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Absolutely continuous invariant measures for some piecewise hyperbolic affine maps

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Abstract. A class of piecewise affine hyperbolic maps on a bounded subset of the plane is considered. It is shown that if a map from this class is sufficiently area-expanding then almost surely this map has an absolutely continuous invariant measure.

1. *Introduction*

In [**12**], Pesin studied a general class of piecewise diffeomorphisms with a hyperbolic attractor. He showed the existence of the Sinai–Ruelle–Bowen measure, or SRB measure for short, and studied the ergodic properties of this measure. If $f : M \to M$ is the system in question then the SRB measure is a weak limit point of the sequence of measures

$$\mu_n = \frac{1}{n} \sum_{k=0}^{n-1} \nu \circ f^{-k},$$

where ν denotes the Lebesgue measure. Pesin showed that the SRB measure has at most countably many ergodic components. This measure is the physically relevant measure as it captures the behavior of the orbits of points from a set of positive Lebesgue measure.

For a more restricted class, Sataev [**13**] showed that there are only finitely many ergodic components. Schmeling and Troubetzkoy studied in [**15**] a more general class than Pesin's and proved the existence of the SRB measure. Their method to deal with the non-invertibility of the system was to lift the system to a higher dimension and get an invertible system on which the calculations were carried out. In this way methods from invertible systems could be used. The result could then be projected back to the original system.

In [**1**], Alexander and Yorke considered a one-parameter class of maps called the fat baker's transformations. These maps are piecewise affine maps of the square with one expanding and one contracting direction. Their results, together with the result of Solomyak in [**16**], imply that, for a positive measure set of parameters, there is an absolutely continuous invariant measure.

The Belykh map, was first introduced in [**2**] by Belykh. Schmeling and Troubetzkoy considered in [**15**] the Belykh map for a wider range of parameters. The fat baker's transformations are a special case of the Belykh map in this wider range of parameters. The Belykh map was further investigated in [**14**].

In this paper we consider a class of piecewise affine hyperbolic maps on a set $K \subset \mathbb{R}^2$, with one contracting and one expanding direction. This class is contained in the class of maps studied in [**15**] and it contains the Belykh maps as well as the fat baker's transformations.

It is shown that if a function from this class is sufficiently area-expanding then almost surely (in the sense of Corollary 3.1) there is an absolutely continuous invariant measure. The method used to show this is a development of the method from [**10**]. Here a new problem arises: the symbolic space changes as the parameters change. In this paper a way to handle this problem is introduced. The different symbolic spaces are embedded in a larger space and certain estimates are carried out that makes this larger space possible to handle.

In [**19**] and [**20**], Tsujii considered two classes of maps in two dimensions and showed that almost all of these maps have absolutely continuous invariant measure. These two classes are different from the class of maps considered in this paper. Tsujii also used the method from [**10**], but in a different way than is used in this paper.

Similar results in two dimensions, but in the case of expanding maps, were independently obtained by Buzzi in [**5**] and Tsujii in [**17**]. The corresponding results for arbitrary dimension are in [**4**] and [**18**].

In the last section we use a result from Chernov [**6**] to conclude that the maps considered in this paper have exponential decay of correlations for Hölder continuous functions, provided that an additional assumption is satisfied.

2. *A class of piecewise hyperbolic maps*

Let $K \subset \mathbb{R}^2$ be compact and connected. Assume that K can be decomposed according to

$$K = \bigcup_{i=1}^{a} \overline{K}_i,$$

where each K_i is an open and non-empty set with the boundary consisting of finitely many C^2 curves, such that for each of these curves the set of points where the curvature is zero has finitely many connected components. Thus, there are closed C^2 curves N_i and M_i such that

$$\bigcup_{i=1}^{a} \partial K_i = \left(\bigcup_{i=1}^{b} N_i\right) \cup \left(\bigcup_{i=1}^{c} M_i\right), \quad \partial K = \bigcup_{i=1}^{c} M_i.$$

Let $\mathcal{Z} = \{K_i\}$ denote the partion of K.

Assume that the sets $N_i \cap N_j$, $M_i \cap M_j$ and $N_i \cap M_k$ consists of finitely many points if $i \neq j$, and there exists a constant H such that if $(t_1, t_2) \in T_p N_i$ then $|t_2/t_1| < H$. Let $N = \bigcup N_i$ and $M = \bigcup M_i$.

See Figure 1 for an example of K, N_i and M_i.

Consider maps $f : K \setminus N \to K$ that satisfy the following two conditions, (A1) and (A2).

(A1) There are numbers $\lambda_1, \ldots, \lambda_a < 1 < \gamma_1, \ldots, \gamma_a$ and $u_1, \ldots, u_a, v_1, \ldots, v_a \in \mathbb{R}$ with $u_i \neq u_j$ whenever $i \neq j$, such that for any $i = 1, \ldots, a$ the map f restricted to K_i is defined by

$$f|_{K_i}(x_1, x_2) = f_i(x_1, x_2) = (\lambda_i x_1 + u_i, \gamma_i x_2 + v_i).$$

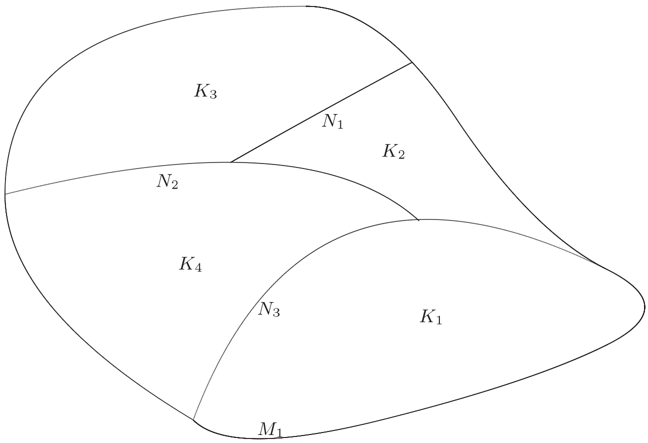

FIGURE 1. An example of the domain K.

The notation $f_{\overline{\lambda},\overline{\gamma},\overline{u},\overline{v}}$ for f will be used to emphasize the dependence on the parameters $\overline{\lambda} = (\lambda_1, \ldots, \lambda_a)$, $\overline{\gamma} = (\gamma_1, \ldots, \gamma_a)$, $\overline{u} = (u_1, \ldots, u_a)$ and $\overline{v} = (v_1, \ldots, v_a)$.

Let $\mathcal{Z}_0 = \mathcal{Z}$ and assume that $\mathcal{Z}_k = \{K_i^{(k)}\}_{i=1}^{a_k}$ is defined. (Note that $a_0 = a$.) Then define the partition $\mathcal{Z}_{k+1} = \{K_i^{(k+1)}\}_{i=1}^{a_{k+1}}$ by

$$\mathcal{Z}_{k+1} = \mathcal{Z}_k \vee \{f^{-1}(K_i^{(k)})\}_{i=1}^{a_k}.$$

The set $\bigcup_{i=1}^{a_k} K_i^{(a_k)}$ is the set of points $x \in K$ such that for each $l = 0, 1, \ldots, k$ the point $f^l(x)$ is defined and $f^l(x) \notin N \cup M$.

Since each K_i has piecewise C^2 boundary so has each $K_j^{(k)}$. There are thus closed C^2 curves $N_i^{(k)}$ such that

$$\bigcup_{i=1}^{a_k} \partial K_i^{(k)} = \bigcup_{i=1}^{b_k} N_i^{(k)}$$

and $N_i^{(k)} \cap N_j^{(k)}$ is a finite set if $i \neq j$.

(A2) There is a number $\tau \geq 1$ such that $(\min \gamma_i)^\tau > D_\tau + 1$ where

$$D_\tau = \max\left\{\#A \;\middle|\; A \subseteq \{1, 2, \ldots, b_\tau\} \text{ such that } \bigcap_{i \in A} N_i^{(\tau)} \neq \emptyset\right\}.$$

That is, D_τ is the maximal number of lines from $\{N_i^{(\tau)}\}$ that cross at one point. The number D_τ is finite since the set $\{N_i^{(\tau)}\}$ is finite.

Remark. Condition (A2) implies that the multiplicity entropy (see [**3**] and [**8**] for a definition) is less than the positive Lyapunov exponent.

3. *The results*

We are going to prove the following theorem.

