Student Problems

Students up to the age of 19 are invited to send solutions to either or both of the following problems to Beth Woollacott, SCH.1.17, Schofield Building, Loughborough University, Loughborough LE11 3TU.

Two prizes will be awarded – a first prize of £25, and a second prize of £20 – to the senders of the most impressive solutions for either problem. It is not necessary to submit solutions to both. Entries should arrive by 20th September 2022 and solutions will be published in the November 2022 edition.

The Mathematical Association and the *Gazette* comply fully with the provisions of the 2018 GDPR legislation. Submissions **must** be accompanied by the SPC permission form which is available on the MA website

https://www.m-a.org.uk/the-mathematical-gazette

Note that if permission is not given, a pupil ***may still participate and will be eligible for a prize in the same way as others****.*

Problem 2022.3 **(Mark Emson)**

Evaluate

$$\int_0^\infty \frac{1}{1 + x^3}\, dx.$$

Problem 2022.4 **(Geoffrey Strickland)**

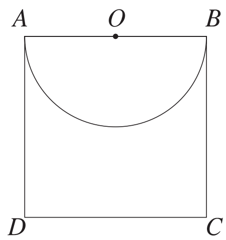
A "nearly isosceles" right-angled triangle with integer side-lengths is defined as one in which the two sides adjacent to the right angle differ in length by just 1 unit. A triangle with side lengths 20, 21 and 29 is an example. Find, with proof, a method for generating a sequence of such triangles.

Solutions to 2022.1 and 2022.2

Both problems were solved by Aditya Gupta of Westminster School, London.

2022.1 **(Geoffrey Strickland)**

Given a square $ABCD$ with a semi-circle, diameter AB and centre O, inscribed inside it, show how to construct the golden points on each side of the square using only a straight edge and a pencil.



Golden points are points which divide a side internally in the Golden Ratio. There are thus two such points on each side.

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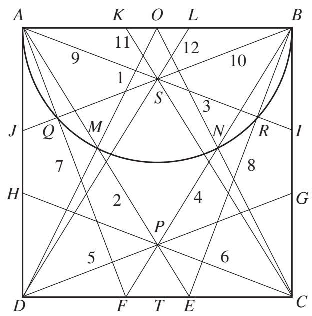
Solution (written by Geoffrey Strickland)

The solution is shown in the diagram in which $ABCD$ is a unit square. The twelve straight lines being numbered in the order in which they were drawn.



- Points E, F, G, H, I, J, K and L are the required golden points.
- Points M and N are the intersections of OD and OC respectively with the semi-circle.
- Points Q and R are the intersections of AF and BE respectively with the semi-circle.
- Point P is the intersection of AE and BF.
- Point S is the intersection of AR and BQ.
- Point T is the midpoint of DC (not actually constructed but needed for reference later).

Applying Pythagoras to ΔAOD gives $OD = \frac{1}{2}\sqrt{5}$ and with $OM = \frac{1}{2}$ it follows that $MD = (\sqrt{5} - 1)/2 = 1/\phi$ where ϕ is the golden ratio $\frac{1}{2}(\sqrt{5} + 1)$, ΔAOM is isosceles with $OA = OM$ and $\angle OAM = \angle OMA$. but

$$\angle OAM = \angle MED \qquad \text{(alternate)}$$

$$\text{and} \qquad \angle OMA = \angle EMD \qquad \text{(vertically opposite).}$$

Hence ΔMDE is isosceles with $DM = DE = 1/\phi$ and E is a golden point of side DC. Similarly, golden point F is obtained by symmetry using OC and BF.

Note that the lengths of line segments DE, DF and FE are easily shown to be $1/\phi$, $1/\phi^2$ and $1/\phi^3$ respectively.

