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PRIMITIVE IDEAL SPACE OF $C^*(R_+) \rtimes R^\times$

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Abstract

For an integral domain R satisfying certain conditions, we characterise the primitive ideal space and its Jacobson topology for the semigroup crossed product $C^*(R_+) \rtimes R^\times$. We illustrate the result by the example $R = \mathbb{Z}[\sqrt{-3}]$.

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1. Introduction

Motivated by the pioneering paper of Bost and Connes [2], Cuntz in [8] constructed the first ring C^*-algebra. Cuntz and Li [11] generalised the work of [8] to an integral domain with finite quotients. Eventually, Li [18] generalised the work of [8] to arbitrary rings. There is more than one way of studying C^*-algebras associated to rings. Hirshberg [12], Larsen and Li [17], and Kaliszewski *et al.* [13] independently investigated C^*-algebras from p-adic rings. Li [19] defined the notion of semigroup C^*-algebras and proved that the $ax + b$-semigroup C^*-algebra of a ring is an extension of the ring C^*-algebra. When the ring is the ring of integers of a field, Li [19] proved that the $ax + b$-semigroup C^*-algebra is isomorphic to another construction due to Cuntz *et al.* [9]. Very recent work due to Bruce and Li [5, 6] and Bruce *et al.* [4] on algebraic dynamical systems and their associated C^*-algebras solves quite a few open problems.

For an integral domain R, denote by R_+ the additive group $(R, +)$ and by $R^\times$ the multiplicative semigroup $(R \setminus \{0\}, \cdot)$. There is a natural unital and injective action of $R^\times$ on $C^*(R_+)$ by multiplication. Thus, we obtain a semigroup crossed product $C^*(R_+) \rtimes R^\times$. We characterise the primitive ideal space and its Jacobson topology for the semigroup crossed product $C^*(R_+) \rtimes R^\times$ under certain conditions. Our main example is $R = \mathbb{Z}[\sqrt{-3}]$. The semigroup crossed product $C^*(R_+) \rtimes R^\times$ is closely

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related to other constructions. In the Appendix, we show that $C^*(R_+) \rtimes R^\times$ is an extension of the boundary quotient of the opposite semigroup of the $ax+b$-semigroup of the ring and that when the ring is a greatest common divisor (GCD) domain, $C^*(R_+) \rtimes R^\times$ is isomorphic to the boundary quotient of the opposite semigroup of the $ax+b$-semigroup of the ring. There are only a few investigations of the opposite semigroup C^*-algebra of the $ax+b$-semigroup of a ring (see for example [10, 20, 21]).

Standing assumptions. Throughout the paper, any semigroup is assumed to be discrete, countable, unital and left cancellative; any group is assumed to be discrete and countable; any subsemigroup of a semigroup is assumed to inherit the unit of the semigroup; any ring is assumed to be countable and unital with $0 \neq 1$; and any topological space is assumed to be second countable.

2. Laca's dilation theorem revisited

Laca [14] proved an important theorem which dilates a semigroup dynamical system (A, P, α) to a C^*-dynamical system (B, G, β) so that the semigroup crossed product $A \rtimes_\alpha^e P$ is Morita equivalent to the crossed product $B \rtimes_\beta G$. In this section, we revisit Laca's theorem when A is a unital commutative C^*-algebra.

NOTATION 2.1. Let P be a subsemigroup of a group G satisfying $G = P^{-1}P$. For $p, q \in P$, define $p \leq q$ if $qp^{-1} \in P$. Then, $\leq$ is a reflexive, transitive and directed relation on P.

THEOREM 2.2 (See [14, Theorem 2.1]). *Let P be a subsemigroup of a group G satisfying $G = P^{-1}P$, let $A = C(X)$, where X is a compact Hausdorff space, and let $\alpha : P \to \mathrm{End}(A)$ be a semigroup homomorphism such that α_p is unital and injective for all $p \in P$. Then, there exists a dynamical system (X_∞, G, γ) (where X_∞ is compact Hausdorff) such that $A \rtimes_\alpha^e P$ is Morita equivalent to $C(X_\infty) \rtimes_\gamma G$.*

PROOF. By [14, Theorem 2.1], there exists a C^*-dynamical system (A_∞, G, β) such that $A \rtimes_\alpha^e P$ is Morita equivalent to $A_\infty \rtimes_\beta G$. We cite the proof of [14, Theorem 2.1] to sketch the construction of A_∞ and the definition of β.

For $p \in P$, define $A_p := A$. For $p, q \in P$ with $p \leq q$, define $\alpha_{p,q} : A_p \to A_q$ to be $\alpha_{qp^{-1}}$. Then, $\{(A_p, \alpha_{p,q}) : p, q \in P, p \leq q\}$ is an inductive system. Let $A_\infty := \lim_p (A_p, \alpha_{p,q})$, let $\alpha^p : A_p \to A_\infty$ be the natural unital embedding for all $p \in P$ and let $\beta : G \to \mathrm{Aut}(A_\infty)$ be the homomorphism satisfying $\beta_{p_0} \circ \alpha^{pp_0} = \alpha^p$ for all $p_0, p \in P$.