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THEOREM 3.1. *Assume that for all $t \in I = (t_0, t_1)$, the maps $f_{t\overline{\lambda},\overline{\gamma},\overline{u},\overline{u}}$ satisfy the conditions (A1) and (A2) with a uniform τ. If*

$$\frac{t_0 \min\{\lambda_i\} \min\{\gamma_i^2\}}{\max\{\gamma_i\}} > 1,$$

and one of the following conditions is satisfied:

$$\begin{aligned}
&(1) \qquad t_1\lambda_{\max} < 0.5, \qquad \frac{\max\{|u_i - u_j|, |u_i|\}}{\min\{|u_i - u_j| : u_i \neq u_j\}} < \frac{1 - t_1\lambda_{\max}}{t_1\lambda_{\max} - 2(t_1\lambda_{\max})^3},\\
&(2) \qquad t_1\lambda_{\max} < 0.61, \qquad \frac{\max\{|u_i - u_j|, |u_i|\}}{\min\{|u_i - u_j| : u_i \neq u_j\}} < \frac{1 - t_1\lambda_{\max}}{t_1\lambda_{\max} - 2(t_1\lambda_{\max})^4},\\
&(3) \qquad t_1\lambda_{\max} < 0.68, \qquad \frac{\max\{|u_i - u_j|, |u_i|\}}{\min\{|u_i - u_j| : u_i \neq u_j\}} < \frac{1 - t_1\lambda_{\max}}{t_1\lambda_{\max} - 2(t_1\lambda_{\max})^5},
\end{aligned}$$

where $\lambda_{\max} = \max\{\lambda_i\}$, then for almost every $t \in I$ there exists an $f_{t\overline{\lambda},\overline{\gamma},\overline{u},\overline{v}}$-invariant measure, absolutely continuous with respect to Lebesgue measure.

Remark. If the maps $f_{t\overline{\lambda},\overline{\gamma},\overline{u},\overline{u}}$ do not satisfy the assumptions of Theorem 3.1, but some iterate of the maps do, then the conclusion of Theorem 3.1 is still true. It is however not clear whether the three conditions can be exchanged by the condition $t_1\lambda_{\max} < 1$, since it is not clear that the second half of the conditions are satisfied for sufficiently high iterates of the maps.

We can reformulate Theorem 3.1 in the following way.

COROLLARY 3.1. *Let P be the set of parameters $(\overline{\lambda}, \overline{\gamma}, \overline{u}, \overline{v})$ such that*

$$\frac{\min\{\lambda_i\} \min\{\gamma_i^2\}}{\max\{\gamma_i\}} > 1,$$

and one of the following conditions is satisfied:

$$\begin{aligned}
&(1) \qquad \max\{\lambda_i\} < 0.5, \qquad \frac{\max\{|u_i - u_j|, |u_i|\}}{\min\{|u_i - u_j| : u_i \neq u_j\}} < \frac{1 - \max\{\lambda_i\}}{\max\{\lambda_i\} - 2(\max\{\lambda_i\})^3},\\
&(2) \qquad \max\{\lambda_i\} < 0.61, \qquad \frac{\max\{|u_i - u_j|, |u_i|\}}{\min\{|u_i - u_j| : u_i \neq u_j\}} < \frac{1 - \max\{\lambda_i\}}{\max\{\lambda_i\} - 2(\max\{\lambda_i\})^4},\\
&(3) \qquad \max\{\lambda_i\} < 0.68, \qquad \frac{\max\{|u_i - u_j|, |u_i|\}}{\min\{|u_i - u_j| : u_i \neq u_j\}} < \frac{1 - \max\{\lambda_i\}}{\max\{\lambda_i\} - 2(\max\{\lambda_i\})^5}.
\end{aligned}$$

If $f_{\overline{\lambda},\overline{\gamma},\overline{u},\overline{u}}$ satisfies the conditions (A1) and (A2) then for Lebesgue almost every $(\overline{\lambda}, \overline{\gamma}, \overline{u}, \overline{v}) \in P$, there is an $f_{\overline{\lambda},\overline{\gamma},\overline{u},\overline{v}}$-invariant measure, absolutely continuous with respect to Lebesgue measure.

4. *A condition on transversality for power series*

The corollary of the following lemma will be used to prove Theorem 3.1. The lemma appears in a somewhat less general form in [**10**]. The proof from [**10**] works here as well.

LEMMA 4.1. *Let $C \geq 1$. Then there is a constant $\delta > 0$ such that, for any function g of the form*

$$g(x) = 1 + \sum_{k=1}^{\infty} b_k x^k, \quad b_k \in [-C, C], \tag{1}$$

the following implication holds true for $n = 2, 3, 4$:

$$g(x) \leq \delta, \quad x \in (0, Q_n), \quad C < f_n(x) \Longrightarrow g'(x) \leq -\delta,$$

where $f_n(x) = (1-x)/(x - 2x^{n+1})$, $Q_2 < 0.5$, $Q_3 = 0.61$ and $Q_4 = 0.68$.

Proof. Let

$$h_n(x) = 1 - C\sum_{k=1}^{n} x^k + C\sum_{k=n+1}^{\infty} x^k = 1 - C\frac{x - 2x^{n+1}}{1-x}$$

and

$$f_n(x) = \frac{1-x}{x - 2x^{n+1}}.$$

Let $Q_2 = 0.5$, $Q_3 = 0.61$ and $Q_4 = 0.68$. This implies that there is a $\delta > 0$ such that, for $n = 2, 3, 4$:

(i) $h_n(x) > \delta$ if $C < f_n(x)$; and

(ii) $h_n'(x) < -\delta$ if $x \in (0, Q_n)$.

Thus $h_n(x) > \delta$ and $h_n'(x) < -\delta$ provided $x < Q_n$ and $C < f_n(x)$.

Let $g(x)$ be of the form (1). There are $c_k \geq 0$ such that the function $G_n(x) = g(x) - h_n(x)$ can be written as $G_n(x) = \sum_{k=1}^{n} c_k x^k - \sum_{k=n+1}^{\infty} c_k x^k$. By the above-mentioned properties of h_n we have, for any $x \in (0, Q_n)$, $C < f_n(x)$,

$$g(x) < \delta \Longrightarrow G_n(x) < 0 \Longrightarrow G_n'(x) < 0 \Longrightarrow g'(x) < -\delta.$$

The second implication is proved by

$$G_n(x) < 0 \Longrightarrow \sum_{k=1}^{n} c_k x^k < \sum_{k=n+1}^{\infty} c_k x^k \Longrightarrow \sum_{k=1}^{\infty} k c_k x^k < \sum_{k=n+1}^{\infty} k c_k x^k \Longrightarrow G_n'(x) < 0.$$

□

COROLLARY 4.1. *Let $n \in \{2, 3, 4\}$ and let Q_n and f_n be as in Lemma 4.1. Let s_k be a sequence with $s_k \in [-C, C]$. Then for any $l > 0$ the set*

$$\left\{ q \in (q_0, Q_n) \,\middle|\, C < f_n(q), \left| q^l + \sum_{k=l+1}^{\infty} s_k q^k \right| < r \right\}$$

is contained in an interval of length at most $2\delta^{-1} q_0^{-l} r$.

Proof. Note that $\left|q^l + \sum_{k=l+1}^{\infty} s_k q^k\right| < r$ implies $\left|1 + \sum_{k=l+1}^{\infty} s_k q^{k-l}\right| < r q_0^{-l}$ if $q \geq q_0$. Lemma 4.1 implies that on the set $\{q \in (0, Q_n) \mid C < f_n(q)\}$ the graph of the function $q \mapsto 1 + \sum_{k=l+1}^{\infty} s_k q^{k-l}$ crosses 0 transversally in at most one point and the slope is at most $-\delta$ around this point. Hence $\left|1 + \sum_{k=l+1}^{\infty} s_k q^{k-l}\right| < r q_0^{-l}$ on an interval of length not more than $2\delta^{-1} q_0^{-l} r$ and therefore $\left|q^l + \sum_{k=l+1}^{\infty} s_k q^k\right| < r$ can only hold in this interval.

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5. *Proof of Theorem 3.1*

We will use the method from [**14**] to prove Theorem 3.1. This method is based on [**10**]. This method has also been used in a similar way in [**9**] and [**11**]. The idea is to integrate the density of the measure and then integrate with respect to the parameter. If this integral is finite then almost surely the density is integrable and so the measure is absolutely continuous with respect to Lebesgue measure. To prove that this is the case it is necessary to control how the measure changes with the parameter.

We begin with some notation and general theory and then make the estimates needed later in the proof.

5.1. *Notation and general theory.*

In [**15**], Schmeling and Troubetzkoy considered maps $f: K \setminus N \to K$, where K is an open, bounded and connected subset of a manifold and $N \subset K$ is a closed set, the discontinuity set. The set $K \setminus N$ is a finite union of open sets K_i on each of which f is a C^2 diffeomorphism. Moreover, it is assumed that there are invariant stable and unstable cones on which f acts uniformly contracting and expanding and that these cones span the tangent manifold at each point; see [**15**] for details. The maps considered in this paper fit into this setting and we are going to use results from [**15**].

The difference between the maps considered by Schmeling and Troubetzkoy, and the maps studied by Pesin in [**12**] is that Schmeling and Troubetzkoy allowed the images of the sets K_i to have non-empty intersection.

Pesin showed that under some conditions on the map there exists an invariant measure of which the conditional measures on unstable manifolds are absolutely continuous with respect to the Lebesgue measure on the unstable manifolds. These measures are called SRB measures or Gibbs u-measures. They have the property that the set of typical points has positive Lebesgue measure.

Schmeling and Troubetzkoy showed that their maps have invariant measures with the property that the set of typical points has positive Lebesgue measure. These measures were called SRB measures. We will stick to this terminology in this paper.

