Global Euler obstruction, global Brasselet numbers and critical points

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Let $X \subset \mathbb{C}^n$ be an equidimensional complex algebraic set and let $f : X \to \mathbb{C}$ be a polynomial function. For each $c \in \mathbb{C}$, we define the global Brasselet number of f at c, a global counterpart of the Brasselet number defined by the authors in a previous work, and the Brasselet number at infinity of f at c. Then we establish several formulas relating these numbers to the topology of X and the critical points of f.

1. Introduction

The local Euler obstruction is an invariant defined by MacPherson in [**22**] as one of the main ingredients in his proof of the Deligne-Grothendieck conjecture on the existence of Chern classes for singular varieties. The local Euler obstruction at $0 \in X$, where X is a sufficiently small representative of the equidimensional analytic germ $(X, 0)$, is denoted by $\mathrm{Eu}_X(0)$. After MacPhersons pioneer work, the Euler obstruction has been studied by many authors. Let us mention briefly some of the most important results on this subject. If $\mathcal{V} = \{V_i\}_{i=1}^t$ is a Whitney stratification of X, then Brylinski, Dubson and Kashiwara [**6**] proved a famous formula that relates the $\mathrm{Eu}_{\overline{V_i}}$'s to the Euler characteristic of the normal links of the strata. In [**19**], Lê and Teissier showed that $\mathrm{Eu}_X(0)$ is equal to an alternated sum of multiplicities of generic polar varieties of X at 0. In [**3**], Brasselet, Lê and Seade proved a Lefschetz type formula for $\mathrm{Eu}_X(0)$, i.e. they relate $\mathrm{Eu}_X(0)$ to the topology of the Milnor fibre on X of a generic linear function. There are also integral formulas for $\mathrm{Eu}_X(0)$ in [**11**, **21**].

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In [**4**], Brasselet, Massey, Parameswaran and Seade defined a new invariant, introducing information for a function f defined on the variety X, called the Euler obstruction of the function and denoted by $\mathrm{Eu}_{f,X}(0)$. They prove a Lefschetz type formula for this invariant. The Euler obstruction of a function can be seen as a generalization of the Milnor number ([**4**, **15**, **29**]). For instance, in [**29**], Seade, Tibăr and Verjovsky showed that $\mathrm{Eu}_{f,X}(0)$ is equal up to sign to the number of critical points of a Morsefication of f lying on the regular part of X.

In [**12**], in order to study topological properties of functions defined on analytic complex varieties, we defined an invariant called the Brasselet number, denoted by $\mathrm{B}_{f,X}(0)$, and we proved that it satisfies a relative version of the multiplicity formula of Lê and Teissier. We also proved a relative version of the Brylinski-Dubson-Kashiwara formula which involves this invariant and moreover, we established an integral formula for $\mathrm{B}_{f,X}(0)$, which is a relative version of the integral formula for $\mathrm{Eu}_X(0)$ proved in [**11**]. When f has isolated singularity, we have $\mathrm{B}_{f,X}(0) = \mathrm{Eu}_X(0) - \mathrm{Eu}_{f,X}(0)$ and in the general case, $\mathrm{B}_{f,X}(0) = \mathrm{Eu}_X(0) - \mathrm{D}_{f,X}(0)$, where $\mathrm{D}_{f,X}(0)$ is the defect defined and studied in §6 of [**4**] (see remark 2.18 below). The defect is related to the Lê numbers (proposition 6.1 in [**4**]) when X is smooth and to the Lê-Vogel numbers (theorem 6.2 in [**4**]) in the general case. Therefore, in view of these relations and the result of Seade, Tibăr and Verjovsky mentioned above, the defect $\mathrm{D}_{f,X}(0)$ appears to be a good singular version of the Milnor number, whereas the Brasselet number appears to be a good relative version of the Euler obstruction.

In a manner similar to the local case, in [**28**], working with an affine equidimensional singular variety $X \subset \mathbb{C}^N$, Seade, Tibăr and Verjovsky defined the global Euler obstruction, denoted by $\mathrm{Eu}(X)$. When X is smooth the global Euler obstruction of X coincides with the Euler characteristic of X. They prove a global version of the Lê-Teissier polar multiplicities formula. Later, this formula was generalized in [**27**] to an index formula for MacPherson cycles.

As the Euler obstruction of a function and the Brasselet number are useful to study the singularities of f in the local case, we introduce in this work the global Brasselet numbers and the Brasselet numbers at infinity, in order to investigate the topological behaviour of the singularities, globally and at infinity, of a given polynomial function f defined on an algebraic variety $X \subset \mathbb{C}^N$. The main references we use in this paper about the study of singularities at infinity are [**7**, **8**, **37**].

In §2 we give prerequisities on the topology of complex algebraic sets: stratified Morse functions, the complex link and the normal Morse datum, constructible functions, the local Euler obstruction and the Brasselet number, the global Euler obstruction. In §3 we recall the notions of t-regularity at infinity and ρ-regularity at infinity, some basic results and we adapt them to the stratified setting.

In §4 we define the global Brasselet numbers and the Brasselet numbers at infinity. We compare the global Brasselets number of f with the global Euler obstruction of the fibres of f. The relation presented in corollary 4.9 can be seen as a global relative version of the local index formula of Brylinsky, Dubson and Kashiwara [**6**].

In §5 we prove several formulas that relate the number of critical points of a Morsefication of a polynomial function f on an algebraic set X, to the global Brasselet numbers and the Brasselet numbers at infinity of f. The main result in this section is theorem 5.2. From this result, we obtain many interesting corollaires.

Corollary 5.7, for instance, is a Brylinski–Dubson–Kashiwara type formula for the total Brasselet number at infinity. We also prove a relative version of the polar multiplicity formula of Seade, Tibăr and Verjovsky (corollary 5.15).

Theorem 5.2 is proved via integration with respect to the Euler characteristic. This result is interesting for two main reasons. Firstly it provides a stratified version of results about the topology of fibres of polynomial functions obtained in the 80s and 90s (see for instance [**41**] for a detailed account of this topics). Moreover, using the machinery of constructible functions, it enables us to improve the results of Seade, Tibăr and Verjovsky [**28**] in different directions: the polynomial function f can have singularities at infinity and it is not supposed to admit only stratified Morse critical points. We also obtain equalities for all fibres of f, not only for the generic ones.

2. Prerequisites on the topology of complex algebraic sets

In this section, we work with a reduced complex algebraic set $X \subset \mathbb{C}^n$ of dimension d. We assume that X is equipped with a finite Whitney stratification $\mathcal{V}$ whose strata are connected. We denote by X_{reg} the regular part of X, i.e. the union of all the strata of dimension d.

2.1. Stratified Morse functions

The main reference for this subject is [**14**].

DEFINITION 2.1 ([**14**], p. 44). Let x be a point in X and let V_b be the stratum that contains it. A degenerate tangent plane of the stratification $\mathcal{V}$ is an element T (of an appropriate Grassmannian) such that $T = \lim_{x_i \to x} T_{x_i} V_a$, where V_a is a stratum that contains V_b in its frontier and where the x_i's belong to V_a.

DEFINITION 2.2 ([**14**], p. 44). A degenerate covector of $\mathcal{V}$ at a point $x \in X$ is a covector which vanishes on a degenerate tangent plane of $\mathcal{V}$ at x, i.e., an element η of $T_x^*\mathbb{C}^n$ such that there exists a degenerate tangent plane T of the stratification at x with $\eta(T) = 0$.

Let $f : X \to \mathbb{C}$ be an analytic function. We assume that f is the restriction to X of an analytic function $F : \mathbb{C}^n \to \mathbb{C}$, i.e. $f = F_{|X}$. A point x in X is a critical point of f if it is a critical point of $F_{|V(x)}$, where $V(x)$ is the stratum containing x.

DEFINITION 2.3 ([**14**], p. 52). Let x be a critical point of f. We say that f is general at x with respect to the stratification $\mathcal{V}$ if $DF(q)$ is not a degenerate covector of $\mathcal{V}$ at x.

We say that f is general with respect to $\mathcal{V}$ if it is general at all critical points with respect to $\mathcal{V}$.

DEFINITION 2.4 ([**14**], p. 52). Let x be a critical point of f. We say that x is a stratified Morse critical point of f if f is general at x and the function $f_{|V(x)} : V(x) \to \mathbb{C}$ has a non-degenerate critical point at x when $\dim V(x) > 0$.

We say that that f is a stratified Morse function if it admits only stratified Morse critical points.

2.2. The complex link and the normal Morse datum

The complex link is an important object in the study of the topology of complex analytic sets. It is analogous to the Milnor fibre and was studied first in [**17**]. It plays a crucial role in complex stratified Morse theory (see [**14**]) and appears in general bouquet theorems for the Milnor fibre of a function with isolated singularity (see [**18**, **30**, **35**]).

Let V be a stratum of the stratification $\mathcal{V}$ of X and let x be a point in V. Let $g : (\mathbb{C}^n, x) \to (\mathbb{C}, 0)$ be an analytic complex function-germ such that the differential form $Dg(x)$ is not a degenerate covector of $\mathcal{V}$ at x. Let $N^{\mathbb{C}}_{x,V}$ be a normal slice to V at x, i.e. $N^{\mathbb{C}}_{x,V}$ is a closed complex submanifold of $\mathbb{C}^n$ which is transversal to V at x and $N^{\mathbb{C}}_{x,V} \cap V = \{x\}$.

DEFINITION 2.5 ([**14**], p. 161). The complex link $\mathcal{L}^X_V$ of V is defined by

$$\mathcal{L}^X_V = X \cap N^{\mathbb{C}}_{x,V} \cap B_\epsilon(x) \cap \{g = \delta\},$$

where $0 < |\delta| \ll \epsilon \ll 1$. Here $B_\epsilon(x)$ is the closed ball of radius ϵ centred at x.

The normal Morse datum $\mathrm{NMD}(V)$ of V is the pair of spaces

$$\mathrm{NMD}(V) = \left(X \cap N^{\mathbb{C}}_{x,V} \cap B_\epsilon(x), X \cap N^{\mathbb{C}}_{x,V} \cap B_\epsilon(x) \cap \{g = \delta\} \right).$$

The fact that these two notions are well-defined, i.e. independent of all the choices made to define them, is explained in [**14**].

2.3. Constructible functions

We start with a presentation of Viro's method of integration with respect to the Euler characteristic with compact support [**42**]. We work in a semi-algebraic setting.

DEFINITION 2.6. Let $Y \subset \mathbb{R}^n$ be a semi-algebraic set. A constructible function $\alpha : Y \to \mathbb{Z}$ is a $\mathbb{Z}$-valued function that can be written as a finite sum:

$$\alpha = \sum_{i \in I} m_i \mathbf{1}_{Y_i},$$

where Y_i is a semi-algebraic subset of Y and $\mathbf{1}_{Y_i}$ is the characteristic function on Y_i.

The sum and the product of two constructible functions on Y are again constructible. The set of constructible functions on Y is thus a commutative ring, denoted by $F(Y)$.

DEFINITION 2.7. If $\alpha \in F(Y)$ and $W \subset Y$ is a semi-algebraic set then the Euler characteristic $\chi(W, \alpha)$ is defined by

$$\chi(W, \alpha) = \sum_{i \in I} m_i \chi_c(W \cap Y_i),$$

where $\alpha = \sum_{i \in I} m_i \mathbf{1}_{Y_i}$ and χ_c is the Euler characteristic of Borel-Moore homology.

The Euler characteristic $\chi(W, \alpha)$ is also called the Euler integral of α and denoted by $\int_W \alpha d\chi_c$. Here we follow the terminology and notations used in [**4**, **12**, **27**].

DEFINITION 2.8. Let $f : Y \to Z$ be a continuous semi-algebraic map and let $\alpha : Y \to \mathbb{Z}$ be a constructible function. The pushforward $f_* \alpha$ of α along f is the function $f_* \alpha : Z \to \mathbb{Z}$ defined by:

$$f_* \alpha(z) = \chi(f^{-1}(z), \alpha).$$

PROPOSITION 2.9. *The pushforward of a constructible function is a constructible function.*

THEOREM 2.10 (Fubini's theorem, [**42**], p. 128). *Let $f : Y \to Z$ be a continuous semi-algebraic map and let α be a constructible function on Y. Then we have:*

$$\chi(Z, f_* \alpha) = \chi(Y, \alpha).$$

Let us go back to the complex situation. Here we write $\mathcal{V} = \{V_1, \ldots, V_t\}$ for the Whitney stratification of X.

DEFINITION 2.11. A constructible function with respect to the stratification $\mathcal{V}$ of X is a function $\alpha : X \to \mathbb{Z}$ which is constant on each stratum V of the stratification.

