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Nonlinear elliptic problems on singularly perturbed domains

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We consider how the shape of a domain affects the number of positive solutions of a nonlinear elliptic problem. In fact, we show that if a bounded domain Ω is sufficiently close to a union of disjoint bounded domains $\Omega^1, \dots, \Omega^m$, the number of positive solutions of a nonlinear elliptic problem on Ω is at least $2^m - 1$.

1. Introduction

In this paper we consider how the shape of the domain Ω affects the number of solutions of

$$\left.\begin{aligned} \Delta u + u^p = 0 \qquad & \text{in } \Omega, \\ u > 0 \qquad & \text{in } \Omega, \\ u = 0 \qquad & \text{on } \partial\Omega. \end{aligned}\right\} \tag{1.1}$$

Here, Ω is a bounded smooth domain in $\mathbb{R}^N$ and $1 < p < (N+2)/(N-2)$ for $N \geqslant 3$, $1 < p < \infty$ for $N = 2$. The main result in this paper is, roughly speaking, that if Ω is sufficiently close to a union of disjoint domains $\Omega^1, \dots, \Omega^m$ (for a precise description, see conditions (d0)–(d2) in § 1), the number of solutions of problem (1.1) is at least $2^m - 1$. In the special case that the domains $\{\Omega^i\}_{i=1}^m$ are balls, the same conclusion was obtained by Dancer [3].

When Ω is a ball, it is known that problem (1.1) has a unique solution, which is linearly non-degenerate (cf. [3, 7, 11]). Here, a solution v of (1.1) is called linearly non-degenerate if an equation $\Delta u + pv^{p-1}u = 0$ has only a trivial solution in $H_0^{1,2}(\Omega)$. Let $\{B^i\}_{i=1}^m$ be mutually disjoint balls. We define $B_0 \equiv \cup_{i=1}^m B^i$. Let $S(B_0)$ be the set of non-trivial solutions of

$$\left.\begin{aligned} \Delta u + u^p = 0 \qquad & \text{in } B_0, \\ u \geqslant 0 \qquad & \text{in } B_0, \\ u = 0 \qquad & \text{on } \partial B_0. \end{aligned}\right\} \tag{1.2}$$

Then the number of elements of $S(B_0)$ is exactly $2^m - 1$, and each $w \in S(B_0)$ is linearly non-degenerate. Then, using a degree argument, Dancer proved in [3] that for each approximate solution $w \in S(B_0)$ of (1.1), there exists a solution u of (1.1) which is close to w when Ω is sufficiently close to B_0. This implies that there exist at least $2^m - 1$ solutions of problem (1.1) when Ω is sufficiently close to B_0. The

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above result was extended by Dancer [6] to some cases using a homotopy-index (Conley-index) argument instead of a degree argument. On the other hand, it is not easy to find the exact number of solutions of (1.1) when Ω is close to B_0. In his paper [4], Dancer showed that the number of solutions of (1.1) is exactly $2^m - 1$ for certain types of Ω. On the other hand, for a $\mathbb{Z}_2$-symmetric dumbbell-type domain $\Omega \subset \mathbb{R}^2$, it is proved in [4,5] that there exists a very large solution of (1.1) which is not close to any solutions of (1.2) when the dumbbell-type domain is sufficiently close to a disjoint union of two balls. Thus, in this case, the number of solutions is strictly greater than three. The result was extended to the high-dimensional case by the author [1]. Recently, Wei and Zhang proved an existence of a large solution of (1.1) in [15] without the $\mathbb{Z}_2$-symmetry assumption on a dumbbell-type domain $\Omega \subset \mathbb{R}^2$.

In this paper we will study the structure of solutions of (1.1) when Ω is close to a disjoint union of general domains $\Omega^1, \dots, \Omega^m$. Define $\Omega_0 = \bigcup_{i=1}^m \Omega^i$. Let w^i be a non-trivial solution of (1.2) when B_0 is replaced by Ω^i. Let $w \equiv w^{i_1} + \cdots + w^{i_l}$ for some $1 \leqslant i_1 < \cdots < i_l \leqslant m$. Our concern is whether or not there is a solution of (1.1) close to an approximate solution w when Ω is close to Ω_0. In general, it is not easy to check whether or not w is linearly non-degenerate or the homotopy index of a component containing w is not trivial. Thus we cannot apply a degree argument to show that there exists a solution of (1.1) close to an approximate solution w when Ω is close to Ω_0. On the other hand, by combining variational methods developed in [2,10,13], we prove in this paper that, if C^{i_j} is an isolated component (containing w^{i_j}) of the minimal energy solutions on Ω^{i_j} for $j = 1, \dots, m$, there exists a solution of (1.1) close to $C^{i_1} + \cdots + C^{i_l}$ when Ω is close to Ω_0. This implies that when Ω is close to Ω_0, there exist at least $2^m - 1$ solutions of (1.1). This shows that in our problem we can glue together minimal energy solutions without any non-degeneracy conditions.

This paper is organized as follows. In § 2, several assumptions and notations will be stated. Then our main results will be stated. In § 3, we prove our main result via a series of propositions.

2. Preliminary

We consider the following problem,

$$\left.\begin{aligned} \Delta u + f(u) &= 0 && \text{in } \Omega_\sigma, \\ u &> 0 && \text{in } \Omega_\sigma, \\ u &= 0 && \text{on } \partial\Omega_\sigma, \end{aligned}\right\} \tag{2.1}$$

where Ω_σ, $\sigma > 0$, is a bounded smooth domain in $\mathbb{R}^N$. We assume that the function f satisfies

(f1) $f \in C^1(\mathbb{R})$, $f(t) = 0$ for $t \leqslant 0$,

(f2) $|f(t)| + |f'(t)t| \leqslant C|t|^p$ for some $C > 0$, where $p \in (1, (N+2)/(N-2))$ for $N \geqslant 3$ and $p \in (1, \infty)$ for $N = 2$,

(f3) there exists a constant $\theta \in (0, 1)$ such that $0 < f(t) < \theta f'(t)t$ for all $t > 0$.

