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A Graph-Grabbing Game

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Two players share a connected graph with non-negative weights on the vertices. They alternately take the vertices (one in each turn) and collect their weights. The rule they have to obey is that the remaining part of the graph must be connected after each move. We conjecture that the first player can get at least half of the weight of any tree with an even number of vertices. We provide a strategy for the first player to get at least 1/4 of an even tree. Moreover, we confirm the conjecture for subdivided stars. The parity condition is necessary: Alice gets nothing on a three-vertex path with all the weight at the middle. We suspect a kind of general parity phenomenon, namely, that the first player can gather a substantial portion of the weight of any 'simple enough' graph with an even number of vertices.

1. Introduction and conjecture

A graph-grabbing game is played on a finite connected graph with non-negative weights on the vertices (from now on, simply a graph). There are two players: Alice and Bob. Starting with Alice, they take the vertices alternately one by one and collect their weights. The vertices taken are removed from the graph. The choice of a vertex to be played in each move is restricted by the rule that after each move the remaining vertices form a connected subgraph. In particular, playing on a tree, the players pick and remove leaves. Both players aim at maximizing their outcomes at the end of the game, when all vertices have been taken.

The first problem in Winkler's puzzle book [7] is to prove that Alice can guarantee herself at least half of the weight of any path on an even number of vertices. The strategy is very simple: two-colour the vertices and gather the heavier colour. On the other hand, Alice cannot expect any positive score for odd paths: just consider a path on three vertices with all the weight at the middle. We believe this dichotomy of parities holds for all trees.

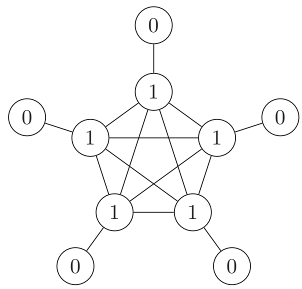

Figure 1. $\mathbf{G}_5$: Alice cannot secure more than 1.

Conjecture 1.1. *Alice can secure at least $\frac{1}{2}$ of the weight of any tree with an even number of vertices.*

Further on we present a strategy for Alice to collect $\frac{1}{4}$ of the weight of any tree with an even number of vertices (Theorem 2.1). We also confirm the conjecture for subdivided stars, that is, trees with at most one vertex of degree greater than 2 (Theorem 2.2). Note that any odd tree on at least three vertices can be weighted so that Alice gets nothing (it suffices to put the whole weight at one non-leaf vertex).

This parity phenomenon does not hold for all graphs in the graph-grabbing game. There is no positive lower bound for the fraction of total weight that Alice can guarantee herself on any graph with an even number of vertices. Example 1.2 describes a sequence of even graphs on which Alice's guaranteed gain tends to zero. Nevertheless, all such sequences known by the authors contain arbitrarily large cliques. There might exist a function $f(n) > 0$ such that Alice can secure at least $f(n)$ of the weight of any graph with an even number of vertices and with clique-size at most n.

Example 1.2. $\mathbf{G}_n = (V, E, w)$ is a weighted graph with $2n$ vertices $V = \{a_1, \ldots, a_n, b_1, \ldots, b_n\}$. The b_i form a clique: $b_i \, E \, b_j$ for all $i \neq j$. The only neighbour of each a_i is b_i. The weights are distributed on the b_i: $w(a_i) = 0$ and $w(b_i) = 1$, and thus the total weight is n. Alice has no strategy to gather more than 1 from $\mathbf{G}_n$.

The graph-grabbing game on a cycle has been studied as the so-called pizza game: vertices are seen as slices of a pizza. The two-colouring argument shows that Alice can get at least half of an even pizza, and there are examples of odd pizzas where she can get only $\frac{4}{9}$ (see Figure 2). Winkler [6] conjectured in 2008 that Alice can secure at least $\frac{4}{9}$ of any pizza, and this has been proved by two independent groups of researchers [1, 3].

In the pizza game not only the remaining part but also the taken part is connected throughout the game. This leads to another generalization for graphs, in which the taken part is required to be connected. In this variant Alice starts by taking any vertex and then the players alternately pick vertices adjacent to already taken ones. The parity of the number of vertices makes a difference in this game as well, but this time odd graphs are better for Alice. In particular, Alice has a strategy to get at least $\frac{1}{4}$ of any odd tree, while even trees can be arbitrarily bad for her [4].