For $p \in P$, denote by $f_p : X \to X$ the unique surjective continuous map induced from α_p and set $X_p := X$. For $p, q \in P$ with $p \leq q$, denote by $f_{q,p} : X_q \to X_p$ the unique surjective continuous map induced from $\alpha_{p,q}$. Since $\alpha_{p,q} = \alpha_{qp^{-1}}$, we have $f_{q,p} = f_{qp^{-1}}$. Then, $\{(X_p, f_{q,p}) : p, q \in P, p \leq q\}$ is an inverse system. Set

$$X_\infty := \Big\{(x_p)_{p \in P} \in \prod_{p \in P} X_p : f_{q,p}(x_q) = x_p \text{ for all } p \leq q\Big\}, \tag{2.1}$$

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which is the inverse limit of the inverse system. By [1, Example II.8.2.2(i)], $A_\infty \cong C(X_\infty)$. For $p \in P$, denote by $f^p : X_\infty \to X_p$ the unique projection induced from α^p. Then, $f_{q,p} \circ f^q = f^p$ for all $p, q \in P, p \leq q$. For $p, p_0 \in P, f \in C(X_\infty)$, denote by $\gamma_{p_0} : X_\infty \to X_\infty$ the unique homeomorphism such that $\beta_{p_0}(f) = f \circ \gamma_{p_0}^{-1}$.

From this construction, (X_∞, G, γ) is a dynamical system with $C(X_\infty) \rtimes_\gamma G \cong A_\infty \rtimes_\beta G$. Hence, $A \rtimes_\alpha^e P$ is Morita equivalent to $C(X_\infty) \rtimes_\gamma G$. □

NOTATION 2.3. We give an explicit description of X_∞ and the action of G on X_∞ given in Theorem 2.2. We start with the definition of X_∞ in (2.1). Then, for $p_0, p, q \in P$ with $q \geq p_0, p$, and for $(x_p)_{p \in P} \in X_\infty$, we have

$$(p_0 \cdot (x_p))(p) = x_{pp_0}, \quad (p_0^{-1} \cdot (x_p))(p) = f_{q,p}(x_{qp_0^{-1}}).$$

In particular, when G is abelian, we have a simpler form of the group action given by

$$\frac{p_0}{q_0} \cdot (x_p) = (f_{q_0}(x_{pp_0})).$$

Our goal is to apply Theorem 2.2 to characterise the primitive ideal space of the semigroup crossed product $C^*(R_+) \rtimes R^\times$ of an integral domain. Since $R^\times$ is abelian, we will need the following version of Williams' theorem.

DEFINITION 2.4. Let G be an abelian group, let X be a locally compact Hausdorff space and let $\alpha : G \to \mathrm{Homeo}(X)$ be a homomorphism. For $x, y \in X$, define $x \sim y$ if $\overline{G \cdot x} = \overline{G \cdot y}$. Then, $\sim$ is an equivalence relation on X. For $x \in X$, define $[x] := \overline{G \cdot x}$, called the *quasi-orbit* of x. The quotient space $Q(X/G)$ by the relation $\sim$ is called the *quasi-orbit space*. For $x \in X$, define $G_x := \{g \in G : g \cdot x = x\}$, called the *isotropy group* (or *stability group*) at x. For $([x], \phi), ([y], \psi) \in Q(X/G) \times \widehat{G}$, define $([x], \phi) \approx ([y], \psi)$ if $[x] = [y]$ and $\phi|_{G_x} = \psi|_{G_x}$. Then, $\approx$ is an equivalence relation on $Q(X/G) \times \widehat{G}$.

THEOREM 2.5 [16, Theorem 1.1]. *Let G be an abelian group, let X be a locally compact Hausdorff space and let $\alpha : G \to \mathrm{Homeo}(X)$ be a homomorphism. Then,* $\mathrm{Prim}(C_0(X) \rtimes_\alpha G) \cong (Q(X/G) \times \widehat{G})/ \approx$.

3. Primitive ideal structure of $C^*(R_+) \rtimes R^\times$

In this section, we characterise the primitive ideal space and its Jacobson topology for the semigroup crossed product $C^*(R_+) \rtimes R^\times$ under certain conditions.