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Let $\{\Omega^i\}_{i=1}^m$ be mutually disjoint bounded smooth domains in $\mathbb{R}^N$. We assume that for $1 \leqslant i \neq j \leqslant m$,

(d0) the set $\bar{\Omega}^i \cap \bar{\Omega}^j$ is of capacity zero, that is, there exist $\{\phi_n\}_{n=1}^\infty \subset C_0^1(\mathbb{R}^N)$ such that $\phi_n(x) = 1$ for $x \in \bar{\Omega}^i \cap \bar{\Omega}^j$, and that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla \phi_n|^2 \,\mathrm{d}x = 0.$$

We define

$$\Omega_0 \equiv \bigcup_{i=1}^m \Omega^i.$$

On the Ω_σ, $\sigma > 0$, we assume that there exists a compact set E of measure zero in $\mathbb{R}^N$ such that the following properties hold

(d1) for any compact subset K of Ω_0, $K \subset \Omega_\sigma$ for sufficiently small $\sigma > 0$,

(d2) for any open set G containing $E \cup \overline{\bigcup_{i=1}^m \Omega^i}$, $\Omega_\sigma \subset G$ for sufficiently small $\sigma > 0$.

We say that the family $\{\Omega_\sigma\}_{\sigma>0}$ represents a singular perturbation of Ω_0 when (d0)–(d2) are satisfied (cf. [3]). From (d2), we can assume that for some $C > 0$,

$$\bigcup_{\sigma>0} \bar{\Omega}_\sigma \subset B(0, C) \equiv B.$$

Then we can regard an element u of $H_0^{1,2}(\Omega_\sigma)$ as that of $H_0^{1,2}(B)$ by defining $u \equiv 0$ on $B \setminus \Omega_\sigma$. Define a norm $\|u\|$ on $H_0^{1,2}(B)$ by

$$\|u\|^2 = \int_B |\nabla u|^2 \,\mathrm{d}x.$$

For each $i \in \{1, \dots, m\}$, we consider the following problem:

$$\left.\begin{aligned} \Delta u + f(u) &= 0 && \text{in } \Omega^i, \\ u &> 0 && \text{in } \Omega^i, \\ u &= 0 && \text{on } \partial\Omega^i. \end{aligned}\right\} \tag{4-i}$$

Define an energy functional

$$\Gamma(u) \equiv \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 \,\mathrm{d}x - \int_{\mathbb{R}^N} F(u) \,\mathrm{d}x$$

on $H_0^{1,2}(B)$, where

$$F(u) = \int_0^u f(t) \,\mathrm{d}t.$$

Let S_i be the set of solutions of (4-i). By condition (f2) and an estimate [9, equation [(2.12)], there exists a constant $C > 0$ such that, for any $u \in S_i$,

$$\int_{\Omega^i} |\nabla u|^2 \,\mathrm{d}x \leqslant C\biggl(\int_{\Omega^i} |\nabla u|^2 \,\mathrm{d}x\biggr)^{(p+1)/2}.$$

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Thus there exists a constant $\delta > 0$ such that

$$\|u\| > 2\delta \quad \text{for any } u \in S_i, \quad i = 1, \dots, m. \tag{2.2}$$

Define

$$c^i \equiv \inf\{\Gamma(u) \mid u \in S_i\}$$

and

$$S_i^{\alpha} \equiv \{u \in S_i \mid \Gamma(u) \leqslant \alpha\}.$$

We note that the set $S_i^{c^i}$ of minimal energy solutions of (4-i) is not empty for each $i = 1, \dots, m$, and that, from condition (f3),

$$c^i = \inf\left\{\Gamma(u) \;\middle|\; \int_{\Omega^i} (\nabla u \cdot \nabla u - f(u)u)\, \mathrm{d}x = 0,\ u \in H_0^{1,2}(\Omega^i) \setminus \{0\}\right\}. \tag{2.3}$$

Our last assumption follows.

(b-i) Let L_i be a compact subset $S_i^{c^i}$ such that, for some constant $\varepsilon^i > 0$,

$$\{u \in S_i^{c^i} \mid \|u - L_i\| \leqslant 2\varepsilon^i\} \setminus L_i = \emptyset.$$

Remark 2.1. Since Γ satisfies the Palais–Smale condition, we see that $S_i^{c^i}$ is compact in $H_0^{1,2}(\Omega^i)$. Then, taking $L_i = S_i$, we see that condition (b-i) holds. Moreover, if L_i is an isolated component of $S_i^{c^i}$, condition (b$-i$) holds for some $\varepsilon^i > 0$.

Now we see our main result in this paper.

Theorem 2.2. *Let $\{i_1, \dots, i_l\} \subset \{1, \dots, m\}$. Assume that conditions (f1)-(f3), (d0)–(d2) and (b-i_1)–(b-i_l) hold. Define $\varepsilon = \min\{\delta, \varepsilon^{i_1}, \dots, \varepsilon^{i_l}\}$. Then, for sufficiently small $\sigma > 0$, there exists a solution u_σ of (2.1) such that*

$$\|u_\sigma - (L_{i_1} + \dots L_{i_l})\| < \varepsilon.$$

Moreover, it holds that

$$\lim_{\sigma \to 0} \|u_\sigma - (L_{i_1} + \dots L_{i_l})\| = 0.$$

From remark 2.1 and theorem 2.2, we easily deduce the following corollaries.