This means that there exist integers n_i, $i \in \{1, \ldots, t\}$, such that $\alpha = \sum_{i=1}^{t} n_i \cdot \mathbf{1}_{V_i}$. In most of the cases that we will consider, we can use the topological Euler characteristic χ instead of χ_c. First since each V_i is an even-dimensional submanifold, by Poincaré duality $\chi_c(V_i)$ is equal to $\chi(V_i)$ and so $\chi(X, \alpha) = \sum_{i=1}^{t} n_i \chi(V_i)$. Now let $B \subset \mathbb{C}^n$ be a euclidian closed ball that intersects X transversally (in the stratified sense). We will give four equalities for $\chi(X \cap B, \alpha)$. By additivity of χ_c, we have

$$\chi(X \cap B, \alpha) = \chi(X \cap \mathring{B}, \alpha) + \chi(X \cap \partial B, \alpha).$$

But $X \cap \partial B$ is Whitney stratified by odd-dimensional strata and so $\chi(X \cap \partial B) = 0$ (see lemma 5.0.3 in [**26**] or proposition 1.6 in [**24**]). Therefore, we have

$$\chi(X \cap B, \alpha) = \chi(X \cap \mathring{B}, \alpha) = \sum_{i=1}^{t} n_i \chi_c(V_i \cap \mathring{B}),$$

and by Poincaré duality,

$$\chi(X \cap B, \alpha) = \sum_{i=1}^{t} n_i \chi(V_i \cap \mathring{B}).$$

But each $V_i \cap B$ is a manifold with boundary, so $\chi(V_i \cap \mathring{B}) = \chi(V_i \cap B)$ and

$$\chi(X \cap B, \alpha) = \sum_{i=1}^{t} n_i \chi(V_i \cap B).$$

Similarly, if $E = \mathbb{C}^n \setminus \mathring{B}$ then

$$\chi(X \cap E, \alpha) = \sum_{i=1}^{t} n_i \chi_c(V_i \cap E) = \sum_{i=1}^{t} n_i \chi_c(V_i \cap \mathring{E})$$

$$= \sum_{i=1}^{t} n_i \chi(V_i \cap \mathring{E}) = \sum_{i=1}^{t} n_i \chi(V_i \cap E).$$

If the radius of B is sufficiently big, then $X \cap \partial B$ is homeomorphic to the link at infinity of X, denoted by $\mathrm{Lk}^\infty(X)$, and $X \cap B$ is a retract by deformation of X which implies that $\chi(X) = \chi(X \cap B)$. Since $X \cap B$ is compact, $\chi(X \cap B) = \chi_c (X \cap B)$ and so, by additivity, $\chi_c(X) = \chi(X) + \chi_c(X \cap \mathring{E})$. But $X \cap \mathring{E}$ is homeomorphic to the product of $\mathrm{Lk}^\infty(X)$ and an open interval in $\mathbb{R}$. Since $\chi_c(\mathrm{Lk}^\infty(X)) = 0$, by multiplicativity of χ_c we obtain that $\chi_c(X \cap \mathring{E}) = 0$ and finally that $\chi(X) = \chi_c(X)$.

DEFINITION 2.12 ([**27**], definition 2.2). Let $\alpha : X \to Z$ be a constructible function with respect to the stratification $\mathcal{V}$. Its normal Morse index $\eta(V, \alpha)$ along V is defined by

$$\eta(V, \alpha) = \chi(\mathrm{NMD}(V), \alpha) = \chi(X \cap N_{x,V}^{\mathbb{C}} \cap B_\epsilon(x), \alpha) - \chi(\mathcal{L}_V^X, \alpha),$$

where x is a point in V.

If $Z \subset X$ is a closed union of strata, then $\eta(V, \mathbf{1}_Z) = 1 - \chi(\mathcal{L}_V \cap Z)$.

2.4. The local Euler obstruction and the Brasselet number

Here we assume that X is equidimensional. The Euler obstruction at $x \in X$, denoted by $\mathrm{Eu}_X(x)$, was defined by MacPherson, using 1-forms and the Nash blow-up (see [**22**] for the original definition). An equivalent definition of the Euler obstruction was given by Brasselet and Schwartz in the context of vector fields [**2**]. Roughly speaking, $\mathrm{Eu}_X(x)$ is the obstruction for extending a continuous stratified radial vector field around x in X to a non-zero section of the Nash bundle over the Nash blow-up of X.

The Euler obstruction is a constructible function and there are two distinguished bases for the free abelian group of constructible functions: the characteristic functions $\mathbf{1}_{\overline{V}}$ and the Euler obstruction $\mathrm{Eu}_{\overline{V}}$ of the closure $\overline{V}$ of all strata V. Moreover, the key role of the Euler obstruction comes from the following identities (see [**27**] p. 34 or [**26**] p. 292 and p. 323–324):

$$\eta(V', \mathrm{Eu}_{\overline{V}}) = 1 \text{ if } V' = V,$$

and:

$$\eta(V', \mathrm{Eu}_{\overline{V}}) = 0 \text{ if } V' \neq V.$$

In [**3**], Brasselet, Lê and Seade study the Euler obstruction using hyperplane sections, following ideas of Dubson and Kato. Let us assume that 0 belongs to X.

THEOREM 2.13 ([**3**], theorem 3.1). *For each generic linear form l, there is ϵ_0 such that for any ϵ with $0 < \epsilon < \epsilon_0$, the Euler obstruction of $(X, 0)$ is equal to:*

$$\mathrm{Eu}_X(0) = \chi\big(X \cap B_\epsilon(0) \cap l^{-1}(\delta), \mathrm{Eu}_X\big),$$

where $0 < |\delta| \ll \epsilon \ll 1$.

Let $f : X \to \mathbb{C}$ be a holomorphic function. We assume that f has an isolated singularity (or an isolated critical point) at 0, i.e. that f has no critical point in a punctured neighbourhood of 0 in X.

In [**4**] Brasselet, Massey, Parameswaran and Seade introduced an invariant which measures, in a way, how far the equality given in theorem 2.13 is from being true if we replace the generic linear form l with some other function on X with at most an isolated stratified critical point at 0. This number is called the Euler obstruction of a function and denoted by $\mathrm{Eu}_{f,X}(0)$. The following result is the Brasselet, Massey, Parameswaran and Seade formula [**4**] that compares, in the same point, the local Euler obstruction with the Euler obstruction of a function.

THEOREM 2.14 ([**4**], theorem 3.1). *Let $f : X \to \mathbb{C}$ be a function with an isolated singularity at 0. For $0 < |\delta| \ll \varepsilon \ll 1$ we have:*

$$\mathrm{Eu}_X(0) - \mathrm{Eu}_{f,X}(0) = \chi\big(X \cap B_\epsilon(0) \cap f^{-1}(\delta), \mathrm{Eu}_X\big),$$

where $0 < |\delta| \ll \epsilon \ll 1$.

In [**29**], J. Seade, Tibăr and Verjovsky show that the Euler obstruction of f is closely related to the number of Morse points of a Morsefication of f, as it is stated in the next proposition.

PROPOSITION 2.15 ([**29**], proposition 2.3). *Let $f : X \to \mathbb{C}$ be an analytic function with isolated singularity at the origin. Then:*

$$\mathrm{Eu}_{f,X}(0) = (-1)^d n_{\mathrm{reg}},$$

where n_{reg} is the number of Morse points on X_{reg} in a stratified Morsefication of f lying in a small neighbourhood of 0.

DEFINITION 2.16 ([**23**], p. 971). A good stratification of X relative to f is a stratification $\mathcal{V}$ of X which is adapted to X^f, (i.e. X^f is a union of strata), where $X^f = X \cap f^{-1}(0)$, such that $\{V_i \in \mathcal{V}; V_i \not\subset X^f\}$ is a Whitney stratification of $X \setminus X^f$ and such that for any pair of strata (V_a, V_b) such that $V_a \not\subset X^f$ and $V_b \subset X^f$, the (a_f)-Thom condition is satisfied.

Let us now recall the definition of the Brasselet number, defined in [**12**].

DEFINITION 2.17 ([**12**], definition 3.18). Let $\mathcal{V}$ be a good stratification of X relative to f. We define $\mathrm{B}_{f,X}(0)$ by:

$$\mathrm{B}_{f,X}(0) = \chi\big(X \cap B_\epsilon(0) \cap f^{-1}(\delta), \mathrm{Eu}_X\big),$$

where $0 < |\delta| \ll \epsilon \ll 1$.

REMARK 2.18. Note that if f has a stratified isolated singularity at the origin then, by theorem 2.14, we have that $\mathrm{B}_{f,X}(0) = \mathrm{Eu}_X(0) - \mathrm{Eu}_{f,X}(0)$. Moreover in the general case, $\mathrm{B}_{f,X}(0) = \mathrm{Eu}_X(0) - \mathrm{D}_{f,X}(0)$, where $\mathrm{D}_{f,X}(0)$ is the defect defined in §6 of [**4**]. Indeed this defect is defined by

$$\mathrm{D}_{f,X}(0) = \mathrm{Eu}_X(0) - \sum_i \chi\big(V_i \cap B_\epsilon(0) \cap f^{-1}(\delta)\big)\mathrm{Eu}_X(V_i),$$

where $\{V_i\}$ is a Whitney stratification of X. But $\sum_i \chi\big(V_i \cap B_\epsilon(0) \cap f^{-1}(\delta)\big)\mathrm{Eu}_X(V_i)$ is exactly $\chi(X \cap B_\epsilon(0) \cap f^{-1}(\delta), \mathrm{Eu}_X)$, since by additivity of the Euler characteristic, the Euler integral of a constructible function does not depend on the stratification used for its computation.

2.5. Global Euler obstruction

Here we assume that $X \subset \mathbb{C}^n$ is a reduced equidimensional algebraic set of dimension d, equipped with a finite Whitney stratification $\mathcal{V} = \{V_1, \ldots, V_t\}$. In [**28**], Seade, Tibăr and Verjovsky introduced a global analogous of the Euler obstruction called the global Euler obstruction and denoted by $\mathrm{Eu}(X)$. Let $\widetilde{X} \xrightarrow{\nu} X$ denote the Nash modification of X, and let us consider a stratified real vector field v on a subset $V \subset X$: this means that the vector field is continuous and tangent to the strata. The restriction of v to V has a well-defined canonical lifting $\widetilde{v}$ to $\nu^{-1}(V)$ as a section of the real bundle underlying the Nash bundle $\widetilde{T} \to \widetilde{X}$.

DEFINITION 2.19 ([**28**], definition 2.1). We say that the stratified vector field v on X is radial-at-infinity if it is defined on the restriction to X of the complement of a sufficiently large ball B_M centred at the origin of $\mathbb{C}^N$, and it is transversal to S_R, pointing outwards, for any $R > M$. In particular, v is without zeros on $X \setminus B_M$.

The 'sufficiently large' radius M is furnished by the following well-known result.

LEMMA 2.20 ([**28**], lemma 2.2). *There exists $M \in \mathbb{R}$ such that, for any $R \geqslant M$, the sphere S_R centred at the origin of $\mathbb{C}^N$ and of radius R is stratified transversal to X, i.e. transversal to all strata of the stratification $\mathcal{V}$.*

Using this last lemma and inspired by [**2**, **9**], Seade, Tibăr and Verjovsky defined the global Euler obstruction in [**28**] as follows:

DEFINITION 2.21 ([**28**], definition 2.3). Let $\tilde{v}$ be the lifting to a section of the Nash bundle $\widetilde{T}$ of a radial-at-infinity stratified vector field v over $X \setminus B_R$. We call global Euler obstruction of X, and denote it by $\mathrm{Eu}(X)$, the obstruction for extending $\tilde{v}$ as a nowhere zero section of $\widetilde{T}$ within $\nu^{-1}(X \cap B_R)$.

To be precise, the obstruction to extend $\tilde{v}$ as a nowhere zero section of $\widetilde{T}$ within $\nu^{-1}(X \cap B_R)$ is in fact a relative cohomology class

$$o(\tilde{v}) \in H^{2d}(\nu^{-1}(X \cap B_R), \nu^{-1}(X \cap S_R)) \simeq H_c^{2d}(\widetilde{X}).$$

The global Euler obstruction of X is the evaluation of $o(\tilde{v})$ on the fundamental class of the pair $(\nu^{-1}(X \cap B_R), \nu^{-1}(X \cap S_R))$. Thus $\mathrm{Eu}(X)$ is an integer and does

not depend on the radius of the sphere defining the link at infinity of X. Since two radial-at-infinity vector fields are homotopic as stratified vector fields, it does not depend on the choice of v either.

REMARK 2.22. The global Euler obstruction has the following properties (see [**28**] p. 396):

1. if X is non-singular, then $\mathrm{Eu}(X) = \chi(X)$,

2. $\mathrm{Eu}(X) = \chi(X, \mathrm{Eu}_X)$.

3. Regularity conditions at infinity

A natural question is if the concepts of the Euler obstruction and the Brasselet number of a function could be extended to the global setting, as Seade, Tibăr and Verjovsky did for the local Euler obstruction, and what kind of information we could obtain with these possible new global invariants.

But, before extending the local notions of the Euler obstruction and Brasselet number of a function, we recall in this section some definitions and results about the study of singularities at infinity and we adapted some results to the stratified setting. The main definitions and results first appeared in [**31**, **37**] in the case of functions and were extended for maps in [**8**]. We refer to these papers and also to [**7**] for details.

We consider $X \subset \mathbb{C}^N$ a reduced algebraic set of dimension d. We use coordinates $(x_1, \ldots, x_N)$ for the space $\mathbb{C}^N$ and coordinates $[x_0 : x_1 : \cdots : x_N]$ for the projective space $\mathbb{P}^N$. We consider the algebraic closure $\overline{X}$ of X in the complex projective space $\mathbb{P}^N$ and we denote by

$$H^\infty = \big\{ [x_0 : x_1 : \cdots : x_N] \mid x_0 = 0 \big\},$$

the hyperplane at infinity of the embedding $\mathbb{C}^N \subset \mathbb{P}^N$.