NOTATION 3.1. Let R be an integral domain. Denote by Q the field of fractions of R, by R_+ the additive group $(R, +)$, by $\widehat{R_+}$ the dual group of R_+, by $R^\times$ the multiplicative semigroup $(R \setminus \{0\}, \cdot)$, by $Q^\times$ the enveloping group $(Q \setminus \{0\}, \cdot)$ of $R^\times$, by $\{u_r\}_{r \in R_+}$ the family of unitaries generating $C^*(R_+)$ and by $\alpha : R^\times \to \mathrm{End}(C^*(R_+))$ the homomorphism such that $\alpha_p(u_r) = u_{pr}$ for all $p \in R^\times, r \in R_+$. Observe that for any $p \in R^\times, \alpha_p$ is unital and injective, and the map $f_p : \widehat{R_+} \to \widehat{R_+}, \phi \mapsto \phi(p\cdot)$ is the unique surjective continuous map induced from α_p. Denote by

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$$X_\infty(R) := \Big\{\phi = (\phi_p)_{p\in R^\times} \in \prod_{p\in R^\times} \widehat{R_+} : \phi_q\Big(\frac{q}{p}\cdot\Big) = \phi_p, \text{ whenever } p \mid q\Big\}.$$

Then, $(p_0/q_0)\cdot(\phi_p) = (\phi_{pp_0}(q_0\cdot))$.

Lemma 3.2. *Let R be an integral domain. Fix $(\phi_p)_{p\in R^\times} \in X_\infty(R)$. If $(\phi_p)_{p\in R^\times} \neq (1)_{p\in R^\times}$, then $Q^\times_\phi = \{1_R\}$. If $(\phi_p)_{p\in R^\times} = (1)_{p\in R^\times}$, then $Q^\times_\phi = Q^\times$.*

Proof. To prove the first statement, suppose for a contradiction that there exists $p_0/q_0 \in Q^\times$ with $p_0/q_0 \neq 1$ and such that $(p_0/q_0)\cdot\phi = \phi$. Since $(\phi_p)_{p\in R^\times} \neq (1)_{p\in R^\times}$, there exists $p_1 \in R^\times$ such that $\phi_{p_1} \neq 1$. Then, $\phi_p = \phi_{pp_0}(q_0\cdot)$ for any $p \in R^\times$. Since $\phi_{pp_0}(p_0\cdot) = \phi_p$ for any $p \in R^\times$, we deduce that $\phi_{pp_0}(p_0\cdot) = \phi_{pp_0}(q_0\cdot)$ for all $p \in R^\times$. So $\phi_{pp_0}((p_0-q_0)\cdot) = 1$ for any $p \in R^\times$. Hence, $\phi_{pp_0}((p_0-q_0)p_0\cdot) = 1$ for any $p \in R^\times$. When $p = p_1(p_0-q_0)$, we get $\phi_{p_1} = \phi_{p_1(p_0-q_0)p_0}(((p_0-q_0)p_0\cdot) = 1$, which is a contradiction. Therefore, $Q^\times_\phi = \{1_R\}$.

To prove the second statement, suppose that $(\phi_p)_{p\in R^\times} = (1)_{p\in R^\times}$. For $p_0/q_0 \in Q^\times$, we have $(p_0/q_0)\cdot(1)_{p\in R^\times} = (p_0/q_0)\cdot(\phi_p)_{p\in R^\times} = (\phi_{pp_0}(q_0\cdot))_{p\in R^\times} = (1)_{p\in R^\times}$. So $Q^\times_\phi = Q^\times$. □

Lemma 3.3. *Let R be an integral domain. Suppose that for $\epsilon > 0$, $(1)_{p\in R^\times} \neq (\phi_p)_{p\in R^\times} \in X_\infty(R)$, $\pi \in \widehat{R_+}$, $P \in R^\times$ and $r_1, r_2, \ldots, r_n \in R_+$, there exist $p, q \in R^\times$ with $P \mid p$ such that $|\phi_p(qr_i) - \pi(r_i)| < \epsilon$, $i = 1, 2, \ldots, n$. Then, $Q(X_\infty(R)/Q^\times)$ consists of only two points with the only nontrivial closed subset $\{[(1)_{p\in R^\times}]\}$.*

Proof. Since $\overline{Q^\times\cdot(1)_{p\in R^\times}} = \overline{(1)_{p\in R^\times}} = (1)_{p\in R^\times}$, we have $[(\phi_p)_{p\in R^\times}] \neq [(1)_{p\in R^\times}]$ whenever $(1)_{p\in R^\times} \neq (\phi_p)_{p\in R^\times} \in X_\infty(R)$.

Fix $(\phi_p)_{p\in R^\times}, (\psi_p)_{p\in R^\times} \in X_\infty(R)$ such that $(\phi_p)_{p\in R^\times}, (\psi_p)_{p\in R^\times} \neq (1)_{p\in R^\times}$. We aim to show that $[(\phi_p)_{p\in R^\times}] = [(\psi_p)_{p\in R^\times}]$. It suffices to show that $(\psi_p)_{p\in R^\times} \in \overline{Q^\times\cdot(\phi_p)_{p\in R^\times}}$ since $(\phi_p)_{p\in R^\times} \in \overline{Q^\times\cdot(\psi_p)_{p\in R^\times}}$ follows from the same argument. Fix $\epsilon > 0$, $p_1, p_2, \ldots, p_n \in R^\times$ and $r_1, r_2, \ldots, r_n \in R$. By the condition imposed in the lemma, there exist $p_0, q_0 \in R^\times$ such that

$$|\phi_{p_1p_2\cdots p_np_0}(q_0p_1\cdots p_{i-1}p_{i+1}\cdots p_nr_j) - \psi_{p_1p_2\cdots p_n}(p_1\cdots p_{i-1}p_{i+1}\cdots p_nr_j)| < \epsilon$$

for $1 \le i,j \le n$. So $|\phi_{p_ip_0}(q_0r_j) - \psi_{p_i}(r_j)| < \epsilon$ for $1 \le i,j \le n$. Hence, $(\psi_p)_{p\in R^\times} \in \overline{Q^\times\cdot(\phi_p)_{p\in R^\times}}$. Therefore, $[(\phi_p)_{p\in R^\times}] = [(\psi_p)_{p\in R^\times}]$.

