ON VARIOUS DEFINITIONS OF CAPACITY AND RELATED NOTIONS

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To Professor Kiyoshi Noshiro on the occasion of his 60th birthday

Introduction. The electric capacity of a conductor in the 3-dimensional euclidean space is defined as the ratio of a positive charge given to the conductor and the potential on its surface. The notion of capacity was defined mathematically first by N. Wiener [7] and developed by C. de la Vallée Poussin, O. Frostman and others. For the history we refer to Frostman's thesis [2]. Recently studies were made on different definitions of capacity and related notions. We refer to M. Ohtsuka [4] and G. Choquet [1], for instance. In the present paper we shall investigate further some relations among various kinds of capacity and related notions. A part of the results was announced in a lecture of the author in 1962.[1)]

1. Let E and F be locally compact Hausdorff spaces and $\Phi(x, y)$ be a lower semicontinuous function on $E \times F$, satisfying $-\infty < \Phi(x, y) \leqq \infty$. This function is called a kernel. As measures we shall consider only non-negative Radon measures with compact support in E or in F. The potential $\int \Phi(x, y) d\mu(y)$ $\left(\int \Phi(x, y) d\nu(x) \text{ resp.}\right)$ of a measure μ (ν resp.) will be denoted by $\Phi(x, \mu)$($\Phi(\nu, y)$ resp.) and the double integral $\iint \Phi(x, y) d\mu(y) d\nu(x) = \int \Phi(x, \mu) d\nu(x)$ by $\Phi(\nu, \mu)$.

Let X be any non-empty set in E and μ be a measure in F. We set

$$V(X, \mu) = \sup_{x \in X} \Phi(x, \mu) \quad \text{and} \quad U(X, \mu) = \inf_{x \in X} \Phi(x, \mu).$$

Let Y be any non-empty set in F, and denote by $\mathscr{U}_Y$ the class of unit measures with compact support in Y. We put

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1) Capacity, Symposium on potential theory, Hakone, 1962.

$$V_X(Y) = \inf_{\mu \in \mathscr{U}_Y} V(X, \mu) \quad \text{and} \quad U_X(Y) = \sup_{\mu \in \mathscr{U}_Y} U(X, \mu) .$$

Similarly we define $\check{V}_Y(X)$ and $\check{U}_Y(X)$ by $\inf_{\nu \in \mathscr{U}_X} \sup_{y \in Y} \Phi(\nu, y)$ and $\sup_{\nu \in \mathscr{U}_X} \inf_{y \in Y} \Phi(\nu, y)$ respectively. B. Fuglede [3] proved the identity $V_E(K) = \check{U}_K(E)$, where K is a non-empty compact subset of F.

In the special case $E = F$ we set

$$W_i(X) = \inf_{\mu \in \mathscr{U}_X} \Phi(\mu, \mu), \; V(X) = \inf_{\mu \in \mathscr{U}_X} V(S_\mu, \mu) \quad \text{and} \quad U(X) = \sup_{\mu \in \mathscr{U}_X} U(S_\mu, \mu).$$

If the adjoint kernel $\check{\Phi}(x, y) = \Phi(y, x)$ is considered, the corresponding quantities will be denoted by $\check{W}_i(X)$, $\check{V}(X)$ and $\check{U}(X)$. We shall establish

THEOREM 1. *Suppose $E = F$ and let K be a non-empty compact set in E. Then*

$$W_i(K) = \check{W}_i(K) \leqq V(K) = \check{V}(K) \leqq \begin{Bmatrix} V_K(K) = \check{U}_K(K) \\ \\ \check{V}_K(K) = U_K(K) \end{Bmatrix} \leqq \begin{cases} V_E(K) = \check{U}_K(E) \\ \check{V}_E(K) = U_K(E) \\ U(K) = \check{U}(K) \; , \end{cases}$$

and these relations can not be improved in general.

Proof. The equalities $V_E(K) = \check{U}_K(E)$ and $\check{V}_E(K) = U_K(E)$ are special cases of the above quoted identity due to Fuglede. The equalities $V_K(K) = \check{U}_K(K)$ and $\check{V}_K(K) = U_K(K)$ are further special cases. The equalities $V(K) = \check{V}(K)$ and $U(K) = \check{U}(K)$ were found by Ohtsuka [5]; cf. [6] too. It is evident that $W_i(K) = \check{W}_i(K)$. Thus all equalities are justified.