One may endow $\overline{X}$ with a semi-algebraic Whitney stratification $\mathcal{W}$ such that X_{reg} is a stratum and the part at infinity $\overline{X} \cap H^\infty$ is a union of strata.

Since $\overline{X}$ is projective and since the stratification of $\overline{X}$ is locally finite, it follows that $\mathcal{W}$ has finitely many strata. We denote by X_{sing} the set of singular points of X, i.e. $X_{\mathrm{sing}} = X \setminus X_{\mathrm{reg}}$.

In order to recall the definition of the t-regularity, let us recall first the definition of the conormal spaces.

DEFINITION 3.1 ([**8**], definition 2.1). We denote by $C(X)$ the conormal modification of X, defined as:

$$C(X) = \mathrm{closure}\big\{ (x, H) \in X_{\mathrm{reg}} \times \check{\mathbb{P}}^{N-1} \mid T_x X_{\mathrm{reg}} \subset H \big\} \subset \overline{X} \times \check{\mathbb{P}}^{N-1}.$$

Let $\pi : C(X) \to \overline{X}$ denote the projection $\pi(x, H) = x$.

DEFINITION 3.2 ([**8**], definition 2.2). Let $g : X \to \mathbb{C}$ be an analytic function defined in some neighbourhood of X in $\mathbb{C}^N$. Let X_0 denote the subset of X_{reg} where g is a

submersion. The relative conormal space of g is defined as follows:

$$C_g(X) = \text{closure}\Big\{(x, H) \in X_0 \times \check{\mathbb{P}}^{N-1} \mid T_x g^{-1}\left((g(x)) \subset H\right\} \subset \overline{X} \times \check{\mathbb{P}}^{N-1},$$

together with the projection $\pi : C_g(X) \to \overline{X}$, $\pi(x, H) = x$.

Let $f : X \to \mathbb{C}$ be a function such that $f = F_{|X}$, where $F : \mathbb{C}^N \to \mathbb{C}$ is a polynomial function.

Let $\mathbb{X} = \overline{\text{graph} f}$ be the closure of the graph of f in $\mathbb{P}^N \times \mathbb{C}$ and let $\mathbb{X}^\infty = \mathbb{X} \cap (H^\infty \times \mathbb{C})$. One has the isomorphism $\text{graph} f \simeq X$.

We consider the affine charts $U_j \times \mathbb{C}$ of $\mathbb{P}^N \times \mathbb{C}$, where

$$U_j = \{[x_0 : \cdots : x_N] \mid x_j \neq 0\}, \quad j = 0, 1, \dots, N.$$

Identifying the chart U_0 with the affine space $\mathbb{C}^N$, we have $\mathbb{X} \cap (U_0 \times \mathbb{C}) = \mathbb{X} \setminus \mathbb{X}^\infty = \text{graph} f$, and $\mathbb{X}^\infty$ is covered by the charts $U_1 \times \mathbb{C}, \dots, U_N \times \mathbb{C}$.

If g denotes the projection to the variable x_0 in some affine chart $U_j \times \mathbb{C}$, then the relative conormal $C_g(\mathbb{X} \setminus \mathbb{X}^\infty \cap U_j \times \mathbb{C}) \subset \mathbb{X} \times \check{\mathbb{P}}^N$ is well defined.

With the projection $\pi(y, H) = y$, let us then consider the space $\pi^{-1}(\mathbb{X}^\infty)$, which is well defined for every chart $U_j \times \mathbb{C}$ as a subset of $C_g(\mathbb{X} \setminus \mathbb{X}^\infty \cap U_j \times \mathbb{C})$.

DEFINITION 3.3 ([**8**], definition 2.4). We call space of characteristic covectors at infinity the set $C^\infty = \pi^{-1}(\mathbb{X}^\infty)$. For some $p_0 \in \mathbb{X}^\infty$, we denote $C^\infty_{p_0} := \pi^{-1}(p_0)$.

By lemma 2.8 in [**37**], these notions are well-defined, i.e. they do not depend on the chart U_j.

Let $\tau : \mathbb{P}^N \times \mathbb{C} \to \mathbb{C}$ denote the second projection. One defines the relative conormal space $C_\tau(\mathbb{P}^N \times \mathbb{C})$ as in definition 3.2 where the function g is replaced by the mapping τ.

DEFINITION 3.4 ([**8**], definition 2.5). We say that f is t-regular at $p_0 \in \mathbb{X}^\infty$ if

$$C_\tau(\mathbb{P}^N \times \mathbb{C}) \cap C^\infty_{p_0} = \emptyset.$$

We say that $f^{-1}(t_0)$ is t-regular if f is t-regular at all points $p_0 \in \mathbb{X}^\infty \cap \tau^{-1}(t_0)$.

Let us now recall the definition of ρ-regularity. Let $K \subset \mathbb{C}^N$ be some compact (eventually empty) set and let $\rho : \mathbb{C}^N \setminus K \to \mathbb{R}_{\geqslant 0}$ be a proper analytic submersion.

DEFINITION 3.5 (ρ-regularity at infinity, [**8**], definition 5.2). We say that f is ρ-regular at $p_0 \in \mathbb{X}^\infty$ if there is an open neighbourhood $U \subset \mathbb{P}^N \times \mathbb{C}$ of p_0 and an open neighbourhood $D \subset \mathbb{C}$ of $\tau(p_0)$ such that, for all $t \in D$, the fibre $f^{-1}(t) \cap X_{\text{reg}} \cap U$ intersects all the levels of the restriction $\rho_{|U \cap X_{\text{reg}}}$ and this intersection is transversal.

We say that the fibre $f^{-1}(t_0)$ is ρ-regular at infinity if f is ρ-regular at all points $p_0 \in \mathbb{X}^\infty \cap \tau^{-1}(t_0)$. We say that t_0 is an asymptotic ρ-non-regular value if $f^{-1}(t_0)$ is not ρ-regular at infinity.

The next proposition relates t-regularity to ρ_E-regularity, where ρ_E denotes the Euclidian norm.

PROPOSITION 3.6. *If f is t-regular at $p_0 \in \mathbb{X}^\infty$, then f is ρ_E-regular at p_0.*

Proof. This is just an adaptation to our setting of the proof of proposition 2.11 in [**37**]. $\qquad \square$

The next corollary has been proved in the more general case of pencils in [**38**] and in [**32**].

COROLLARY 3.7. *The set of asymptotic non-ρ_E-regular values of f is finite.*

Proof. It is enough to prove that there are only finitely many values t_0 such that $f^{-1}(t_0)$ is not t-regular. The proof of this fact is as in corollary 2.12 in [**37**]. We can equip $\mathbb{X}$ with a Whitney stratification such that $\mathbb{X}^\infty$ is a union of strata and such that any pair of strata (V, W), with $V \subset \mathbb{X} \setminus \mathbb{X}^\infty$, $W \subset \mathbb{X}^\infty$ and $W \subset \overline{V} \setminus V$, satisfies the Thom (a_g)-regularity condition for some function g defining $\mathbb{X}^\infty$ in $\mathbb{X}$. If $\tau^{-1}(t_0)$ is transversal to $\mathbb{X}^\infty$ in the stratified sense, then $f^{-1}(t_0)$ is t-regular. But the mapping $\tau_{|\mathbb{X}^\infty} : \mathbb{X}^\infty \to \mathbb{C}$ has a finite number of critical values (in the stratified sense). $\qquad \square$

PROPOSITION 3.8. *Let $l : X \to \mathbb{C}$ be a generic linear projection then for all $p_0 \in \mathbb{X}^\infty$, l is t-regular at p_0.*

Proof. With U_j defined as before, let us work in the chart $U_j \times \mathbb{C}$ and with $C_g(\mathbb{X} \setminus \mathbb{X}^\infty \cap U_j \times \mathbb{C})$ with $\pi(x, H) = x$, as defined above.

Let us suppose that l is not t-regular at $p_0 = (q_0, t_0) \in \mathbb{X}^\infty$. It means that there exists a sequence $p_n = (q_n, l(q_n)) \to p_0$, with $q_n \in X_{\mathrm{reg}}$ such that

$$T_{p_n} \mathrm{graph}(l) \cap T_{p_n} g^{-1}(g(p_n)) \subset H_n,$$

and $H_n \to H$ and a sequence of hyperplanes $\{L_n\}$ such that $\mathbb{C}^N \times \{0\} \subset L_n$ and $L_n \to H$, where $(p_0, H) \in \mathcal{C}_\tau(\mathbb{P}^N \times \mathbb{C}) \cap \mathbb{C}^\infty_{p_0}$.

Since in fact each $L_n = \mathbb{C}^N \times \{0\}$, we conclude that $H = \mathbb{C}^N \times \{0\}$. Note also that,

$$T_{p_n} \mathrm{graph}(l) \cap T_{p_n} g^{-1}(g(p_n))$$
$$= \mathrm{graph}(l : T_{q_n} X_{\mathrm{reg}} \to \mathbb{C}) \cap T_{q_n} g^{-1}(g(q_n)) \times \mathbb{C}.$$

This implies that $\lim_{n \to +\infty} l(u_n) = 0$ for any bounded sequence (u_n) of vectors such that $u_n \in T_{q_n} X_{\mathrm{reg}} \cap T_{q_n} g^{-1}(g(q_n))$. As in the previous corollary, we can equip $\overline{X}$ with a Whitney stratification such that $\overline{X} \cap H^\infty$ is a union of strata and such that any pair of strata (V, W), with $V \subset X$, $W \subset \overline{X} \cap H^\infty$ and $W \subset \overline{V} \setminus V$, satisfies the Thom (a_g)-regularity. Therefore we see that the axis of the pencil defined by l is not transversal to $\overline{X} \cap H^\infty$. By lemma 3.1 in [**28**], this is not possible if l is generic. So we conclude that l generic is t-regular. $\qquad \square$

We assume now that X is equipped with a finite Whitney stratification $\mathcal{V} = \{V_i\}_{i=1}^t$ such that $V_1, \ldots, V_{t-1}$ are connected, $\overline{V}_1, \ldots, \overline{V}_t$ are reduced and $V_t = X_{\mathrm{reg}}$. For $i = \{1, \ldots, t\}$, let $f_i : \overline{V}_i \to \mathbb{C}$ be the restriction to $\overline{V}_i$ of the polynomial function F. Note that $f_t = f$.

DEFINITION 3.9 (stratified *t*-regularity at infinity). We say that f is stratified *t*-regular at $p_0 \in \mathbb{X}^\infty$ if for $i = 1, \ldots, t$, f_i is *t*-regular at p_0.

We say that $f^{-1}(t_0)$ is stratified *t*-regular if f is stratified *t*-regular at all points $p_0 \in \mathbb{X}^\infty \cap \tau^{-1}(t_0)$.

DEFINITION 3.10 (stratified ρ-regularity at infinity). We say that f is stratified ρ-regular at $p_0 \in \mathbb{X}^\infty$ if for $i = 1, \ldots, t$, f_i is ρ-regular at p_0.

We say that the fibre $f^{-1}(t_0)$ is stratified ρ-regular at infinity if f is stratified ρ-regular at all points $p_0 \in \mathbb{X}^\infty \cap \tau^{-1}(t_0)$. We say that t_0 is a stratified asymptotic ρ-non-regular value if $f^{-1}(t_0)$ is not stratified ρ-regular at infinity.

The following statements are easy consequences of the definitions of stratified *t*-regularity and stratified ρ-regularity.

PROPOSITION 3.11. *Stratified t-regularity implies stratified ρ_E-regularity.*

COROLLARY 3.12. *The set of stratified asymptotic non-ρ_E-regular values of f is finite.*

COROLLARY 3.13. *Let $l : X \to \mathbb{C}$ be a generic linear projection, then for all $p_0 \in \mathbb{X}^\infty$, l is stratified t-regular (and therefore stratified ρ_E-regular) at p_0. Moreover, the set of stratified asymptotic non-ρ_E-regular values of l is empty.*

We note that the results of proposition 3.8 and corollary 3.13 can be deduced from proposition 3.1 in [**27**].

4. Global Brasselet numbers and Brylinsky-Dubson-Kashiwara formulas

Let $X \subset \mathbb{C}^n$ be a reduced algebraic set of dimension d, equipped with a finite Whitney stratification $\mathcal{V} = \{V_i\}_{i=1}^t$. We assume that $V_1, \ldots, V_{t-1}$ are connected, $\overline{V_1}, \ldots, \overline{V_t}$ are reduced and that $\overline{V_t} = X_{\text{reg}}$, where X_{reg} has dimension d. Let $f : X \to \mathbb{C}$ be a complex polynomial, restriction to X of a polynomial function $F : \mathbb{C}^n \to \mathbb{C}$, i.e. $f = F_{|X}$. We assume that f has a finite number of critical points, which means that for $i = 1, \ldots, t$, $F_{|V_i}$ has a finite number of critical points. We denote by $\{q_1, \ldots, q_s\}$ the set of critical points of f and by $\{a_1, \ldots, a_r\}$ the set of stratified asymptotic non-ρ_E-regular values of f.