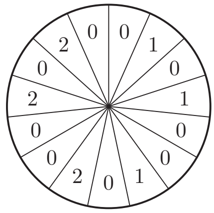

Figure 2. Alice can guarantee herself at most $\frac{4}{9}$ of the pizza.

Independent research on generalizations of pizza game, also leading to the two aforementioned variants, has been carried out by Cibulka, Kynčl, Mészáros, Stolař and Valtr [2]. They focus on connectivity and computational complexity issues. In particular, they provide a sequence of graphs of arbitrarily high connectivity and Alice's guaranteed gain tending to zero. They also prove that it is PSPACE-complete to decide who can get half of the weight of the graph in the graph-grabbing game. The complexity of the problem when input graphs are restricted to trees is unknown.

The name *graph-grabbing game* follows Moshe Rosenfeld, who proposed the game for trees and called it the *gold-grabbing game* [5].

2. Results

Theorem 2.1. *Alice can secure at least $\frac{1}{4}$ of the weight of any tree with an even number of vertices.*

Proof. A *two-coloured tree* is a tree whose vertices are coloured black or white so that no two adjacent vertices have the same colour. We are going to prove that given a two-coloured tree $\mathbf{T}$ with an even number of vertices, Alice can secure at least $\frac{1}{2}$ of the total weight of a colour she chooses. This yields at least $\frac{1}{4}$ of the total weight of $\mathbf{T}$ for Alice.

The proof goes by induction on the number of vertices of $\mathbf{T}$ (only even numbers). For a tree with two vertices the statement is trivial. To get through the induction step we construct a strategy for Alice which leads the game to a point (after some move of Bob) at which the weight of black vertices she has taken is at least the weight of black vertices taken by Bob. We distinguish two cases.

Case 1: $\mathbf{T}$ has a black leaf.

This is an easy case. Alice takes the heaviest available black vertex. With this move she does not uncover any new black vertex, so Bob can only respond with a white vertex or a non-heavier black vertex. In both scenarios Alice gains no less than Bob from the black part, and for the rest of the tree the induction hypothesis can be applied.

Case 2: All leaves of $\mathbf{T}$ are white.

First observe that there must be a black vertex of degree greater than 2: if all black vertices have degree 2 then the total number of edges is even, so $\mathbf{T}$ has an odd number of vertices, which contradicts the assumption. Let $\mathbf{K}$ be the subtree of $\mathbf{T}$ spanned by all

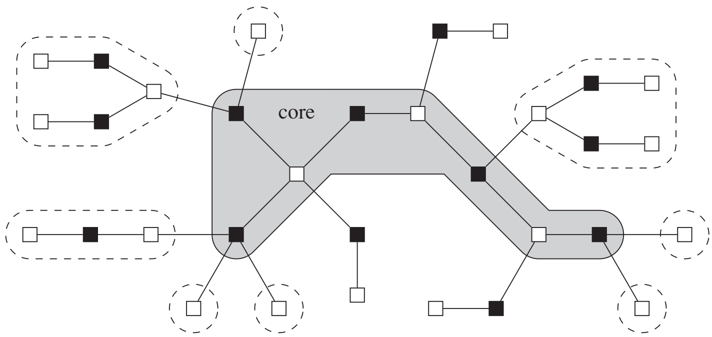

Figure 3. Illustration for Case 2. Odd components (marked with dashed lines) have white roots, even components have black roots.

black vertices of degree greater than 2. We call $\mathbf{K}$ the *core* of $\mathbf{T}$, and we call connected components of $\mathbf{T}-\mathbf{K}$ simply *components*. The *root* of a component is its only vertex adjacent to the core. A similar argument with counting edges shows that a component with a white root has an odd number of vertices and a component with a black root has an even number of vertices (see Figure 3). Since all leaves of the core are black and have degree greater than 2, at least two of their (white) neighbours must be roots of components. Therefore, taking a vertex from the core is possible only after at least two components with an odd number of vertices have been entirely shared out.

Let $\mathbf{C}_1,\ldots,\mathbf{C}_k$ be the components, and assume $\mathbf{C}_1$ is a component with an odd number of vertices and the least total weight of black vertices among all components with an odd number of vertices.

Alice starts with any vertex from $\mathbf{C}_1$. The following invariants are to be kept till the moment the induction hypothesis is applied:

(i) for $i \neq 1$, all vertices in $\mathbf{C}_i$ available for Bob are white,

(ii) no component with an odd number of vertices other than $\mathbf{C}_1$ has been entirely shared out.