It is now necessary to obtain the height TP of ΔFPE which is similar to ΔAPB. Applying simple ratio theory to the two triangles gives $TP = (1/\phi^3)/(1 + 1/\phi^3) = 1/2\phi^2$ from which it follows that if DP is extended to meet side BC at G then $CG = 1/\phi^2$ and therefore G is a golden point of side BC.

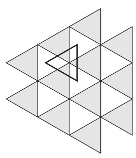
Similarly extending CP to meet side AD at H gives a golden point on AD.

Next, join AR and extend it to I on side BC. Note that AI is perpendicular to BE and so, by rotational symmetry, I is the second golden point on side BC.

The second golden point, J, on side AD, is similarly obtained.

Point S has now been located, symmetrical with point P, so finally joining CS and DS and extending them to meet AB at K and L respectively establishes the golden points on side AB.

***Problem 2022.2* (Paul Stephenson)**



A regular triangular tiling is two-coloured so that adjacent triangles have opposite colours. The grey triangles point left. A free triangle of the same size, also pointing left, may be translated anywhere on the tiling.

What is the smallest fraction of the free triangle which can be grey?

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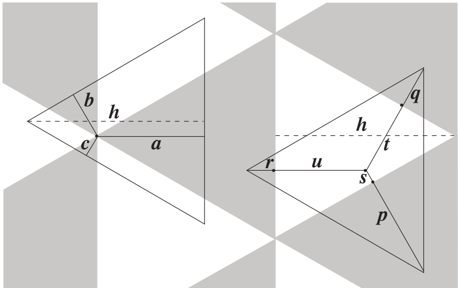
Solution (written by Paul Stephenson)

There are two sites where the free triangle is $\frac{1}{3}$ grey. At one, the centroid of the triangle coincides with a vertex of the grid; at the other it coincides with that of a white triangle.

Accordingly, we can consider two loci. In the first (left), the free triangle contains a grid vertex (or the vertex lies on a boundary of the free triangle). In the second (right), a white triangle contains the centroid of the free triangle (or the centroid lies on the boundary of the white triangle).



Let a triangle have unit area. Let the fraction of grey contained be f_1 on the left, f_2 on the right.

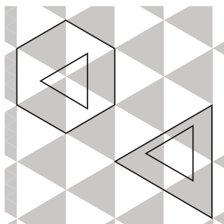
By similarity, $f_1 = \left(a^2 + b^2 + c^2\right)/h^2$. This expression is symmetrical in a, b, c. Therefore, any value of a which gives a minimum value for f_1 must equal that of b and c. From the geometry, therefore, we see that $a = b = c = \frac{1}{3}h$, and the required minimum is $\frac{1}{3}$. Again by similarity, $f_2 = \left(p^2 + q^2 + r^2\right)/h^2$. Since the distance from the centroid to the apex is $\frac{2}{3}$ of the height of the triangle, we have $s + p = t + q = u + r = \frac{2}{3}h$.

Arguing as above, when $p = q = r$, $s = t = u$, and furthermore, $p = q = r = s = t = u = \frac{1}{3}h$, and we again find a minimum value of $\frac{1}{3}$.

In the figure below, we see that, in the first case, the locus of the boundary of the free triangle is a hexagon, with a side equal to a grid triangle side.

In the second case, we have extended the locus so that the boundary is that of a triangle with twice the edge of a grid triangle. It represents all of the positions where the free triangle does not contain a grid vertex. We note here that, when the centroid of the free triangle moves into a grey zone, the amount of grey it contains becomes at least $\frac{1}{2}$, that is, greater than $\frac{1}{3}$.

Since the hexagon allows all positions of the free triangle where it contains a grid vertex, and the big triangle allows all positions of the free triangle where it does not contain a grid vertex, between them, the hexagon and the big triangle allow all possible positions of the free triangle. Since in each case the grey fraction is not less than a third, this local minimum must be the global minimum.



Prizewinner

The first prize of £25 is awarded to Aditya Gupta with thanks for the brilliant solutions.

10.1017/mag.2022.91 *Beth Woollacott*

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