For simplicity, we will write B_R for the ball $B_R(0)$ and S_R for ∂B_R. The next lemma has been proved in the more general case of pencils and at the level of homology in [**32**, **38**, **39**]. We give a proof at the level of Euler characteristics, which is enough for our purpose.

LEMMA 4.1. *Let $\alpha : X \to \mathbb{Z}$ be a constructible function with respect to $\mathcal{V}$. The function $c \mapsto \chi(f^{-1}(c), \alpha)$ is constant on $\mathbb{C} \setminus (\{f(q_1), \ldots, f(q_s)\} \cup \{a_1, \ldots, a_r\})$.*

Proof. Let $c \in \mathbb{C} \setminus (\{f(q_1), \ldots, f(q_s)\} \cup \{a_1, \ldots, a_r\})$ and let us choose $R_c > 0$ such that $f^{-1}(c) \cap \{\rho_E \geqslant R_c\}$ does not contain any critical point of $\rho_{E|f^{-1}(c)}$ (here $f^{-1}(c)$ is equipped with the Whitney stratification $\sqcup V_i \cap f^{-1}(c)$). This implies

that $f^{-1}(c) \cap B_R$ is a retract by deformation of $f^{-1}(c)$ and that $\chi(f^{-1}(c)) = \chi(f^{-1}(c) \cap B_R)$ for any $R \geqslant R_c$. Since $f^{-1}(c)$ is stratified ρ_E-regular at infinity, there is $R \geqslant R_c$ and $\epsilon > 0$ such that the mapping

$$(\rho_E, f) : f^{-1}(D_\epsilon(c)) \cap \{\rho \geqslant R\} \to \mathbb{R} \times \mathbb{C},$$

where $D_\epsilon(c)$ is the closed disc of radius ϵ centred at c in $\mathbb{C}$, is a stratified submersion and so for $c' \in D_\epsilon(c)$, $f^{-1}(c') \cap B_R$ is also a retract by deformation of $f^{-1}(c')$. But since c is a regular value of f,

$$\chi(f^{-1}(c') \cap B_R) = \chi(f^{-1}(c) \cap B_R),$$

for c' in a small neighbourhood of c. Hence the result is proved for the function $c \mapsto \chi(f^{-1}(c))$, i.e. when $\alpha = \mathbf{1}_X$ (see the discussion after definition 2.11).

Let V be a stratum of X. By additivity, we have

$$\chi_c(\bar{V} \cap f^{-1}(c)) = \chi_c(V \cap f^{-1}(c)) + \chi_c(\bar{V} \setminus V \cap f^{-1}(c)),$$

and by the arguments after definition 2.11, we get that

$$\chi(\bar{V} \cap f^{-1}(c)) = \chi(V \cap f^{-1}(c)) + \chi(\bar{V} \setminus V \cap f^{-1}(c)).$$

But since $\bar{V}$ and $\bar{V} \setminus V$ are algebraic subsets of X stratified by strata of $\mathcal{V}$, c is a regular value and $f^{-1}(c)$ is stratified ρ_E-regular at infinity for $f : \bar{V} \to \mathbb{C}$ and $f : \bar{V} \setminus V \to \mathbb{C}$. Therefore the functions

$$c \mapsto \chi(\bar{V} \cap f^{-1}(c)) \text{ and } c \mapsto \chi((\bar{V} \setminus V) \cap f^{-1}(c)),$$

are constant on $\mathbb{C} \setminus (\{f(q_1), \dots, f(q_s)\} \cup \{a_1, \dots, a_r\})$ and so is the function $c \mapsto \chi(V \cap f^{-1}(c))$. Hence for $i \in \{1, \dots, t\}$, the function $c \mapsto \chi(V_i \cap f^{-1}(c))$ is constant on $\mathbb{C} \setminus (\{f(q_1), \dots, f(q_s)\} \cup \{a_1, \dots, a_r\})$. This implies that $c \mapsto \chi(f^{-1}(c), \alpha)$ is also constant on $\mathbb{C} \setminus (\{f(q_1), \dots, f(q_s)\} \cup \{a_1, \dots, a_r\})$ for any constructible function α. □

DEFINITION 4.2. When X is equidimensional, we define the global Brasselet number of f at c by

$$\mathrm{B}_{f,c}^X = \chi(f^{-1}(c), \mathrm{Eu}_X),$$

and the global Euler obstruction of f at c by

$$\mathrm{Eu}_{f,c}^X = \mathrm{Eu}(X) - \mathrm{B}_{f,c}^X.$$

Let $a \in \mathbb{C}$ and let $R_a > 0$ be such that $f^{-1}(a) \cap B_{R_a}$ does not contain any critical point of $\rho_{E|f^{-1}(a)}$. Then there exists $\delta_1 > 0$ such that

$$f : f^{-1}\left(D_{\delta_1}(a) \setminus \{a\}\right) \to D_{\delta_1}(a) \setminus \{a\},$$

is a locally trivial topological fibration (this is just a singular version of the Milnor-Lê fibration) and so $\chi(f^{-1}(c) \cap B_{R_a})$ is constant for $c \in D_{\delta_1}(a) \setminus \{a\}$.

Since $\chi(f^{-1}(c))$ is constant on $\mathbb{C} \setminus (\{f(q_1), \ldots, f(q_s)\} \cup \{a_1, \ldots, a_r\})$, there exists $\delta_2 > 2$ such that $\chi(f^{-1}(c))$ is constant for $c \in D_{\delta_2}(a) \setminus \{a\}$. Since

$$\chi(f^{-1}(c)) = \chi(f^{-1}(c) \cap B_{R_a}) + \chi(f^{-1}(c) \cap \{\rho_E \geqslant R_a\}) - \chi(f^{-1}(c) \cap S_{R_a})$$
$$= \chi(f^{-1}(c) \cap B_{R_a}) + \chi(f^{-1}(c) \cap \{\rho_E \geqslant R_a\}),$$

we see that $\chi(f^{-1}(c) \cap \{\rho_E \geqslant R_a\})$ is constant for c in $D_\delta(a) \setminus \{a\}$, where $\delta = \min\{\delta_1, \delta_2\}$.

Let $R'_a > 0$ be such that $f^{-1}(a) \cap B_{R'_a}$ does not contain any critical point of $\rho_{E|f^{-1}(a)}$. Then there exists $\delta' > 0$ such that $\chi(f^{-1}(c) \cap \{\rho_E \geqslant R'_a\})$ is constant for c in $D_{\delta'}(a) \setminus \{a\}$. We can suppose that $R'_a > R_a$. Since there are no critical points of $\rho_{E|f^{-1}(a)}$ on $\{R_a \leqslant \rho_E \leqslant R'_a\}$, a is a regular value of $f : \{R_a \leqslant \rho_E \leqslant R'_a\} \to \mathbb{C}$ (note that a critical point of f on $f^{-1}(a)$ is also a critical point of $\rho_{E|f^{-1}(a)}$). Hence there exists $\nu > 0$ such that

$$f : f^{-1}(D_\nu(a)) \cap \{R_a \leqslant \rho_E \leqslant R'_a\} \to D_\nu(a),$$

is a locally trivial topological fibration. Let $\nu' = \min\{\delta, \delta', \nu\}$. For c in $D_{\nu'}(a) \setminus \{a\}$, we have

$$\chi(f^{-1}(c) \cap \{\rho_E \geqslant R_a\}) = \chi(f^{-1}(c) \cap \{\rho_E \geqslant R'_a\}) + \chi(f^{-1}(c) \cap \{R_a \leqslant \rho_E \leqslant R'_a\})$$
$$= \chi(f^{-1}(c) \cap \{\rho_E \geqslant R'_a\}) + \chi(f^{-1}(a) \cap \{R_a \leqslant \rho_E \leqslant R'_a\}).$$

But $\rho : f^{-1}(a) \cap \{R_a \leqslant \rho_E \leqslant R_{a'}\} \to \mathbb{C}$ is a stratified submersion and so

$$\chi(f^{-1}(a) \cap \{R_a \leqslant \rho_E \leqslant R_{a'}\}) = \chi(f^{-1}(a) \cap \{\rho_E = R_a\}) = 0.$$

We have proved that

$$\chi(f^{-1}(c) \cap \{\rho_E \geqslant R_a\}) = \chi(f^{-1}(c) \cap \{\rho_E \geqslant R'_a\}),$$

if c is sufficiently close to a.

DEFINITION 4.3. Let $\alpha : X \to \mathbb{Z}$ be a constructible function with respect to $\mathcal{V}$. For any $a \in \mathbb{C}$, we set

$$\mathrm{B}_{f,a}^{X,\infty}(\alpha) = \lim_{c \to a} \chi(f^{-1}(c) \cap \{\rho_E \geqslant R_a\}, \alpha) \quad \text{and} \quad \lambda_{f,a}^{X,\infty} = \mathrm{B}_{f,a}^{X,\infty}(\mathbf{1}_X),$$

where $R_a > 0$ is such that $f^{-1}(a) \cap \{\rho_E \geqslant R_a\}$ does not contain any critical point of $\rho_{E|f^{-1}(a)}$.

Note that $\mathrm{B}_{f,a}^{X,\infty}(\alpha)$ is well-defined since, by the previous considerations,

$$\lim_{c \to a} \chi(\overline{V_i} \cap f^{-1}(c) \cap \{\rho_E \geqslant R_a\}),$$

is well-defined and so is

$$\lim_{c \to a} \chi(V_i \cap f^{-1}(c) \cap \{\rho_E \geqslant R_a\}).$$

We note that similar jumps at infinity have been considered in [**33**].

LEMMA 4.4. *Let* $\alpha : X \to \mathbb{Z}$ *be a constructible function with respect to* $\mathcal{V}$. *If* $c \in \mathbb{C}$ *is such that* $f^{-1}(c)$ *is stratified* ρ_E-*regular at infinity then* $\mathrm{B}_{f,c}^{X,\infty}(\alpha) = 0$.

Proof. Let us prove first that $\lambda_{f,c}^{X,\infty} = 0$. Since $f^{-1}(c)$ is stratified ρ_E-regular at infinity, there is $R_c > 0$ and $\epsilon > 0$ such that the mapping

$$(\rho_E, f) : f^{-1}(D_\epsilon(c)) \cap \{\rho_E \geqslant R_c\} \to \mathbb{R} \times \mathbb{C},$$

is a stratified submersion. Let $c' \in f^{-1}(D_\epsilon(c))$. Then the mapping

$$\rho_E : \{\rho_E \geqslant R_c\} \cap f^{-1}(c') \to \mathbb{R},$$

is a proper stratified submersion and so

$$\chi(f^{-1}(c') \cap \{\rho_E \geqslant R_c\}) = \chi(f^{-1}(c') \cap \{\rho_E = R_c\}) = 0.$$

If α is constructible with respect to $\mathcal{V}$ then it has a presentation of the form $\alpha = \sum_i m_i \mathbf{1}_{\overline{V_i}}$ and so

$$\mathrm{B}_{f,c}^{X,\infty}(\alpha) = \sum_i m_i \lambda_{f,c}^{\overline{V_i},\infty} = 0. \quad \square$$

DEFINITION 4.5. *Let* $\alpha : X \to \mathbb{Z}$ *be a constructible function with respect to* $\mathcal{V}$. *We set*

$$\mathrm{B}_f^{X,\infty}(\alpha) = \sum_{c \in \mathbb{C}} \mathrm{B}_{f,c}^{X,\infty}(\alpha) \quad \text{and} \quad \lambda_f^{X,\infty} = \mathrm{B}_f^{X,\infty}(\mathbf{1}_X).$$

DEFINITION 4.6. *When* X *is equidimensional, we define the Brasselet numbers at infinity of* f *by:*

$$\mathrm{B}_{f,c}^{X,\infty} = \mathrm{B}_{f,c}^{X,\infty}(\mathrm{Eu}_X),$$

for $c \in \mathbb{C}$, *and the total Brasselet number at infinity of* f *by:*

$$\mathrm{B}_f^{X,\infty} = \mathrm{B}_f^{X,\infty}(\mathrm{Eu}_X).$$

We start comparing the global Brasselet numbers of f and the Euler obstructions of the fibres of f.