Subsequent moves of Alice depend on what Bob has just played. Observe that by (ii) no vertex from the core is available. Thus, Bob can choose only from the vertices in the components.

Case 1: Bob takes a vertex from $\mathbf{C}_1$.

Alice takes another vertex of $\mathbf{C}_1$. Such a vertex exists as $\mathbf{C}_1$ has an odd number of vertices. Clearly, both invariants are preserved.

Case 2: Bob takes a vertex v from $\mathbf{C}_i$ with $i \neq 1$.

If v is not the root of $\mathbf{C}_i$ then Bob's move uncovers exactly one black vertex in $\mathbf{C}_i$, which is now a new leaf. Alice takes it. It may be the root of $\mathbf{C}_i$ but then $\mathbf{C}_i$ has an even number of vertices, so taking the last vertex of $\mathbf{C}_i$ does not violate (ii). All vertices that remain available in $\mathbf{C}_i$ are white exactly as (i) states.

If v is the root of $\mathbf{C}_i$ then $\mathbf{C}_i$ has an odd number of vertices and all of them have been taken. Note that all black vertices taken by Bob are in $\mathbf{C}_1$, and Alice has taken all black

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vertices from $\mathbf{C}_i$. Since the weight of black vertices in $\mathbf{C}_i$ is at least the weight of black vertices in $\mathbf{C}_1$, Alice has collected at least as much as Bob from the black part. For the remaining tree the induction hypothesis is applied. □

Theorem 2.2. *Alice can secure at least* $\frac{1}{2}$ *of the weight of any subdivided star with an even number of vertices.*

Proof. Let $\mathbf{T}$ be the given subdivided star and w be the weight function. The proof goes by induction on the number of vertices of $\mathbf{T}$, just like the proof of Theorem 2.1. If $\mathbf{T}$ is a path then the statement is easy, as discussed in the Introduction. Thus suppose $\mathbf{T}$ is not a path.

Let v be the vertex of $\mathbf{T}$ of degree $d \geqslant 3$. Other vertices of $\mathbf{T}$ form d paths $u_{k1}u_{k2}\ldots u_{k\ell_k}$ for $k = 1,\ldots,d$, which we call *arms* of $\mathbf{T}$. Here $u_{11},\ldots,u_{d1}$ are the d leaves of $\mathbf{T}$ and $u_{1\ell_1},\ldots,u_{d\ell_d}$ are the d neighbours of v.

We define the *alternating weight* of any prefix $u_{k1}\ldots u_{kj}$ of the kth arm to be

$$\hat{w}(u_{k1}\ldots u_{kj}) = \sum_{i=1}^{j}(-1)^i w(u_{ki}).$$

The idea behind this definition is as follows. Suppose at some point of the game the players start to play on the kth arm and they alternately take the vertices $u_{k1},\ldots,u_{kj}$. Then $\hat{w}(u_{k1}\ldots u_{kj})$ measures the relative gain of the second player from this line of play.

We say that $u_{k1}\ldots u_{kj}$ has the *even prefix property* if we have $\hat{w}(u_{k1}\ldots u_{ki}) > 0$ for any even $i \leqslant j$. The kth arm is called *bad* if it has the even prefix property and its length ℓ_k is even. Otherwise the kth arm is *good*. The intuition is that it is (almost) never profitable to start playing in a bad arm, because the other player can always continue in the same arm and gain more.

We will present two complementary strategies: one for the case of at most two good arms, and one for the other case. Only the latter uses induction.

Strategy 1: $\mathbf{T}$ has at most two good arms.

Since each bad arm has even length and there is an even number of vertices in total, exactly one arm has odd length. Assume without loss of generality that arm 1 has odd length and that arms $3,\ldots,d$ are bad. Two-colour the vertices of $\mathbf{T}$ properly with black or white. One of the colours, say black, carries at least half of the total weight of $\mathbf{T}$. We will show that Alice can secure at least the total weight of all black vertices.

Case 1: u_{11} is black and $u_{21},\ldots,u_{d1},v$ are white.

In this case the strategy for Alice is just to take the only black leaf available. This leads to a position where all leaves are white. Bob is forced to take a white leaf and thus to uncover a new black leaf for Alice. This way Alice gathers all black vertices.

Case 2: u_{11} is white and $u_{21},\ldots,u_{d1},v$ are black.

