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MULTIPLE SOLUTIONS OF SUBLINEAR QUASILINEAR SCHRÖDINGER EQUATIONS WITH SMALL PERTURBATIONS

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Abstract In this paper, we consider the existence of multiple solutions for the quasilinear Schrödinger equation

$$\begin{cases} -\Delta u - \Delta(|u|^{\alpha})|u|^{\alpha-2}u = g(x,u) + \theta h(x,u), & x \in \Omega \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$ $(N \geq 1)$, $\alpha \geq 2$ and θ is a parameter. Under the assumption that $g(x, u)$ is sublinear near the origin with respect to u, we study the effect of the perturbation term $h(x, u)$, which may break the symmetry of the associated energy functional. With the aid of critical point theory and the truncation method, we show that this system possesses multiple small negative energy solutions.

Keywords: quasilinear Schrödinger equations; multiple solutions; critical point theory

2010 *Mathematics Subject Classification:* Primary 35J20; 35J65; 35Q55

1. Introduction and main results

This paper deals with the quasilinear Schrödinger equation

$$\begin{cases} -\Delta u - \Delta(|u|^{\alpha})|u|^{\alpha-2}u = g(x,u) + \theta h(x,u), & x \in \Omega \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$ $(N \geq 1)$, θ is a parameter and $\alpha \geq 2$. The quasilinear elliptic equation appears naturally in several physical models, for instance in the superfluid film equation in plasma physics. For more physical motivations and detailed

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information on applications, we refer readers to [12, 13, 22] and the references therein. In recent years, the quasilinear elliptic problem in a bounded domain or whole space has been widely studied for both its importance in applications and its mathematical interest; see, e.g. [6, 7, 9, 15–18, 20, 23].

When $g(x, t)$ is odd in t and $h \equiv 0$, (1.1) possesses a natural $\mathbb{Z}_2$ symmetry, and some results of multiple solutions for quasilinear Schrödinger equations in bounded domains or whole space have been obtained with g satisfying various conditions, see [14, 25, 26, 32, 33] and the references therein. It is worth pointing out that the $\mathbb{Z}_2$ symmetry plays a crucial part in these works. In this paper, we mainly focus on the situation when the symmetry of (1.1) is broken by the effects of non-odd term h. To be more precise, if $g(x, t)$ is odd and $h(x, t)$ is not odd in t, a natural question is whether multiple solutions persist for (1.1) in the absence of symmetry. As far as we know, this perturbation problem for quasilinear Schrödinger equations has not been much investigated, and we are aware of only one paper [29] in this direction. In [29], we proved that if $g, h \in C(\bar{\Omega} \times \mathbb{R})$ and $g(x, t)$ is only locally superlinear with respect to t at the origin, then for any $j \in \mathbb{N}$, there exists $\theta_j > 0$ such that if $|\theta| \le \theta_j$, (1.1) has at least j distinct weak solutions.

In the present work, we consider the perturbation problem in the sublinear case. The objective of this paper is to prove the existence of multiple solutions for (1.1) under the assumption that g satisfies a sublinear growth condition around the origin. Our main approach is based on minimax methods and the truncation technique. Roughly speaking, the main idea of our proof is to find a suitable truncation of the energy functional of (1.1), in order to obtain a modified functional that has almost the same small critical values as the original functional; then, we can obtain multiple solutions for (1.1). Now we are ready to state our main results, as follows.

Theorem 1.1. *Assume that g and h satisfy the following conditions:*

(W_1) $g(x, t) = g_1(x, t) + g_2(x, t)$, $g_1 \in C(\bar{\Omega} \times \mathbb{R})$ *and there exist constants* $C_0 > 0$ *and* $1 < p < 2\alpha$ *such that*

$$|g_1(x,t)| \le C_0|t|^{p-1}, \quad \forall\,(x,t) \in \bar{\Omega} \times \mathbb{R}; \tag{1.2}$$

(W_2) *there exists a constant* $1 < \mu < 2$ *such that*

$$g_1(x,t)t \le \mu G_1(x,t), \quad \forall\,(x,t) \in \bar{\Omega} \times \mathbb{R},$$

where $G_1(x, t) := \int_0^t g_1(x, s)\,\mathrm{d}s$;

(W_3) $G_1(x, t) \ge 0$, $(x, t) \in \Omega \times \mathbb{R}$ *and*

$$\lim_{t\to 0} \frac{g_1(x,t)}{t} = +\infty \quad \textit{uniformly for } x \in \Omega; \tag{1.3}$$

(W_4) $g_1(x, t) = -g_1(x, -t)$, $\forall\,(x, t) \in \bar{\Omega} \times \mathbb{R}$;

(W_5) g_2 *is a continuous function defined on* $\bar{\Omega} \times [-\delta_0, \delta_0]$ *with some* $\delta_0 > 0$ *and there exist constants* $C_1 > 0$ *and* $\alpha_1 > 2\alpha$ *such that*

$$|g_2(x,t)| \le C_1|t|^{\alpha_1-1} \quad \textit{for } (x,t) \in \bar{\Omega} \times [-\delta_0, \delta_0];$$

(W_6) $g_2(x, t) = -g_2(x, -t)$ *for* $(x, t) \in \bar{\Omega} \times [-\delta_0, \delta_0]$;

(H_1) $h(x, t)$ *is a continuous function defined on* $\bar{\Omega} \times [-\delta_1, \delta_1]$ *for some* $\delta_1 > 0$.

Then, for any $j \in \mathbb{N}$, *there exists* $\theta_j > 0$ *such that if* $|\theta| \le \theta_j$, *(1.1) possesses at least* j *distinct weak solutions.*

Remark 1.1. Since we only assume the perturbation term h is continuous without restricting the growth range with extra bounds, the classical perturbation methods in [**2**–**4**, **10**, **11**, **19**, **24**, **27**, **28**, **30**, **31**] cannot be employed to solve our problem directly. We develop a new variational method based on the minimax methods. Moreover, our method can also be applied to the perturbation from symmetry problem of elliptic systems and Hamiltonian systems.

Corollary 1.1. *Assume that* g *and* h *satisfy* (W_1)–(W_6), (H_1) *and the following condition:*

(H_2) $h(x, t) = -h(x, -t)$ *for* $\bar{\Omega} \times [-\delta_1, \delta_1]$.

Then (1.1) possesses a sequence of small negative energy solutions approaching zero.

The rest of this paper is organized as follows. In § 2, we introduce a cut-off function to define a modified functional φ_θ, and some useful estimates for φ_θ are given. Then we prove φ_θ satisfies the Palais–Smale condition and construct several minimax sequences related to the critical values of φ_θ, after which we can prove multiple critical values of φ_θ and show that φ_θ shares the same small critical values as the energy functional of (1.1). Last we give an example to illustrate our result in § 4.

Notation. Throughout the paper we shall denote C_i various positive constants which may vary from line to line but are not essential to our proofs.

2. Some preliminary lemmas

First we introduce some functional spaces which will be useful in the sequel. As usual, for $1 \le \nu < +\infty$, let

$$\|u\|_\nu = \left(\int_\Omega |u(x)|^\nu \, \mathrm{d}x \right)^{1/\nu}, \quad u \in L^\nu(\Omega).$$

Throughout this paper, we denote by E the usual Sobolev space $H_0^1(\Omega)$ equipped with the following inner product and norm

$$(u, v) = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x, \quad \|u\| = (u, u)^{1/2}, \quad \forall\, u, v \in H_0^1(\Omega).$$

It is well known that E is continuously embedded into $L^\nu(\Omega)$ for any $1 \le \nu \le 2^*$, i.e. there exists $\tau_\nu > 0$ such that

$$\|u\|_\nu \le \tau_\nu \|u\|, \quad \forall\, u \in E. \tag{2.1}$$

Moreover, E is compactly embedded into $L^\nu(\Omega)$ only for any $1 \le \nu < 2^*$.