The inequality $W_i(K) \leqq V(K)$ follows from

$$W_i(K) \leqq \int \Phi(x, \mu) d\mu(x) \leqq \sup_{x \in S_\mu} \Phi(x, \mu)$$

which is valid for any $\mu \in \mathscr{U}_K$. The inequalities $V(K) \leqq V_K(K) \leqq V_E(K)$ and $U_K(K) \leqq U(K)$ are clear.

We shall give examples in which the inequalities are strict. Consider first the space E consisting of two points x_1 and x_2. If the kernel Φ is given by the matrix $\begin{pmatrix} 1 & 1 \\ \frac{1}{2} & 1 \end{pmatrix}$, then $W_i(K) = 7/8$ and $V(K) = 1$ for $K = E$. If we consider the symmetric kernel given by $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$, $V(K) = 1$ but $V_K(K) = 3/2$. If K consists of one point x_1 and Φ is given by $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$, then $U(K) = V_K(K) = 1$ but $V_E(K) = 2$.

If K consists of two points and Φ is given by $\begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix}$, then $V_E(K)=V_K(K)=1$ but $U(K)=2$. Our proof will be completed if we can find a kernel for which $V_K(K)<\check{V}_K(K)$. This is possible, because $V_K(K)=1$ but $\check{V}_K(K)=2$ for K consisting of two points and $\Phi=\begin{pmatrix}1 & 2\\ 1 & 2\end{pmatrix}$.

2. Suppose still $E=F$. We define $D_n(X)$ by

$$\frac{1}{n(n-1)}\inf_{x_1,\ldots\ldots,x_n\in X}\sum_{i\neq j}\Phi(x_i, x_j)\ .$$

This increases as $n\to\infty$. In fact, if we exclude the terms containing x_k and denote the remaining sum by $\sum_{i\neq j}^{(k)}$, then

$$\sum_{i\neq j}\Phi(x_i, x_j)=\frac{1}{n-2}\sum_{k=1}^{n}\sum_{i\neq j}^{(k)}\Phi(x_i, x_j)\geqq\frac{1}{n-2}\sum_{k=1}^{n}(n-1)(n-2)D_{n-1}(X)$$
$$=n(n-1)D_{n-1}(X)\ .$$

We set

$$\lim_{n\to\infty}D_n(X)=D(X)\ .$$

It is a known result that $D(K)=W_i(K)$; see, for instance, Choquet [1]. In case K is a compact set in E_3 and Φ is Newtonian, $1/D(K)$ is called the transfinite diameter of K.

We come back to the general case where E and F may not be the same. Consider two non-empty sets X and Y in E and F respectively. We set

$$nR_n(X,Y)=\sup_{y_1,\ldots\ldots,y_n\in Y}\ \inf_{x\in X}\sum_{i=1}^{n}\Phi(x, y_i)\ .$$

We shall assume $R_1(X,Y)>-\infty$ and show that $\lim_{n\to\infty}R_n(X,Y)$ exists. Choose $y_1\in Y$ such that $\inf_{x\in X}\Phi(x, y_1)>-\infty$. Then $nR_n(X,Y)\geqq\inf_{x\in X}n\Phi(x, y_1)>-\infty$. If $y_1, \ldots\ldots, y_n, \eta_1, \ldots\ldots, \eta_m\in Y$, then

$$(n+m)R_{n+m}(X,Y)\geqq\inf_{x\in X}\left\{\sum_{i=1}^{n}\Phi(x, y_i)+\sum_{j=1}^{m}\Phi(x, \eta_j)\right\}$$
$$\geqq\inf_{x\in X}\sum_{i=1}^{n}\Phi(x, y_i)+\inf_{x\in X}\sum_{j=1}^{m}\Phi(x, \eta_j)\ ,$$

from which it follows that

$$(n+m)R_{n+m}(X,Y)\geqq nR_n(X,Y)+mR_m(X,Y)\ .$$

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On the other hand,

$$nkR_n(X, Y) = k \sup_{y_1, \cdots\cdots, y_n \in Y} \inf_{x \in X} \sum_{i=1}^{n} \Phi(x, y_i)$$

$$\leqq \sup_{\eta_1, \cdots\cdots, \eta_{nk} \in Y} \inf_{x \in X} \sum_{j=1}^{nk} \Phi(x, \eta_j) = nkR_{nk}(X, Y).$$

