

6. Conclusion

In conclusion, it might be interesting to try to classify all of the values of $S(A,B,C;2)$ which can be treated using the ideas in Section 5. It would also be extremely interesting if the methods from Section 3 could be used to say something about 3-dimensional lattice sums such as the Madelung constant.

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