Fix $\overline{\lambda}$, $\overline{\gamma}$, $\overline{u}$ and $\overline{v}$. Let $\lambda_{\min} = \min\{\lambda_i\}$, $\lambda_{\max} = \max\{\lambda_i\}$, $\gamma_{\min} = \min\{\gamma_i\}$ and $\gamma_{\max} = \max\{\gamma_i\}$. We will use the shorter notation f_t to denote $f_{t\overline{\lambda},\overline{\gamma},\overline{u},\overline{u}}$.

Let $\hat{K} = K \times [0, 1]$ and $\hat{K}_i = K_i \times [0, 1]$. The sets $\hat{N}, \hat{M}, \ldots$ are defined analogously. We use the idea from [**15**] and lift the map f_t to an injective map $\hat{f}_t$ on $\hat{K}$ by

$$\hat{f}_t|_{\hat{K}_i}(x_1, x_2, x_3) = (f_t(x_1, x_2), \theta x_3 + i/(a + 1)),$$

where $0 < \theta < 1/(a + 1)$. The map $\pi : \hat{K} \to K$, $(x_1, x_2, x_3) \mapsto (x_1, x_2)$ is the projection of $\hat{K}$ on K. It satisfies $\pi(\hat{f}_t(x_1, x_2, x_3)) = f_t(\pi(x_1, x_2, x_3))$.

Let $\hat{D}_t^+ = \{\hat{p} \in \hat{K} \mid \hat{f}_t^n(\hat{p}) \notin \hat{N} \cup \hat{M},\ \forall n \in \mathbb{N}\}$ and $\hat{D}_t = \bigcap_{n=0}^{\infty} \hat{f}_t^n(\hat{D}_t)$. The set $\hat{\Lambda}_t = \overline{\hat{D}_t}$ is the attractor of $\hat{f}_t$.

The condition (A2) is a more general version of condition (H9) in [**12**]. It appears in [**15**]. Theorem 6.1 in [**15**] can be applied to conclude that there are constants $c_t > 0$ such that, for any $\varepsilon > 0$ and any $n \in \mathbb{N}$,

$$\hat{\nu}(\hat{f}_t^{-n}(U(\varepsilon, \hat{N} \cup \hat{M}))) < c_t \varepsilon,$$

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where $U(\varepsilon, \hat{N} \cup \hat{M})$ denotes the ε-neighborhood of $\hat{N} \cup \hat{M}$. This shows that the functions $\hat{f}_t$ are in the class of functions from [**12**]. Moreover, the map f_t also satisfies the conditions in [**15**]. This gives us the following results.

(i) Let $V^{\mathrm{u}} \subset K$ be a curve in the unstable direction, i.e. there are numbers ρ, σ_1 and σ_2 such that $V^{\mathrm{u}} = \{(x_1, x_2) \in K \mid x_1 = \rho, \sigma_1 < x_2 < \sigma_2\}$. Let $\hat{V}^{\mathrm{u}} = \pi^{-1}(V^{\mathrm{u}})$ be the corresponding manifold in $\hat{K}$. Let $\nu_{V^{\mathrm{u}}}$ and $\hat{\nu}_{\hat{V}^{\mathrm{u}}}$ denote the normalized Lebesgue measure on V^{u} and $\hat{V}^{\mathrm{u}}$ respectively. The sequence of measures $\hat{\mu}_n^t = (1/n)\sum_{k=0}^{n-1} \hat{\nu}_{\hat{V}^{\mathrm{u}}} \circ \hat{f}_t^{-k}$ converges weakly to an SRB measure $\hat{\mu}_{\mathrm{SRB}}^t$. The projection of this measure $\hat{\mu}_{\mathrm{SRB}}^t \circ \pi^{-1}$ is an SRB measure for f_t and we thus write $\mu_{\mathrm{SRB}}^t = \hat{\mu}_{\mathrm{SRB}}^t \circ \pi^{-1}$.

(ii) Given $\varepsilon > 0$, the set

$$\hat{D}_{t,\varepsilon,c} = \{\hat{x} \in \hat{\Lambda}_t \mid d(\hat{f}_t^n(\hat{x}), \hat{N}) \geq ce^{-\varepsilon n}\}$$

is non-empty if c is sufficiently small and the set $\hat{D}_{t,\varepsilon} = \bigcup_{i=1}^{\infty} \hat{D}_{t,\varepsilon,i^{-1}}$ has full $\hat{\mu}_{\mathrm{SRB}}^t$-measure, $\hat{\mu}_{\mathrm{SRB}}^t(\hat{D}_{t,\varepsilon}) = 1$.

(iii) The conditional measures of $\hat{\mu}_{\mathrm{SRB}}^t$ on the unstable manifolds are absolutely continuous with respect to Lebesgue measure.

(iv) The entropy of the measure $\hat{\mu}_{\mathrm{SRB}}^t$ is

$$h_{\hat{\mu}_{\mathrm{SRB}}^t} = \int_{\hat{\Lambda}_t} \log \chi_t(\hat{x})\, d\hat{\mu}_{\mathrm{SRB}}^t(\hat{x}),$$

where $\chi_t(\hat{x})$ is the positive Lyapunov exponent at the point $\hat{x}$ for the map $\hat{f}_t$. In particular, $\log(\gamma_{\min}) \leq h_{\hat{\mu}_{\mathrm{SRB}}^t} \leq \log(\gamma_{\max})$.

(v) The measure $\hat{\mu}_{\mathrm{SRB}}^t$ has at most countably many ergodic components.

The results (i)–(v) make it possible to define stable manifolds for $\hat{\mu}_{\mathrm{SRB}}^t$-almost every $\hat{x} \in \hat{K}$. If $\hat{x} = (x_1, x_2, x_3) \in \hat{D}_{t,-\log(t\lambda_{\max})}$ then there is a $c = c(\hat{x})$ such that $\hat{x} \in \hat{D}_{t,-\log(t\lambda_{\max}),c}$, that is $d(\hat{f}_t^n(\hat{x}), \hat{N}) \geq c(t\lambda_{\max})^n$ for all $n \geq 0$. If $\hat{y} = (y_1, y_2, y_3) \in \hat{K}$ with $|y_1 - x_1| < c$ and $y_2 = x_2$ then

$$|\hat{f}_t^n(\hat{y}) - \hat{f}_t^n(\hat{x})|_1 \leq |y_1 - x_1|(t\lambda_{\max})^n < c(t\lambda_{\max})^n \leq d(\hat{f}_t^n(\hat{x}), \hat{N}),$$

where $|\cdots|_1$ denotes the modulus of the difference in the first coordinate. Hence the points $\hat{f}_t^n(\hat{y})$ and $\hat{f}_t^n(\hat{x})$ are never separated by a discontinuity and we say that $\hat{y}$ is in the stable manifold of $\hat{x}$. The stable manifold of $\hat{x}$ is thus defined to be the set

$$\hat{W}^{t,\mathrm{s}}(\hat{x}) = \{\hat{y} \in \hat{K} \mid |x_1 - y_1| < c,\ y_2 = x_2\},$$

where c is the largest constant such that $\hat{x} \in \hat{D}_{t,-\log(t\lambda_{\max}),c}$. This defines the stable manifold of $\hat{\mu}_{\mathrm{SRB}}^t$-a.e. point $\hat{x} \in \hat{K}$ since $\hat{\mu}_{\mathrm{SRB}}^t(\hat{D}_{t,\log(t\lambda_{\max})}) = 1$.

The stable manifold $W^{t,\mathrm{s}}(x)$ is defined as the projection of corresponding stable manifold in $\hat{K}$. All the stable manifolds will therefore be parallel line segments but the length of the manifolds are only measurable [**12**].

Similarly the unstable manifolds can be defined. They consist of parallel line segments, orthogonal to the stable manifolds, and their length is measurable.

The partition of K into stable manifolds is thus measurable and the conditional measures on these manifolds can be defined. Take $x \in \Lambda = \pi(\hat{\Lambda})$ and let $\mu_{\mathrm{SRB}}^{t,\mathrm{s},x}$ denote the conditional measure on the stable manifold $W^{t,\mathrm{s}}(x)$.

The sequence of measures $\hat{\mu}_n^t$ may converge to a measure which is not ergodic, but if $\hat{V}_t^{\mathrm{u}} \subseteq \hat{W}^{t,\mathrm{u}}(\hat{x}_t)$ for some $\hat{x}_t \in \hat{\Lambda}_t$ then $\hat{\mu}_n^t$ will converge to a unique ergodic component, since then $\hat{V}_t^{\mathrm{u}} \subseteq \hat{\Lambda}_t$. We choose $\hat{x}_t \in \hat{\Lambda}_t$ so that $\hat{x}_t$ depends continuously on t. This is possible; see [**13**].