Initially arms 1, 2 are marked *active* and arms $3,\ldots,d$ are marked *passive*. Alice starts by taking u_{21}. After Bob takes u_{ki}, Alice responds in the same arm by taking $u_{k,i+1}$ unless

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$i = \ell_k$. In the latter case Alice chooses any remaining passive arm, marks it active and takes its available vertex. If there are no more passive arms, the remaining part of **T** is a path and therefore Alice can easily collect all remaining black vertices. This strategy guarantees that Alice takes black and Bob takes white vertices, except that playing in arms $3,\ldots,n$ when passive Alice takes white vertices and Bob takes black ones. Vertices played when an arm is passive form an even prefix of it. Since arms $3,\ldots,n$ are bad, the weight of white is greater than the weight of black in these prefixes. Therefore, Alice scores at least the total weight of all black vertices.

Strategy 2: T has at least three good arms.

The strategy we are going to present now leads the game to a point at which Alice has gathered at least as much as Bob after some move of Bob. Then we apply the induction hypothesis to show that Alice can secure at least half of the remaining part of **T**.

For any good arm k let r_k be such that

- r_k is odd;
- $u_{k1}\ldots u_{kr_k}$ has the even prefix property;
- $\hat{w}(u_{k1}\ldots u_{kr_k})$ is maximized.

Assume without loss of generality that arm 1 minimizes $\hat{w}(u_{k1}\ldots u_{kr_k})$ among all good arms k. Alice starts by taking u_{11}. Then, each time Bob takes a vertex u_{ki}, Alice responds in the same arm by taking $u_{k,i+1}$ until

(a) Bob takes u_{1i_1} such that $\hat{w}(u_{11}\ldots u_{1i_1}) \leqslant 0$, or
(b) Bob takes u_{kr_k} from a good arm $k \neq 1$.

We will prove that

- when (a) or (b) happens, Alice has just gathered at least as much as Bob and can therefore play inductively in the remaining part of **T**;
- if Bob takes u_{ki} and neither of (a), (b) happens, then $i < \ell_k$, so the vertex $u_{k,i+1}$ exists;
- Bob cannot take v.

Suppose that (a) happens. We may assume without loss of generality that vertices were taken from arms $1,\ldots,c$ for some $1 \leqslant c \leqslant d$. Let $u_{2i_2},\ldots,u_{ci_c}$ be the last vertices taken (by Alice) from arms $2,\ldots,c$ respectively. For good arms k we have $i_k < r_k$, as otherwise (b) would happen before. Therefore, by the even prefix property in the definitions of r_k and of a bad arm, $\hat{w}(u_{k1}\ldots u_{ki_k}) > 0$ for every $k = 2,\ldots,c$. Observe that $\hat{w}(u_{k1}\ldots u_{ki_k})$ is the difference between Alice's and Bob's gain in arm k, and the difference between Alice's and Bob's gain in arm 1 equals $-\hat{w}(u_{11}\ldots u_{1i_1}) \geqslant 0$.

Now, suppose that (b) happens. The same argument as above shows that Alice has scored no less than Bob in arms other than 1 and k. Let i_1 be the last vertex taken (by Alice) from arm 1. Since (a) did not happen before, $u_{11}\ldots u_{1i_1}$ has the even prefix property. Therefore,

$$\hat{w}(u_{11}\ldots u_{1i_1}) \leqslant \hat{w}(u_{11}\ldots u_{1r_1}) \leqslant \hat{w}(u_{k1}\ldots u_{kr_k}).$$

It suffices to observe that the difference between Alice's and Bob's gain in arms 1 and k together is $\hat{w}(u_{k1}\ldots u_{kr_k}) - \hat{w}(u_{11}\ldots u_{1i_1}) \geqslant 0$.

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Suppose that Bob takes $u_{k\ell_k}$ and neither of (a), (b) happens. If $k = 1$ then ℓ_1 is even, and $u_{11} \ldots u_{1\ell_1}$ has the even prefix property, as otherwise (a) would happen now or before. This contradicts the fact that arm 1 is good. If $k \neq 1$ then ℓ_k is odd, so arm k is good. Therefore, (b) would happen now or before.

Finally, to show that Bob cannot take v, observe that taking v is possible only if all arms except one are entirely taken out. However, there are at least three good arms, of which only arm 1 can be entirely taken out before (b) happens. □

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