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In view of (W_5) and (H_1) in Theorem 1.1, the terms g_2 and h are only locally defined, so we cannot apply the variational methods directly. To overcome this difficulty, we use a cut-off method to modify $g_2(x,\, t)$ and $h(x,\, t)$ for t outside a neighbourhood of the origin. In detail, we have the following lemma.

Lemma 2.1. *Assume that (W_5) and (H_1) are satisfied. Then there exist two new functions $\widetilde{g}_2(x,\, t)$ and $\widetilde{h}(x,\, t)$ possessing the following properties:*

(i) *$\widetilde{g}_2 \in C(\bar{\Omega} \times \mathbb{R})$ and there exists a constant $2\alpha < \alpha_1' < 2^*\alpha$ such that*

$$|\widetilde{g}_2(x,t)| \le C_1 |t|^{\alpha_1' - 1}, \quad \forall\,(x,t) \in \Omega \times \mathbb{R},$$

where $2^* := 2N/(N-2)$ if $N \ge 3$ and $2^* := \infty$ if $N = 1,\, 2$;

(ii) *there exists a positive constant $\delta_0' \le \min\{\delta_0/2,\, 1/2\}$ such that*

$$\widetilde{g}_2(x,t) = g_2(x,t) \quad \text{for } (x,t) \in \bar{\Omega} \times [-\delta_0', \delta_0'];$$

(iii) *$\widetilde{h} \in C(\bar{\Omega} \times \mathbb{R})$ and $|\widetilde{h}(x,\, t)| \le C_2,\ \forall\,(x,\, t) \in \bar{\Omega} \times \mathbb{R}$, where C_2 is a positive constant;*

(iv) *there exists a positive constant $\delta_1' \le \delta_1/2$ such that*

$$\widetilde{h}(x,t) = h(x,t) \quad \text{for } (x,t) \in \bar{\Omega} \times [-\delta_1', \delta_1'].$$

Proof. First we prove (i) and (ii). Choose a constant $\delta_0' = \min\{\delta_0/2,\, 1/2\}$. Define a cut-off function $\chi_0 \in C^1(\mathbb{R}, \mathbb{R})$ such that $\chi_0(t) = 1$ for $t \le 1$, $\chi_0(t) = 0$ for $t \ge 2$ and $-2 \le \chi_0'(t) < 0$ for $1 < t < 2$. Set

$$\widetilde{g}_2(x,t) = \chi_0(t^2/{\delta_0'}^2)\, g_2(x,t), \quad \forall\,(x,t) \in \bar{\Omega} \times \mathbb{R}. \tag{2.2}$$

By (W_5), (W_6) and (2.2), it is easy to verify that (i) and (ii) hold and

$$\widetilde{g}_2(x,t) = -\widetilde{g}_2(x,-t), \quad \forall\,(x,t) \in \bar{\Omega} \times \mathbb{R}. \tag{2.3}$$

Next we prove (iii) and (iv). Choose a constant $\delta_1' = \delta_1/2$, define

$$\widetilde{h}(x,t) = \chi_0(t^2/{\delta_1'}^2)\, h(x,t), \quad \forall\,(x,t) \in \bar{\Omega} \times \mathbb{R}. \tag{2.4}$$

Since $h(x,\, t) \in C(\bar{\Omega} \times \mathbb{R})$, (2.4) implies (iii) and (iv). The proof is completed.

Next we introduce the following modified nonlinear Schrödinger equation

$$\begin{cases} -\Delta u - \Delta(|u|^\alpha)|u|^{\alpha-2}u = \widetilde{g}(x,u) + \theta \widetilde{h}(x,u), & x \in \Omega \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{2.5}$$

where $\widetilde{g} := g_1 + \widetilde{g}_2$, $\widetilde{g}_2$ and $\widetilde{h}$ are defined by (2.2) and (2.4).

By direct computation, (2.5) is the Euler–Lagrange equation associated with the energy functional $J_\theta : \mathbb{R} \times E \to \mathbb{R}$ given by

$$J_\theta(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x + \frac{1}{2\alpha}\int_\Omega |\nabla(|u|^\alpha)|^2 \,\mathrm{d}x - \int_\Omega \widetilde{G}(x,u)\,\mathrm{d}x - \theta \int_\Omega \widetilde{H}(x,u)\,\mathrm{d}x, \tag{2.6}$$

where $\widetilde{G}(x,\, t) := \int_0^t \widetilde{g}(x,\, s)\,\mathrm{d}s$ and $\widetilde{H}(x,\, t) := \int_0^t \widetilde{h}(x,\, s)\,\mathrm{d}s$. Since J_θ is not well defined in $\mathbb{R} \times E$, we employ a dual approach as in [**6**, **16**] to overcome this difficulty. Precisely

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speaking, the idea of the dual approach is that the quasilinear Schrödinger equation (1.1) can be reduced to a semilinear equation by the use of a suitable function f, then the classical Sobolev space framework can be used as the working space. In the spirit of the transformation introduced in [1], we make the change of variables by $v = f^{-1}(u)$, where the function f can be defined by

$$f'(t) = (1 + \alpha|f(t)|^{2(\alpha-1)})^{-1/2}, t \in [0, +\infty) \quad \text{and} \quad f(-t) = -f(t), t \in (-\infty, 0].$$

Next we collect some properties of the function f, which will be very useful in the remainder of the paper. The detailed proof can be found in [1]. □

Lemma 2.2. *The function f and its derivative have the following properties:*

(f_1) *f is a uniquely defined C^∞ function and invertible;*

(f_2) $0 < f'(t) \leq 1$ *and* $|f(t)| \leq |t|, \ \forall\, t \in \mathbb{R}$;

(f_3) $\lim_{t\to 0}((|f(t)|)/(|t|)) = 1$ *and* $\lim_{t\to\infty}((|f(t)|^\alpha)/(|t|)) = \sqrt{\alpha}$;

(f_4) *there exists a positive constant C such that* $|f(t)|^{\alpha-1} f'(t) \leq C, \ \forall\, t \in \mathbb{R}$;

(f_5) $f''(t)f(t) = (\alpha - 1)(f'(t))^2((f'(t))^2 - 1), \ \forall\, t \in \mathbb{R}$.

Therefore, by a change of variable and (2.6), we obtain the following functional

$$\begin{aligned} I_\theta(v) := J_\theta(f(v)) = \tfrac{1}{2}\int_\Omega |\nabla v|^2 \,\mathrm{d}x - \int_\Omega \widetilde{G}(x, f(v)) \,\mathrm{d}x \\ - \theta \int_\Omega \widetilde{H}(x, f(v)) \,\mathrm{d}x, \quad (\theta, v) \in \mathbb{R} \times E. \end{aligned}$$

By Lemmas 2.1 and 2.2, for fixed $\theta_0 \in \mathbb{R}$, $I_{\theta_0} \in C^1(E, \mathbb{R})$ and

$$\langle I'_{\theta_0}(v), w\rangle = (v, w) - \int_\Omega \widetilde{g}(x, f(v)) f'(v) w \,\mathrm{d}x - \theta_0 \int_\Omega \widetilde{h}(x, f(v)) f'(v) w \,\mathrm{d}x$$

for any v, $w \in E$. It is obvious that the critical points of I_θ are the weak solutions of the following problem

$$\begin{cases} -\Delta v = (1 + \alpha|f(v)|^{2(\alpha-1)})^{-1/2}(\widetilde{g}(x, f(v)) + \theta\widetilde{h}(x, f(v))), & x \in \Omega, \\ v = 0, \quad x \in \partial\Omega. \end{cases} \tag{2.7}$$

Arguing similarly to the proof of Lemma 2.6 and Remark 2.7 in [1], if $v_0 \in E$ is a critical point of the functional I_{θ_0}, then $u_0 = f(v_0) \in E$ is a weak solution of problem (2.5) with $\theta = \theta_0$. Next we introduce a modified functional φ_θ. When θ is small enough, we can show that the functional φ_θ possesses the same multiple critical values as I_θ.