To get control over $\hat{\mu}_{\mathrm{SRB}}^t$ for almost all $t \in I$ argue as follows. Since the size of the unstable manifolds depends only measurably on t, it may not be possible to get a uniform length of $\hat{V}_t^{\mathrm{u}} \subseteq \hat{\Lambda}_t$. However, given $\varepsilon_0 > 0$ there is a set $J_0 \subset I$ with $\nu(I \setminus J_0) < \varepsilon_0$ such that the sets $\hat{V}_t^{\mathrm{u}} \subseteq \hat{W}^{t,\mathrm{u}}(\hat{x})$ can be chosen to have equal length for all $t \in J_0$.

This construction has the following consequence. For any $t \in J_0$ the measure $\hat{\mu}_n^t$ converges to an ergodic measure and it is not necessary to take a subsequence. Indeed, if it is necessary to take a subsequence then there exists a set A such that

$$\liminf_{n\to\infty} \hat{\mu}_n^t(A) < \limsup_{n\to\infty} \hat{\mu}_n^t(A).$$

But this would contradict that $\hat{\mu}_n^t$ converges to a unique ergodic component.

Given a sequence $\{i_n\} \in \{1, 2, \ldots, a\}^{\mathbb{Z}}$ and integers l, m we define the cylinder

$$\hat{C}_l^m(\{i_n\}, t) = \bigcap_{k=l}^{m} \hat{f}_t^{-k}(\hat{K}_{i_k})$$

or

$$\hat{C}_l^m(i_0, i_1, \ldots, i_{m-l}, t) = \bigcap_{k=l}^{m} \hat{f}_t^{-k}(\hat{K}_{i_{k-l}}).$$

Let $\Sigma_t = \{\{i_k\} \mid \hat{C}_l^m(\{i_k\}, t) \neq \emptyset \; \forall\, l, m \in \mathbb{Z}\}$ and let $\rho_t \colon \Sigma_t \to \hat{\Lambda}_t$ be the natural identification of sequences in Σ_t and points in $\hat{\Lambda}_t$.

5.2. *The measure's dependence on the parameters.* The first step is to estimate how the measure $\hat{\mu}_{\mathrm{SRB}}^t$ changes with the parameter t. This is done by using that $\hat{\mu}_n^t$ converges weakly to $\hat{\mu}_{\mathrm{SRB}}^t$.

Let $L_1 > 0$. The measure $\hat{\mu}_n^t = (1/n) \sum_{k=0}^{n-1} \hat{\nu}_{\hat{V}_t^{\mathrm{u}}} \circ \hat{f}_t^{-k}$ converges weakly to $\hat{\mu}_{\mathrm{SRB}}^t$. Hence there is a number $n_0(t, L_1)$ such that for any cylinder of length L_1

$$\frac{1}{2} \leq \frac{\hat{\mu}_{\mathrm{SRB}}^t(\hat{C}_{-L_1}^0)}{\hat{\mu}_n^t(\hat{C}_{-L_1}^0)} \leq 2,$$

for all $n \geq n_0$. Since $n_0(t, L_1)$ is measurable with respect to t, Lusin's theorem implies that for any $\varepsilon_1 > 0$ there is a set $J_1 \subset J_0$ with $\nu(J_0 \setminus J_1) < \varepsilon_1$ and a number n_1 such that $n_1 > n_0(t, L_1)$ for all $t \in J_1$.

For a fixed cylinder $\hat{C}_{-L_1}^0$, the measure $\hat{\mu}_{n_0}^t(\hat{C}_{-L_1}^0)$ depends continuously on t, because the measure $\hat{\mu}_{n_0}^t$ involves taking finitely many preimages with respect to $\hat{f}_t$ and these preimages depend continuously on t, and the measure $\hat{\nu}_{\hat{V}_t^{\mathrm{u}}}$ depends continuously on t. It is therefore possible to partition I into finitely many subintervals $I = \bigcup_{k=1}^m I_k$ such that when $t_1, t_2 \in I_k$ then

$$\frac{1}{2} \leq \frac{\hat{\mu}_{n_0}^{t_1}(\hat{C}_{-L_1}^0)}{\hat{\mu}_{n_0}^{t_2}(\hat{C}_{-L_1}^0)} \leq 2,$$

for any cylinder of length L_1. This implies that

$$\frac{1}{4} \le \frac{\hat{\mu}^{t_1}_{\mathrm{SRB}}(\hat{C}^0_{-L_1})}{\hat{\mu}^{t_2}_{\mathrm{SRB}}(\hat{C}^0_{-L_1})} \le 4,$$

for any cylinder of length L_1, provided $t_1, t_2 \in I_k \cap J_1$ for some k.

For each I_k, choose a $t_k \in I_k \cap J_1$ and a cylinder $\hat{C}^0_{-L_1}(\{x_n(t_k)\}, t_k)$ with $m_k = \hat{\mu}^{t_k}_{\mathrm{SRB}}(\hat{C}^0_{-L_1}(\{x_n(t_k)\}, t_k)) > 0$.

For every $t \in I_k \cap J_1$, define

$$\hat{\Omega}_{0,t} = \hat{C}^0_{-L_1}(\{x_n(t_k)\}, t).$$

Then

$$\hat{\mu}^t_{\mathrm{SRB}}(\hat{\Omega}_{0,t}) \ge \tfrac{1}{4} m_k \tag{2}$$

for all $t \in I_k \cap J_1$.

5.3. *Entropy.* It will be necessary to control the number of cylinders and the measure of the cylinders. As already noted, the general theory gives that the entropy of the measure $\hat{\mu}^t_{\mathrm{SRB}}$ satisfies $\log(\gamma_{\min}) \le h_{\hat{\mu}^t_{\mathrm{SRB}}} \le \log(\gamma_{\max})$.

The Shannon–McMillan–Breiman theorem implies that for $t \in J_1 \cap I_k$ and $\varepsilon_2 > 0$ there is a constant $A(t)$ such that

$$\hat{\mu}^t_{\mathrm{SRB}}\bigg(\bigcap_{L>0}\{\hat{x} \mid \exists\, \hat{C}^0_{-L}(\{x_k\}, t) \ni \hat{x} \text{ with}$$
$$A(t)^{-1}(\gamma_{\max} + \varepsilon_2)^{-L} < \hat{\mu}^t_{\mathrm{SRB}}(\hat{C}^0_{-L}) < A(t)(\gamma_{\min} - \varepsilon_2)^{-L}\}\bigg) > 1 - \frac{1}{8} m_k.$$

This shows that by making a different choice of $A(t)$ a similar estimate on the stable manifolds will be valid,

$$\hat{\mu}^t_{\mathrm{SRB}}\bigg(\bigcap_{L>0}\{\hat{x} \mid \exists\, \hat{C}^0_{-L}(\{x_k\}, t) \ni \hat{x} \text{ with}$$
$$A(t)^{-1}(\gamma_{\max} + \varepsilon_2)^{-L} < \hat{\mu}^{t,s,x}_{\mathrm{SRB}}(\hat{C}^0_{-L}) < A(t)(\gamma_{\min} - \varepsilon_2)^{-L}\}\bigg) > 1 - \frac{1}{8} m_k. \tag{3}$$

Let $\hat{\Omega}_{\mathrm{SMB},t}$ be the set whose measure is estimated above.

An application of Lusin's theorem shows that given $\varepsilon_3 > 0$ there exists a set $J_2 \subset J_1$ and numbers A_k such that

$$\nu(J_1 \setminus J_2) < \varepsilon_3,$$
$$A(t) \le A_k, \quad \text{whenever } t \in I_k \cap J_2.$$

For all $t \in I_k \cap J_2$, define $\hat{\Omega}_t = \hat{\Omega}_{0,t} \cap \hat{\Omega}_{\mathrm{SMB},t}$. The estimates (2) and (3) show that $\hat{\mu}^t_{\mathrm{SRB}}(\hat{\Omega}_t) \ge \frac{1}{8} m_k$.

It follows by (3) that the number of cylinders of length L in $\hat{\Omega}_t$ satisfies

$$N_t(L) \le A_k(\gamma_{\max} + \varepsilon_2)^L. \tag{4}$$

Let $L_2 > 0$ and consider for each $t \in I_k$ the set of words of length $L_2 + 1$:

$$\mathcal{C}_t = \{x_0, x_1, \ldots, x_{L_2} \mid x_i \in \{1, 2, \ldots, a\}, \hat{C}^{L_2}_0(x_0, x_1, \ldots, x_{L_2}, t) \ne \emptyset\}.$$

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The cylinders $\hat{C}_0^{L_2}(\{x_k\}, t)$ change continuously with t, i.e. the cylinders are sets with smooth boundaries and the boundaries depend continuously on t. This follows from the fact that iterates of the maps are piecewise affine and they depend continuously on t. The fact that the sets $\hat{K}_i$ have piecewise C^2 boundaries with finitely many connected components with zero curvature allows us to draw the following conclusion. There is a partition of I_k into finitely many intervals $I_{k,l}(L_2)$ such that $\mathcal{C}_t = \mathcal{C}_{t'}$ if $t', t \in I_{k,l}(L_2)$ for some l. Indeed, when t runs over I_k, each cylinder appears and disappears only finitely many times. Let the finite set of t, for which some cylinder $\hat{C}_0^{L_2}(\{x_k\}, t)$ appears and disappears, define the endpoints of the intervals $I_{k,l}(L_2)$.