PROPOSITION 4.7. *Let* $a \in \mathbb{C}$, *we have*

$$\mathrm{B}_{f,a}^X = \mathrm{Eu}(f^{-1}(a)) + \sum_{j \mid f(q_j)=a} \mathrm{Eu}_X(q_j) - \mathrm{Eu}_{f^{-1}(a)}(q_j).$$

Proof. By definition,

$$\mathrm{B}^X_{f,a} = \sum_{i=1}^{t} \chi(V_i \cap f^{-1}(a))\mathrm{Eu}_X(V_i).$$

For each $i \in \{1, \ldots, t\}$, let us denote by Γ_i the set consisting of the q_j's such that $q_j \in V_i \cap f^{-1}(a)$. The partition

$$f^{-1}(a) = \left(\sqcup_{i \mid \Gamma_i=\emptyset} V_i \cap f^{-1}(a)\right) \bigcup \left(\sqcup_{i \mid \Gamma_i\neq\emptyset} V_i \cap f^{-1}(a) \setminus \Gamma_i\right) \bigcup \left(\sqcup_{i \mid \Gamma_i\neq\emptyset}\Gamma_i\right),$$

gives a Whitney stratification of $f^{-1}(a)$, and

$$\mathrm{Eu}(f^{-1}(a)) = \sum_{i \mid \Gamma_i=\emptyset} \chi(V_i \cap f^{-1}(a))\mathrm{Eu}_{f^{-1}(a)}(V_i \cap f^{-1}(a))$$

$$+ \sum_{i \mid \Gamma_i\neq\emptyset} \chi(V_i \cap f^{-1}(a) \setminus \Gamma_i)\mathrm{Eu}_{f^{-1}(a)}(V_i \cap f^{-1}(a) \setminus \Gamma_i)$$

$$+ \sum_{i \mid \Gamma_i\neq\emptyset} \sum_{q\in\Gamma_i} \mathrm{Eu}_{f^{-1}(a)}(q).$$

If Γ_i is empty then the intersection $V_i \cap f^{-1}(a)$ is transverse (necessarily dim $V_i > 0$) and by [**10**], proposition IV. 4.1.1

$$\mathrm{Eu}_X(V_i) = \mathrm{Eu}_{f^{-1}(a)}(V_i \cap f^{-1}(a)).$$

If Γ_i is not empty and dim$V_i > 0$, then

$$\chi(V_i \cap f^{-1}(a)) = \chi(V_i \cap f^{-1}(a) \setminus \Gamma_i) + \#\Gamma_i,$$

and $\mathrm{Eu}_X(V_i) = \mathrm{Eu}_{X\cap f^{-1}(a)}(V_i \cap f^{-1}(a) \setminus \Gamma_i)$ because outside Γ_i, $f^{-1}(a)$ intersects V_i transversally. If Γ_i is not empty and dim$V_i = 0$, then

$$\chi(V_i \cap f^{-1}(a)) = 1 \quad \text{and} \quad \chi(V_i \cap f^{-1}(a) \setminus \Gamma_i) = 0.$$

Therefore we get

$$\mathrm{B}^X_{f,a} = \sum_{i \mid \Gamma_i=\emptyset} \chi(V_i \cap f^{-1}(a))\mathrm{Eu}_{f^{-1}(a)}(V_i \cap f^{-1}(a))$$

$$+ \sum_{i \mid \Gamma_i\neq\emptyset} \chi(V_i \cap f^{-1}(a) \setminus \Gamma_i)\mathrm{Eu}_{f^{-1}(a)}(V_i \cap f^{-1}(a) \setminus \Gamma_i)$$

$$+ \sum_{i \mid \Gamma_i\neq\emptyset} \sum_{q\in\Gamma_i} \mathrm{Eu}_X(q)$$

$$= \mathrm{Eu}(f^{-1}(a)) + \sum_{i \mid \Gamma_i\neq\emptyset} \sum_{q\in\Gamma_i} \mathrm{Eu}_X(q) - \mathrm{Eu}_{f^{-1}(a)}(q). \qquad \square$$

Note that for a regular value c of f, $\mathrm{B}^X_{f,c} = \mathrm{Eu}(f^{-1}(c))$. Furthermore, if $X = \mathbb{C}^n$ then $\mathrm{Eu}_X(q_j) = 1$ and $\mathrm{Eu}_{f^{-1}(a)}(q_j) = 1 + (-1)^{n-2}\mu'(f, q_j)$, where $\mu'(f, q_j)$ is the first Milnor-Teissier number of f at q_j, so

$$\mathrm{B}^{\mathbb{C}^n}_{f,a} = \chi(f^{-1}(a)) = \mathrm{Eu}(f^{-1}(a)) + (-1)^{n-1} \sum_{j \mid f(q_j)=a} \mu'(f, q_j),$$

which is Equality (3.3) in [**40**]. So the formula of proposition 4.7 extends this equality to the case of singular algebraic sets.

REMARK 4.8. In the equality of proposition 4.7, we can distinguish between two kinds of critical points: those lying in $V = X_{\mathrm{reg}}$ and those lying in a lower dimensional stratum. Note that the collection of lower dimensional strata gives a Whitney stratification of X_{sing}, the singular locus of X. Hence the critical points of f that lie on X_{sing} depend on the stratification of X_{sing}. However, the formula of proposition 4.7 implies that the sum

$$\sum_{\substack{j \mid f(q_j)=a \\ q_j \in X_{\mathrm{sing}}}} \mathrm{Eu}_X(q_j) - \mathrm{Eu}_{f^{-1}(a)}(q_j)$$

does not depend on the stratification of X_{sing}.

A direct corollary of the previous proposition is a global relative version of the local index formula of Brylinski, Dubson and Kashiwara. Its proof relies on the global version of the index formula established in [**40**].

COROLLARY 4.9. *Let $\alpha : X \to \mathbb{Z}$ be a constructible function with respect to $\mathcal{V}$. For any $a \in \mathbb{C}$, we have*

$$\chi(f^{-1}(a), \alpha) = \sum_{i=1}^t \mathrm{B}^{\overline{V_i}}_{f,a} \eta(V_i, \alpha).$$

Proof. We keep the notations of the previous proposition and apply Equality (0.2) of [**40**] to get

$$\chi(f^{-1}(a)) = \sum_{i \mid \Gamma_i=\emptyset} \mathrm{Eu}(\overline{V_i \cap f^{-1}(a)}) \left(1 - \chi\left(\mathcal{L}^{f^{-1}(a)}_{V_i \cap f^{-1}(a)}\right)\right)$$

$$+ \sum_{i \mid \Gamma_i \neq \emptyset} \mathrm{Eu}(\overline{V_i \cap f^{-1}(a) \setminus \Gamma_i}) \left(1 - \chi\left(\mathcal{L}^{f^{-1}(a)}_{V_i \cap f^{-1}(a) \setminus \Gamma_i}\right)\right)$$

$$+ \sum_{i \mid \Gamma_i \neq \emptyset} \sum_{q \in \Gamma_i} 1 - \chi\left(\mathcal{L}^{f^{-1}(a)}_{\{q\}}\right)$$

$$= \sum_{i \mid \Gamma_i \neq V_i} \mathrm{Eu}(\overline{V_i} \cap f^{-1}(a)) \left(1 - \chi\left(\mathcal{L}^{f^{-1}(a)}_{V_i \cap f^{-1}(a) \setminus \Gamma_i}\right)\right)$$

$$+ \sum_{i \mid \Gamma_i \neq \emptyset} \sum_{q \in \Gamma_i} 1 - \chi\left(\mathcal{L}^{f^{-1}(a)}_{\{q\}}\right),$$

because $\Gamma_i = V_i$ if and only if V_i is just a 0-dimensional stratum and in this case, $V_i \cap f^{-1}(a) \setminus \Gamma_i = \emptyset$. By the previous proposition, we obtain the equality

$$\chi(f^{-1}(a)) = \sum_{i \mid \Gamma_i \neq V_i} \left[B_{f,a}^{\overline{V_i}} - \sum_{j \mid f(q_j)=a} -\mathrm{Eu}_{\overline{V_i}}(q_j) + \mathrm{Eu}_{\overline{V_i} \cap f^{-1}(a)}(q_j) \right]$$
$$\times \left(1 - \chi\left(\mathcal{L}_{V_i \cap f^{-1}(a) \setminus \Gamma_i}^{f^{-1}(a)} \right) \right) + \sum_{i \mid \Gamma_i \neq \emptyset} \sum_{q \in \Gamma_i} 1 - \chi\left(\mathcal{L}_{\{q\}}^{f^{-1}(a)} \right),$$

that we rewrite

$$\chi(f^{-1}(a)) = \sum_{i \mid \Gamma_i \neq V_i} \left[B_{f,a}^{\overline{V_i}} - \sum_{j \mid f(q_j)=a} -\mathrm{Eu}_{\overline{V_i}}(q_j) + \mathrm{Eu}_{\overline{V_i} \cap f^{-1}(a)}(q_j) \right]$$
$$\times \left(1 - \chi\left(\mathcal{L}_{V_i \cap f^{-1}(a) \setminus \Gamma_i}^{f^{-1}(a)} \right) \right) + \sum_{j \mid f(q_j)=a} 1 - \chi(\mathcal{L}_{\{q_j\}}^{f^{-1}(a)}).$$

If $\Gamma_i \neq V_i$ then $f^{-1}(a)$ intersects $V_i \setminus \Gamma_i$ transversally and so

$$\chi\left(\mathcal{L}_{V_i \cap f^{-1}(a) \setminus \Gamma_i}^{f^{-1}(a)} \right) = \chi(\mathcal{L}_{V_i}^X).$$

Hence we have

$$\chi(f^{-1}(a)) = \sum_{i \mid \Gamma_i \neq V_i} B_{f,a}^{\overline{V_i}} \left(1 - \chi(\mathcal{L}_{V_i}^X) \right) + \sum_{j \mid f(q_j)=a} \left[1 - \chi\left(\mathcal{L}_{\{q_j\}}^{f^{-1}(a)} \right) \right.$$
$$+ \sum_{i \mid \Gamma_i \neq V_i} -\mathrm{Eu}_{\overline{V_i}}(q_j) \left(1 - \chi(\mathcal{L}_{V_i}^X) \right)$$
$$+ \mathrm{Eu}_{\overline{V_i} \cap f^{-1}(a)}(q_j) \left(1 - \chi\left(\mathcal{L}_{V_i \cap f^{-1}(a) \setminus \Gamma_i}^{f^{-1}(a)} \right) \right) \Bigg].$$

Let us evaluate the second part of this sum and fix q a critical point of f such that $f(q) = a$. Two cases are possible.

If q belongs to a stratum V_k with $\Gamma_k \neq V_k$ then we add the stratum $V_0 = \{q\}$ to the Whitney stratification of X. By the Brylinski, Dubson and Kashiwara index formula ([**6**] or [**26**], p. 294), we know that

$$1 = \sum_{i=0}^{t} \mathrm{Eu}_{\overline{V_i}}(q) \left(1 - \chi(\mathcal{L}_{V_i}^X) \right),$$

and so

$$1 = \sum_{i \mid \Gamma_i \neq V_i} \mathrm{Eu}_{\overline{V_i}}(q) \left(1 - \chi(\mathcal{L}_{V_i}^X) \right) + \mathrm{Eu}_{V_0}(q) \left(1 - \chi(\mathcal{L}_{V_0}^X) \right),$$

because $q \notin \overline{V_i}$ if $\Gamma_i = V_i$ $(i \geqslant 1)$. But $\mathrm{Eu}_{V_0}(q) = 1$ and $\chi(\mathcal{L}_{V_0}^X) = 1$ because a generic linear form is a stratified submersion at q (see [**13**], p. 90 for details). The same

index formula applied to $f^{-1}(a)$ gives

$$1 = \sum_{i=1}^{t} \mathrm{Eu}_{\overline{V_i \cap f^{-1}(a) \setminus \Gamma_i}}(q) \left(1 - \chi\left(\mathcal{L}^{f^{-1}(a)}_{V_i \cap f^{-1}(a) \setminus \Gamma_i}\right)\right) + 1 - \chi(\mathcal{L}^{f^{-1}(a)}_{\{q\}}).$$

Therefore we get that

$$\chi(\mathcal{L}^{f^{-1}(a)}_{\{q\}}) = \sum_{i \; | \; \Gamma_i \neq V_i} \mathrm{Eu}_{\overline{V_i} \cap f^{-1}(a)}(q) \left(1 - \chi\left(\mathcal{L}^{f^{-1}(a)}_{V_i \cap f^{-1}(a) \setminus \Gamma_i}\right)\right),$$

and so the contribution of q in the second summand of the above sum is zero.

If q belongs to a stratum V_k with $\Gamma_k = V_k$ then, actually $V_k = \{q\}$. By the index formula and the same arguments, we find that

$$1 = \sum_{i \; | \; \Gamma_i \neq V_i} \mathrm{Eu}_{\overline{V_i}}(q)\left(1 - \chi(\mathcal{L}^X_{V_i})\right) + 1 - \chi\left(\mathcal{L}^X_{\{q\}}\right),$$

and

$$1 = \sum_{i \; | \; \Gamma_i \neq V_i} \mathrm{Eu}_{\overline{V_i} \cap f^{-1}(a)}(q) \left(1 - \chi\left(\mathcal{L}^{f^{-1}(a)}_{V_i \cap f^{-1}(a) \setminus \Gamma_i}\right)\right) + 1 - \chi\left(\mathcal{L}^{f^{-1}(a)}_{\{q\}}\right).$$

Hence the contribution of q in the above second summand is $1 - \chi(\mathcal{L}^X_{\{q\}})$, which we can write $\mathrm{B}^{\overline{V_k}}_{f,a}(1 - \chi(\mathcal{L}^X_{V_k}))$. Finally, we have proved that

$$\chi(f^{-1}(a)) = \sum_{i=1}^{t} \mathrm{B}^{\overline{V_i}}_{f,a}\left(1 - \chi(\mathcal{L}^X_{V_i})\right).$$

Hence the result is true for any constructible function of the form $\mathbf{1}_{\overline{V_j}}$. If α is a constructible function with respect to $\mathcal{V}$, then it has a presentation of the form $\alpha = \sum_j m_j \mathbf{1}_{\overline{V_j}}$. We conclude with the equalities $\chi(f^{-1}(a), \alpha) = \sum_j m_j \chi(f^{-1}(a), \mathbf{1}_{\overline{V_j}})$ and $\eta(V_i, \alpha) = \sum_j m_j \eta(V_i, \mathbf{1}_{\overline{V_j}})$, for $i = 1, \ldots, t$. $\qquad\square$

5. Global Brasselet numbers and critical points

In this section, we prove several formulas that relate the number of critical points of a Morsefication of a polynomial function f on an algebraic set X, to the global Brasselet numbers and the Brasselet numbers at infinity of f. We note that when $X = \mathbb{C}^n$, similar formulas have already appeared in the literature ([**1**, **5**, **16**, **25**, **31**, **34**, **36**, **37**]). Most of these formulas are presented in the monograph [**41**].