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First we introduce a cut-off function $\zeta \in C^{\infty}(\mathbb{R}, \mathbb{R})$ satisfying

$$\begin{cases} \zeta(t) = 1, & t \in (-\infty, 1] \\ 0 \le \zeta(t) \le 1, & t \in (1, 2) \\ \zeta(t) = 0, & t \in [2, \infty) \\ |\zeta'(t)| \le 2, & t \in \mathbb{R}. \end{cases} \tag{2.8}$$

With the help of this cut-off function ζ, we define

$$k(v) = \zeta\left(\frac{\|v\|^2}{T_0}\right), \quad \forall\, v \in E, \tag{2.9}$$

where T_0 is a positive constant independent of v determined by both (2.24) and (2.26).

Lemma 2.3. *The functional k given by (2.9) is of $C^1(E, \mathbb{R})$ and*

$$\left|\left\langle k'(v), \frac{f(v)}{f'(v)} \right\rangle\right| \le C_3, \quad \forall\, v \in E, \tag{2.10}$$

where C_3 is a positive constant independent of v.

Proof. By (2.9) and direct calculation, we have

$$\langle k'(v), w\rangle = 2\zeta'\left(\frac{\|v\|^2}{T_0}\right)\frac{(v, w)}{T_0}, \quad \forall\, v,\, w \in E. \tag{2.11}$$

Assume that $v_n \to v_0$ in E. In view of (2.11), for any $w \in E$, we obtain

$$\begin{aligned} |\langle k'(v_n) - k'(v_0), w\rangle| &= 2\left|\zeta'\left(\frac{\|v_n\|^2}{T_0}\right)\frac{(v_n, w)}{T_0} - \zeta'\left(\frac{\|v_0\|^2}{T_0}\right)\frac{(v_0, w)}{T_0}\right| \\ &\le 2{T_0}^{-1}\|w\|\left[\left|\zeta'\left(\frac{\|v_n\|^2}{T_0}\right)\right|\|v_n - v_0\| \right. \\ &\quad \left. + \left|\zeta'\left(\frac{\|v_n\|^2}{T_0}\right) - \zeta'\left(\frac{\|v_0\|^2}{T_0}\right)\right|\|v_0\|\right], \end{aligned}$$

which implies that $\|k'(v_n) - k'(v_0)\|_{E^*} \to 0$, $n \to \infty$. This means that $k \in C^1(E, \mathbb{R})$. By (f_5) in Lemma 2.2 and direct computation, there exists a positive constant C_4 independent of v such that

$$\left\|\frac{f(v)}{f'(v)}\right\| \le C_4\|v\|, \quad \forall\, v \in E. \tag{2.12}$$

In combination with (2.8), (2.11) and (2.12), we get

$$\left|\left\langle k'(v), \frac{f(v)}{f'(v)}\right\rangle\right| \le 2C_4\left|\zeta'\left(\frac{\|v\|^2}{T_0}\right)\right|\frac{\|v\|^2}{T_0} \le 8C_4, \quad \forall\, v \in E,$$

which implies that (2.10) holds. This completes the proof.

Next we introduce a modified functional φ_θ on $\mathbb{R} \times E$ as follows:

$$\varphi_\theta(v) = \tfrac{1}{2}\|v\|^2 - \int_\Omega G_1(x, f(v))\,\mathrm{d}x - k(v)\int_\Omega \widetilde{G}_2(x, f(v))\,\mathrm{d}x - \theta\int_\Omega \widetilde{H}(x, f(v))\,\mathrm{d}x, \tag{2.13}$$

where $G_1(x, t) := \int_0^t g_1(x, s)\,\mathrm{d}s$ and $\widetilde{G}_2(x, t) := \int_0^t \widetilde{g}_2(x, s)\,\mathrm{d}s$. Under assumptions of Theorem 1.1, by Lemmas 2.1 and 2.3, for fixed $\theta_0 \in \mathbb{R}$, it is easy to verify that $\varphi_{\theta_0} \in C^1(E, \mathbb{R})$ and

$$\begin{aligned}\langle \varphi'_{\theta_0}(v), w\rangle = (v, w) &- \int_\Omega g_1(x, f(v))f'(v)w\,\mathrm{d}x - \langle k'(v), w\rangle \int_\Omega \widetilde{G}_2(x, f(v))\,\mathrm{d}x \\ &- k(v)\int_\Omega \widetilde{g}_2(x, f(v))f'(v)w\,\mathrm{d}x - \theta_0\int_\Omega \widetilde{h}(x, f(v))f'(v)w\,\mathrm{d}x \end{aligned}\tag{2.14}$$

for any $v, w \in E$. Next we give some properties of φ_θ which will be useful in the sequel. □

Lemma 2.4. *Suppose that* (W_1), (W_2) *and* (W_5) *are satisfied. Then:*

(i) *for every* $\theta \in \mathbb{R}$, φ_θ *satisfies the Palais–Smale condition. Moreover, there exists a positive constant* C_5 *independent of* v *such that*

$$|\varphi_0(v) - \varphi_\theta(v)| \le C_5|\theta|, \quad \forall\, v \in E; \tag{2.15}$$

(ii) φ_0 *has no critical point with positive critical value on* E *and* $K_0 = \{0\}$, *where* $K_0 := \{v \in E : \varphi_0(v) = 0,\ \varphi'_0(v) = 0\}$.

Proof. In view of (f_3) in Lemma 2.2, there exist positive constants M_0 and C_6 such that

$$|f(t)| \le C_6|t|^{1/\alpha}, \quad |t| \ge M_0. \tag{2.16}$$

Since $\alpha \ge 2$, in combination with (f_3) in Lemma 2.2 and (2.16), there exists a positive constant C_7 independent of t such that

$$|f(t)| \le C_7|t|^{1/\alpha}, \quad t \in \mathbb{R}. \tag{2.17}$$

It follows from (2.4) that there exists a positive constant C_8 independent of v such that

$$\left|\int_\Omega \widetilde{H}(x, f(v))\,\mathrm{d}x\right| \le C_8, \quad \forall\, v \in E. \tag{2.18}$$

By (1.2), (2.9), (2.13), (2.17) and (2.18), when $\|v\|^2 > 2T_0$, for any $\theta \in \mathbb{R}$ we have

$$\varphi_\theta(v) \ge \tfrac{1}{2}\|v\|^2 - C_9\|v\|^{p/\alpha} - C_8|\theta|. \tag{2.19}$$

Since $1 < p < 2\alpha$, for any $\theta \in \mathbb{R}$, (2.19) implies that $\varphi_\theta(v) \to +\infty$ as $\|v\| \to +\infty$.