Any sequence in $\Sigma_t \cap \rho_t^{-1}(\hat{\Omega}_t)$ can be written as a concatenation of words from $\mathcal{C}_t$. Together with (4) this implies that for each $I_{k,l}$ the number of words of length n in $\bigcup_{t\in I_{k,l}\cap J_2} \Sigma_t \cap \rho_t^{-1}(\hat{\Omega}_t)$ does not exceed $(A_k^{1/L_2}(\gamma_{\max} + \varepsilon_2))^{n+L_2}$ for any n. Hence, we have the following lemma.

LEMMA 5.1. *For any $\varepsilon_4 > 0$ there is a number $L_3 = L_3(\varepsilon_4)$ such that if $L_2 > L_3$ then for each $I_{k,l}(L_2)$ the symbolic space $\Sigma_{I_{k,l}(L_2)} = \bigcup_{t\in I_{k,l}(L_2)\cap J_2} \Sigma_t \cap \rho_t^{-1}(\hat{\Omega}_t)$ satisfies*

$$N_{I_{k,l}(L_2)}(n) \le B_{k,l}(\gamma_{\max} + \varepsilon_2 + \varepsilon_4)^n,$$

for some $B_{k,l}$, where $N_{I_{k,l}(L_2)}(n)$ denotes the number of words of length n in $\Sigma_{I_{k,l}(L_2)}$.

In order to make use of Lemma 5.1, choose $L_2 > L_3$.

5.4. *Integrability of the densities.* The conditional measures of μ_{SRB}^t on unstable manifolds are absolutely continuous with respect to Lebesgue measure. We will prove that the conditional measures on the stable manifolds are almost surely absolutely continuous with respect to Lebesgue measure. The local product structure of μ_{SRB}^t then implies that μ_{SRB}^t is absolutely continuous with respect to Lebesgue measure.

Take $x \in \Lambda = \pi(\hat{\Lambda})$ and let $W_r^{t,\text{s}}(y, x) = \{z \in W^{t,\text{s}}(x) \mid d(y, z) \le r\}$. The derivative of $\mu_{\text{SRB}}^{t,\text{s},x}$ at y is the limit

$$D(\mu_{\text{SRB}}^{t,\text{s},x}, y) = \liminf_{r\to 0} \frac{\mu_{\text{SRB}}^{t,\text{s},x}(W_r^{t,\text{s}}(y, x))}{2r}.$$

If the function $D(\mu_{\text{SRB}}^{t,\text{s},x}, y)$ is integrable on $W^{t,\text{s}}(x)$ then the measure $\mu_{\text{SRB}}^{t,\text{s},x}$ is absolutely continuous with respect to Lebesgue measure.

Let k be fixed. We want to prove that for a.e. $t \in I_k \cap J_2$

$$\int_{\Omega_t}\int_{\Omega_t} D(\mu_{\text{SRB}}^{t,\text{s},x}|_{\Omega_t}, y)\, d\mu_{\text{SRB}}^{t,\text{s},x}(y)\, d\mu_{\text{SRB}}^t(x) < \infty. \tag{5}$$

This implies that the measure $\mu_{\text{SRB}}^{t,\text{s},x}$ restricted to the set Ω_t is absolutely continuous for a.e. $x \in \Omega_t$. Since the conditional measures on the unstable manifolds are absolutely continuous with respect to Lebesgue measure, this implies that $\mu_{\text{SRB}}^t|_{\Omega_t}$ is absolutely continuous with respect to Lebesgue measure. Since $\mu_{\text{SRB}}^t(\Omega_t) > 0$, ergodicity then implies that this also holds for the measure μ_{SRB}^t. Since k is arbitrary this implies that μ_{SRB}^t is absolutely continuous with respect to Lebesgue for a.e. $t \in I \cap J_2$.

Fatou's lemma implies that in order to prove (5) it suffices to prove that

$$\liminf_{r\to 0} \frac{1}{r} \int_{\Omega_t} \int_{\Omega_t} \mu_{\mathrm{SRB}}^{t,s,x}(\Omega_t \cap W_r^{t,s}(y,x))\, d\mu_{\mathrm{SRB}}^{t,s,x}(y)\, d\mu_{\mathrm{SRB}}^{t}(x) < \infty.$$

We may rewrite this as

$$\liminf_{r\to 0} \frac{1}{r} \int_{\Omega_t} \int_{\Omega_t} \int_{\Omega_t} \chi_{\{|y_1 - z_1| < r\}}\, d\mu_{\mathrm{SRB}}^{t,s,x}(z)\, d\mu_{\mathrm{SRB}}^{t,s,x}(y)\, d\mu_{\mathrm{SRB}}^{t}(x) < \infty. \tag{6}$$

To prove that this holds for a.e. $t \in I_k \cap J_2$ we prove that for any l

$$\liminf_{r\to 0} \frac{1}{r} \int_{I_{k,l}\cap J_2} \int_{\Omega_t} \int_{\Omega_t} \int_{\Omega_t} \chi_{\{|y_1 - z_1| < r\}}\, d\mu_{\mathrm{SRB}}^{t,s,x}(z)\, d\mu_{\mathrm{SRB}}^{t,s,x}(y)\, d\mu_{\mathrm{SRB}}^{t}(x)\, dt < \infty. \tag{7}$$

This then implies that $\mu_{\mathrm{SRB}}^t \ll \nu$ for a.e. $t \in I_k \cap J_2$. Instead of proving (7) we use that $\mu_{\mathrm{SRB}}^t = \hat{\mu}_{\mathrm{SRB}}^t \circ \pi^{-1}$ and prove the equivalent condition

$$\liminf_{r\to 0} \frac{1}{r} \int_{I_{k,l}\cap J_2} \int_{\hat{\Omega}_t} \int_{\hat{\Omega}_t} \int_{\hat{\Omega}_t} \chi_{\{|y_1 - z_1| < r\}}\, d\hat{\mu}_{\mathrm{SRB}}^{t,s,\hat{x}}(\hat{z})\, d\hat{\mu}_{\mathrm{SRB}}^{t,s,\hat{x}}(\hat{y})\, d\hat{\mu}_{\mathrm{SRB}}^{t}(\hat{x})\, dt < \infty. \tag{8}$$

Recall that $\rho_t \colon \Sigma_t \to \hat{\Lambda}_t$ maps sequences in Σ_t to points in $\hat{\Lambda}_t$ in the natural way. Put $\mu_\Sigma^t = \hat{\mu}_{\mathrm{SRB}}^t \circ \rho_t$ and rewrite (8) as

$$\liminf_{r\to 0} \frac{1}{r} \int_{I_{k,l}\cap J_2} \int_{\hat{\Omega}_t} \int_{\rho_t^{-1}(\hat{\Omega}_t)} \int_{\rho_t^{-1}(\hat{\Omega}_t)} \chi_{\{|\rho(\{i_n\}) - \rho(\{j_n\})|_1 < r\}} \\ d\mu_\Sigma^{t,s,\hat{x}}(\{i_n\})\, d\mu_\Sigma^{t,s,\hat{x}}(\{j_n\})\, d\hat{\mu}_{\mathrm{SRB}}^{t}(\hat{x})\, dt < \infty, \tag{9}$$

where $|\cdots|_1$ denotes the difference in the first coordinate.

Embed all subshifts Σ_t, $t \in I_{k,l} \cap J_1$, into the larger subshift $\Sigma_{I_{k,l}}$ as in Lemma 5.1. The measures μ_Σ^t extend from Σ_t to $\Sigma_{I_{k,l}}$ in a natural way since Σ_t is a subset of $\Sigma_{I_{k,l}}$. A cylinder in $\Sigma_{I_{k,l}}$ will be denoted by

$${}_l[\{i_n\}]_m = {}_l[i_l \cdots i_m]_m = \{\{j_n\} \in \Sigma_{I_{k,l}} \mid j_n = i_n,\ n = l, \ldots, m\}.$$

To prove (9) we estimate the quantity

$$T_r(\{\hat{\Omega}_t \mid t \in I_{k,l} \cap J_2\}) \\ = \sum_{L > L_2} \sum_{\substack{{}_{-L}[i_{-L},\ldots,i_0]_0 \\ \subset \Sigma_{I_{k,l}}}} \sum_{\substack{1 \le l_1, l_2 \le a \\ l_1 \ne l_2}} \int_{I_{k,l}\cap J_2} \int_{\hat{\Omega}_t} \int_{\rho_t^{-1}(\hat{\Omega}_t) \cap [l_1, i_{-L},\ldots,i_0]} \int_{\rho_t^{-1}(\hat{\Omega}_t) \cap [l_2, i_{-L},\ldots,i_0]} \\ \chi_{\{|\rho_t(\{i_n\}) - \rho_t(\{j_n\})|_1 < r\}}\, d\mu_\Sigma^{t,s,\hat{x}}(\{j_n\})\, d\mu_\Sigma^{t,s,\hat{x}}(\{i_n\})\, d\hat{\mu}_{\mathrm{SRB}}^{t}(\hat{x})\, dt \tag{10}$$

and show that $T_r < \eta r$ for all $r > 0$ and some constant η. This implies (9) as follows. The product $\Sigma_{I_{k,l}} \times \Sigma_{I_{k,l}}$ can be written as

$$\Sigma_{I_{k,l}} \times \Sigma_{I_{k,l}} = \bigcup_{L} \bigcup_{\substack{{}_{-L}[i_{-L},\ldots,i_0]_0 \\ \subset \Sigma_{I_{k,l}}}} \bigcup_{\substack{1 \le l_1, l_2 \le a \\ l_1 \ne l_2}} \\ {}_{-L-1}[l_1, i_{-L}, \ldots, i_0]_0 \times {}_{-L-1}[l_2, i_{-L}, \ldots, i_0]_0,$$