The setting is the same as in the previous section: $X \subset \mathbb{C}^n$ is a reduced algebraic set of dimension d, equipped with a finite Whitney stratification $\mathcal{V} = \{V_i\}_{i=1}^{t}$ such that $V_1, \ldots, V_{t-1}$ are connected, $\overline{V_1}, \ldots, \overline{V_t}$ are reduced and $V_t = X_{\mathrm{reg}}$; $f : X \to \mathbb{C}$ is a complex polynomial, restriction to X of a polynomial function $F : \mathbb{C}^n \to \mathbb{C}$. We assume that f has a finite number of critical points, which means that for $i = 1, \ldots, t$, $F_{|V_i}$ has a finite number of critical points. We denote by $\{q_1, \ldots, q_s\}$ the set of critical points of f and by $\{a_1, \ldots, a_r\}$ the set of stratified asymptotic non-ρ_E-regular values of f.

DEFINITION 5.1. We say that $\tilde{f} : X \to \mathbb{C}$ is a Morsefication of f if $\tilde{f}$ is a small deformation of f which is a local (stratified) Morsefication at all isolated critical points of f.

Let $\tilde{f}$ be a Morsefication of f. We can take $\tilde{f}$ of the form $f + tl$ where t is a sufficiently small complex number and l is the restriction to X of a generic linear form. This is explained in [**18**], theorem 2.2, in the local case. The argument holds in our situation since it does not depend on the value taken by f at the considered critical point and there are a finite number of critical points. Note that $\tilde{f}$ has two kinds of critical points: those appearing in a small neighbourhood of a critical point of f and those appearing at infinity, i.e. outside a ball of sufficiently big radius. We will only consider the first ones.

Let n_i, $i = 1, \ldots, t$, be the number of critical points of $\tilde{f}$ appearing in a small neighbourhood of a critical point of f on the stratum V_i. Note that

$$n_i \geqslant \mu^T(f_{|V_i}) = \sum_{j \ | \ q_j \in V_i} \mu(f_{|V_i}, q_j),$$

where $\mu(f_{|V_i}, q_j)$ is the Milnor number of $f_{|V_i}$ at q_j, since we do not assume that f is general with respect to $\mathcal{V}$.

The following theorem relates the number of stratified critical points of $\tilde{f}$ appearing on the stratum V_i to the topology of X and a generic fibre of f. It can be proved using results on general pencils, namely corollaries 9.2.3 and 9.2.6 in [**41**]. We give a different proof, enough for our purpose, based on the method of integration with respect to the Euler characteristic.

THEOREM 5.2. *Let $c \in \mathbb{C}$ be a regular value of f, which is not a stratified asymptotic non-ρ_E-regular value. We have*

$$\chi(X) - \chi(f^{-1}(c)) = \sum_{i=1}^{t} (-1)^{d_i} n_i \left(1 - \chi(\mathcal{L}_{V_i}^X)\right) - \lambda_f^{X, \infty}.$$

Moreover, if f is general with respect to $\mathcal{V}$, then we have

$$\chi(X) - \chi(f^{-1}(c)) = \sum_{i=1}^{t} (-1)^{d_i} \mu^T(f_{|V_i}) \left(1 - \chi(\mathcal{L}_{V_i}^X)\right) - \lambda_f^{X, \infty}.$$

Proof. For $x \in X$, let $\varphi(x) = \chi(f^{-1}(\tilde{x}) \cap B_\epsilon(x)) = \chi_c(f^{-1}(\tilde{x}) \cap B_\epsilon(x))$, where $\tilde{x}$ is a regular value of f close to $f(x)$ and $0 \leqslant |\tilde{x} - f(x)| \ll \epsilon \ll 1$. Note that $\varphi(x) = 1$ if x is not a critical point of f and so φ is a constructible function. By Fubini theorem, we have

$$\chi(X, \varphi) = \chi(\mathbb{C}, f_* \varphi),$$

that we can rewrite

$$\int_X \varphi(x) \, \mathrm{d}\chi_c(x) = \int_{\mathbb{C}} \left[\int_{f^{-1}(y)} \varphi(x) \, \mathrm{d}\chi_c(x) \right] \mathrm{d}y.$$

Let us compute the integral $\int_{f^{-1}(y)} \varphi(x) d\chi_c(x)$ for y in $\mathbb{C}$. Let $\tilde{y}$ be a regular value of f, which is not a stratified asymptotic non-ρ_E-regular value. Let $R_y \gg 1$ be such that $f^{-1}(y) \cap B_{R_y}$ is a retract by deformation of $f^{-1}(y)$ and let us denote by $z_1, \ldots, z_l$ the critical points of f in $f^{-1}(y)$. On the one hand, we have

$$\chi(f^{-1}(\tilde{y}) \cap B_{R_y}) = \chi\left(f^{-1}(\tilde{y}) \cap B_{R_y} \setminus (\cup_{i=1}^{l} \mathring{B}_\epsilon(z_i))\right) + \sum_{i=1}^{l} \chi(f^{-1}(\tilde{y}) \cap B_\epsilon(z_i))$$

$$= \chi_c\left(f^{-1}(y) \cap B_{R_y} \setminus (\cup_{i=1}^{l} B_\epsilon(z_i))\right) + \sum_{i=1}^{l} \varphi(z_i)$$

$$= \chi_c\left(f^{-1}(y) \cap B_{R_y} \setminus \{z_1, \ldots, z_l\}\right) + \sum_{i=1}^{l} \varphi(z_i)$$

$$= \int_{f^{-1}(y) \cap B_{R_y}} \varphi(x) \, d\chi_c(x).$$

On the other hand, we have

$$\int_{f^{-1}(y) \cap B_{R_y}} \varphi(x) \, d\chi_c(x) = \int_{f^{-1}(y)} \varphi(x) \, d\chi_c(x) - \int_{f^{-1}(y) \cap \{\rho_E > R_y\}} \varphi(x) \, d\chi_c(x)$$

$$= \int_{f^{-1}(y)} \varphi(x) d\chi_c(x) - \chi_c(f^{-1}(y) \cap \{\rho_E > R_y\}).$$

But the function $f^{-1}(y) \cap \{\rho_E > R_y\} \to]R_y, +\infty[$, $x \mapsto \rho_E(x)$ is a proper stratified submersion, so

$$\chi_c(f^{-1}(y) \cap \{\rho_E > R_y\}) = \chi_c(f^{-1}(y) \cap S_R) \times \chi_c(]R_y, +\infty[),$$

where $R \in]R_y, +\infty[$. Since $\chi_c(f^{-1}(y) \cap S_R) = 0$, we find that

$$\int_{f^{-1}(y)} \varphi(x) d\chi_c(x) = \chi(f^{-1}(\tilde{y}) \cap B_{R_y})$$

$$= \chi_c(f^{-1}(\tilde{y}) \cap B_{R_{\tilde{y}}})) - \chi_c(f^{-1}(\tilde{y}) \cap \{R_y < \rho_E \leqslant R_{\tilde{y}}\})$$

$$= \chi(f^{-1}(\tilde{y}) \cap B_{R_{\tilde{y}}})) - \chi(f^{-1}(\tilde{y}) \cap \{R_y \leqslant \rho_E \leqslant R_{\tilde{y}}\})$$

$$= \chi(f^{-1}(\tilde{y})) - \lambda_{f,y}^{X,\infty},$$

because $\chi(f^{-1}(\tilde{y}) \cap \{R_y \leqslant \rho_E \leqslant R_{\tilde{y}}\}) = \chi(f^{-1}(\tilde{y}) \cap \{R_y \leqslant \rho_E\})$. Therefore, we get

$$\int_X \varphi(x) \, d\chi_c(x) = \chi(X) - \sum_{j=1}^{s} 1 - \varphi(q_j) = \int_\mathbb{C} \chi(f^{-1}(\tilde{y})) \, dy - \int_\mathbb{C} \lambda_{f,y}^{X,\infty} \, dy$$

$$= \chi(f^{-1}(c)) - \sum_{j=1}^{r} \lambda_{f,a_j}^{X,\infty} = \chi(f^{-1}(c)) - \lambda_f^{X,\infty},$$

and so,

$$\chi(X) - \chi(f^{-1}(c)) = \sum_{j=1}^{s} 1 - \chi(f^{-1}(\tilde{q}_j) \cap B_\epsilon(q_j)) - \lambda_f^{X,\infty}.$$

By theorem 3.2 in [**23**] applied to the sheaf $\mathbb{Z}_X^\bullet$, we know that

$$1 - \chi(f^{-1}(\tilde{q}_j) \cap B_\epsilon(q_j)) = \sum_{i=1}^{t} (-1)^{d_i} n_{ij}(1 - \chi(\mathcal{L}_{V_i}^X)), \qquad (*)$$

where n_{ij} is the number of critical points of a Morsefication of f that lie on V_i in a small neighbourhood of q_j. Summing over all the critical points of f, we obtain the result. $\qquad \square$

COROLLARY 5.3. *Let* $\alpha : X \to \mathbb{Z}$ *be a constructible function with respect to* $\mathcal{V}$ *and let* $c \in \mathbb{C}$ *be a regular value of* f, *which is not a stratified asymptotic non-ρ_E-regular value. We have*

$$\chi(X, \alpha) - \chi(f^{-1}(c), \alpha) = \sum_{i=1}^{t} (-1)^{d_i} n_i \eta(V_i, \alpha) - \mathrm{B}_f^{X,\infty}(\alpha).$$

Moreover, if f *is general with respect to* $\mathcal{V}$, *then we have*

$$\chi(X, \alpha) - \chi(f^{-1}(c), \alpha) = \sum_{i=1}^{t} (-1)^{d_i} \mu^T(f_{|V_i}) \eta(V_i, \alpha) - \mathrm{B}_f^{X,\infty}(\alpha).$$

Proof. By the previous theorem, the result is true for $\alpha = \mathbf{1}_{\overline{V_i}}$. Since both sides of the equality are linear in α, we see that the result is valid for any constructible function α. $\qquad \square$

If X is equidimensional then by [**29**], proposition 2.3, the term $(-1)^{d_i} n_{ij}$ that appears in Equality ($*$) is equal to $\mathrm{Eu}_{f,\overline{V_i}}(q_j)$. Hence the above corollary can be refined.

COROLLARY 5.4. *Assume that* X *is equidimensional. Let* $\alpha : X \to \mathbb{Z}$ *be a constructible function with respect to* $\mathcal{V}$ *and let* $c \in \mathbb{C}$ *be a regular value of* f, *which is not a stratified asymptotic non-ρ_E-regular value. We have*

$$\chi(X, \alpha) - \chi(f^{-1}(c), \alpha) = \sum_{i=1}^{t} (-1)^{d_i} n_i \eta(V_i, \alpha) - \mathrm{B}_f^{X,\infty}(\alpha)$$

$$= \sum_{i=1}^{t} \left(\sum_{j=1}^{q} \mathrm{Eu}_{f,\overline{V_i}}(q_j) \right) \eta(V_i, \alpha) - \mathrm{B}_f^{X,\infty}(\alpha).$$

Moreover, if f *is general with respect to* $\mathcal{V}$, *then we have*

$$\chi(X, \alpha) - \chi(f^{-1}(c), \alpha) = \sum_{i=1}^{t} (-1)^{d_i} \mu^T(f_{|V_i}) \eta(V_i, \alpha) - \mathrm{B}_f^{X,\infty}(\alpha).$$

An interesting application occurs when $\alpha = \mathrm{Eu}_X$.