Corollary 2.3. *Assume that (f1)–(f3) and (d0)–(d2) are satisfied. Then the number of solutions of (2.1) is at least $2^m - 1$ for sufficiently small $\sigma > 0$.*

Corollary 2.4. *Assume that (f1)–(f3) and (d0)–(d2) are satisfied. Denote by q_i the number of isolated connected components of $S_i^{c^i}$ and by n_σ the number of solutions of (2.1). Then it holds that*

$$\liminf_{\sigma \to 0} n_\sigma \geqslant \prod_{i=1}^{m} (q_i + 1) - 1.$$

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REMARK 2.5. Condition (d0) is inevitable in theorem 2.2. In fact, defining

$$\Omega^1 \equiv \{(x,y) \in \mathbb{R}^2 \mid -1 < x < 0,\ 0 < y < 1\},$$
$$\Omega^2 \equiv \{(x,y) \in \mathbb{R}^2 \mid 0 < x < 1,\ 0 < y < 1\}$$

and

$$\Omega_\sigma \equiv \{(x,y) \in \mathbb{R}^2 \mid -1-\sigma < x < 1+\sigma,\ -\sigma < y < 1+\sigma\},$$

conditions (d1), (d2) are satisfied with $m = 2$ and $E = \emptyset$. It is not difficult to see that the intersection

$$\bar{\Omega}^1 \cap \bar{\Omega}^2 = \{(0,y) \in \mathbb{R}^2 \mid 0 \leqslant y \leqslant 1\}$$

is not of capacity 0 in $\mathbb{R}^2$. On the other hand, it is proved in [3] that problem (1.1) has a unique solution when the domain $\Omega \subset \mathbb{R}^2$ is convex and invariant under the reflections $(x,y) \to (-x,y)$, $(x,y) \to (x,-y)$. Since Ω_σ is invariant under the reflections, problem (1.1) with $\Omega = \Omega_\sigma$ has a unique solution for each $\sigma > 0$. This shows that condition (d0) is necessary in theorem 2.2.

3. Proof of theorem 2.2

For any $u \in H_0^{1,2}(B)$, let $P_{m+1}(u) \in H_0^{1,2}(B)$ be a solution of the equation

$$\begin{aligned} \Delta v &= 0 \quad \text{in } \Omega_0, \\ v &= u \quad \text{in } B \setminus \Omega_0. \end{aligned}$$

For each $i \in \{1,\dots,m\}$, we define

$$P_i(u) \equiv \begin{cases} u - P_{m+1}(u) & \text{on } \Omega^i, \\ 0 & \text{on } B \setminus \Omega^i. \end{cases}$$

Then we see that $P_i(u) \in H_0^{1,2}(\Omega^i)$ for $i = 1,\dots,m$. Moreover, we see that $u = P_1(u) + \cdots + P_m(u) + P_{m+1}(u)$, and that

$$\int_B \nabla P_i(u) \cdot \nabla P_j(u)\,\mathrm{d}x = 0 \quad \text{for } 1 \leqslant i \neq j \leqslant m+1.$$

Thus it follows that

$$\|u\|^2 = \sum_{i=1}^{m+1} \|P_i(u)\|^2. \tag{3.1}$$

Define

$$L \equiv L_{i_1} + \cdots + L_{i_l} = \{u_1 + \cdots + u_l \mid u_k \in L_{i_k},\ k = 1,\dots,l\}$$

and

$$\varepsilon = \min\{\delta, \varepsilon^{i_1}, \dots, \varepsilon^{i_l}\}.$$

We define $S_i^* = S_i^{c^i} \cup \{0\}$. Then, from (2.2) and condition (b-j), $j = i_1,\dots,i_l$, we see that

$$\{u \in H_0^{1,2}(B) \mid \|u - L\| \leqslant \varepsilon\} \cap \{S_{i_1}^* + \cdots + S_{i_l}^*\} = L. \tag{3.2}$$

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We define

$$J_k(u) = \int_{\Omega_\sigma} (\nabla u \cdot \nabla P_{i_k}(u) - f(u) P_{i_k}(u)) \, \mathrm{d}x, \quad k = 1, \dots, l,$$

for $u \in H_0^{1,2}(\Omega_\sigma)$. Then, from (f2), we see that $J_k \in C^1(H_0^{1,2}(\Omega_\sigma))$, $k = 1, \dots, l$. We define

$$W_\sigma = W_\sigma(i_1, \dots, i_l) \equiv \{u \in H_0^{1,2}(\Omega_\sigma) \mid u \neq 0 \text{ and } J_1 = \cdots = J_l(u) = 0\}.$$

Denote

$$\Omega_0(d) \equiv \{x \in \Omega_0 \mid |x - \partial\Omega_0| > d\}.$$

For sufficiently small $d > 0$, $\Omega_0(d)$ is an open set in $\mathbb{R}^N$ with smooth boundary. For each small $d > 0$, $\Omega_0(d) \subset \Omega_\sigma$ if $\sigma > 0$ is sufficiently small. For each $\sigma > 0$, we define

$$d(\sigma) \equiv \inf\{d \mid \Omega_0(d) \subset \Omega_\sigma\}.$$

Then we see that

$$\lim_{\sigma \to 0} d(\sigma) = 0 \quad \text{and} \quad \Omega_0(d(\sigma)) \subset \Omega_\sigma.$$

For each $u \in L$, we define $Q_\sigma(u)$ the unique solution of

$$\begin{aligned} -\Delta v &= f(u) \quad \text{in } \Omega_0(d(\sigma)), \\ v &= 0 \qquad\ \text{on } \partial\Omega_0(d(\sigma)). \end{aligned}$$

We note that $Q_\sigma(u) \in H_0^{1,2}(\Omega_\sigma)$ and that $Q_\sigma(u) > 0$ in $\Omega_0(d(\sigma))$. From elliptic estimates [8], it is easy to see that $\lim_{\sigma\to 0} \|u - Q_\sigma(u)\| = 0$. We note that for each $k = 1, \dots, l$,

$$P_{i_k}(Q_\sigma(u)) = Q_\sigma(u)|_{\Omega^{i_k}},$$

where $v|_A$ means a restriction of v on A. Then, from elliptic estimates [8] and the compactness of L, we see that

$$\lim_{\sigma \to 0} J_k(Q_\sigma(u)) = 0$$

uniformly with respect to $u \in L$. Then, from (2.2) and condition (f3), we deduce that for each $u \in L$, there exist $t_\sigma^k(u) \in \mathbb{R}$, $k = 1, \dots, l$, such that

$$Q_\sigma^*(u) \equiv \sum_{k=1}^{l} t_\sigma^k(u) Q_\sigma(u)|_{\Omega^{i_k}} \in W_\sigma,$$

and that

$$\lim_{\sigma \to 0} t_\sigma^k(u) = 1, \quad k = 1 \dots, l,$$

uniformly with respect to $u \in L$. Therefore, we conclude the following proposition.