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i.e. $\Sigma_{I_{k,l}} \times \Sigma_{I_{k,l}}$ is the union over L of the set of pair of sequences with the first L letters equal. This implies that

$$\liminf_{r\to 0} \frac{1}{r} \int_{I_{k,l}\cap J_2} \int_{\hat{\Omega}_t} \int_{\rho_t^{-1}(\hat{\Omega}_t)} \int_{\rho_t^{-1}(\hat{\Omega}_t)} \chi_{\{|\rho(\{i_n\})-\rho(\{j_n\})|_1<r\}} \, d\mu_\Sigma^{t,s,\hat{x}}(\{i_n\})\, d\mu_\Sigma^{t,s,\hat{x}}(\{j_n\})\, d\hat{\mu}_{\mathrm{SRB}}^t(\hat{x})\, dt = \liminf_{r\to 0} \frac{1}{r} T_r(\{\hat{\Omega}_t \mid t \in I_{k,l} \cap J_2\}),$$

so (10) implies (9).

It remains to show (10). This will be done in §5.6. To do this, the estimate in §5.5 is needed.

5.5. *An estimate on power series.* The expression $|\rho_t(\{i_n\}) - \rho_t(\{j_n\})|_1$ appearing in (10) can be expressed as a power series. The following estimate on this power series is an important part in proving (10).

If $(x_1, x_2, x_3) \in \hat{\Lambda}_t$ and $\{i_n\} \in \Sigma_{I_{k,l}}$ is the sequence such that $\rho_t(\{i_n\}) = (x_1, x_2, x_3)$, then it is easy to see that

$$x_1 = \sum_{n=1}^{\infty} \prod_{l=1}^{n-1} (t\lambda_{i_{l-n}}) u_{i_{-n}} = \sum_{n=1}^{\infty} \prod_{l=1}^{n-1} \frac{\lambda_{i_{l-n}}}{\lambda_{\max}} u_{i_{-n}} (\lambda_{\max} t)^{n-1}.$$

So the expression in the brackets of the integrand in (10) can be rewritten in the form

$$\left| \sum_{n=L+1}^{\infty} \left(\prod_{l=1}^{n-1} \lambda_{i_{l-n}} u_{i_{-n}} - \prod_{l=1}^{n-1} \lambda_{j_{l-n}} u_{j_{-n}} \right) t^{n-1} \right| < r. \tag{11}$$

Multiplying both sides of (11) with

$$\frac{1}{|u_{l_1} - u_{l_2}|} \frac{1}{\prod_{l=1}^{L} (t\lambda_{i_{l-n}})}$$

we see that (11) holds if

$$\left| 1 + \sum_{n=L+2}^{\infty} \left(\prod_{l=1}^{n-1} \frac{\lambda_{i_{l-n}}}{\lambda_{\max}} u_{i_{-n}} - \prod_{l=1}^{n-1} \frac{\lambda_{j_{l-n}}}{\lambda_{\max}} u_{j_{-n}} \right) \frac{(t\lambda_{\max})^{n-1-L}}{u_{l_1} - u_{l_2}} \right| < \frac{r}{|u_{l_1} - u_{l_2}|} \frac{1}{\prod_{l=1}^{L} (t\lambda_{i_{l-n}})}. \tag{12}$$

The coefficients of $(t\lambda_{\max})^{n-1-L}$ in the sum in (12) are bounded by

$$\max_{\{i_n\},\{j_n\},n} \left(\prod_{l=1}^{n-1} \frac{\lambda_{i_{l-n}}}{\lambda_{\max}} u_{i_{-n}} - \prod_{l=1}^{n-1} \frac{\lambda_{j_{l-n}}}{\lambda_{\max}} u_{j_{-n}} \right) \frac{1}{u_{l_1} - u_{l_2}} \le \frac{\max\{|u_i - u_j|, |u_i|\}}{\min\{|u_i - u_j| \mid u_i \neq u_j\}} =: C,$$

so an application of Corollary 4.1 with C as above shows that

$$\left| \left\{ t\lambda_{\max} \,\middle|\, \left| \sum_{n=L}^{\infty} \left(\prod_{l=1}^{n-1} \lambda_{i_{l-n}} u_{i_{-n}} - \prod_{l=1}^{n-1} \lambda_{j_{l-n}} u_{j_{-n}} \right) t^{n-1} \right| < r \right\} \right| < \delta^{-1} \frac{(t_0\lambda_{\min})^{-L}}{\min\{|u_i - u_j| \mid u_i \neq u_j\}} r = E\lambda_{\max}(t_0\lambda_{\min})^{-L} r, \tag{13}$$

if $t_1\lambda_{\max} < \min\{(1/2C), 0.68\}$.

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5.6. *The final step.* Put $F(t, \{i_n\}, \{j_n\}) = \chi_{\{|\rho_t(\{i_n\})-\rho_t(\{j_n\})|_1<r\}}$ and rewrite (10) as

$$\begin{aligned} T_r(\{\hat{\Omega}_t \mid t \in I_{k,l} \cap J_2\}) \\ = \sum_{L>L_2} \sum_{\substack{_{-L}[i_{-L},\ldots,i_0]_0 \\ \subset \Sigma_{I_{k,l}}}} \sum_{\substack{1\le l_1,l_2\le a \\ l_1\neq l_2}} \int_{I_{k,l}\cap J_2} \int_{\hat{\Omega}_t} \int_{\rho_t(\hat{\Omega}_t)\cap[l_1,i_{-L},\ldots,i_0]} \int_{\rho_t(\hat{\Omega}_t)\cap[l_2,i_{-L},\ldots,i_0]} \\ F(t, \{i_n\}, \{j_n\})\, d\mu_\Sigma^{t,s,\hat{x}}(\{j_n\})\, d\mu_\Sigma^{t,s,\hat{x}}(\{i_n\})\, d\hat{\mu}^t_{\mathrm{SRB}}(\hat{x})\, dt. \end{aligned} \tag{14}$$

To estimate the quantity in (14) we want to change the order of integration to integrate with respect to t first and then use the estimate (13). This cannot be done immediately because the other integrals depend on t. To get around this problem the function $F(t, \{i_n\}, \{j_n\})$ will be bounded by a function $G(t, \{i_n\}, \{j_n\})$ which is constant on cylinders. More precisely we have the following lemma.

LEMMA 5.2. *Assume that the conditions of Theorem 3.1 are satisfied. For each pair of cylinders $[l_1, i_{-L}, \ldots, i_0]$ and $[l_2, i_{-L}, \ldots, i_0]$ appearing in (14) there is a partition into finitely many cylinders*

$$[l_1, i_{-L}, \ldots, i_0] = \bigcup_{p=1}^{n_\alpha} S_\alpha^p, \quad [l_2, i_{-L}, \ldots, i_0] = \bigcup_{q=1}^{n_\beta} S_\beta^q,$$

where S_α^p, $p = 1, \ldots, n_\alpha$, and S_β^q, $q = 1, \ldots, n_\beta$, are cylinders, and functions $G_{S_\alpha^p, S_\beta^q}(t)$ with

$$G_{S_\alpha^p, S_\beta^q}(t) \geq F(t, \{i_n\}, \{j_n\}) \quad \text{for all } t \in I_{k,l} \cap J_1, \text{ for all } \{i_n\} \in S_\alpha^p, \text{ for all } \{j_n\} \in S_\beta^q$$

and

$$\int_{I_{k,l}} G_{S_\alpha^p, S_\beta^q}(t)\, dt < 2E\lambda_{\max}(t_0\lambda_{\min})^{-L} r. \tag{15}$$

Proof. For fixed $\hat{y}$ and $\hat{z}$ the estimate (13) implies that

$$|\{t \mid F(t, \{i_n\}, \{j_n\}) < r\}| < E\lambda_{\max}(t_0\lambda_{\min})^{-L} r$$

and

$$|\{t \mid |\rho_t(\{i_n\}) - \rho_t(\{j_n\})|_1 < 2r\}| < 2E\lambda_{\max}(t_0\lambda_{\min})^{-L} r. \tag{16}$$

The function $(t, \{i_n\}, \{j_n\}) \mapsto |\rho_t(\{i_n\}) - \rho_t(\{j_n\})|_1$ depends continuously on t, $\{i_n\}$ and $\{j_n\}$. Choose L' so large that

$$\frac{2\max\{u_i - u_j\}(t_1\lambda_{\max})^{L'+3}}{1 - t_1\lambda_{\max}} < r.$$

The sets $[l_1, i_{-L}, \ldots, i_0]$ and $[l_1, i_{-L}, \ldots, i_0]$ can be partitioned into cylinders of length $L' + L + 3$:

$$[l_1, i_{-L}, \ldots, i_0] = \bigcup_{p=1}^{n_\alpha} S_\alpha^p = \bigcup_{p=1}^{n_\alpha} [\alpha_{-L'}(p), \ldots, \alpha_0(p), l_1, i_{-L}, \ldots, i_0],$$

$$[l_2, i_{-L}, \ldots, i_0] = \bigcup_{q=1}^{n_\beta} S_\beta^q = \bigcup_{q=1}^{n_\beta} [\beta_{-L'}(q), \ldots, \beta_0(q), l_2, i_{-L}, \ldots, i_0].$$

Note that n_α and n_β can be bounded uniformly by $n_\alpha, n_\beta \leq a^{L'+1}$.