We conclude that $Q(X_\infty(R)/Q^\times)$ consists of only two points. For any $(1)_{p\in R^\times} \neq (\phi_p)_{p\in R^\times} \in X_\infty(R)$, $\overline{Q^\times\cdot(\phi_p)_{p\in R^\times}} = X_\infty(R) \setminus \{(1)_{p\in R^\times}\}$ is open but not closed. Finally, we deduce that $\{[(1)_{p\in R^\times}]\}$ is the only nontrivial closed subset of $Q(X_\infty(R)/Q^\times)$. □

Theorem 3.4. *Let R be an integral domain satisfying the condition of Lemma 3.3. Take an arbitrary element $(\phi_p)_{p\in R^\times} \in X_\infty(R)$ with $(1)_{p\in R^\times} \neq (\phi_p)_{p\in R^\times}$. Then, we have $\mathrm{Prim}(C^*(R_+)\rtimes R^\times) \cong \{[(\phi_p)_{p\in R^\times}]\} \amalg \{[(1)_{p\in R^\times}]\} \times \widehat{Q^\times}$, and the open sets of $\mathrm{Prim}(C^*(R_+)\rtimes R^\times)$ comprise $\{[(\phi_p)_{p\in R^\times}]\} \amalg \{[(1)_{p\in R^\times}]\} \times N$, where N is an open subset of $\widehat{Q^\times}$.*

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PROOF. By Theorem 2.2, $(C^*(R_+) \rtimes R^\times)$ is Morita equivalent to $C(X_\infty(R)) \rtimes Q^\times$. So $\mathrm{Prim}(C^*(R_+) \rtimes R^\times) \cong \mathrm{Prim}(C(X_\infty(R)) \rtimes Q^\times)$. By Theorem 2.5 and Lemma 3.3, $\mathrm{Prim}(C(X_\infty(R)) \rtimes Q^\times) \cong \{[(\phi_p)_{p\in R^\times}], [(1)_{p\in R^\times}]\} \times \widehat{Q^\times}/\approx$. By Lemma 3.2, $Q^\times_{(\phi_p)_{p\in R^\times}} = \{1_R\}$ and $Q^\times_{(1)_{p\in R^\times}} = Q^\times$. So, $\mathrm{Prim}(C(X_\infty(R)) \rtimes Q^\times) \cong \{[(\phi_p)_{p\in R^\times}]\} \amalg \{[(1)_{p\in R^\times}]\} \times \widehat{Q^\times}$. Hence, $\mathrm{Prim}(C^*(R_+) \rtimes R^\times) \cong \{[(\phi_p)_{p\in R^\times}]\} \amalg \{[(1)_{p\in R^\times}]\} \times \widehat{Q^\times}$, and the open sets of $\mathrm{Prim}(C^*(R_+) \rtimes R^\times)$ are $\{[(\phi_p)_{p\in R^\times}]\} \amalg \{[(1)_{p\in R^\times}]\} \times N$, where N is an open subset of $\widehat{Q^\times}$. □

EXAMPLE 3.5. Let $R = \mathbb{Z}$. Then, $\widehat{R_+} = \mathbb{T}$. Fix $\epsilon > 0$, $(1)_{p\in\mathbb{Z}^\times} \neq (\phi_p)_{p\in\mathbb{Z}^\times} \in X_\infty(\mathbb{Z})$, $\pi \in \mathbb{T}$, $P \in \mathbb{Z}^\times$ and $r_1, r_2, \ldots, r_n \in \mathbb{Z}_+$. Take an arbitrary $p_0 \in \mathbb{Z}^\times$ such that $P \mid p_0$ and let $\phi_{p_0} = e^{2\pi i\theta}$ for some $\theta \in (0, 1)$.

Case 1: θ is rational. Then, $\phi^{\mathbb{Z}}_{p_0} = \{e^{2\pi ik/n}\}_{k=0}^{n-1}$ for some $n \geq 1$. Since $\phi^{p}_{pp_0} = \phi_{p_0}$ for any $p \geq 1$, we get $\phi^{\mathbb{Z}}_{pp_0} = \{e^{2\pi ik/pn}\}_{k=0}^{pn-1}$. Choose $p_1 \geq 1$ such that $|e^{2\pi i/p_1 n} - 1| < \epsilon/\sum_{i=1}^n |r_i|$. Then, there exists $q_0 \in \mathbb{Z}^\times$ such that $|\phi^{q_0}_{p_1p_0} - \pi| < \epsilon/\sum_{i=1}^n |r_i|$.