Proposition 3.1. *There exists a map* $Q_\sigma^* : L \to W_\sigma$ *such that*

$$\lim_{\sigma \to 0} \sup_{u \in L} \|u - Q_\sigma^*(u)\| = 0,$$

and that, for each $u \in L$,

$$Q_\sigma^*(u) \geqslant 0 \quad \text{in } \Omega_\sigma.$$

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We define

$$L^\sigma \equiv Q^*_\sigma(L).$$

Proposition 3.1 says that

$$\lim_{\sigma\to 0} \|L - L^\sigma\| = 0. \tag{3.3}$$

It is easy to see that

$$J'_k(u)(P_{i_j}(u)) = 0$$

for $u \in L^\sigma$ and $1 \leqslant k \neq j \leqslant l$. Moreover, we see from (3.1) and condition (f3) that, for $u \in L^\sigma$ and $k = 1, \dots, l$,

$$\begin{aligned} J'_k(u)(P_{i_k}(u)) &= \int_{\Omega_0(d(\sigma))} (2|\nabla P_{i_k}(u)|^2 - f'(u)(P_{i_k}(u))^2 - f(u)P_{i_k}(u))\,\mathrm{d}x \\ &= \int_{\Omega_0(d(\sigma))} (f(P_{i_k}(u))(P_{i_k}(u)) - f'(P_{i_k}(u))(P_{i_k}(u))^2)\,\mathrm{d}x \\ &\leqslant (\theta - 1)\int_{\Omega_0(d(\sigma))} f'(P_{i_k}(u))(P_{i_k}(u))^2\,\mathrm{d}x \\ &< 0. \end{aligned}$$

We define a map $J : H^{1,2}_0(\Omega_\sigma) \to \mathbb{R}^l$ by $J(u) = (J_1(u), \dots, J_l(u))$. Then there exists a constant $R > 0$, independent of small $\sigma > 0$, such that a map

$$\nabla J(u) : H^{1,2}_0(\Omega_\sigma) \to \mathbb{R}^l$$

is surjective for any $u \in \{v \in W_\sigma | \|v - L^\sigma\| < R\}$. This implies the following proposition.

PROPOSITION 3.2. *There exists a constant $R > 0$, independent of small $\sigma > 0$, such that $\{u \in W_\sigma \mid \|u - L^\sigma\| < R\}$ is a smooth (C^1) Hilbert manifold of codimension l.*

We define

$$\Gamma^\alpha \equiv \{u \in H^{1,2}_0(B) \mid \Gamma(u) \leqslant \alpha\}$$

and

$$l^\sigma \equiv \max\{\Gamma(u) \mid u \in L^\sigma\}.$$

It is not difficult to see that

$$\lim_{\sigma\to 0} l^\sigma = c^{i_1} + \dots + c^{i_l}.$$

We also define

$$L^\sigma_R \equiv \{u \in W_\sigma \mid \|u - L^\sigma\| < R\}$$

and

$$D^\sigma_R \equiv \{u \in H^{1,2}_0(\Omega_\sigma) \mid \|u - L^\sigma\| < R\}.$$

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From (3.2) and proposition 3.1, we see that, for $0 < r < R < \varepsilon$,

$$(D_R^\sigma \setminus D_r^\sigma) \cap \{S_1^* + \dots S_m^*\} = \emptyset$$

if $\sigma > 0$ is sufficiently small. Then we have the following proposition.

PROPOSITION 3.3. *Let $0 < r < R < \varepsilon$. Then there exists a positive constant α such that, if $\sigma > 0$ is sufficiently small,*

$$|\Gamma'(u)| \geqslant \alpha$$

for $u \in (D_R^\sigma \setminus D_r^\sigma) \cap \Gamma^{l^\sigma}$.

Prior to proving proposition 3.3, we prove the following lemma.

LEMMA 3.4. *Let $u \in H_0^{1,2}(\Omega)$. Suppose that $u \equiv 0$ on $B \setminus \Omega_0$. Then its restriction $u_i \equiv u|_{\Omega^i}$ to Ω^i belongs to $H_0^{1,2}(\Omega^i)$ for each $i = 1, \dots, m$.*

Proof. Let Ω' be the set of interior points of $\bar{\Omega}_0$. Then we see that $u \in H_0^{1,2}(\Omega')$ (cf. [9, lemma 3.3]). Take $\psi_k \in C_0^\infty(\Omega')$, $k = 1, 2, \dots$, such that

$$\lim_{k\to\infty} \|u - \psi_k\| = 0.$$

Since the capacity of $\bigcup_{1\leqslant\, i\neq j\leqslant\, m} \bar{\Omega}^i \cap \bar{\Omega}^j$ is 0, there exist functions $\{\varphi_n\}_n \subset C_0^1(\mathbb{R}^N)$ such that $\varphi_n(x) = 1$ for any $x \in \bigcup_{1\leqslant\, i\neq j\leqslant\, m} \bar{\Omega}^i \cap \bar{\Omega}^j$, and that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla \varphi_n|^2 \, \mathrm{d}x = 0.$$

We can assume that the support of φ_n is contained in B for each n. Then we see that for some constant $c(k)$,

$$\begin{aligned} \|\psi_k - \psi_k(1-\varphi_n)\|^2 &= \|\psi_k \varphi_n\|^2 \\ &= \int_{\mathbb{R}^N} (|\nabla \psi_k|^2 (\varphi_n)^2 + 2\nabla\psi_k \cdot \nabla\varphi_n \psi_k \varphi_n + (\psi_k)^2 |\nabla \varphi_n|^2) \, \mathrm{d}x \\ &\leqslant c(k) \int_{\mathbb{R}^N} (|\nabla\varphi_n|^2 + (\varphi_n)^2) \, \mathrm{d}x. \end{aligned}$$