COROLLARY 5.5. *Assume that X is equidimensional. Let $c \in \mathbb{C}$ be a regular value of f, which is not a stratified asymptotic non-ρ_E-regular value. We have*

$$\mathrm{Eu}^X_{f,c} = \mathrm{Eu}(X) - \mathrm{B}^X_{f,c} = (-1)^d n_t - \mathrm{B}^{X,\infty}_f = \sum_{j=1}^s \mathrm{Eu}_{f,X}(q_j) - \mathrm{B}^{X,\infty}_f.$$

Moreover, if f is general with respect to $\mathcal{V}$, then we have

$$\mathrm{Eu}^X_{f,c} = \mathrm{Eu}(X) - \mathrm{B}^X_{f,c} = \sum_{j \;|\; q_j \in X_{\mathrm{reg}}} (-1)^d \mu(f_{|X_{\mathrm{reg}}}, q_j) - \mathrm{B}^{X,\infty}_f.$$

Proof. By definition, $\chi(X \cap f^{-1}(c), \mathrm{Eu}_X) = \mathrm{B}^X_{f,c}$ and $\mathrm{B}^{X,\infty}_f(\mathrm{Eu}_X) = \mathrm{B}^{X,\infty}_f$. By remark 2.22, $\chi(X, \mathrm{Eu}_X) = \mathrm{Eu}(X)$. But if $V_i \neq V_t$, then $\eta(V_i, \mathrm{Eu}_X) = 0$ and $\eta(V_t, \mathrm{Eu}_X) = 1$. □

REMARK 5.6. We note that only the Morse critical points lying in X_{reg} appear in the first formula of corollary 5.5. However some of them may emerge in a small neighbourhood of a critical point of f lying in a lower dimensional stratum, because f is not supposed to be general. This is a difference with Equality (2), page 401 in [**28**], where only the critical points of a generic linear form lying in X_{reg} contribute to the equality. Nevertheless, as in remark 4.8, we can conclude that the sum $\sum_{j \;|\; q_j \in X_{\mathrm{sing}}} \mathrm{Eu}_{f,X}(q_j)$ does not depend on the stratification of the singular locus of X.

Another corollary is a Brylinski-Dubson-Kashiwara type formula for the global Brasselet number at infinity.

COROLLARY 5.7. *Assume that X is equidimensional and let $\alpha : X \to \mathbb{Z}$ be a constructible function with respect to $\mathcal{V}$. We have*

$$\mathrm{B}^{X,\infty}_f(\alpha) = \sum_{i=1}^t \mathrm{B}^{\overline{V_i},\infty}_f \eta(V_i, \alpha).$$

Proof. Applying corollaries 5.4 and 5.5 to each set $\overline{V_i}$, we obtain that

$$\chi(X, \alpha) - \chi(f^{-1}(c), \alpha) + \mathrm{B}^{X,\infty}_f(\alpha)$$

$$= \sum_{i=1}^t \left[\mathrm{Eu}(\overline{V_i}) - \mathrm{B}^{\overline{V_i}}_{f,c} - \mathrm{B}^{\overline{V_i},\infty}_f \right] \eta(V_i, \alpha).$$

But we know that $\chi(X, \alpha) = \sum_{i=1}^t \mathrm{Eu}(\overline{V_i}) \eta(V_i, \alpha)$ (see [**11**], corollary 5.4) and that $\chi(f^{-1}(c), \alpha) = \sum_{i=1}^t \mathrm{B}^{\overline{V_i}}_{f,c} \eta(V_i, \alpha)$. □

Let us study what happens if we replace the generic regular value c with any value a. First, we do not assume that X is equidimensional. For $i = 1, \ldots, t$, let n^a_i be the number of critical points of $\tilde{f}$ on V_i appearing in a small neighbourhood of

a critical point of f, but that do not lie in a small neighbourhood of a critical point q of f such that $f(q) = a$. Similarly, we set

$$\mu_a^T(f_{|V_i}) = \sum_{j \mid f(q_j) \neq a} \mu(f_{|V_j}, q_j).$$

PROPOSITION 5.8. *Let $\alpha : X \to \mathbb{Z}$ be a constructible function with respect to $\mathcal{V}$ and let $a \in \mathbb{C}$. We have*

$$\chi(X, \alpha) - \chi(f^{-1}(a), \alpha) = \sum_{i=1}^{t} (-1)^{d_i} n_i^a \eta(V_i, \alpha) - \mathrm{B}_f^{X,\infty}(\alpha) + \mathrm{B}_{f,a}^{X,\infty}(\alpha).$$

Moreover if f is general with respect to $\mathcal{V}$ then we have

$$\chi(X, \alpha) - \chi(f^{-1}(a), \alpha) = \sum_{i=1}^{t} (-1)^{d_i} \mu_a^T(f_{|V_i}) \eta(V_i, \alpha) - \mathrm{B}_f^{X,\infty}(\alpha) + \mathrm{B}_{f,a}^{X,\infty}(\alpha).$$

Proof. Let c be a generic value (i.e., regular and not stratified asymptotic non-ρ_E-regular value) of f close to a. Let $R_a \gg 1$ be such that $f^{-1}(a) \cap B_{R_a}$ is a deformation retract of $f^{-1}(a)$. We have

$$\chi(f^{-1}(c) \cap B_{R_a}) = \chi(f^{-1}(c) \cap B_{R_a} \setminus (\cup_{j \mid f(q_j)=a} \mathring{B}_\epsilon(q_j)))$$
$$+ \sum_{j \mid f(q_j)=a} \chi(f^{-1}(c) \cap B_\epsilon(q_j))$$
$$= \chi_c(f^{-1}(a) \cap B_{R_a} \setminus (\cup_{j \mid f(q_j)=a} \mathring{B}_\epsilon(q_j)))$$
$$+ \sum_{j \mid f(q_j)=a} \chi(f^{-1}(c) \cap B_\epsilon(q_j))$$
$$= \chi_c(f^{-1}(a) \cap B_{R_a} \setminus (\cup_{j \mid f(q_j)=a} B_\epsilon(q_j)))$$
$$+ \sum_{j \mid f(q_j)=a} \chi(f^{-1}(c) \cap B_\epsilon(q_j))$$
$$= \chi_c(f^{-1}(a) \cap B_{R_a}) - \sum_{j \mid f(q_j)=a} 1 - \chi(f^{-1}(c) \cap B_\epsilon(q_j))$$
$$= \chi(f^{-1}(a) \cap B_{R_a}) - \sum_{j \mid f(q_j)=a} 1 - \chi(f^{-1}(c) \cap B_\epsilon(q_j))$$
$$= \chi(f^{-1}(a)) - \sum_{j \mid f(q_j)=a} 1 - \chi(f^{-1}(c) \cap B_\epsilon(q_j)).$$

But

$$\chi(f^{-1}(c)) = \chi(f^{-1}(c) \cap B_{R_a}) + \chi(f^{-1}(c) \cap \{R_a \leqslant \rho_E \leqslant R_c\})$$
$$= \chi(f^{-1}(a)) - \sum_{j \mid f(q_j)=a} 1 - \chi(f^{-1}(c) \cap B_\epsilon(q_j)) + \lambda_{f,a}^{X,\infty}.$$

Combining this with the equality proved in theorem 5.2, we get

$$\chi(X) - \chi(f^{-1}(a)) = \sum_{j \mid f(q_j) \neq a} 1 - \chi(f^{-1}(c) \cap B_\epsilon(q_j)) - \lambda_f^{X,\infty} + \lambda_{f,a}^{X,\infty}.$$

Using Massey's results ([**23**], theorem 3.2), we obtain the result for $\alpha = \mathbf{1}_X$. The general case easily follows because of the linearity in α of both sides of the equality. $\square$

If X is equidimensional, we can refine the above proposition.

COROLLARY 5.9. *Assume that X is equidimensional. Let $\alpha : X \to \mathbb{Z}$ be a constructible function with respect to $\mathcal{V}$ and let $a \in \mathbb{C}$. We have*

$$\chi(X, \alpha) - \chi(f^{-1}(a), \alpha)$$

$$= \sum_{i=1}^{t} (-1)^{d_i} n_i^a \eta(V_i, \alpha) - B_f^{X,\infty}(\alpha) + B_{f,a}^{X,\infty}(\alpha)$$

$$= \sum_{i=1}^{t} \left(\sum_{j \mid f(q_j) \neq a} \mathrm{Eu}_{f, \overline{V_i}}(q_j) \right) \eta(V_i, \alpha) - B_f^{X,\infty}(\alpha) + B_{f,a}^{X,\infty}(\alpha).$$

Moreover, if f is general with respect to $\mathcal{V}$ then we have

$$\chi(X, \alpha) - \chi(f^{-1}(a), \alpha) = \sum_{i=1}^{t} (-1)^{d_i} \mu_a^T(f_{|V_i}) \eta(V_i, \alpha) - B_f^{X,\infty}(\alpha) + B_{f,a}^{X,\infty}(\alpha).$$

As above, we can specify these equalities to the case $\alpha = \mathrm{Eu}_X$.

COROLLARY 5.10. *Assume that X is equidimensional. Let $a \in \mathbb{C}$. We have*

$$\mathrm{Eu}_{f,a}^X = \mathrm{Eu}(X) - B_{f,a}^X = (-1)^d n_t^a - B_f^{X,\infty} + B_{f,a}^{X,\infty}$$

$$= \sum_{j \mid f(q_j) \neq a} \mathrm{Eu}_{f,X}(q_j) - B_f^{X,\infty} + B_{f,a}^{X,\infty}.$$

Moreover if f is general with respect to $\mathcal{V}$, then we have

$$\mathrm{Eu}_{f,a}^X = \mathrm{Eu}(X) - B_{f,a}^X = \sum_{\substack{j \mid q_j \in X_{\mathrm{reg}} \\ f(q_j) \neq a}} (-1)^d \mu(f_{|X_{\mathrm{reg}}}, q_j) - B_f^{X,\infty} + B_{f,a}^{X,\infty}.$$

REMARK 5.11. As in remark 5.6, we see that the critical points of f lying in a lower dimensional stratum can have a contribution in the first equalities of corollary 5.10, when f is not general. However the sum $\sum_{\substack{j \mid f(q_j) \neq a \\ q_j \in X_{\mathrm{sing}}}} \mathrm{Eu}_{f,X}(q_j)$ does not depend on the stratification of the singular locus of X.

We can also give a version of the Brylinski-Dubson-Kashiwara formula for the Brasselet numbers at infinity $B_{f,a}^{X,\infty}$.

COROLLARY 5.12. *Assume that X is equidimensional. Let $\alpha : X \to \mathbb{Z}$ be a constructible function with respect to $\mathcal{V}$ and let $a \in \mathbb{C}$. We have*

$$\mathrm{B}_{f,a}^{X,\infty}(\alpha) = \sum_{i=1}^{t} \mathrm{B}_{f,a}^{\overline{V_i},\infty} \eta(V_i, \alpha).$$

Proof. Apply the previous two corollaries and proceed as in the proof of corollary 5.7. $\qquad\square$

An easy corollary is a relation between $\mathrm{Eu}_{f,a}^{X}$ (resp. $\mathrm{B}_{f,a}^{X}$) and $\mathrm{Eu}_{f,c}^{X}$ (resp. $\mathrm{B}_{f,c}^{X}$) where c is generic.

COROLLARY 5.13. *Assume that X is equidimensional. Let $c \in \mathbb{C}$ be a regular value of f, which is not a stratified asymptotic non-ρ_E-regular value, and let $a \in \mathbb{C}$. We have*

$$\mathrm{Eu}_{f,c}^{X} - \mathrm{Eu}_{f,a}^{X} = \mathrm{B}_{f,a}^{X} - \mathrm{B}_{f,c}^{X} = (-1)^d (n_t - n_t^a) - \mathrm{B}_{f,a}^{X,\infty}$$

$$= \sum_{j \mid f(q_j)=a} \mathrm{Eu}_{f,X}(q_j) - \mathrm{B}_{f,a}^{X,\infty}.$$

Moreover, if f is general with respect to $\mathcal{V}$, then we have

$$\mathrm{Eu}_{f,c}^{X} - \mathrm{Eu}_{f,a}^{X} = \mathrm{B}_{f,a}^{X} - \mathrm{B}_{f,c}^{X} = \sum_{\substack{j \mid q_j \in X_{\mathrm{reg}} \\ f(q_j)=a}} (-1)^d \mu(f_{|X_{\mathrm{reg}}}, q_j) - \mathrm{B}_{f,a}^{X,\infty}.$$

When f has no stratified asymptotic non-ρ_E-regular values then in all the above equalities, the terms $\lambda_f^{X,\infty}$, $\lambda_{f,a}^{X,\infty}$, $\mathrm{B}_f^{X,\infty}(\alpha)$, $\mathrm{B}_{f,a}^{X,\infty}(\alpha)$, $\mathrm{B}_f^{X,\infty}$ and $\mathrm{B}_{f,a}^{X,\infty}$ vanish. From now on, we assume that X is equidimensional. If $f = l$ is the restriction to X of a generic linear function $L : \mathbb{C}^N \to \mathbb{C}$, then l has no stratified asymptotic non-ρ_E-regular values and moreover l is a stratified Morse function (see [**28**], lemma 3.1).

Keeping the notations introduced in [**28**], we denote by $\alpha_X^{(d)}$ the number of (Morse) critical points of l on X_{reg} and by $\alpha_{X,a}^{(d)}$ those not occurring on $l^{-1}(a)$. By corollary 5.10, if c is a regular value of l then $\mathrm{Eu}_{l,c}^{X} = (-1)^d \alpha_X^{(d)}$ and if a is a critical value of l, then $\mathrm{Eu}_{l,a}^{X} = (-1)^d \alpha_{X,a}^{(d)}$. By the relation between $\mathrm{B}_{l,a}^{X}$ and $\mathrm{Eu}(l^{-1}(a))$ proved in proposition 4.7, we obtain

$$\mathrm{Eu}(X) - \mathrm{Eu}(l^{-1}(a)) = (-1)^d \alpha_{X,a}^{(d)} + \sum_{j \mid l(q_j)=a} \mathrm{Eu}_X(q_j) - \mathrm{Eu}_{l^{-1}(a)}(q_j),$$

where the q_j's are the critical points of l. For a regular value c of l, this gives

$$\mathrm{Eu}(X) - \mathrm{Eu}(l^{-1}(c)) = (-1)^d \alpha_X^{(d)},$$

and so we have extended Equality (2), page 401 in [**28**]. Based on this equality, Seade, Tibăr and Verjovsky could express the global Euler obstruction as an alternating sum of global polar invariants. In the sequel, we will establish a relative

version of this result for the global Brasselet number and the Brasselet numbers at infinity.