Therefore

$$(nk+m)R_{nk+m}(X, Y) \geqq nkR_{nk}(X, Y) + mR_m(X, Y) \geqq nkR_n(X, Y) + mR_m(X, Y)$$

and hence

$$(1) \qquad R_{nk+m}(X, Y) \geqq \frac{nk}{nk+m} R_n(X, Y) + \frac{m}{nk+m} R_m(X, Y).$$

Given ε, $0 < \varepsilon < 1$, we choose n_0 such that

$$R_{n_0}(X, Y) \geqq \begin{cases} \overline{\lim\limits_{n\to\infty}} R_n(X, Y) - \varepsilon & \text{if } \overline{\lim\limits_{n\to\infty}} R_n(X, Y) < \infty, \\ 1/\varepsilon & \text{if } \overline{\lim\limits_{n\to\infty}} R_n(X, Y) = \infty. \end{cases}$$

Next we choose k_0 such that, for any $k \geqq k_0$ and every m $(0 \leqq m \leqq n_0 - 1)$, it holds that

$$\frac{n_0 k}{n_0 k + m} > 1 - \varepsilon \quad \text{and} \quad \frac{m}{n_0 k + m} R_m(X, Y) > -\varepsilon.$$

In case $\overline{\lim\limits_{n\to\infty}} R_n(X, Y) = \infty$, (1) yields

$$R_{n_0 k + m}(X, Y) \geqq \frac{1-\varepsilon}{\varepsilon} - \varepsilon$$

for any $k \geqq k_0$ and every m, $0 \leqq m \leqq n_0 - 1$. It follows that $\underline{\lim\limits_{n\to\infty}} R_n(X, Y) = \infty$.

In case $\overline{\lim\limits_{n\to\infty}} R_n(X, Y) < \infty$, we choose m_k $(0 \leqq m_k \leqq n_0 - 1)$ such that

$$(2) \qquad R_{n_0 k + m_k}(X, Y) \leqq \begin{cases} \underline{\lim\limits_{n\to\infty}} R_n(X, Y) + \varepsilon & \text{if } \underline{\lim\limits_{n\to\infty}} R_n(X, Y) > -\infty, \\ -1/\varepsilon & \text{if } \underline{\lim\limits_{n\to\infty}} R_n(X, Y) = -\infty. \end{cases}$$

It holds on account of (1) that

$$\underline{\lim_{k\to\infty}} R_{n_0 k + m_k}(X, Y) \geqq R_{n_0}(X, Y) \geqq \overline{\lim_{n\to\infty}} R_n(X, Y) - \varepsilon.$$

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This and (2) yield $\underline{\lim}_{n\to\infty} R_n(X,Y) \geqq \overline{\lim}_{n\to\infty} R_n(X,Y)$. Thus $\lim_{n\to\infty} R_n(X,Y)$ exists. We shall denote this limit by $R(X,Y)$.

Remark. There is an example in which $\lim_{n\to\infty} R_n(X,Y)$ does not exist. Take the x-axis as $X=E$ and $\{1, 2, \ldots\ldots\}$ as $Y=F$. We define $\Phi(x,n)$ by $(-1)^n x$. Then $R_n(X,Y)=-\infty$ if n is odd and $R_n(X,Y)=0$ if n is even.

Let us establish

THEOREM 2. *Let K be a non-empty compact set in E, and Y be any non-empty set in F. Then $R(K,Y)$ exists and*

$$R(K,Y)=U_K(Y) .$$

Proof. First we note that $R_1(K,Y)=\sup_{y\in Y} \inf_{x\in K} \Phi(x,y) > -\infty$, whence $R(K,Y)=\lim_{n\to\infty} R_n(K,Y)$ exists. For each n

$$R_n(K,Y)=\frac{1}{n} \sup_{y_1,\ldots\ldots,y_n\in Y} \inf_{x\in K} \sum_{i=1}^{n} \Phi(x,y_i) \leqq U_K(Y) ,$$

so that $R(K,Y)\leqq U_K(Y)$. To prove the inverse inequality take $\mu\in\mathscr{U}_Y$. Given $\varepsilon>0$, we can find a continuous function $\Phi_\varepsilon(x,y)$ on $K\times S_\mu$ such that $\Phi_\varepsilon(x,y)\leqq\Phi(x,y)$ on $K\times S_\mu$ and

$$\min_{x\in K} \Phi_\varepsilon(x,\mu) \geqq \min_{x\in K} \Phi(x,\mu)-\varepsilon .$$

