Canad. Math. Bull. Vol. 63 (4), 2020 pp. 744–754
http://dx.doi.org/10.4153/S0008439519000791

Concordance, Crossing Changes, and Knots in Homology Spheres

Christopher W. Davis

Abstract. Any knot in S^3 can be reduced to a slice knot by crossing changes. Indeed, this slice knot can be taken to be the unknot. In this paper we study the question of when the same holds for knots in homology spheres. We show that a knot in a homology sphere is nullhomotopic in a smooth homology ball if and only if that knot is smoothly concordant to a knot that is homotopic to a smoothly slice knot. As a consequence, we prove that the equivalence relation on knots in homology spheres given by cobounding immersed annuli in a homology cobordism is generated by concordance in homology cobordisms together with homotopy in a homology sphere.

1 Introduction and Statement of Results

Classically, a knot K is an isotopy class of smooth embeddings of the circle S^1 into the 3-sphere S^3. A knot is called *slice* if it forms the boundary of a smoothly embedded 2-disk D in the 4-ball. This disk D is called a slice disk for K. This notion was first considered by Fox and Milnor in 1957 in the study of singularities of surfaces in 4-manifolds [7, 8]. The question of which knots are slice is closely related to local obstructions arising in a surgery-theoretic attempt to classify 4-manifolds [2]. Since then the question of what knots admit slice disks has been at the heart of the study of 4-manifold topology.

While not every knot in S^3 is slice, every knot can be transformed into a slice knot by a finite sequence of crossing changes. Indeed, that slice knot can be taken to be the unknot. In the case of knots in a non-simply connected homology sphere, the situation is more subtle. A knot representing a nontrivial class in the fundamental group cannot be reduced to the unknot by any sequence of crossing changes. The main goal of this paper is to ask when a knot in a homology sphere can be homotoped to a new knot that bounds a smoothly embedded disk in homology ball. We will consider the following question.

Question 1.1 Let Y be a homology sphere and let K be a knot in Y. Does there exist a homotopy transforming K to a new knot K' in Y that bounds a smoothly embedded disk in a smooth homology ball bounded by Y?

If one allows for topological homology balls and locally flat embedded disks, then the answer to this question is affirmative for all knots. In [1], Austin and Rolfsen proved that any knot in a homology sphere admits a homotopy to a knot with

Received by the editors August 6, 2019; revised December 8, 2019.
Published online on Cambridge Core December 16, 2019.
AMS subject classification: **57M25**.

Alexander polynomial 1. By the work of Freedman and Quinn [9, Theorem 11.7B] such a knot bounds a locally flat embedded disk in a contractible 4-manifold.

In the smooth setting there are obstructions to Question 1.1 having an affirmative answer. First, not every homology sphere bounds a homology ball. For example, Y might be the Poincaré homology sphere or any other homology sphere with nonzero Rohlin invariant. See [10, Definition 5.7.16] for a brief discussion of the Rohlin invariant. Secondly, if K is homotopic in Y to a knot that bounds an embedded disk in a homology ball, then K is nullhomotopic in that homology ball. By the work of Daemi [3, Remark 1.6], there exists a knot K in a homology sphere Y such that Y bounds a homology ball and yet K is not nullhomotopic in any homology ball bounded by Y. Such a knot cannot be homotopic to a knot that bounds a smoothly embedded disk.

We give a name to the property of bounding a smoothly embedded disk in a homology ball. Let Y be a homology sphere and let K be a knot in Y. We say that (Y, K) is *homology slice* if there exists a smooth homology ball bounded by Y in which K bounds a smoothly embedded disk. Thus, Question 1.1 asks whether the homotopy class of K in Y contains a homology slice representative.

The notions of sliceness and homology sliceness extend to equivalence relations. Two knots K and J in S^3 are called *concordant* if $K \times \{1\} \subseteq S^3 \times \{1\}$ and $J \times \{0\} \subseteq S^3 \times \{0\}$ cobound a smoothly embedded annulus in $S^3 \times [0,1]$. We call knots K and J in homology spheres Y and X *homology concordant* if there exists a smooth homology cobordism from Y to X in which $K \subseteq Y$ and $J \subseteq X$ cobound a smoothly embedded annulus. Similar to the relationship between classical concordance and slice knots, the homology concordance class containing the unknot in S^3 is precisely the set of homology slice knots. Importantly, homology concordance allows one to compare knots that do not lie in the same 3-manifold. The quotient of the set of knots in homology spheres by homology concordance is the topic of study of [4, 5, 11, 13] amongst others.

Our first main result says that every knot in the boundary of a contractible 4-manifold is homology concordant to a knot for which the answer to Question 1.1 is affirmative.