Case 2: θ is irrational. Then, by the properties of an irrational rotation, $\{\phi^z_{p_0}\}_{z\in\mathbb{Z}}$ is a dense subset of $\mathbb{T}$. So, there exists $q_0 \in \mathbb{Z}^\times$ such that $|\phi^{q_0}_{p_0} - \pi| < \epsilon/\sum_{i=1}^n |r_i|$.

In both cases, there exist $p, q \in \mathbb{Z}^\times$ with $P \mid p$ such that $|\phi^q_p - \pi| < \epsilon/\sum_{i=1}^n |r_i|$. For $1 \leq i \leq n$, we may assume that $r_i \geq 0$ and we calculate that

$$|\phi_p(qr_i) - \pi(r_i)| = |\phi_p^{qr_i} - \pi^{r_i}| = |\phi^q_p - \pi|\left|\sum_{j=0}^{r_i-1} \phi_p^{q(r_i-1-j)}\pi^j\right| \leq |\phi^q_p - \pi|\sum_{j=0}^{r_i-1} |\phi_p^{q(r_i-1-j)}\pi^j|$$
$$< \epsilon r_i \Big/ \sum_{i=1}^n |r_i| < \epsilon.$$

So, $\mathbb{Z}$ satisfies the condition of Lemma 3.3.

EXAMPLE 3.6. Let $R = \mathbb{Z}[\sqrt{-3}]$. Then, $\mathbb{Z}[\sqrt{-3}]_+ \cong \mathbb{Z}^2$ and $\widehat{\mathbb{Z}[\sqrt{-3}]_+} \cong \mathbb{T}^2$. Fix $\epsilon > 0$, $((1,1))_{p\in R^\times} \neq ((a_p, b_p))_{p\in R^\times} \in X_\infty(\mathbb{Z}[\sqrt{-3}])$, $(\pi, \rho) \in \mathbb{T}^2$, $P \in R^\times$ and $r_i + s_i\sqrt{-3} \in \mathbb{Z}[\sqrt{-3}]_+$ for $i = 1, 2 \ldots, n$. Take an arbitrary $P \mid p_0 \in R^\times$ such that $(a_{p_0}, b_{p_0}) \neq (1, 1)$. There exist $p, q = q_1 + q_2\sqrt{-3} \in R^\times$ with $P \mid p$ such that $|a_p^{q_1} b_p^{q_2} - \pi|, |a_p^{-3q_2} b_p^{q_1} - \rho| < \epsilon/\sum_{i=1}^n(|r_i| + |s_i|)$. For $1 \leq i \leq n$, we may assume that $r_i \geq 0$ and we estimate

$$\begin{aligned}
&|(a_p, b_p)(q(r_i + s_i\sqrt{-3})) - (\pi, \rho)(r_i + s_i\sqrt{-3})| \\
&= |(a_p^{q_1} b_p^{q_2})^{r_i}(a_p^{-3q_2} b_p^{q_1})^{s_i} - \pi^{r_i}\rho^{s_i}| \\
&= |((a_p^{q_1} b_p^{q_2})^{r_i} - \pi^{r_i})(a_p^{-3q_2} b_p^{q_1})^{s_i} + \pi^{r_i}((a_p^{-3q_2} b_p^{q_1})^{s_i} - \rho^{s_i})| \\
&\leq |(a_p^{q_1} b_p^{q_2})^{r_i} - \pi^{r_i}| + |(a_p^{-3q_2} b_p^{q_1})^{s_i} - \rho^{s_i}| \\
&< \frac{\epsilon|r_i|}{\sum_{i=1}^n |r_i| + |s_i|} + \frac{\epsilon|s_i|}{\sum_{i=1}^n |r_i| + |s_i|} \leq \epsilon.
\end{aligned}$$

So, $\mathbb{Z}[\sqrt{-3}]$ satisfies the condition of Lemma 3.3.

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By a similar argument to this example, we conclude that any (concrete) order of a number field satisfies the condition of Lemma 3.3. (For the background about number fields, one may refer to [22].)

Appendix. The relationship between $C^*(R_+) \rtimes R^\times$ and semigroup C^*-algebras

In this appendix, we show that $C^*(R_+) \rtimes R^\times$ is an extension of the boundary quotient of the opposite semigroup of the $ax + b$-semigroup of the ring and that when the ring is a GCD domain, $C^*(R_+) \rtimes R^\times$ is isomorphic to the boundary quotient of the opposite semigroup of the $ax + b$-semigroup of the ring.