There exist a finite subdivision $S_\mu=\bigcup_{i=1}^{k} Y_i$ into mutually disjoint Borel sets $Y_1,\ldots\ldots,$ Y_k and points $y_1\in Y_1, \ldots\ldots, y_k\in Y_k$ such that

$$|\Phi_\varepsilon(x,y)-\Phi_\varepsilon(x,y_i)|<\varepsilon$$

whenever $x\in K$ and $y\in Y_i$ for each i. We have

$$\left|\sum_i \Phi_\varepsilon(x,y_i)\mu(Y_i)-\Phi_\varepsilon(x,\mu)\right| \leqq \sum_i \int_{Y_i} \left|\Phi_\varepsilon(x,y_i)-\Phi_\varepsilon(x,y)\right| d\mu(y) \leqq \varepsilon$$

on K and hence

$$\min_{x\in K} \sum_i \Phi_\varepsilon(x,y_i)\mu(Y_i) \geqq \min_{x\in K} \Phi_\varepsilon(x,\mu)-\varepsilon \geqq \min_{x\in K} \Phi(x,\mu)-2\varepsilon .$$

We approximate each $\mu(Y_i)$ by a non-negative rational number r_i such that $\sum_i r_i=1$ and

$$\min_{x\in K}\sum_i \Phi_\varepsilon(x, y_i)\mu(Y_i) \leqq \min_{x\in K}\sum_i \Phi_\varepsilon(x, y_i)r_i+\varepsilon \leqq \min_{x\in K}\sum_i \Phi(x, y_i)r_i+\varepsilon\ .$$

Set $r_i=p_i/q$ with integers $p_i \geqq 0$ and $q>0$, and consider

$$\frac{1}{q}\left\{p_1\Phi(x, y_1)+p_2\Phi(x, y_2)+\ldots\ldots+p_k\Phi(x, y_k)\right\}$$

Its minimum on K is not greater than $R_q(K, Y)$. Thus

$$\min_{x\in K}\Phi(x, \mu) \leqq R_q(K, Y)+3\varepsilon.$$

Since we can take q arbitrarily large, $\min_{x\in K}\Phi(x, \mu) \leqq R(K, Y)+3\varepsilon$, whence $\min_{x\in K}\Phi(x, \mu) \leqq R(K, Y)$. Because of the arbitrariness of $\mu\in\mathscr{U}_Y$, we have $U_K(Y) \leqq R(K, Y)$, which gives the equality.

3. Finally we prove

THEOREM 3. *Let X be a non-empty set in E and L be a non-empty compact set in F. In order that there be $\mu\in\mathscr{U}_L$ such that $\Phi(x, \mu)=\infty$ for every $x\in X$, it is necessary and sufficient that $U_X(L)=\infty$.*

Proof. Suppose that there is a measure $\mu\in\mathscr{U}_L$ such that $\Phi(x, \mu)=\infty$ for every $x\in X$. Then

$$U_X(L)=\sup_{\mu\in\mathscr{U}_L}\ \inf_{x\in X}\Phi(x, \mu)=\infty\ .$$

Conversely assume $U_X(L)=\infty$. For each k there is $\mu_k\in\mathscr{U}_L$ such that $\Phi(x, \mu_k)>2^k$ on X. Naturally $\sum_{k=1}^{\infty}2^{-k}\mu_k\in\mathscr{U}_L$ and

$$\Phi\left(x, \sum_{k=1}^{\infty}2^{-k}\mu_k\right)=\infty \qquad \text{for every } x\in X.$$

Using Theorem 2 we obtain the following generalization of the so-called Evans-Selberg's theorem.

COROLLARY. *Let K and L be non-empty compact sets in E and F respectively. In order that there be $\mu\in\mathscr{U}_L$ such that $\Phi(x, \mu)=\infty$ for every $x\in K$, it is necessary and sufficient that $R(K, L)=\infty$.*

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