As in (11) the expression $|\,|\rho_t(\{i_n^{(1)}\}) - \rho_t(\{j_n^{(1)}\})|_1 - |\rho_t(\{i_n^{(2)}\}) - \rho_t(\{j_n^{(2)}\})|_1\,|$ can be written as

$$\begin{aligned}
&|\,|\rho_t(\{i_n^{(1)}\}) - \rho_t(\{j_n^{(1)}\})|_1 - |\rho_t(\{i_n^{(2)}\}) - \rho_t(\{j_n^{(2)}\})|_1\,| \\
&\quad = \bigg|\bigg|\sum_{n=1}^{\infty}\bigg(\prod_{l=1}^{n-1}\lambda_{i_{l-n}^{(1)}}u_{i_{-n}^{(1)}} - \prod_{l=1}^{n-1}\lambda_{j_{l-n}^{(1)}}u_{j_{-n}^{(1)}}\bigg)t^{n-1}\bigg| \\
&\qquad - \bigg|\sum_{n=1}^{\infty}\bigg(\prod_{l=1}^{n-1}\lambda_{i_{l-n}^{(2)}}u_{i_{-n}^{(2)}} - \prod_{l=1}^{n-1}\lambda_{j_{l-n}^{(2)}}u_{j_{-n}^{(2)}}\bigg)t^{n-1}\bigg|\bigg|.
\end{aligned}$$

If $\{i_n^{(1)}\}, \{i_n^{(2)}\} \in S_\alpha^p$, $\{j_n^{(1)}\}, \{j_n^{(2)}\} \in S_\beta^q$ for some p and q, then since $\{i_n^{(1)}\}$ and $\{i_n^{(2)}\}$, respectively $\{j_n^{(1)}\}$ and $\{j_n^{(2)}\}$, are equal on the first $L' + L + 3$ letters, the first $L' + L + 3$ terms in this power series are zero. Hence

$$\begin{aligned}
&|\,|\rho_t(\{i_n^{(1)}\}) - \rho_t(\{j_n^{(1)}\})|_1 - |\rho_t(\{i_n^{(2)}\}) - \rho_t(\{j_n^{(2)}\})|_1\,| \\
&\quad = \bigg|\bigg|\sum_{n=L+L'+4}^{\infty}\bigg(\prod_{l=1}^{n-1}\lambda_{i_{l-n}^{(1)}}u_{i_{-n}^{(1)}} - \prod_{l=1}^{n-1}\lambda_{j_{l-n}^{(1)}}u_{j_{-n}^{(1)}}\bigg)t^{n-1}\bigg| \\
&\qquad - \bigg|\sum_{n=L+L'+4}^{\infty}\bigg(\prod_{l=1}^{n-1}\lambda_{i_{l-n}^{(2)}}u_{i_{-n}^{(2)}} - \prod_{l=1}^{n-1}\lambda_{j_{l-n}^{(2)}}u_{j_{-n}^{(2)}}\bigg)t^{n-1}\bigg|\bigg| \\
&\quad \le \sum_{n=L+L'+4}^{\infty} 2\max\{u_i - u_j\}(\lambda_{\max}t)^{n-1} \\
&\quad \le \frac{2\max\{u_i - u_j\}(\lambda_{\max}t_1)^{L+L'+3}}{1-\lambda_{\max}t_1} < r. \qquad (17)
\end{aligned}$$

The last inequality follows from the choice of L' that

$$\frac{2\max\{u_i - u_j\}(t_1\lambda_{\max})^{L'+3}}{1 - t_1\lambda_{\max}} < r.$$

For each pair S_α^p and S_β^q, take $\{i_n^{(p)}\} \in S_\alpha^p$ and $\{j_n^{(q)}\} \in S_\beta^q$. Put

$$G_{S_\alpha^p, S_\beta^q}(t) = \chi_{\{|\rho_t(\{i_n^{(p)}\}) - \rho_t(\{j_n^{(q)}\})|_1 < 2r\}}.$$

Then the estimates (16) and (17) imply that

$$G_{S_\alpha^p, S_\beta^q}(t) \ge F(t, \{i_n\}, \{j_n\}) \quad \text{for all } t \in I_{k,l} \cap J_1,\ \text{for all } \{i_n\} \in S_\alpha^p,\ \text{for all } \{j_n\} \in S_\beta^q$$

and

$$\int_{I_{k,l}} G_{S_\alpha^p, S_\beta^q}(t)\,dt < 2E\lambda_{\max}(t_0\lambda_{\min})^{-L}r.$$

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Lemma 5.2 is used to estimate the integrals in (14) in the following way:

$$\begin{aligned}
\mathcal{I} :=& \int_{I_{k,l}\cap J_2} \int_{\hat{\Omega}_t} \int_{\rho_t^{-1}(\hat{\Omega}_t)\cap[l_1,i_{-L},\ldots,i_0]} \int_{\rho_t^{-1}(\hat{\Omega}_t)\cap[l_2,i_{-L},\ldots,i_0]} \\
& F_{L,i_{-L},\ldots,i_0}(t,\{i_n\},\{j_n\})\, d\mu_\Sigma^{t,s,\hat{x}}(\{j_n\})\, d\mu_\Sigma^{t,s,\hat{x}}(\{i_n\})\, d\hat{\mu}_{\mathrm{SRB}}^t(\hat{x})\, dt \\
=& \sum_{p,q} \int_{I_{k,l}\cap J_2} \int_{\hat{\Omega}_t} \int_{\rho_t^{-1}(\hat{\Omega}_t)\cap S_\alpha^p} \int_{\rho_t^{-1}(\hat{\Omega}_t)\cap S_\beta^q} \\
& F_{L,i_{-L},\ldots,i_0}(t,\{i_n\},\{j_n\})\, d\mu_\Sigma^{t,s,\hat{x}}(\{j_n\})\, d\mu_\Sigma^{t,s,\hat{x}}(\{i_n\})\, d\hat{\mu}_{\mathrm{SRB}}^t(\hat{x})\, dt \\
\le& \sum_{p,q} \int_{I_{k,l}\cap J_2} \int_{\hat{\Omega}_t} \int_{\rho_t^{-1}(\hat{\Omega}_t)\cap S_\alpha^p} \int_{\rho_t^{-1}(\hat{\Omega}_t)\cap S_\beta^q} \\
& G_{S_\alpha^p,S_\beta^q}(t)\, d\mu_\Sigma^{t,s,\hat{x}}(\{j_n\})\, d\mu_\Sigma^{t,s,\hat{x}}(\{i_n\})\, d\hat{\mu}_{\mathrm{SRB}}^t(\hat{x})\, dt \\
=& \sum_{p,q} \int_{I_{k,l}\cap J_2} \int_{\hat{\Omega}_t} G_{S_\alpha^p,S_\beta^q}(t) \\
& \mu_\Sigma^{t,s,\hat{x}}(\rho_t^{-1}(\hat{\Omega}_t)\cap S_\alpha^p)\mu_\Sigma^{t,s,\hat{x}}(\rho_t^{-1}(\hat{\Omega}_t)\cap S_\beta^q)\, d\hat{\mu}_{\mathrm{SRB}}^t(\hat{x})\, dt.
\end{aligned}$$

This can be estimated by

$$\begin{aligned}
\mathcal{I} \le& \max_{p,q}\bigg(\int_{I_{k,l}\cap J_1} G_{S_\alpha^p,S_\beta^q}(t)\, dt\bigg) \\
& \times \sup_t\bigg(\sum_{p,q}\int_{\hat{\Omega}_t} \mu_\Sigma^{t,s,\hat{x}}(\rho_t^{-1}(\hat{\Omega}_t)\cap S_\alpha^p)\mu_\Sigma^{t,s,\hat{x}}(\rho_t^{-1}(\hat{\Omega}_t)\cap S_\beta^q)\, d\hat{\mu}_{\mathrm{SRB}}^t(\hat{x})\bigg).
\end{aligned}$$