Thus, from Poincaré's inequality, we see that

$$\lim_{n\to\infty} \|\psi_k - \psi_k(1-\varphi_n)\| = 0$$

for each k. This implies that for some $n(k)$,

$$\lim_{k\to\infty} \|u - \psi_k(1-\varphi_{n(k)})\| = 0.$$

We note that $\theta_k \equiv \psi_k(1-\varphi_{n(k)}) \in C_0^1(B)$ and $\theta_k(x) = 0$ for $x \notin \Omega_0$. Hence we see that for each $i = 1, \dots, m$, the restriction of θ_k on Ω^i, $\theta_k^i \equiv \theta_k|_{\Omega^i}$, is continuously differentiable in Ω^i and vanishes on $\partial\Omega^i$. The function ψ_k^i can be approximated by functions in $C_0^\infty(\Omega^i)$ with respect to the norm

$$\|w\| = \left(\int_\Omega |\nabla w|^2 \, \mathrm{d}x \right)^{1/2}.$$

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Therefore, u_i is contained in $\overline{C_0^\infty(\Omega^i)} = H_0^{1,2}(\Omega^i)$, $i = 1, \dots, m$. This completes the proof. □

Proof of proposition 3.3. We argue by contradiction. First assume that, for some $0 < r < R < \varepsilon$, there exist $\{u_{\sigma_j}\}_{j=1}^\infty \subset H_0^{1,2}(B)$ such that

$$\lim_{j\to\infty} \sigma_j = 0, \quad \lim_{j\to\infty} \Gamma'(u_{\sigma_j}) = 0 \quad \text{and} \quad u_{\sigma_j} \in (D_R^{\sigma_j} \setminus D_r^{\sigma_j}) \cap \Gamma^{l^\sigma}.$$

Taking a subsequence if necessary, we can assume that the u_{σ_j} converges weakly to some $u_0 \in H_0^{1,2}(B)$. For the sake of convenience, we write simply σ for σ_j. Since the $\{\Omega_\sigma\}$ represents a singular perturbation of Ω_0, we see that $u_0(x) = 0$ for any $x \notin \bar{\Omega}_0$. From lemma 3.4, we see that the restricted function $u_0^i \equiv u_0|_{\Omega^i}$ belongs to $H_0^{1,2}(\Omega^i)$ for each $i = 1, \dots, m$. Since $\lim_{\sigma\to 0} \Gamma'(u_\sigma) = 0$, we see that for each $i = 1, \dots, m$,

$$\Delta u_0^i + f(u_0^i) = 0 \quad \text{in } \Omega^i.$$

Thus u_0^i is a solution of (4-i) or, identically, 0 on Ω^i. Since u_σ converges weakly to u_0 in $H_0^{1,2}(\mathbb{R}^N)$ as $\sigma \to 0$, it holds that

$$\lim_{\sigma\to 0} \int_B f(u_\sigma)u_\sigma \,\mathrm{d}x = \int_B f(u_0)u_0 \,\mathrm{d}x$$

(cf. [12, proposition B.10 in appendix B]). Since

$$\int_B |\nabla u_0|^2 \,\mathrm{d}x = \int_B f(u_0)u_0 \,\mathrm{d}x$$

and

$$\lim_{\sigma\to 0} \int_{\Omega_\sigma} (|\nabla u_\sigma|^2 - f(u_\sigma)u_\sigma) \,\mathrm{d}x = 0,$$

it follows that

$$\lim_{\sigma\to 0} \int_{\Omega_\sigma} |\nabla u_\sigma|^2 \,\mathrm{d}x = \int_{\Omega_\sigma} |\nabla u_0|^2 \,\mathrm{d}x.$$

This implies that u_σ converges to u_0 in $H_0^{1,2}(B)$ as $\sigma \to 0$. Then $u_0 \in H_0^{1,2}(\Omega_0)$ and $\Gamma(u_0) = c^{i^1} + \dots + c^{i^l}$. From (3.3), we see that

$$\|u_0 - L\| = \lim_{\sigma\to 0} \|u_\sigma - L^\sigma\| \leqslant R.$$

Since $R < \varepsilon \leqslant \delta$, it follows that $u_0^i \equiv 0$ for $i \in \{1, \dots, m\} \setminus \{i_1, \dots, i_l\}$. Then, from (2.2) and conditions (b-i_1)–(b-i_l), it follows that $\|u_0 - L\| = 0$. This contradicts that for sufficiently small $\sigma > 0$, $u_\sigma \notin D_r^\sigma$. This completes the proof. □

Denote

$$M_R^\sigma \equiv \inf\{\Gamma(u) \mid u \in L_R^\sigma\}.$$

Then we have the following estimate on M_R^σ.

Proposition 3.5. *For sufficiently small $R > 0$, it holds that*

$$\lim_{\sigma\to 0} M_R^\sigma = c^{i_1} + \dots + c^{i_l}.$$

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Proof. From conditions (b-i_1)–(b-i_l), we easily see that

$$\limsup_{\sigma\to 0} M_R^\sigma \leqslant c^{i_1} + \cdots + c^{i_l}.$$

Thus it suffices to show that, for sufficiently small $R > 0$,

$$\liminf_{\sigma\to 0} M_R^\sigma \geqslant c^{i_1} + \cdots + c^{i_l}.$$

Since 0 is a local minimum of $\varGamma$ on $H_0^{1,2}(B)$, there exists a constant $R' \in (0, \varepsilon)$ such that

$$\varGamma(u) \geqslant 0 \quad \text{for any } u \in \{v \in H_0^{1,2}(B) \mid \|v\| < R'\}.$$