Next we show that for any $\theta \in \mathbb{R}$, φ_θ satisfies the Palais–Smale condition. Assume that $\{v_n\}_{n\in\mathbb{N}} \subset E$ is a (Palais–Smale) sequence of φ_θ, i.e. $\{\varphi_\theta(v_n)\}_{n\in\mathbb{N}}$ is bounded and $\varphi'_\theta(v_n) \to 0$ as $n \to +\infty$. We need to prove that $\{v_n\}$ has a convergent subsequence. For any $\theta \in \mathbb{R}$, φ_θ is coercive. Then $\{v_n\}$ is bounded; passing to subsequence, also denoted

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by $\{v_n\}$, it can be assumed that $v_n \rightharpoonup v_0,\ n \to \infty$. Since $v_n \rightharpoonup v_0$, by (i) in Lemma 2.1, (f_2) in Lemma 2.2, (1.2) and (2.17), we obtain

$$\int_\Omega g_1(x, f(v_n))f'(v_n)(v_n - v_0)\,\mathrm{d}x \to 0, \quad n \to \infty \tag{2.20}$$

and

$$\int_\Omega \widetilde{g}_2(x, f(v_n))f'(v_n)(v_n - v_0)\,\mathrm{d}x \to 0, \quad n \to \infty. \tag{2.21}$$

In view of (iii) in Lemma 2.1 and (f_2) in Lemma 2.2, we have

$$\int_\Omega \widetilde{h}(x, f(v_n))f'(v_n)(v_n - v_0)\,\mathrm{d}x \to 0, \quad n \to \infty. \tag{2.22}$$

Moreover, combining (i) in Lemma 2.1, (2.1), (2.11) and (2.17), we get

$$\begin{aligned}\left|\langle k'(v_n), v_n - v_0\rangle \int_\Omega \widetilde{G}_2(x, f(v_n))\,\mathrm{d}x\right| \\ \le 2^{(\alpha_1'+2\alpha)/(2\alpha)} C_1 C_7^{\alpha_1'} \tau_{\alpha_1'/\alpha}^{\alpha_1'/\alpha} T_0^{(\alpha_1'-2\alpha)/(2\alpha)} \|v_n - v_0\|^2 + o_n(1).\end{aligned} \tag{2.23}$$

Since $\alpha_1' > 2\alpha$, we can choose T_0 small enough such that

$$2^{(\alpha_1'+2\alpha)/(2\alpha)} C_1 C_7^{\alpha_1'} \tau_{\alpha_1'/\alpha}^{\alpha_1'/\alpha} T_0^{(\alpha_1'-2\alpha)/(2\alpha)} < 2^{-1}. \tag{2.24}$$

For any $\theta \in \mathbb{R}$, it follows from (2.14) and (2.20)–(2.24) that

$$|\langle \varphi_\theta'(v_n), v_n - v_0\rangle| \ge 2^{-1}\|v_n - v_0\|^2 + o_n(1),$$

which implies that $v_n \to v_0$, $n \to \infty$. Moreover, by (2.13) and (2.18), we have that (2.15) holds.

Next we prove (ii) by contradiction. If v_0 is a critical point of φ_0 with $\varphi_0(v_0) > 0$, by (i) in Lemma 2.1, (W_1) and (2.13), we get $v_0 \neq 0$. Without loss of generality, we can assume $\|v_0\|^2 \le 2T_0$. Otherwise, by (2.9) and (2.11), we have $k(v_0) = 0$ and $k'(v_0) = 0$. Then it follows from (W_2), (2.13) and (2.14) that

$$0 < \varphi_0(v_0) = \varphi_0(v_0) - \mu^{-1}\left\langle \varphi_0'(v_0), \frac{f(v_0)}{f'(v_0)} \right\rangle \le \frac{\mu - 2}{2\mu}\|v_0\|^2 < 0,$$

which yields a contradiction, so $\|v_0\|^2 \le 2T_0$. In view of (i) in Lemma 2.1, (W_2), (2.1), (2.10), (2.13), (2.14) and (2.17), we obtain

$$\begin{aligned}0 < \varphi_0(v_0) &= \varphi_0(v_0) - \mu^{-1}\left\langle \varphi_0'(v_0), \frac{f(v_0)}{f'(v_0)} \right\rangle \\ &\le \frac{\mu - 2}{2\mu}\|v_0\|^2 + (C_3 + 1) C_1 C_7^{\alpha_1'} \tau_{\alpha_1'/\alpha}^{\alpha_1'/\alpha} \|v_0\|^{\alpha_1'/\alpha}.\end{aligned} \tag{2.25}$$

In view of $\alpha_1' > 2\alpha$, we can choose T_0 small enough such that if $\|v\|^2 \le 2T_0$,

$$(C_3 + 1) C_1 C_7^{\alpha_1'} \tau_{\alpha_1'/\alpha}^{\alpha_1'/\alpha} \|v\|^{\alpha_1'/\alpha} < \frac{2 - \mu}{4\mu}\|v\|^2. \tag{2.26}$$

By (2.25) and (2.26), we have a contradiction. So φ_0 has no critical point with positive critical value on E. Moreover, by a similar proof and direct computation, $K_0 = \{0\}$. The proof is completed. □

3. Proofs of main results

It is well known that the eigenvalue problem for the following equation

$$\begin{cases} -\Delta u = \lambda u, & x \in \Omega \\ u = 0, & x \in \partial\Omega, \end{cases}$$

has a sequence of eigenvalues λ_n (counted with multiplicity) and $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots \to \infty$. The corresponding system of normalized eigenfunctions $\{e_n : n \in \mathbb{N}\}$ forms an orthogonal basis in E. Hereafter, let $E_n := \operatorname{span}\{e_1, \ldots, e_n\}$ and let $E_n^\perp$ be the orthogonal complement of E_n in E.

By this normalized orthogonal sequence $\{e_n\}_{n=1}^\infty$, we define some subspaces as follows:

$$B^n = \{v \in E_n;\ \|v\| \le 1\}, \quad S^n = \{v \in E_n;\ \|v\| = 1\}$$

and

$$S_+^{n+1} = \{v = w + te_{n+1};\ \|v\| = 1,\ w \in B^n,\ 0 \le t \le 1\}.$$

With the help of these subspaces, we can introduce some continuous maps and a minimax sequence of φ_0 as follows

$$\Lambda_n := \{\gamma \in C(S^n, E);\ \gamma \text{ is odd}\} \tag{3.1}$$

and

$$b_n := \inf_{\gamma \in \Lambda_n} \max_{v \in S^n} \varphi_0(\gamma(v)). \tag{3.2}$$

In view of (3.1)–(3.2), it is easy to get $b_n \le b_{n+1}$, $n \in \mathbb{N}$. Next we give some useful estimates for the sequence of minimax values b_n.

Lemma 3.1. *Assume that* (W_3), (W_4) *and* (W_5) *hold. Then for any* $n \in \mathbb{N}$, $b_n < 0$.

Proof. Since E_n is a finite-dimensional space, there exists $\varrho_n > 0$ such that

$$\|v\| \le \varrho_n \|v\|_2, \quad \forall\, v \in E_n. \tag{3.3}$$

By direct computation and the definition of f, there exists a positive constant C_{10} such that

$$|f(t)| \ge C_{10}|t|, \quad |t| \le 1. \tag{3.4}$$

In view of (1.3), we can choose $0 < r_0 \le 1$ such that

$$g_1(x,t) \ge 8\varrho_n^2 C_{10}^{-2} t \tag{3.5}$$

for all $x \in \Omega$ and $0 \le t \le r_0$. By (3.5) and direct computation, we obtain

$$G_1(x,t) \ge 4\varrho_n^2 C_{10}^{-2} t^2 \tag{3.6}$$

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for all $x \in \Omega$ and $0 \le t \le r_0$. In view of (W_4), $G_1(x, t)$ is an even function in t. Combining this with (f_2) in Lemma 2.2, (3.4) and (3.6), we get

$$G_1(x, f(t)) \ge 4\varrho_n^2 C_{10}^{-2} f^2(t) \ge 4\varrho_n^2 t^2, \quad x \in \Omega \text{ and } |t| \le r_0. \tag{3.7}$$

Since E_n is finite dimensional, we claim that there exists a constant $\kappa > 0$ such that

$$\tfrac{1}{2}\int_\Omega |v(x)|^2 \,\mathrm{d}x \ge \int_{|v|>r_0} |v(x)|^2 \,\mathrm{d}x, \quad \forall\, v \in E_n \text{ with } \|v\| \le \kappa. \tag{3.8}$$