The maximum is estimated by (15) and the sum can be eliminated using that $\{S_\alpha^p\}$ and $\{S_\beta^q\}$ are partitions of the cylinders $[l_1, i_{-L}, \ldots, i_0]$ and $[l_1, i_{-L}, \ldots, i_0]$ respectively:

$$\begin{aligned}
\mathcal{I} \le& 2E\lambda_{\max}(t_0\lambda_{\min})^{-L} r \sup_t\bigg(\int_{\hat{\Omega}_t} \mu_\Sigma^{t,s,\hat{x}}(\rho_t^{-1}(\hat{\Omega}_t)\cap[l_1,i_{-L},\ldots,i_0]) \\
& \times \mu_\Sigma^{t,s,\hat{x}}(\rho_t^{-1}(\hat{\Omega}_t)\cap[l_2,i_{-L},\ldots,i_0])\, d\hat{\mu}_{\mathrm{SRB}}^t(\hat{x})\bigg).
\end{aligned}$$

The measure of the cylinders is estimated with (3):

$$\begin{aligned}
\mathcal{I} \le& 2E\lambda_{\max}(t_0\lambda_{\min})^{-L} r \sup_t\bigg(\int_{\hat{\Omega}_t} A_k^2(\gamma_{\min}-\varepsilon_2)^{-2(L+1)}\, d\hat{\mu}_{\mathrm{SRB}}^t(\hat{x})\bigg) \\
\le& 2E\lambda_{\max}(t_0\lambda_{\min})^{-L} r A_k^2(\gamma_{\min}-\varepsilon_2)^{-2(L+1)}.
\end{aligned}$$

Thus

$$\begin{aligned}
&T_r(\{\hat{\Omega}_t \mid t\in I_{k,l}\cap J_2\}) \\
&\quad\le \sum_{L=L_2}^{\infty}\sum_{\underline{i}}\sum_{l=1}^{m}\sum_{\substack{1\le l_1,l_2\le a\\ l_1\ne l_2}} 2Er(t_0\lambda_{\min})^{-L}A_k^2(\gamma_{\min}-\varepsilon_2)^{-2(L+1)} \\
&\quad\le \sum_{L=L_2}^{\infty} 2ErB_{k,l}(\gamma_{\max}+\varepsilon_2+\varepsilon_4)^L m(a^2-a)(t_0\lambda_{\min})^{-L}A_k^2(\gamma_{\min}-\varepsilon_2)^{-2L} \\
&\quad= \sum_{L=L_2}^{\infty} 2EA_k^2B_{k,l}m(a^2-a)r\bigg(\frac{\gamma_{\max}+\varepsilon_2+\varepsilon_4}{t_0\lambda_{\min}(\gamma_{\min}-\varepsilon_2)^2}\bigg)^L,
\end{aligned}$$

where $B_{k,l}$ is defined in Lemma 5.1.

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If $t_0\lambda_{\min}\gamma_{\min}^2/\gamma_{\max} > 1$ then it is possible to choose ε_2 and ε_4 so small that

$$\tau = \frac{\gamma_{\max} + \varepsilon_2 + \varepsilon_4}{t_0\lambda_{\min}(\gamma_{\min} - \varepsilon_2)^2} < 1.$$

Then $T_r \leq 2EA_k^2 B_{k,l}(a^2 - a)\tau^{L_2}[1/(1-\tau)]r$. This implies that μ_{SRB}^t is absolutely continuous with respect to Lebesgue measure for a.e. $t \in I \cap J_2$. Let $\varepsilon_0, \varepsilon_1, \varepsilon_3 \to 0$. Then $\nu(I \cap J_2) > \nu(I) - \varepsilon_0 - \varepsilon_1 - \varepsilon_3 \to \nu(I)$ and this shows that $\hat{\mu}_{\mathrm{SRB}}^t$ is absolutely continuous with respect to Lebesgue measure for a.e. $t \in I_k$.

6. *Application to fat Belykh maps*

The Belykh maps are defined as follows. Let $-1 < k < 1$ and put $K = [-1, 1]^2$, $K_1 = \{(x_1, x_2) \in (-1, 1)^2 \mid x_2 > kx_1\}$ and $K_2 = \{(x_1, x_2) \in (-1, 1)^2 \mid x_2 < kx_1\}$. The discontinuity set is $N = \{(x_1, x_2) \mid x_2 = kx_1\}$ and the border is $M = \{(x_1, x_2) \in K \mid x_1 = \pm 1 \text{ or } x_2 = \pm 1\}$. The Belykh maps $f_{\lambda,\gamma,k} \colon K_1 \cup K_2 \to K$ are defined by

$$f_{\lambda,\gamma,k}|_{K_1}(x_1, x_2) = (\lambda x_1 + (1-\lambda), \gamma x_2 - (\gamma - 1)),$$
$$f_{\lambda,\gamma,k}|_{K_2}(x_1, x_2) = (\lambda x_1 - (1-\lambda), \gamma x_2 + (\gamma - 1)),$$

where $0 < \lambda < 1$ and $1 < \gamma \leq 2/(1 + |k|)$.

These maps were first introduced for $\lambda < 1/2$ by Belykh in [**2**] as a model of a Poincaré map from phase synchronization. It was investigated in the case $\lambda < 1/2$ in [**12**] and [**13**]. Schmeling and Troubetzkoy studied in [**15**] the Belykh maps when $\lambda > 1/2$ and called this case the fat Belykh map.

The Belykh maps satisfy the conditions (A1) and (A2). Here

$$C = 1 \text{ and } C < f_4(x) \text{ if } x < Q_4 = 0.61$$

and we conclude.

THEOREM 6.1. *Let* $P = \{(\lambda, \gamma, k) \mid \gamma\lambda > 1, \lambda < 0.61\}$. *For Lebesgue almost all* $(\gamma, \lambda, k) \in P$ *the fat Belykh map* $f_{\lambda,\gamma,k}$ *has an absolutely continuous invariant measure.*

7. *Decay of correlations*

Applying Young's scheme from [**21**], Chernov proved in [**6**] the exponential decay of correlations for Hölder continuous functions for a class of piecewise hyperbolic systems with singularities in arbitrary dimensions. This result can be used in the following way.

Let $H_\eta = \{\phi \colon K \to \mathbb{R} \mid \exists\, C : |\phi(x) - \phi(y)| \leq Cd(x, y)^\eta, \forall\, x, y \in K\}$ be the set of Hölder continuous functions on K.

COROLLARY 7.1. *Assume that* $f \colon K \to K$ *satisfies the assumptions (A1) and (A2) and assume that* $(f^n, \mu_{\mathrm{SRB}})$ *is ergodic for every* $n \geq 1$, *where* μ_{SRB} *is the* SRB *measure. For every* $\eta > 0$ *there exists a constant* $\theta \in (0, 1)$ *such that for every* $\phi, \psi \in H_\eta$ *there is a constant* $C(\phi, \psi)$ *such that*

$$\left| \int_K \phi \circ f^n \cdot \psi \, d\mu_{\mathrm{SRB}} - \int_K \phi \, d\mu_{\mathrm{SRB}} \int_K \psi \, d\mu_{\mathrm{SRB}} \right| \leq C\theta^n$$

for all $n \in \mathbb{N}$.

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Proof. Lift $f: K \to K$ to $\hat{f}: \hat{K} \to \hat{K}$ as in §5.1. The lift $\hat{f}$ is just a model for the natural extension of f. The natural extension is ergodic if and only if the original system is ergodic; see [**7**, Theorem 1 in Chapter 10, §4]. Hence $(\hat{f}^n, \hat{\mu}_{\mathrm{SRB}})$ is ergodic for every $n \geq 1$.

The function $\phi: K \to \mathbb{R}$ is lifted to $\hat{\phi}: \hat{K} \to \mathbb{R}$ by $\hat{\phi} = \phi \circ \pi$ and $\hat{\psi}$ is lifted in the same way.

Note that $\hat{\phi}, \hat{\psi} \in \hat{H}_\eta = \{\hat{\phi}: \hat{K} \to \mathbb{R} \mid \exists\, C : |\hat{\phi}(x) - \hat{\phi}(y)| \leq C d(x, y)^\eta, \forall\, x, y \in \hat{K}\}$ and

$$\int_K \phi \circ f^n \cdot \psi \, d\mu_{\mathrm{SRB}} = \int_{\hat{K}} \hat{\phi} \circ \hat{f}^n \cdot \hat{\psi} \, d\hat{\mu}_{\mathrm{SRB}},$$

$$\int_K \phi \, d\mu_{\mathrm{SRB}} = \int_{\hat{K}} \hat{\phi} \, d\hat{\mu}_{\mathrm{SRB}}, \qquad \int_K \psi \, d\mu_{\mathrm{SRB}} = \int_{\hat{K}} \hat{\psi} \, d\hat{\mu}_{\mathrm{SRB}}.$$

Theorem 1.1 in [**6**] states that $(\hat{f}, \hat{\mu}_{\mathrm{SRB}})$ has exponential decay of correlations so this implies that (f, μ_{SRB}) has exponential decay of correlations.

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