Suppose that, for sufficiently small $R \in (0, R')$,

$$\liminf_{\sigma\to 0} M_R^\sigma < c^{i_1} + \cdots + c^{i_l}.$$

We take a $u_\sigma \in L_R^\sigma$ such that

$$\liminf_{\sigma\to 0} \varGamma(u_\sigma) = \liminf_{\sigma\to 0} M_R^\sigma.$$

Since $\{u_\sigma\}_\sigma$ is bounded in $H_0^{1,2}(B)$, we can assume that, taking a subsequence if necessary, u_σ converges weakly to some u_0 in $H_0^{1,2}(B)$ as $\sigma \to 0$. Since $u \in H_0^{1,2}(\varOmega_\sigma)$ and $\{\varOmega_\sigma\}$ represents a singular perturbation of $\varOmega_0$, it follows that

$$u_0(x) = 0 \quad \text{for } x \notin \bar{\varOmega}_0.$$

Then we see that $P_{m+1}(u_\sigma)$ converges weakly to 0 in $H_0^{1,2}(B)$ as $\sigma \to 0$. We note that the imbedding $H_0^{1,2}(B) \hookrightarrow L^q(B)$ is compact for $1 < q < 2N/(N-2)$. Thus we see that

$$\begin{aligned}\lim_{\sigma\to 0}\int_{\varOmega_\sigma} F(u_\sigma)\,\mathrm{d}x &= \lim_{\sigma\to 0}\int_{\varOmega_\sigma} F(P_1(u_\sigma) + \cdots + P_m(u_\sigma) + P_{m+1}(u_\sigma))\,\mathrm{d}x \\ &= \lim_{\sigma\to 0}\int_{\varOmega_\sigma} F(P_1(u_\sigma) + \cdots + P_m(u_\sigma))\,\mathrm{d}x,\end{aligned}$$

and that for each $i = 1, \dots, m$,

$$\begin{aligned}\lim_{\sigma\to 0}\int_{\varOmega_\sigma} f(u_\sigma)P_i(u_\sigma)\,\mathrm{d}x &= \lim_{\sigma\to 0}\int_{\varOmega_\sigma} f(P_i(u_\sigma) + P_{m+1}(u_\sigma))P_i(u_\sigma)\,\mathrm{d}x \\ &= \lim_{\sigma\to 0}\int_{\varOmega_\sigma} f(P_i(u_\sigma))P_i(u_\sigma)\,\mathrm{d}x\end{aligned}$$

(see also [12, proposition B.10 in the appendix]). Thus it follows that

$$\lim_{\sigma\to 0} \varGamma(u_\sigma) = \lim_{\sigma\to 0}\int_{\varOmega_\sigma}\biggl(\frac{1}{2}\sum_{i=1}^{m+1}|\nabla P_i(u_\sigma)|^2 - F(P_1(u_\sigma) + \cdots + P_m(u_\sigma))\biggr)\,\mathrm{d}x,$$

and that

$$\begin{aligned}&\lim_{\sigma\to 0}\int_{\varOmega_0^{i_k}} (|\nabla P_{i_k}(u_\sigma)|^2 - f(P_{i_k}(u_\sigma))P_{i_k}(u_\sigma))\,\mathrm{d}x \\ &\qquad = \lim_{\sigma\to 0}\int_{\varOmega_0^{i_k}} (\nabla u_\sigma \cdot \nabla P_{i_k}(u_\sigma) - f(u_\sigma)P_{i_k}(u_\sigma))\,\mathrm{d}x = 0.\end{aligned}$$

From (3.1), we see that, for any $u \in H_0^{1,2}(B)$,

$$\int_B |\nabla P_{m+1}(u)|^2 \, \mathrm{d}x \leqslant \int_B |\nabla u|^2 \, \mathrm{d}x.$$

Thus P_i is a continuous linear map from $H_0^{1,2}(B)$ to $H_0^{1,2}(B)$. We note that for $k = 1, \dots, l$ and $u \in L^\sigma$, $P_{i_k}(u) = u|_{\Omega^{i_k}}$. Then, since L^σ is bounded away from 0, it follows that for sufficiently small $R > 0$, $\{P_{i_k}(u_\sigma)\}_\sigma$ is bounded away from 0 for each $i = 1, \dots, l$. Then, from condition (f3), there exists a $t_k(\sigma) \in \mathbb{R}$, $k = 1 \dots, l$, such that, for each $k = 1, \dots, l$,

$$\lim_{\sigma \to 0} t_k(\sigma) = 1$$

and

$$\int_{\Omega_0^{i_k}} (|\nabla t_k(\sigma) P_{i_k}(u_\sigma)|^2 - f(t_k(\sigma) P_{i_k}(u_\sigma)) t_k(\sigma) P_{i_k}(u_\sigma)) \, \mathrm{d}x = 0.$$

Moreover, if $R > 0$ is sufficiently small,

$$\|P_i(u_\sigma)\| < R'$$

for $i \in \{1, \dots, m\} \setminus \{i_1, \dots, i_l\}$. Thus, from (2.3), we see that

$$\begin{aligned}
\lim_{\sigma \to 0} \Gamma(u_\sigma) &= \lim_{\sigma \to 0} \int_{\Omega_\sigma} \left(\frac{1}{2} \sum_{i=1}^{m+1} |\nabla P_i(u_\sigma)|^2 - F(P_1(u_\sigma) + \cdots + P_m(u_\sigma)) \right) \mathrm{d}x \\
&\geqslant \lim_{\sigma \to 0} \sum_{k=1}^{l} \int_{\Omega_\sigma} (\tfrac{1}{2} |\nabla P_{i_k}(u_\sigma)|^2 - F(P_{i_k}(u_\sigma))) \, \mathrm{d}x \\
&\qquad + \lim_{\sigma \to 0} \sum_{i \in \{1, \dots, m\} \setminus \{i_1, \dots, i_k\}} \int_{\Omega_\sigma} (\tfrac{1}{2} |\nabla P_i(u_\sigma)|^2 - F(P_i(u_\sigma))) \, \mathrm{d}x \\
&\geqslant \lim_{\sigma \to 0} \sum_{k=1}^{l} \int_{\Omega_\sigma} (\tfrac{1}{2} |\nabla t_k(\sigma) P_{i_k}(u_\sigma)|^2 - F(t_k(\sigma)(P_{i_k}(u_\sigma)))) \, \mathrm{d}x \\
&\geqslant c^{i_1} + \cdots + c^{i_l} \\
&> \liminf_{\sigma \to 0} M_R^\sigma.
\end{aligned}$$

This is a contradiction. This completes the proof. □

We will show that the energy functional Γ restricted to W_σ has a critical point in $L_R^\sigma \cap \Gamma^{l^\sigma}$ for small $\sigma > 0$. Prior to showing this, we need the following result.