If (3.8) does not hold true, there exists a sequence of $\{v_k\} \subset E_n\backslash\{0\}$ such that $v_k \to 0$ in E_n and

$$\tfrac{1}{2}\int_\Omega |v_k(x)|^2 \,\mathrm{d}x < \int_{|v_k|>r_0} |v_k(x)|^2 \,\mathrm{d}x, \quad \forall\, k \in \mathbb{N}. \tag{3.9}$$

Set $u_k = \|v_k\|_2^{-1} v_k$, $k \in \mathbb{N}$. By (3.3) and (3.9), $\{u_k\}_{k\in\mathbb{N}}$ is bounded and

$$\tfrac{1}{2} < \int_{|v_k|>r_0} |u_k(x)|^2 \,\mathrm{d}x, \quad \forall\, k \in \mathbb{N}. \tag{3.10}$$

On the other hand, since E_n is a finite-dimensional space, we can assume that $u_k \to u_0$ in E_n. So $u_k \to u_0$ in $L^2(\Omega)$. Moreover, in view of $v_k \to 0$ in E_n, we have

$$\operatorname{meas}\{x \in \Omega : \, |v_k(x)| > r_0\} \to 0 \quad \text{as } k \to \infty. \tag{3.11}$$

Therefore, by (3.11), we obtain

$$\int_{|v_k|>r_0} |u_k(x)|^2 \,\mathrm{d}x \le 2\int_\Omega |u_k(x) - u_0(x)|^2 \,\mathrm{d}x + 2\int_{|v_k|>r_0} |u_0(x)|^2 \,\mathrm{d}x \to 0, \quad k \to \infty,$$

which contradicts (3.10). So (3.8) holds.

By (i) in Lemma 2.1 and (2.13), there exists a constant $\kappa' > 0$ such that

$$\varphi_0(v) \le \|v\|^2 - \int_\Omega G_1(x, f(v)) \,\mathrm{d}x, \quad \forall\, v \in E_n \text{ with } \|v\| \le \kappa'. \tag{3.12}$$

In combination with (W_3), (3.3), (3.7)–(3.8), (3.12), if $v \in E_n$ with $\|v\| \le \min\{\kappa, \kappa'\}$, we get

$$\begin{aligned} \varphi_0(v) &\le \|v\|^2 - \int_\Omega G_1(x, f(v)) \,\mathrm{d}x \\ &\le \|v\|^2 - \int_{\Omega_{r_0}} G_1(x, f(v)) \,\mathrm{d}x \\ &\le \|v\|^2 - 4\varrho_n^2 \int_{\Omega_{r_0}} |v(x)|^2 \,\mathrm{d}x \\ &= \|v\|^2 - 4\varrho_n^2 \left(\int_\Omega |v(x)|^2 \,\mathrm{d}x - \int_{\Omega\backslash\Omega_{r_0}} |v(x)|^2 \,\mathrm{d}x \right) \\ &\le -\|v\|^2, \end{aligned} \tag{3.13}$$

where $\Omega_{r_0} = \{x \in \Omega : |v(x)| \le r_0\}$. Choose $0 < \rho_0 < \min\{\kappa, \kappa'\}$, and let $\gamma_0(v) = \rho_0 v$, $v \in S^n$. In view of (3.2) and (3.13), $b_n < 0$. This completes the proof. □

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Lemma 3.2. *Suppose that* (W_1) *and* (W_5) *are satisfied. Then there exists a positive constant* C_{11} *independent of* n *such that for all* n *large enough*

$$b_n \geq -C_{11} n^{(-2p)/(N(2\alpha-p))}. \tag{3.14}$$

Proof. For any $\gamma \in \Lambda_n$ $(n \geq 2)$, if $0 \notin \gamma(S^n)$, then the genus $\vartheta(\gamma(S^n))$ is well defined and $\vartheta(\gamma(S^n)) \geq \vartheta(S^n) = n$. By [**21**, Proposition 7.8], $\gamma(S^n) \cap E_{n-1}^{\perp} \neq \emptyset$. Otherwise, $0 \in \gamma(S^n)$, and we have $0 \in \gamma(S^n) \cap E_{n-1}^{\perp}$. So for any $\gamma \in \Lambda_n$ $(n \geq 2)$, $\gamma(S^n) \cap E_{n-1}^{\perp} \neq \emptyset$. Therefore, for any $\gamma \in \Lambda_n$ $(n \geq 2)$, we obtain

$$\max_{v \in S^n} \varphi_0(\gamma(v)) \geq \inf_{v \in E_{n-1}^{\perp}} \varphi_0(v). \tag{3.15}$$

In view of (i) in Lemma 2.1, (1.2), (2.9), (2.13) and (2.26), we conclude that

$$\varphi_0(v) \geq \ \tfrac{1}{4}\|v\|^2 - C_{12}\|v\|_2^{p/\alpha}, \quad \forall\, v \in E. \tag{3.16}$$

When $v \in E_{n-1}^{\perp}$, $\lambda_n \|v\|_2^2 \leq \|v\|^2$. If $v \in E_{n-1}^{\perp}$, by (3.16), we get

$$\varphi_0(v) \geq \tfrac{1}{4}\|v\|^2 - C_{12}\lambda_n^{-p/(2\alpha)}\|v\|^{p/\alpha}. \tag{3.17}$$

Combining this with (3.2), (3.15) and (3.17), for $n \geq 2$, we have

$$\begin{aligned} b_n &\geq \inf_{t \geq 0} \left\{ \tfrac{1}{4}t^2 - C_{12}\lambda_n^{-p/(2\alpha)} t^{p/\alpha} \right\} \\ &= -C_{13}\lambda_n^{-p/(2\alpha-p)}, \end{aligned} \tag{3.18}$$

where C_{13} is a positive constant independent of n and λ_n. When n is large enough, it is well known that $\lambda_n \geq C_{14} n^{2/N}$. In view of (3.18), (3.14) holds. The proof is completed. □

For any $\delta > 0$, set

$$\Gamma_n(\delta) = \{\gamma \in \Gamma_n;\ \varphi_0(\gamma(v)) \leq b_n + \delta,\ v \in S^n\}, \tag{3.19}$$

where $\Gamma_n := \left\{\gamma \in C(S_+^{n+1}, E);\ \gamma|_{S^n} \in \Lambda_n\right\}$. For any $\delta > 0$, by (3.1), there exists $\gamma_0 \in \Lambda_n$ such that $\varphi_0(\gamma_0(v)) \leq b_n + \delta$, $\forall\, v \in S_n$. So we can extend γ_0 to S_+^{n+1} as a continuous function. Hence $\Gamma_n(\delta)$ is non-empty. Moreover, we have the following lemma.

Lemma 3.3. *Assume that* (W_1), (W_2), (W_3) *and* (W_5) *are satisfied. Then for any* $n \in \mathbb{N}$ *and any* $\delta > 0$, *there exists* $\gamma_n \in \Gamma_n(\delta)$ *such that*

$$\max_{v \in S_+^{n+1}} \varphi_0(\gamma_n(v)) < 0. \tag{3.20}$$

Proof. In view of (3.19), for fixed $n \in \mathbb{N}$, when $0 < \delta < \delta'$, we have $\Gamma_n(\delta) \subset \Gamma_n(\delta')$. By Lemma 3.2, $b_n < 0$, $n \in \mathbb{N}$. So we only need to find $\gamma_n \in \Gamma_n(\delta)$ with $\delta \in (0, |b_n|)$ such that (3.20) holds. For any $\delta \in (0, |b_n|)$, it follows from (3.2) that there exists $\gamma_0 \in \Lambda_n$ such that $\max_{v \in S^n} \varphi_0(\gamma_0(v)) \leq b_n + \delta/2$. Since $\gamma_0(S^n)$ is a compact set in E, there exists a positive integer m_0 such that

$$\max_{v \in S^n} \varphi_0((P_{m_0} \circ \gamma_0)v) \leq b_n + \delta, \tag{3.21}$$

where P_{m_0} denotes the orthogonal projective operator from E to E_{m_0}.