Theorem 1.2 *Let Y be a homology sphere that bounds a smooth contractible 4-manifold. Let K be a knot in Y. There exists a knot K' in a homology sphere Y' such that (Y, K) is homology concordant to (Y', K') and K' is homotopic in Y' to a third knot K'' that bounds a smoothly embedded disk in a smooth contractible 4-manifold bounded by Y'.*

Said another way, if we let $\simeq_h^3$ and $\simeq_c$ denote homotopy in a homology sphere and homology concordance, respectively, and we let U denote the unknot in S^3, then Theorem 1.2 concludes

$$(Y, K) \simeq_c (Y', K') \simeq_h^3 (Y', K'') \simeq_c (S^3, U).$$

Thus, the equivalence relation generated by homology concordance and homotopy equates every knot in the boundary of a contractible 4-manifold with the unknot.

If a knot K in a homology sphere Y is not nullhomotopic in any homology ball bounded by Y, then K cannot be related to a homology slice knot by any sequence of

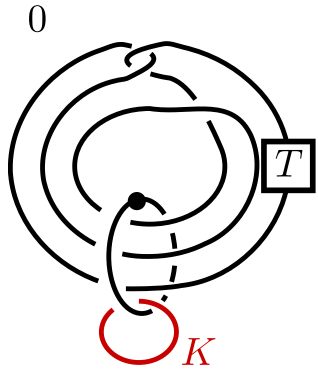

Figure 1: A knot K in the boundary of a contractible 4-manifold. For a suitable knot T, this knot is not homology concordant to any knot in S^3 [13, Theorem 1.1].

homotopies and concordances. Such a knot exists by [3, Remark 1.6]. The following theorem reveals that this is the only obstruction.

Theorem 1.3 *Let Y be a homology sphere that bounds a smooth homology ball W. Let K be a knot in Y that is nullhomotopic in W. There exists a knot K' in a homology sphere Y' such that (Y, K) is homology concordant to (Y', K') and K' is homotopic in Y to a third knot K'' that bounds a smoothly embedded disk in a smooth homology ball bounded by Y'.*

Remark 1.4 In [14, Proposition 2.1], Strle and Owens proved a quantitative version of Theorem 1.2 for knots in S^3. They showed that if a knot in S^3 bounds an immersed disk in the 4-ball that has n self-intersections, then K is concordant to another knot K' (in S^3) that can be reduced to a slice knot after n crossing changes. While we will not do so in this document, by marrying their techniques with ours, we expect the same can be proven of knots in homology spheres.

The work of Kojima [12] implies that if K is a knot in a homology sphere Y that bounds a homology ball W that has a handle structure lacking any 1-handles, then K is homotopic in Y to a knot that bounds a smoothly embedded disk in W. Thus, in some cases, the homology concordances of Theorems 1.2 and 1.3 can be removed and Question 1.1 has an affirmative answer for every knot in Y. Our next theorem reaches a similar conclusion when W admits a handle structure that has no 3-handles.

Theorem 1.5 *Let Y be a homology sphere that bounds a smooth homology ball W that has a handle structure with no 3-handles. Let K be a knot in Y that is nullhomotopic in W. Then K is homotopic in Y to a knot K' that bounds a smoothly embedded disk in W.*

As an application, consider the knot in a homology sphere (Y, K) of Figure 1. By [13, Theorem 1.1], this knot does not bound a piecewise linear embedded disk in any smooth homology ball. As a consequence, (Y, K) is not homology concordant to any knot in S^3. However, notice that Y bounds a contractible 4-manifold that has one 0-handle, one 1-handle, one 2-handle, and most importantly no 3-handles. Thus, by Theorem 1.5, K is homotopic to another knot in Y that is homology slice.

There is an extension of the equivalence relations $\simeq_h^3$ and $\simeq_c$. Given knots K and J in homology spheres Y and X, if there exists a smooth homology cobordism from Y to X in which K and J are homotopic, then we write $(Y,K) \simeq_h^4 (X,J)$. Equivalently and more geometrically we say $(Y,K) \simeq_h^4 (X,J)$ if there exists a smooth homology cobordism from Y to X in which K and J cobound an immersed annulus. It is clear that if (Y,K) and (X,J) are related by a sequence of homotopies and homology concordances, then they are related under $\simeq_h^4$. The following theorem reveals the converse: if $(Y,K) \simeq_h^4 (X,J)$, then (Y,K) is related to (X,J) by a sequence consisting of one homotopy and two homology concordances. Thus, the equivalence relation $\simeq_h^4$ is generated by homotopy and homology concordance.

Theorem 1.6 *Let (Y,K) and (X,J) be knots in homology spheres. If there exists a smooth homology cobordism from Y to X in which K is homotopic to J, then there exist knots K' and J' in some homology sphere Z such that*

$$(Y,K) \simeq_c (Z,K') \simeq_h^3 (Z,J') \simeq_c (X,J).$$

Outline of Paper

In Section 2, we consider the case that a knot K in a homology sphere Y is nullhomotopic in a homology ball that has no 3-handles and prove Theorem 1.5. In Section 3, we manipulate handle structures of homology balls in order to separate the 1- and 3-handles. We go on to prove Theorems 1.2 and 1.5. In Section 4, we take advantage of the group structure on $\mathcal{C}$ to prove Theorem 1.6.