Proposition 3.6. *Let Γ_w be a restriction of Γ to W_σ. Assume that a positive constant R is sufficiently small. Then, for each $r \in (0, R)$, there exists a constant $\alpha_w > 0$ such that, for sufficiently small $\sigma > 0$ and $u \in (L_R^\sigma \setminus L_r^\sigma) \cap \Gamma^{l^\sigma}$,*

$$|\Gamma_w'(u)| \geqslant \alpha_w.$$

Proof. From the continuity of P_i, $i = 1, \dots, m$, we see that, for sufficiently small $R > 0$, there exists a constant $\beta > 0$, independent of $\sigma > 0$, with the following property: for any $u \in L_R^\sigma$ and $k = 1, \dots, l$,

$$\|P_{i_k}(u)\| \geqslant \beta.$$

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We note that if $R > 0$ is sufficiently small,

$$J_k'(u)(P_{i_k}(u)) \neq 0$$

for any $u \in L_R^\sigma$ and $k = 1, \dots, l$. Thus we see that

$$P_{i_1}(u), \dots, P_{i_l}(u) \notin T_u L_R^\sigma$$

for any $u \in L_R^\sigma$. Here, $T_u L_R^\sigma$ means a tangent space of L_R^σ at $u \in L_R^\sigma$. It is easy to see that the $\{P_{i_1}(u), \dots, P_{i_k}\}$ are linearly independent. This implies that

$$T_u(H_0^{1,2}(\Omega_\sigma)) = T_u L_R^\sigma \oplus \langle \{P_{i_1}(u), \dots, P_{i_l}(u)\} \rangle,$$

where $\langle A \rangle$ means a space spanned by elements of A. Since we have $\Gamma'(u) P_{i_k}(u) = 0$, $k = 1, \dots, l$, we conclude from proposition 3.3 that there exists a constant $\alpha_w > 0$ such that $|\Gamma_w'(u)| \geqslant \alpha_w$ for any $u \in (L_R^\sigma \setminus L_r^\sigma) \cap \Gamma^{l^\sigma}$. □

Proposition 3.7. *For sufficiently small $R \in (0, \varepsilon)$, there exists a constant $\sigma_0 > 0$ such that Γ_w has a critical point $u_\sigma \in L_R^\sigma \cap \Gamma^{l^\sigma}$ for each $\sigma < \sigma_0$.*

Proof. From the above propositions, we can take sufficiently small $R \in (0, \varepsilon)$ so that L_R^σ is a C^1-manifold, and that

$$\lim_{\sigma \to 0} M_R^\sigma = C^{i_1} + \cdots + C^{i_l}.$$

Suppose that there exist no critical points of Γ_w on $L_R^\sigma \cap \Gamma^{l^\sigma}$ for sufficiently small $\sigma > 0$. Then there exists a pseudo-gradient vector field V for Γ_w such that, for any $u \in L_R^\sigma \cap \Gamma^{l^\sigma}$,

$$V(u) \in T_u L_R^\sigma, \qquad \|V(u)\| \leqslant 2\|\Gamma_w'(u)\|$$

and

$$\Gamma_w'(u) V(u) \geqslant \|\Gamma_w'(u)\|^2$$

(cf. [12, appendix A]). Set

$$g(u) \equiv \frac{\|u - \complement L_R^\sigma\|}{\|u - \complement L_R^\sigma\| + \|u - L_{R/2}^\sigma\|},$$

where $\complement L_R^\sigma \equiv W_\sigma \setminus L_R^\sigma$. Define

$$h(s) = \begin{cases} s & \text{for } 0 \leqslant s \leqslant 1, \\ 1/s & \text{for } s \geqslant 1. \end{cases}$$

We set

$$\Theta(u) \equiv g(u) h(\|V(u)\|) V(u).$$

Then we consider a Cauchy problem

$$\frac{\mathrm{d}\eta}{\mathrm{d}t} = -\Theta(\eta), \qquad \eta(0, u) = u. \tag{3.4}$$

The Cauchy problem (3.4) has a unique global solution. By proposition 3.6, there exists a constant $\alpha_w \in (0, 1)$, independent of $\sigma > 0$, such that

$$\|\Theta(u)\| \geqslant \alpha_w \quad \text{for } u \in (L_{R/2}^\sigma \setminus L_{R/4}^\sigma) \cap \Gamma^{l^\sigma}.$$

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We take $v_\sigma \in L^\sigma \subset L^\sigma_{R/4}$ such that $\Gamma_w(v_\sigma) = l^\sigma$. Since Γ_w has no critical points in $L^\sigma_R \cap \Gamma^{l^\sigma}$ and satisfies the Palais–Smale condition (cf. [14]), there exists a constant $c(\sigma) > 0$ such that

$$|\Gamma_w'(u)| \geqslant c(\sigma) \quad \text{for } u \in L^\sigma_{R/2} \cap \Gamma^{l^\sigma}.$$