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For any $c \in \mathbb{R}$, let $\varphi_0^c = \{v \in E : \varphi_0(v) \le c\}$. Choose $\bar{\varepsilon} = -(b_n + \delta)/2 > 0$. By a similar argument to that in Lemma 3.1, there exists $\rho_{m_0+1} > 0$ such that if $v \in \bar{B}(0, \rho_0) \cap E_{m_0+1}$, $\varphi_0(v) \le 0$, where $B(x_0, \rho)$ denotes the open ball of radius ρ centred at x_0 in E, and $\bar{B}(x_0, \rho)$ denotes the closure of $B(x_0, \rho)$ in E. Since $\varphi_0 \in C^1(E, \mathbb{R})$ and $\varphi_0(0) = 0$, we have $\mathrm{dist}(0, \varphi_0^{-\bar{\varepsilon}}) > 0$. Define $\rho_0' = \min\left\{\rho_{m_0+1},\ \mathrm{dist}(0, \varphi_0^{-\bar{\varepsilon}})\right\}$, then $\rho_0' > 0$. By the deformation theorem (see [**21**, Theorem A.4]), there exists $\varepsilon \in (0, \bar{\varepsilon})$ and a continuous map $\eta \in C([0, 1] \times E, E)$ such that

$$\eta(1, v) = v, \quad \text{if } \varphi_0(v) \notin [-\bar{\varepsilon}, \bar{\varepsilon}] \tag{3.22}$$

and

$$\eta(1, \varphi_0^{\varepsilon} \backslash B(0, \rho_0')) \subset \varphi_0^{-\varepsilon}, \tag{3.23}$$

where $B(0, \rho_0')$ is a neighbourhood of K_0 given by (ii) in Lemma 2.4.

By (3.1), $P_{m_0} \circ \gamma_0 \in C(S^n, E_{m_0})$. Since E_{n+1} is a metric space with the norm $\|\cdot\|$ and S^n is a closed subset in E_{n+1}, there exists an extension $\widetilde{P_{m_0} \circ \gamma_0} : E_{n+1} \to E_{m_0}$ of $P_{m_0} \circ \gamma_0$ by the Dugundji extension theorem (see [**8**, Theorem 4.1]); furthermore,

$$((\widetilde{P_{m_0} \circ \gamma_0}) E_{n+1}) \subset \mathrm{co}((P_{m_0} \circ \gamma_0) S^n), \tag{3.24}$$

where the symbol co denotes the convex hull. Since $(P_{m_0} \circ \gamma_0) S^n$ is a compact set in E_{m_0}, by the definition of a convex hull, $\mathrm{co}((P_{m_0} \circ \gamma_0) S^n)$ is a bounded set in E_{m_0}. Then there exists a constant ν such that $\varphi_0(v) \le \nu$, $\forall\, v \in \mathrm{co}((P_{m_0} \circ \gamma_0) S^n)$. It follows from (3.24) that

$$\varphi_0((\widetilde{P_{m_0} \circ \gamma_0}) v) \le \nu, \quad v \in E_{n+1}. \tag{3.25}$$

Next we consider two possible cases.

Case 1. $\nu \le \varepsilon$. Since $\widetilde{P_{m_0} \circ \gamma_0} \in C(E_{n+1}, E_{m_0})$, by (3.25), we have

$$(\widetilde{P_{m_0} \circ \gamma_0}) v \in \varphi_{0,\, m_0}^{\varepsilon}, \quad \forall\, v \in E_{n+1}, \tag{3.26}$$

where $\varphi_{0,\, m_0}^{\varepsilon} := \{v \in E_{m_0} :\ \varphi_0(v) \le \varepsilon\}$. Define a map T as follows:

$$T(v) = \begin{cases} v, & v \notin \bar{B}(0, \rho_0') \cap E_{m_0} \\ v + (\rho_0'^2 - \|v\|^2)^{1/2} e_{m_0+1}, & v \in \bar{B}(0, \rho_0') \cap E_{m_0}. \end{cases} \tag{3.27}$$

It is obvious that $T \in C(E_{m_0}, E_{m_0+1})$ and

$$(T \circ (\widetilde{P_{m_0} \circ \gamma_0})) v \cap B(0, \rho_0') = \emptyset, \quad \forall\, v \in E_{n+1}. \tag{3.28}$$

When $v \in E_{n+1}$ and $\|(\widetilde{P_{m_0} \circ \gamma_0}) v\| > \rho_0'$, by (3.26) and (3.27), we get

$$(T \circ (\widetilde{P_{m_0} \circ \gamma_0})) v = (\widetilde{P_{m_0} \circ \gamma_0}) v \in \varphi_{0,\, m_0}^{\varepsilon}. \tag{3.29}$$

Otherwise, if $v \in E_{n+1}$ and $\|(\widetilde{P_{m_0} \circ \gamma_0}) v\| \le \rho_0'$, in view of (3.27), $\|(T \circ (\widetilde{P_{m_0} \circ \gamma_0})) v\| = \rho_0'$. By the definition of ρ_0' and (3.29), we conclude that

$$(T \circ (\widetilde{P_{m_0} \circ \gamma_0})) v \in \varphi_0^{\varepsilon}, \quad \forall\, v \in E_{n+1}. \tag{3.30}$$

Define a map $\gamma_n : E_{n+1} \to E$ as follows:

$$\gamma_n(\cdot) = \eta(1, (T \circ (\widetilde{P_{m_0} \circ \gamma_0}))(\cdot)). \tag{3.31}$$

Next we prove $\gamma_n \in \Gamma_n(\delta)$ and $\max_{v \in S_+^{n+1}} \varphi_0(\gamma_n(v)) < 0$. First, it is obvious that $\gamma_n \in C(S_+^{n+1}, E)$. By the Dugundji extension theorem, we obtain

$$(\widetilde{P_{m_0} \circ \gamma_0})v = (P_{m_0} \circ \gamma_0)v, \quad \forall\, v \in S^n. \tag{3.32}$$

By (3.21), $(P_{m_0} \circ \gamma_0)\ v \in \varphi_0^{-2\bar{\varepsilon}}$, $\forall\, v \in S^n$. By the definition of ρ_0' and $\varphi_0^{-2\bar{\varepsilon}} \subset \varphi_0^{-\bar{\varepsilon}}$, we get

$$\|(P_{m_0} \circ \gamma_0)\ v\| \geq \rho_0', \quad \forall\, v \in S^n. \tag{3.33}$$

It follows from (3.27), (3.32) and (3.33) that

$$(T \circ (\widetilde{P_{m_0} \circ \gamma_0}))\ v = T \circ ((P_{m_0} \circ \gamma_0)\ v) = (P_{m_0} \circ \gamma_0)\ v, \quad \forall\, v \in S^n. \tag{3.34}$$

In view of $(P_{m_0} \circ \gamma_0)\ v \in \varphi_0^{-2\bar{\varepsilon}}$, $\forall\, v \in S^n$, by (3.22) and (3.34), we obtain

$$\gamma_n(v) = \eta(1, (T \circ (\widetilde{P_{m_0} \circ \gamma_0}))\ v) = (P_{m_0} \circ \gamma_0)\ v, \quad \forall\, v \in S^n, \tag{3.35}$$

which implies that $\gamma_n|_{S^n} \in \Lambda_n$. Moreover, in view of (3.21) and (3.35), we have $\gamma_n \in \Gamma_n(\delta)$. Since $S^{n+1} \subset E_{n+1}$, by (3.28) and (3.30), we have $(T \circ (\widetilde{P_{m_0} \circ \gamma_0}))v \cap B(0, \rho_0') = \emptyset$, $\forall\, v \in S_+^{n+1}$ and $(T \circ (\widetilde{P_{m_0} \circ \gamma_0}))v \in \varphi_0^{\varepsilon}$, $\forall\, v \in S_+^{n+1}$. It follows from (3.23) and (3.31) that $\max_{v \in S_+^{n+1}} \varphi_0(\gamma_n(v)) \leq -\varepsilon < 0$.