DEFINITION A.1 ([15, Section 2], [19, Definition 2.13]). Let P be a semigroup, A be a unital C^*-algebra and $\alpha : P \to \mathrm{End}(A)$ be a semigroup homomorphism such that α_p is injective for all $p \in P$. Define the *semigroup crossed product* $A \rtimes_\alpha P$ to be the universal unital C^*-algebra generated by the image of a unital homomorphism $i_A : A \to A \rtimes_\alpha P$ and a semigroup homomorphism $i_P : P \to \mathrm{Isom}(A \rtimes_\alpha P)$ satisfying the following conditions:

(1) $i_P(p)i_A(a)i_P(p)^* = i_A(\alpha_p(a))$ for all $p \in P, a \in A$;
(2) for any unital C^*-algebra B, unital homomorphism $j_A : A \to B$ and semigroup homomorphism $j_P : P \to \mathrm{Isom}(B)$ satisfying $j_P(p)j_A(a)j_P(p)^* = j_A(\alpha_p(a))$, there exists a unique unital homomorphism $\Phi : A \rtimes_\alpha P \to B$ such that $\Phi \circ i_A = j_A$ and $\Phi \circ i_P = j_P$.

REMARK A.2. We have $i_A(1_A) = i_P(1_P) =$ the unit of $A \rtimes_\alpha P$.

If α_p is unital for all $p \in P$, then $i_P(p)$ is a unitary for any $p \in P$. To see this, we calculate that $i_P(p)i_P(p)^* = i_P(p)i_A(1_A)i_P(p)^* = i_A(\alpha_p(1_A)) = i_A(1_A)$.

NOTATION A.3 [3, 19]. Let P be a semigroup. For $p \in P$, we also denote by p the left multiplication map $q \mapsto pq$. The set of *constructible right ideals* is defined to be

$$\mathcal{J}(P) := \{p_1^{-1}q_1 \cdots p_n^{-1}q_nP : n \geq 1, p_1, q_1, \ldots, p_n, q_n \in P\} \cup \{\emptyset\}.$$

A finite subset $F \subset \mathcal{J}(P)$ is called a *foundation set* if for any nonempty $X \in \mathcal{J}(P)$, there exists $Y \in F$ such that $X \cap Y \neq \emptyset$.

DEFINITION A.4 ([3, Remark 5.5], [19, Definition 2.2]). Let P be a semigroup. Define the *full semigroup C^*-algebra* $C^*(P)$ of P to be the universal unital C^*-algebra generated by a family of isometries $\{v_p\}_{p\in P}$ and a family of projections $\{e_X\}_{X\in\mathcal{J}(P)}$ satisfying the following relations:

(1) $v_pv_q = v_{pq}$ for all $p, q \in P$;
(2) $v_pe_Xv_p^* = e_{pX}$ for all $p \in P, X \in \mathcal{J}(P)$;
(3) $e_\emptyset = 0$ and $e_P = 1$;
(4) $e_Xe_Y = e_{X\cap Y}$ for all $X, Y \in \mathcal{J}(P)$.

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Define the *boundary quotient* $\mathcal{Q}(P)$ of $C^*(P)$ to be the universal unital C^*-algebra generated by a family of isometries $\{v_p\}_{p\in P}$ and a family of projections $\{e_X\}_{X\in\mathcal{J}(P)}$ satisfying conditions (1)–(4) and $\prod_{X\in F}(1-e_X)=0$ for any foundation set $F\subset\mathcal{J}(P)$.

DEFINITION A.5 ([3, Definition 2.1], [23, Definition 2.17]). Let P be a semigroup. Then, P is said to be *right* LCM (or to satisfy the *Clifford condition*) if the intersection of two principal right ideals is either empty or a principal right ideal.

NOTATION A.6. Let P be a semigroup. Denote by P^{op} the opposite semigroup of P. Let R be an integral domain. Denote by $R_+\rtimes R^\times$ the $ax+b$-semigroup of R. Denote by $\times$ the multiplication of $(R_+\rtimes R^\times)^{\mathrm{op}}$, that is, $(r_1,p_1)\times(r_2,p_2)=(r_2,p_2)(r_1,p_1)=(r_2+p_2r_1,p_1p_2)$.

REMARK A.7. Let R be an integral domain. We claim that any nonempty element of $\mathcal{J}((R_+\rtimes R^\times)^{\mathrm{op}})$ is a foundation set of $(R_+\rtimes R^\times)^{\mathrm{op}}$. To see this, for any $(r_1,p_1),(r_2,p_2)\in(R_+\rtimes R^\times)^{\mathrm{op}}$, we compute

$$\begin{aligned}(r_1,p_1)\times(p_1r_2,p_2)=(p_1r_2,p_2)(r_1,p_1)&=(p_1r_2+p_2r_1,p_1p_2)\\&=(p_2r_1,p_1)(r_2,p_2)=(r_2,p_2)\times(p_2r_1,p_1).\end{aligned}$$

THEOREM A.8. *Let R be an integral domain. Then, the crossed product $C^*(R_+)\rtimes R^\times$ is an extension of $\mathcal{Q}((R_+\rtimes R^\times)^{\mathrm{op}})$. Moreover, if R is a* GCD *domain (see* [7]*), then we have $C^*(R_+)\rtimes R^\times\cong\mathcal{Q}((R_+\rtimes R^\times)^{\mathrm{op}})$.*