So we consider a polynomial function $f : X \to \mathbb{C}$, restriction to X of a polynomial function $F : \mathbb{C}^N \to \mathbb{C}$. We assume that f has a finite number of critical points, which means that for $i = 1, \ldots, t$, $F_{|V_i}$ has a finite number of critical points. We denote by $\{q_1, \ldots, q_s\}$ the set of critical points of f. For $a \in \mathbb{C}$, we put $X_a = f^{-1}(a)$. The algebraic set X_a is equidimensional and if $q_1, \ldots, q_u$, $u \leqslant s$, are the critical points of f on $f^{-1}(a)$, then

$$\mathcal{V}_a = \left(\sqcup_{i=1}^t V_i \cap f^{-1}(a) \setminus \{q_1, \ldots, q_u\} \right) \cup \left(\sqcup_{j=1}^u \{q_j\} \right),$$

is a Whitney stratification of X_a.

Let $L : \mathbb{C}^N \to \mathbb{C}$ be a linear function and $l : X \to \mathbb{C}$ be its restriction to X. We denote by $\Gamma_{f,l}^X$ the relative polar variety of f and l. It is defined as follows:

$$\Gamma_{f,l}^X = \overline{\{x \in X_{\mathrm{reg}} \mid \mathrm{rank}[\mathrm{d}f(x), \mathrm{d}l(x)] < 2\}}.$$

It is well-known that for L generic, $\Gamma_{f,l}^X$ is a reduced algebraic curve. Moreover, if L is generic, we can assume the following fact:

$$l_{|X_a} : X_a \to \mathbb{C} \text{ is } \rho\text{-regular at infinity and Morse stratified.}$$

Let $I_X(\Gamma_{f,l}^X, X_a)$ be the global intersection multiplicity of $\Gamma_{f,l}^X$ and X_a, namely

$$I_X(\Gamma_{f,l}^X, X_a) = \sum_{p \in \Gamma_{f,l}^X \cap f^{-1}(a)} I_p(\Gamma_{f,l}^X, X_a),$$

where $I_p(\Gamma_{f,l}^X, X_a)$ is the local intersection multiplicity of $\Gamma_{f,l}^X$ and X_a at p. If $\dim(X) = 1$ then $\Gamma_{f,l}^X = X$ and in this case $I_p(\Gamma_{f,l}^X, X_a)$ is the degree of $l : (X, p) \to (\mathbb{C}, a)$, that is the cardinality of $l^{-1}(c) \cap X \cap B_\epsilon(p)$ for $0 < |c - a| \ll \epsilon \ll 1$.

PROPOSITION 5.14. *We have*

$$\mathrm{B}_{f,a}^X - \mathrm{B}_{f,a}^{X \cap H} = (-1)^{d-1} I_X(\Gamma_{f,l}^X, X_a) + \sum_{j=1}^u \mathrm{Eu}_{f,X}(q_j),$$

where H is a generic hyperplane given by $H = L^{-1}(g)$ for a regular value g of $l_{|X_a}$ and $l_{|X}$.

Proof. Let us treat first the case $\dim(X) > 1$. Applying Equality (2) of [28] that we have mentioned above to X_a and $X_a \cap L^{-1}(g)$, we get

$$\mathrm{Eu}(X_a) - \mathrm{Eu}(X_a \cap H) = (-1)^{d-1} \alpha_{X_a}^{(d-1)},$$

where $\alpha_{X_a}^{(d-1)}$ is the number of critical points of $l_{|X_a}$ on $X_{\mathrm{reg}} \cap (f^{-1}(a) \setminus \{q_1, \ldots, q_u\})$. Since g is a regular value of $l_{|X_a}$ then

$$\mathrm{Eu}(X_a \cap H) = \sum_{i=1}^t \chi(V_i \cap f^{-1}(a) \cap H) \mathrm{Eu}_{X_a \cap H}(V_i \cap H \cap f^{-1}(a)).$$

The hyperplane H intersects X transversally. Furthermore, because the intersections $V_i \cap f^{-1}(a) \cap H$ are transverse, we know that

$$\mathrm{Eu}_{X_a \cap H}(V_i \cap H \cap f^{-1}(a)) = \mathrm{Eu}_{X \cap H}(V_i \cap H),$$

which implies that $\mathrm{Eu}(X_a \cap H) = \mathrm{B}_{f,a}^{X \cap H}$. Applying proposition 4.7, we obtain

$$\mathrm{B}_{f,a}^X - \mathrm{B}_{f,a}^{X \cap H} = (-1)^{d-1} \alpha_{X_a}^{(d-1)} + \sum_{j=1}^{u} \mathrm{Eu}_X(q_j) - \mathrm{Eu}_{X_a}(q_j).$$

But $I_X(\Gamma_{f,l}^X, X_a)$ is equal to

$$\alpha_{X_a}^{(d-1)} + \sum_{j=1}^{u} I_{q_j}(\Gamma_{f,l}^X, X_a).$$

By corollary 5.2 in [12], we have

$$(-1)^{d-1} I_{q_j}(\Gamma_{f,l}^X, X_a) = \mathrm{B}_{f,X}(q_j) - \mathrm{B}_{f,X \cap H_j}(q_j),$$

where $H_j = L^{-1}(L(q_j))$. Corollary 6.6 in [12] implies that

$$(-1)^{d-1} I_{q_j}(\Gamma_{f,l}^X, X_a) = \mathrm{B}_{f,X}(q_j) - \mathrm{Eu}_{X_a}(q_j) = \mathrm{Eu}_X(q_j) - \mathrm{Eu}_{f,X}(q_j) - \mathrm{Eu}_{X_a}(q_j),$$

and so

$$\mathrm{Eu}_X(q_j) - \mathrm{Eu}_{X_a}(q_j) = (-1)^{d-1} I_{q_j}(\Gamma_{f,l}^X, X_a) + \mathrm{Eu}_{f,X}(q_j).$$

If $\dim(X) = 1$ then $\mathrm{B}_{f,a}^{X \cap H} = 0$ and

$$\mathrm{B}_{f,a}^X = \sum_{p \in f^{-1}(a)} \mathrm{Eu}_X(p) = \# f^{-1}(a) \setminus \{q_1, \ldots, q_u\} + \sum_{j=1}^{u} \mathrm{Eu}_X(q_j).$$

Applying theorem 3.1 in [4], it is easy to see that

$$\mathrm{Eu}_X(q_j) = \mathrm{Eu}_{f,X}(q_j) + I_{q_j}(\Gamma_{f,l}^X, X_a).$$

Since $I_p(\Gamma_{f,l}^X, X_a) = 1$ if p is a regular point of f, we obtain the result. □

By a standard connectivity argument, $I_X(\Gamma_{f,l}^X, X_a)$ does not depend on the choice of the generic linear function L. Following Tibăr's notation [36], we denote it by $\gamma_{X,a}^{(d-1)}$. Similarly for $i = 2, \ldots, d$, we define

$$\gamma_{X,a}^{(d-i)} = I_{X \cap H^{i-1}}(\Gamma_{f,l}^{X \cap H^{i-1}}, X_a \cap H^{i-1}),$$

where H^{i-1} is a generic linear space of codimension $i - 1$. The following statement is a relative version of the Seade-Tibăr-Verjovsky polar formula for the global Euler obstruction.

COROLLARY 5.15. *We have*

$$\mathrm{B}_{f,a}^X = \sum_{i=1}^{d}(-1)^{d-i}\gamma_{X,a}^{(d-i)} + \sum_{j=1}^{u}\mathrm{Eu}_{f,X}(q_j).$$

Proof. We apply the previous result to $X \cap H^{i-1}$. Note that for $i \geqslant 2$, we can choose H^{i-1} generic enough so that H^{i-1} intersects X_a transversally, which implies that $f^{-1}(a)$ intersects $H^{i-1} \cap X$ transversally. $\qquad\square$

If $a = c$ is a generic value of f, then the above corollary becomes

$$\mathrm{B}_{f,c}^X = \sum_{i=1}^{d}(-1)^{d-i}\gamma_{X,c}^{(d-i)}.$$

If we apply this to $f = l$, the restriction to X of a generic linear function $L : \mathbb{C}^n \to \mathbb{C}$, then for $i = 1, \ldots, d$, $\gamma_{X,c}^{d-i}$ is exactly equal to the number $\alpha_X^{(d-i)}$ defined in [**28**], which is the number of critical points of a generic linear function on $X_{\mathrm{reg}} \cap H^{i-1}$. Combining this fact with the equality

$$\mathrm{B}_{f,c}^X = \mathrm{Eu}(X) - \mathrm{Eu}_{f,c}^X = \mathrm{Eu}(X) - (-1)^d\alpha_X^{(d)},$$

we obtain

$$\mathrm{Eu}(X) = \sum_{i=0}^{d}(-1)^{d-i}\alpha_X^{(d-i)} = \sum_{i=0}^{d}(-1)^i\alpha_X^{(i)},$$

that is, the main result of [**28**].

Another corollary is a characterization of the Brasselet numbers at infinity in terms of critical points of generic linear forms.

COROLLARY 5.16. *Let $a \in \mathbb{C}$ be a stratified asymptotic non-ρ_E-regular value of f and let $c \in \mathbb{C}$ be a generic regular value of f. We have*

$$\mathrm{B}_{f,a}^{X,\infty} = \sum_{i=1}^{d}(-1)^{d-i}\left(\gamma_{X,c}^{(d-i)} - \gamma_{X,a}^{(d-i)}\right).$$

Proof. Use the previous corollary and the equality

$$\mathrm{B}_{f,a}^{X,\infty} = \mathrm{B}_{f,c}^X - \mathrm{B}_{f,a}^X + \sum_{j=1}^{u}\mathrm{Eu}_{f,X}(q_j). \quad \square$$

If $\alpha : X \to \mathbb{Z}$ is a constructible function relative to $\mathcal{V}$, then the previous equality, combined with the Brylinski-Dubson-Kashiwara formula for $B_{f,a}^{X,\infty}$ proved in

corollary 5.12, gives

$$\mathrm{B}_{f,a}^{X,\infty}(\alpha) = \sum_{j=1}^{t} \left(\sum_{i=1}^{d_j} (-1)^{d_j-i} (\gamma_{\overline{V_j},c}^{(d_j-i)} - \gamma_{\overline{V_j},a}^{(d_j-i)}) \right) \eta(V_j, \alpha).$$

In particular for $\alpha = \mathbf{1}_X$, we get

$$\lambda_{f,a}^{X,\infty} = \sum_{j=1}^{t} \left(\sum_{i=1}^{d_j} (-1)^{d_j-i} (\gamma_{\overline{V_j},c}^{(d_j-i)} - \gamma_{\overline{V_j},a}^{(d_j-i)}) \right) (1 - \chi(\mathcal{L}_{V_j}^X)).$$

We end this section with an application. We assume that $\dim(X) \geqslant 2$ and that f is general with respect to $\mathcal{V}$. We also suppose that there exists $l : X \to \mathbb{C}$, restriction to X of a linear form $L : \mathbb{C}^N \to \mathbb{C}$, such that l has no stratified asymptotic non-ρ_E-regular values and is general with respect to $\mathcal{V}$ and such that the mapping $(f,l)_{|X_{\mathrm{reg}}} : X_{\mathrm{reg}} \to \mathbb{C}^2$ is a submersion. In this situation, corollary 5.5 gives $\mathrm{Eu}_{f,c}^X = -\mathrm{B}_f^{X,\infty}$ because f has no critical points on X_{reg}. Similarly $\mathrm{Eu}_{f,c}^{X \cap H} = -\mathrm{B}_f^{X \cap H,\infty}$, where $H = L^{-1}(g)$ for a regular value g of $l : X \to \mathbb{C}$. But, by proposition 5.14 applied to f and l, we find

$$\mathrm{B}_{f,c}^X - \mathrm{B}_{f,c}^{X \cap H} = 0 = \mathrm{Eu}(X) - \mathrm{Eu}_{f,c}^X - \mathrm{Eu}(X \cap H) + \mathrm{Eu}_{f,c}^{X \cap H}.$$

But $\mathrm{Eu}(X) = \mathrm{Eu}(X \cap H)$ because l has no critical points on X_{reg}. Finally, we obtain that $\mathrm{B}_f^{X,\infty} = \mathrm{B}_f^{X \cap H,\infty}$. This extends the fact that for $X = \mathbb{C}^2$, $\lambda_f^{\mathbb{C}^2,\infty} = \lambda_f^{H,\infty}$. Indeed in this situation, $\lambda_f^{\mathbb{C}^2,\infty} = 0$ because the mapping $(f,l) : \mathbb{C}^2 \to \mathbb{C}^2$ is a submersion (see for instance [**20**]) and $\lambda_f^{H,\infty} = 0$ because $f_{|H}$ is proper.

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