Case 2. $\nu > \varepsilon$. Let $\varphi_0|_{E_{m_0}}$ denote the restriction of φ_0 on E_{m_0}. Arguing as in Lemma 2.4, we can prove that $\varphi_0|_{E_{m_0}}$ satisfies the Palais–Smale condition. Moreover, $\varphi_0|_{E_{m_0}}$ has no critical point with positive critical values on E_{m_0}. By the non-critical interval theorem (see [**5**, Theorem 5.1.6]), $\varphi_{0,\,m_0}^{\varepsilon}$ is a strong deformation retraction of $\varphi_{0,\,m_0}^{\nu}$. So there exists a map ς such that $\varsigma \in C(\varphi_{0,\,m_0}^{\nu}, \varphi_{0,\,m_0}^{\varepsilon})$ and $\varsigma(v) = v$, if $v \in \varphi_{0,\,m_0}^{\varepsilon}$. Define a map from $E_{n+1} \to E$ as follows:

$$\bar{\gamma}_n(\cdot) = \eta(1, (T \circ (\varsigma \circ (\widetilde{P_{m_0} \circ \gamma_0})))(\cdot)).$$

By a similar proof to that used in Case 1, $\bar{\gamma}_n \in \Gamma_n(\delta)$ and $\max_{v \in S_+^{n+1}} \varphi_0(\bar{\gamma}_n(v)) \leq -\varepsilon < 0$. This completes the proof. □

Under assumptions of Theorem 1.1, in view of Lemma 3.1, $b_n \leq b_{n+1}$ and (3.14), it is impossible that $b_{n+1} = b_n$ for all large n. So we can construct critical values of φ_θ as follows.

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Lemma 3.4. *Let n be a positive integer satisfying $b_{n+1} > b_n$. For any $\sigma \in (0,\, b_{n+1} - b_n)$ and $|\theta| < \theta_n$, where θ_n is a positive constant depending on n given by (3.40). Define*

$$c_n(\theta, \sigma) = \inf_{\gamma \in \Gamma_n(\sigma)} \max_{v \in S_+^{n+1}} \varphi_\theta(\gamma(v)). \tag{3.36}$$

Then $c_n(\theta,\, \sigma)$ is a critical value of φ_θ. Moreover, for any $\sigma \in (0,\, b_{n+1} - b_n)$ and $|\theta| < \theta_n$, we have

$$b_{n+1} - C_5|\theta| \le c_n(\theta, \sigma) \le c_{n+1} + C_5|\theta|, \tag{3.37}$$

where C_5 is given in (2.15) and c_{n+1} is defined by (3.39).

Proof. By (2.15), for any $(\theta,\, v) \in \mathbb{R} \times E$, we have

$$\varphi_0(v) - C_5|\theta| \le \varphi_\theta(v) \le \varphi_0(v) + C_5|\theta|. \tag{3.38}$$

For any $\sigma \in (0,\, b_{n+1} - b_n)$, by Lemma 3.3, there exists $\gamma_n \in \Gamma_n(\sigma)$ such that

$$c_{n+1} := \max_{v \in S_+^{n+1}} \varphi_0(\gamma_n(v)) < 0. \tag{3.39}$$

We can choose θ_n small enough such that for $\sigma \in (0,\, b_{n+1} - b_n)$ and $|\theta| < \theta_n$,

$$b_n + \sigma + 2C_5|\theta| < b_{n+1}, \quad c_{n+1} + C_5|\theta| < 0. \tag{3.40}$$

In view of (3.36) and (3.38)–(3.40), for any $\sigma \in (0,\, b_{n+1} - b_n)$ and $|\theta| < \theta_n$, we obtain

$$c_n(\theta, \sigma) \le \max_{v \in S_+^{n+1}} \varphi_0(\gamma_n(v)) + C_5|\theta| = c_{n+1} + C_5|\theta| < 0. \tag{3.41}$$

For an arbitrary $\gamma \in \Gamma_n(\sigma)$, we have the natural odd extension of γ on S^{n+1} denoted by $\bar{\gamma}$, i.e. $\bar{\gamma}(v) = \gamma(v)$ for $v \in S_+^{n+1}$ and $\bar{\gamma}(v) = -\gamma(-v)$ for $v \in -S_+^{n+1}$. So we get $\bar{\gamma} \in \Lambda_{n+1}$. Since I_0 is an even functional,

$$\max_{v \in S_+^{n+1}} \varphi_0(\gamma(v)) = \max_{v \in S^{n+1}} \varphi_0(\bar{\gamma}(v)). \tag{3.42}$$

For any $\sigma \in (0,\, b_{n+1} - b_n)$ and $|\theta| < \theta_n$, by (3.2), (3.38) and (3.42), we conclude that

$$\begin{aligned}\max_{v \in S_+^{n+1}} \varphi_\theta(\gamma(v)) &\ge \max_{v \in S_+^{n+1}} \varphi_0(\gamma(v)) - C_5|\theta| \\ &= \max_{v \in S^{n+1}} \varphi_0(\bar{\gamma}(v)) - C_5|\theta| \\ &\ge b_{n+1} - C_5|\theta|.\end{aligned} \tag{3.43}$$

Taking the infimum on $\Gamma_n(\sigma)$ in (3.43), in view of (3.36) and (3.40), for any $\sigma \in (0,\, b_{n+1} - b_n)$ and $|\theta| < \theta_n$, we get

$$c_n(\theta, \sigma) \ge b_{n+1} - C_5|\theta| > b_n + \sigma + C_5|\theta|. \tag{3.44}$$

For any $\sigma \in (0,\, b_{n+1} - b_n)$ and $|\theta| < \theta_n$, we claim that $c_n(\theta,\, \sigma)$ is a critical value of φ_θ. If not, $c_n(\theta,\, \sigma)$ is a regular value. Set $\bar{\varepsilon} = (c_n(\theta,\, \sigma) - b_n - \sigma - C_5|\theta|)/2$; then $\bar{\varepsilon} > 0$

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by (3.44). By the deformation theorem, there exists $\varepsilon \in (0, \bar{\varepsilon})$ and a continuous map $\eta \in C([0, 1] \times E, E)$ such that

$$\eta(1, v) = v \quad \text{if } \varphi_\theta(v) \notin [c_n(\theta,\sigma) - \bar{\varepsilon}, c_n(\theta,\sigma) + \bar{\varepsilon}] \tag{3.45}$$

and

$$\eta(1, \varphi_\theta^{c_n(\theta,\sigma)+\varepsilon}) \subset \varphi_\theta^{c_n(\theta,\sigma)-\varepsilon}. \tag{3.46}$$

By (3.36), there exists $\gamma_0 \in \Gamma_n(\sigma)$ such that

$$\max_{v \in S_+^{n+1}} \varphi_\theta(\gamma_0(v)) < c_n(\theta,\sigma) + \varepsilon. \tag{3.47}$$

Define

$$\bar{\gamma}_0(v) = \eta(1, \gamma_0(v)), \quad v \in S_+^{n+1}. \tag{3.48}$$

It is obvious that $\bar{\gamma}_0 \in C(S_+^{n+1}, E)$. Since $\gamma_0 \in \Gamma_n(\sigma)$, it follows from (3.19), (3.38) and (3.44) that

$$\varphi_\theta(\gamma_0(v)) \leq \varphi_0(\gamma_0(v)) + C_5|\theta| \leq b_n + \sigma + C_5|\theta| = c_n(\theta,\sigma) - 2\bar{\varepsilon}, \quad v \in S^n. \tag{3.49}$$