We note that $\Gamma(\eta(t, v_\sigma)) \leqslant l^\sigma$ for any $t \geqslant 0$. If $\eta(t, v_\sigma)$ lies in $L^\sigma_{R/2}$ for all $t \in (0, \infty)$, it follows that

$$\lim_{t\to\infty} \Gamma(\eta(t, v_\sigma)) = -\infty.$$

This contradicts proposition 3.5. Hence there exist $t_1, t_2 \in (0, \infty)$ with $t_1 < t_2$ such that

$$\eta(t_1, v_\sigma) \in \partial L^\sigma_{R/4}, \qquad \eta(t_2, v_\sigma) \in \partial L^\sigma_{R/2},$$

and that

$$\eta(t, v_\sigma) \in (L^\sigma_{R/2} \setminus L^\sigma_{R/4}) \cap \Gamma^{l^\sigma} \quad \text{for } t \in (t_1, t_2).$$

Then it follows that

$$\begin{aligned}
\Gamma(\eta(t_2, v_\sigma)) - \Gamma(v_\sigma) &\leqslant \Gamma(\eta(t_2, v_\sigma)) - \Gamma(\eta(t_1, v_\sigma)) \\
&= \int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}t} \Gamma_w(\eta(t, v_\sigma))\,\mathrm{d}t \\
&= -\int_{t_1}^{t_2} \Gamma_w'(\eta)\Theta(\eta)\,\mathrm{d}t \\
&\leqslant -\int_{t_1}^{t_2} \min\{1, \|\Gamma_w'(\eta(t, v_\sigma))\|\}\|\Gamma_w'(\eta(t, v_\sigma))\|\,\mathrm{d}t \\
&\leqslant -\alpha_w \int_{t_1}^{t_2} \|\Gamma_w'(\eta(t, v_\sigma))\|\,\mathrm{d}t \\
&\leqslant -\tfrac{1}{2}\alpha_w \int_{t_1}^{t_2} \|V(\eta(t, v_\sigma))\|\,\mathrm{d}t \\
&\leqslant -\tfrac{1}{2}\alpha_w \int_{t_1}^{t_2} \|\Theta(\eta(t, v_\sigma))\|\,\mathrm{d}t \\
&= -\tfrac{1}{2}\alpha_w \int_{t_1}^{t_2} \left\|\frac{\mathrm{d}\eta}{\mathrm{d}t}(t, v_\sigma)\right\|\mathrm{d}t \\
&\leqslant -\tfrac{1}{2}\alpha_w(\tfrac{1}{2}R - \tfrac{1}{4}R) \\
&= -\tfrac{1}{8}R\alpha_w.
\end{aligned}$$

Thus we see that

$$\Gamma(\eta(t_2, v_\sigma)) \leqslant \Gamma(v_\sigma) - \tfrac{1}{8}R\alpha_w = l^\sigma - \tfrac{1}{8}R\alpha_w.$$

This implies that

$$\liminf_{\sigma\to 0} \Gamma(\eta(t_2, v_\sigma)) \leqslant c^{i_1} + \cdots + c^{i_l} - \tfrac{1}{8}R\alpha_w.$$

This contradicts proposition 3.5. Therefore, there exists a constant $\sigma_0 > 0$ such that the functional Γ_w has a critical point $u_\sigma \in L^\sigma_R \cap \Gamma^{l^\sigma}$ for each $\sigma < \sigma_0$. This completes the proof. □

Completion of the proof for theorem 2.2. Since we have that $\Gamma'(u)P_{i_k}(u) = 0$ for $k = 1, \dots, l$ and that

$$T_u(H_0^{1,2}(\Omega_\sigma)) = T_u L_R^\sigma \oplus \langle \{P_{i_1}(u), \dots, P_{i_l}(u)\} \rangle$$

for each $u \in L_R^\sigma$, it follows that the u_σ obtained in proposition 3.7 is a critical point of Γ on $H_0^{1,2}(\Omega_\sigma)$. Thus, by a maximum principle, u_σ is a solution of (2.1) such that

$$\|u_\sigma - (L_{i_1} + \cdots + L_{i_l})\| < \varepsilon \quad \text{and} \quad \Gamma(u_\sigma) \leqslant l^\sigma.$$

It remains to show that

$$\lim_{\sigma \to 0} \|u_\sigma - (L_{i_1} + \cdots + L_{i_l})\| = 0.$$

Suppose that there exist $\{u_{\sigma_k}\}_{k=1}^\infty$ such that

$$\lim_{k \to \infty} \sigma_k = 0 \quad \text{and} \quad \lim_{k \to \infty} \|u_{\sigma_k} - (L_{i_1} + \cdots + L_{i_l})\| > 0.$$

Taking a subsequence if necessary, we can assume that u_{σ_k} converges weakly to some u_0 in $H_0^{1,2}(B)$ as $k \to \infty$. Then it follows that $\Gamma(u_0) \leqslant c^{i^1} + \cdots + c^{i^l}$. Thus, from lemma 3.4 and conditions (b-i_1)–(b-i_l), we deduce that $u_0 \in L_{i_1} + \cdots + L_{i_l}$. Since

$$\begin{aligned} \int_{\Omega_{\sigma_k}} \nabla u_{\sigma_k} \cdot \nabla u_{\sigma_k} \, \mathrm{d}x &= \int_{\Omega_{\sigma_k}} f(u_{\sigma_k}) u_{\sigma_k} \, \mathrm{d}x \\ &\to \int_{\Omega_0} f(u_0) u_0 \, \mathrm{d}x \\ &= \int_{\Omega_0} \nabla u_0 \cdot \nabla u_0 \, \mathrm{d}x \end{aligned}$$

as $k \to \infty$, it follows that

$$\lim_{k \to \infty} \|u_{\sigma_k} - u_0\| = 0.$$

This implies that

$$\lim_{k \to \infty} \|u_{\sigma_k} - (L_{i_1} + \cdots + L_{i_l})\| = 0.$$

This is a contradiction. Thus it holds that

$$\lim_{\sigma \to 0} \|u_\sigma - (L_{i_1} + \cdots + L_{i_l})\| = 0.$$

This completes the proof of theorem 2.2. □

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