PROOF. Denote by $i_A: C^*(R_+)\to C^*(R_+)\rtimes R^\times$ and $i_P: R^\times\to\mathrm{Isom}(C^*(R_+)\rtimes R^\times)$ the canonical homomorphisms generating $C^*(R_+)\rtimes R^\times$. Let $\{v_{(r,p)}:(r,p)\in(R_+\rtimes R^\times)^{\mathrm{op}}\}$ be the family of isometries and $\{e_X: X\in\mathcal{J}((R_+\rtimes R^\times)^{\mathrm{op}})\}$ be the family of projections generating $\mathcal{Q}((R_+\rtimes R^\times)^{\mathrm{op}})$.

For any $(r,p)\in(R_+\rtimes R^\times)^{\mathrm{op}}$, note that $1-v_{(r,p)}v_{(r,p)}^*=1-e_{(r,p)\times(R_+\rtimes R^\times)^{\mathrm{op}}}=0$ because $\{(r,p)\times(R_+\rtimes R^\times)^{\mathrm{op}}\}$ is a foundation set. So each $v_{(r,p)}$ is a unitary.

For $r\in R_+$, define $U_r:=v_{(r,1)}$. For any $r,s\in R_+$,

$$U_rU_s=v_{(r,1)}v_{(s,1)}=v_{(s,1)(r,1)}=v_{(r+s,1)}=v_{(r,1)(s,1)}=v_{(s,1)}v_{(r,1)}=U_sU_r,$$

so $j_A: C^*(R_+)\to\mathcal{Q}((R_+\rtimes R^\times)^{\mathrm{op}}), u_r\mapsto v_{(r,1)}$ is a homomorphism by the universal property of $C^*(R_+)$. For $p\in R^\times$, define $j_P(p):=v_{(0,p)}^*$. For any $p,q\in R^\times$,

$$j_P(p)j_P(q)=v_{(0,p)}^*v_{(0,q)}^*=(v_{(0,q)}v_{(0,p)})^*=(v_{(0,p)(0,q)})^*=v_{(0,pq)}^*=j_P(pq),$$

so $j_P: R^\times\to\mathrm{Isom}(\mathcal{Q}((R_+\rtimes R^\times)^{\mathrm{op}}))$ is a semigroup homomorphism. For any $p\in R^\times$, $r\in R_+$, we compute

$$j_P(p)j_A(u_r)j_P(p)^*=v_{(0,p)}^*v_{(r,1)}v_{(0,p)}=v_{(0,p)}^*v_{(pr,p)}=v_{(pr,1)}=j_A(u_{pr})=j_A(\alpha_p(u_r)).$$

By the universal property of $C^*(R_+)\rtimes R^\times$, there exists a unique homomorphism $\Phi: C^*(R_+)\rtimes R^\times\to\mathcal{Q}((R_+\rtimes R^\times)^{\mathrm{op}})$ such that $\Phi\circ i_A=j_A$ and $\Phi\circ i_P=j_P$. Since $v_{(r,p)}=v_{(0,p)}v_{(r,1)}$ for any $(r,p)\in(R_+\rtimes R^\times)^{\mathrm{op}}$, we see that Φ is surjective. So, $C^*(R_+)\rtimes R^\times$ is an extension of $\mathcal{Q}((R_+\rtimes R^\times)^{\mathrm{op}})$.

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Now, we assume that R is a GCD domain. By [23, Proposition 2.23], $R^\times$ is right LCM. For $(r_1, p_1), (r_2, p_2) \in (R_+ \rtimes R^\times)^{\mathrm{op}}$, suppose that $p_1R^\times \cap p_2R^\times = pR^\times$ for some $p \in R^\times$. We claim that

$$(r_1, p_1) \times (R_+ \rtimes R^\times)^{\mathrm{op}} \cap (r_2, p_2) \times (R_+ \rtimes R^\times)^{\mathrm{op}} = (0, p) \times (R_+ \rtimes R^\times)^{\mathrm{op}}.$$

Indeed, for any $(s_1, q_1), (s_2, q_2) \in (R_+ \rtimes R^\times)^{\mathrm{op}}$, if $(r_1, p_1) \times (s_1, q_1) = (r_2, p_2) \times (s_2, q_2)$, then $(r_1, p_1) \times (s_1, q_1) = (r_2, p_2) \times (s_2, q_2) = (0, p) \times (s_1 + q_1r_1, q_1p_1/p)$. Conversely, for any $(s, q) \in (R_+ \rtimes R^\times)^{\mathrm{op}}$,

$$(0, p) \times (s, q) = (r_1, p_1) \times \Big(s - \frac{pqr_1}{p_1}, \frac{pq}{p_1}\Big) = (r_2, p_2) \times \Big(s - \frac{pqr_2}{p_2}, \frac{pq}{p_2}\Big).$$

This proves the claim. Hence, $(R_+ \rtimes R^\times)^{\mathrm{op}}$ is right LCM as well.