By (3.45), (3.48) and (3.49), we obtain $\bar{\gamma}_0(v) = \gamma_0(v)$, $v \in S^n$, which yields

$$\bar{\gamma}_0|_{S_n} \in \Lambda_n \quad \text{and} \quad \varphi_0(\bar{\gamma}_0(v)) = \varphi_0(\gamma_0(v)) \leq b_n + \sigma, \quad v \in S^n. \tag{3.50}$$

So we have $\bar{\gamma}_0 \in \Gamma_n(\sigma)$ by (3.50). In combination with (3.46)–(3.48), we get

$$\max_{v \in S_+^{n+1}} \varphi_\theta(\bar{\gamma}_0(v)) = \max_{v \in S_+^{n+1}} \varphi_\theta(\eta(1, \gamma_0(v))) \leq c_n(\theta,\sigma) - \varepsilon,$$

which contradicts (3.36). Consequently, for any $\sigma \in (0, b_{n+1} - b_n)$ and $|\theta| < \theta_n$, $c_n(\theta, \sigma)$ is a critical value of φ_θ. Moreover, in view of (3.41) and (3.44), (3.37) holds. This completes the proof. □

Lemma 3.5. *For any $\varepsilon > 0$, there exists $\tau > 0$ such that if $|\theta| \leq \tau$, $\varphi'_\theta(v) = 0$ and $|\varphi_\theta(v)| \leq \tau$, then $\|v\| < \varepsilon$.*

Proof. We prove this lemma by contradiction. If the result is false, there exist sequences $\{v_n\}$ and $\{\theta_n\}$ such that $\theta_n \to 0$, $\varphi'_{\theta_n}(v_n) = 0$, $\varphi_{\theta_n}(v_n) \to 0$ and $\|v_n\| \geq \varepsilon_0$, where $\varepsilon_0 > 0$ is independent of n. By a similar argument to that used in (2.25) in Lemma 2.4, we have

$$\begin{aligned} \varphi_{\theta_n}(v_n) &= \varphi_{\theta_n}(v_n) - \mu^{-1}\left\langle \varphi'_{\theta_n}(v_n), \frac{f(v_n)}{f'(v_n)} \right\rangle \\ &\leq \frac{\mu-2}{4\mu}\|v_n\|^2 + \theta_n\mu^{-1}\left(\int_\Omega \widetilde{h}(x, f(v_n)) f(v_n)\,\mathrm{d}x - \int_\Omega \widetilde{H}(x, f(v_n))\,\mathrm{d}x \right). \end{aligned} \tag{3.51}$$

In combination with (2.4), (2.18) and (3.51), we get

$$\varphi_{\theta_n}(v_n) \leq \frac{\mu-2}{4\mu}\|v_n\|^2 + C_{15}|\theta_n| \leq \frac{\mu-2}{4\mu}\varepsilon_0^2 + C_{15}|\theta_n|, \tag{3.52}$$

where C_{15} is a positive constant independent of n. Since $\theta_n \to 0$, (3.52) contradicts the fact that $\varphi_{\theta_n}(v_n) \to 0$. The proof is completed. □

Proof of Theorem 1.1. In view of Lemma 3.5, there exists $\tau_0 > 0$ such that if $|\theta| \le \tau_0$, $\varphi'_\theta(v) = 0$ and $|\varphi_\theta(v)| \le \tau_0$, then $\|v\|^2 < T_0$. Combining (2.8), (2.9) and (2.11), we have $k(v) = 1$ and $k'(v) = 0$, implying that $I'_\theta(v) = 0$ by (2.14) and v is a weak solution of (2.7). By the facts above, Lemma 3.5 and elliptic regularity theory, we can choose a positive constant $\tau'_0 \le \tau_0$ if $|\theta| < \tau'_0$, $\varphi'_\theta(v) = 0$ and $|\varphi_\theta(v)| \le \tau'_0$; then v is a weak solution of (2.7) and $\|v\|_{C(\bar\Omega)} \le \min\{\delta'_0, \delta'_1\}$, where δ'_0 and δ'_1 are defined as in Lemma 2.1.

In view of Lemmas 3.1 and 3.2, we have $b_n \to 0$, $n \to \infty$. For any $j \in \mathbb{N}$ and τ'_0 defined above, choose strictly increasing integers p_i $(1 \le i \le j)$ such that

$$b_{p_i+1} > b_{p_i} > -\tau'_0,\ 1 \le i \le j \quad \text{and} \quad b_{p_{(i+1)}} > c_{p_i+1},\ 1 \le i \le j-1, \tag{3.53}$$

where c_{p_i+1} is defined as in (3.39). By (3.36), (3.37) and (3.53), we can choose a positive constant $\theta_j \le \tau'_0$ such that $c_{p_i}(\theta, \sigma_i)$ with $1 \le i \le j$ are well defined for $|\theta| \le \theta_j$, where $\sigma_i \in (0, b_{p_i+1} - b_{p_i})$. Moreover, when $|\theta| \le \theta_j$ we also have $-\tau'_0 < b_{p_1} - C_5|\theta|$ and

$$c_{p_i+1} + C_5|\theta| < b_{p_{(i+1)}} - C_5|\theta|, \quad 1 \le i \le j-1. \tag{3.54}$$

In view of (3.37) and (3.54), for $|\theta| \le \theta_j$, φ_θ has at least j critical values

$$-\tau'_0 < c_{p_1}(\theta, \sigma_1) < c_{p_2}(\theta, \sigma_2) < \cdots < c_{p_j}(\theta, \sigma_j) < 0,$$

which implies that for $|\theta| \le \theta_j$, (2.7) has at least j distinct weak solutions $v_1(\theta)$, $v_2(\theta)$, $\ldots$, $v_j(\theta)$ and $\|v_i(\theta)\|_{C(\bar\Omega)} \le \min\{\delta'_0, \delta'_1\}$, $1 \le i \le j$. Then for any $|\theta| \le \theta_j$, (2.5) has at least distinct weak solutions $u_i(\theta) = f(v_i(\theta))$, $1 \le i \le j$. Moreover, by (f_2) in Lemma 2.1, for any θ with $|\theta| \le \theta_j$, we obtain

$$\|u_i(\theta)\|_{C(\bar\Omega)} \le \min\{\delta'_0, \delta'_1\}, \quad 1 \le i \le j. \tag{3.55}$$

Combining (ii), (iv) in Lemma 2.1 and (3.55), $u_1(\theta)$, $u_2(\theta)$, $\ldots$, $u_j(\theta)$ are also j distinct weak solutions for (1.1) with $|\theta| \le \theta_j$. This completes the proof. □

4. Examples

Example 4.1. In (1.1), let Ω be a bounded smooth domain in $\mathbb{R}^4$ and $\alpha = 2$. Define $g(x, t) = a(x)|t|^{-1/2}t\ln(e + t^4)$, $(x, t) \in \Omega \times \mathbb{R}$, where $a(x) \in C(\bar\Omega, \mathbb{R})$ with $\inf_{x\in\bar\Omega} a(x) > 0$ and $h(x, t)$ is a continuous function defined on $\bar\Omega \times [-\delta_0, \delta_0]$ for some $\delta_0 > 0$. Set

$$g_1(x,t) = a(x)|t|^{-1/2}t, \quad g_2(x,t) = a(x)|t|^{-1/2}t(\ln(e + t^4) - 1).$$

It is obvious that $g = g_1 + g_2$. By the Lagrange theorem, $|g_2(x, t)| \le M|t|^{9/2}$, $(x, t) \in \Omega \times \mathbb{R}$, where M is a positive constant. Choose $\mu = p = 3/2$ and $\alpha_1 = 11/2$, so all the conditions of Theorem 1.1 are satisfied. By Theorem 1.1, for any $j \in \mathbb{N}$, there exists $\theta_j > 0$ such that if $|\theta| \le \theta_j$, (1.1) possesses at least j distinct weak solutions.

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