Since $(R_+ \rtimes R^\times)^{\mathrm{op}}$ is right LCM, it follows from [24, Lemma 3.4] that $\mathcal{Q}((R_+ \rtimes R^\times)^{\mathrm{op}})$ is the universal unital C^*-algebra generated by a family of unitaries $\{v_{(r,p)} : (r, p) \in (R_+ \rtimes R^\times)^{\mathrm{op}}\}$ satisfying the conditions:

(1) $v_{(r_1,p_1)}v_{(r_2,p_2)} = v_{(r_1,p_1)\times(r_2,p_2)}$;

(2) $v^*_{(r_1,p_1)}v_{(r_2,p_2)} = v_{(s_1,q_1)}v^*_{(s_2,q_2)}$, whenever $(r_1, p_1) \times (s_1, q_1) = (r_2, p_2) \times (s_2, q_2)$ and $(r_1, p_1) \times (R_+ \rtimes R^\times)^{\mathrm{op}} \cap (r_2, p_2) \times (R_+ \rtimes R^\times)^{\mathrm{op}} = (r_1, p_1) \times (s_1, q_1) \times (R_+ \rtimes R^\times)^{\mathrm{op}}$.

For $(r, p) \in (R_+ \rtimes R^\times)^{\mathrm{op}}$, define $V_{(r,p)} := i_P(p)^*i_A(u_r)$. Finally, we check that $\{V_{(r,p)}\}$ satisfies the above two conditions. For any $(r_1, p_1), (r_2, p_2) \in (R_+ \rtimes R^\times)^{\mathrm{op}}$,

$$\begin{aligned}V_{(r_1,p_1)}V_{(r_2,p_2)} &= i_P(p_1)^*i_A(u_{r_1})i_P(p_2)^*i_A(u_{r_2}) = i_P(p_1)^*i_P(p_2)^*i_A(\alpha_{p_2}(u_{r_1}))i_A(u_{r_2})\\ &= (i_P(p_2)i_P(p_1))^*i_A(u_{p_2r_1})i_A(u_{r_2}) = i_P(p_1p_2)^*i_A(u_{r_2+p_2r_1})\\ &= V_{(r_2+p_2r_1,p_1p_2)} = V_{(r_1,p_1)\times(r_2,p_2)}.\end{aligned}$$

For $(r_1, p_1), (r_2, p_2) \in (R_+ \rtimes R^\times)^{\mathrm{op}}$, suppose that $p_1R^\times \cap p_2R^\times = pR^\times$ for some $p \in R^\times$. By the above claim, $(r_1, p_1) \times (R_+ \rtimes R^\times)^{\mathrm{op}} \cap (r_2, p_2) \times (R_+ \rtimes R^\times)^{\mathrm{op}} = (0, p) \times (R_+ \rtimes R^\times)^{\mathrm{op}}$. It is not hard to see that $(r_1, p_1) \times (-pr_1/p_1, p/p_1) = (r_2, p_2) \times (-pr_2/p_2, p/p_2) = (0, p)$. So,

$$\begin{aligned}V^*_{(r_1,p_1)}V_{(r_2,p_2)} &= i_A(u_{-r_1})i_P(p_1)i_P(p_2)^*i_A(u_{r_2}) = i_A(u_{-r_1})i_P\Big(\frac{p}{p_1}\Big)^* i_P\Big(\frac{p}{p_2}\Big)i_A(u_{r_2}\Big)\\ &= i_P\Big(\frac{p}{p_1}\Big)^* i_A(u_{-pr_1/p_1})i_A(u_{pr_2/p_2})i_P\Big(\frac{p}{p_2}\Big)\\ &= V_{(-pr_1/p_1,p/p_1)}V^*_{(-pr_2/p_2,p/p_2)}.\end{aligned}$$

By the universal property of $\mathcal{Q}((R_+ \rtimes R^\times)^{\mathrm{op}})$, there exists a homomorphism $\Psi : \mathcal{Q}((R_+ \rtimes R^\times)^{\mathrm{op}}) \to C^*(R_+) \rtimes R^\times$ such that $\Psi(v_{(r,p)}) = i_P(p)^*i_A(u_r)$. Since

$$\Phi \circ \Psi(v_{(r,p)}) = \Phi(i_P(p)^*i_A(u_r)) = j_P(p)^*j_A(u_r) = v_{(0,p)}v_{(r,1)} = v_{(r,p)},$$

$$\Psi \circ \Phi(i_A(u_r)) = \Psi(j_A(u_r)) = \Psi(v_{(r,1)}) = i_A(u_r),$$

$$\Psi \circ \Phi(i_P(p)) = \Psi(j_P(p)) = \Psi(v_{(0,p)})^* = i_P(p),$$

we conclude that $C^*(R_+) \rtimes R^\times \cong \mathcal{Q}((R_+ \rtimes R^\times)^{\mathrm{op}})$. □

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