2 In the Absence of 3-handles

Throughout this paper, we make extensive use of handle decompositions of smooth manifolds. A good reference is [10]. We begin with the following proposition revealing that if W is a 4-manifold, $Y \subseteq \partial W$, and (W,Y) has no relative 1-handles, then every homotopy class in $\ker(\pi_1(Y) \to \pi_1(W))$ is represented by a knot that bounds a smoothly embedded disk in W.

Proposition 2.1 *Let W be a smooth, connected, compact, orientable 4-manifold and let Y be a submanifold of ∂W. Suppose that (W,Y) admits a handle structure with no 1-handles. Let $\gamma \in \ker(\pi_1(Y) \to \pi_1(W))$. There exists a knot K in the homotopy class, γ, which bounds a smoothly embedded disk in W.*

Proof Let $\beta_1,\dots,\beta_m \subseteq Y$ be the attaching spheres of the 2-handles of (W,Y) regarded as framed knots. Notice that if we take any collection of framed pushoffs of the various β_i, then the resulting link bounds a collection of disjoint smoothly embedded disks in W. Indeed, these disks are pushoffs of the cores of the 2-handles.

Pick a basepoint q in Y. After a choice of basing arcs, the homotopy classes $[\beta_1],\dots,[\beta_m]$ normally generate $\ker(\pi_1(Y,q) \to \pi_1(W,q))$. By assumption, γ is in

https://doi.org/10.4153/S0008439519000791 Published online by Cambridge University Press

$\ker(\pi_1(Y,q) \to \pi_1(W,q))$. We conclude that γ is a product of conjugates of the β_i:

$$\gamma = \prod_{k=1}^{n} [c_k \beta_{i_k}^{\epsilon_k} c_k^{-1}]$$

for some choice of arcs $c_k \subseteq Y$ running from q to a point on β_k and $\epsilon_k = \pm 1$. Here, $[c_k \beta_{i_k}^{\epsilon_k} c_k^{-1}] \in \pi_1(Y,q)$ is the homotopy class of the result of following c_k, then β_{i_k} (or its reverse if $\epsilon_k = -1$), and finally the reverse of c_k. After a small homotopy we assume that the various c_k are embedded arcs, are disjoint from each other (except for the point at q), and have interiors disjoint from the various β_i.

As a consequence, we can construct a knot K in the the homotopy class γ by starting with a single unknot based at q and banding it with pushoffs of the β_{i_k} (or the reverse $\beta_{i_k}^{-1}$) along bands around the c_k. Finally, we build a smoothly embedded disk bounded by K as follows. For each $k = 1, \dots, n$ let Δ_k be a pushoff of the core of the 2-handle attached along β_{i_k} (or its orientation reverse if $\epsilon_k = -1$). Observe that $\Delta_1, \dots, \Delta_n$ are disjoint smoothly embedded disks. Band each Δ_k to a single disk bounded by the unknot centered at q along the arc c_k. The resulting surface is the required smoothly embedded disk. ∎

Theorem 1.5 amounts to a special case of Proposition 2.1

***Theorem* 1.5** *Let Y be a homology sphere that bounds a smooth homology ball W that has a handle structure with no 3-handles. Let K be a knot in Y that is nullhomotopic in W. Then K is homotopic in Y to a knot K' that bounds a smoothly embedded disk in W.*

Proof Let Y be a smooth homology sphere that bounds a smooth homology ball W that has a handle structure with no 3-handles. Turning this handle structure upside-down gives a handle structure on (W,Y) with no 1-handles. Thus, Proposition 2.1 applies, so that since the homotopy class of K is in $\ker(\pi_1(Y) \to \pi_1(W))$, we conclude that K is homotopic in Y to a knot that bounds a smoothly embedded disk in W. ∎

3 Separating 3-handles from 1-handles

In the case of a homology ball W that has 3-handles, we split W into two pieces separating the 1- and 3-handles.

Proposition 3.1 *Let W be a smooth homology ball with boundary $\partial W = Y$. Then there exists a smooth homology ball W' with $\partial W' = Y'$ and a smooth homology cobordism V from Y' to Y such that*

- (i) $W' \cup_{Y'} V = W$;
- (ii) *W' has a handle structure with neither* 3- *nor* 4-*handles and so (W',Y') has neither* 0- *nor* 1-*handles;*
- (iii) *(V,Y') has a handle structure with neither* 0- *nor* 1-*handles, so that (V,Y) has neither* 3- *nor* 4-*handles.*

If W is a contractible 4-manifold, then W' can be arranged to also be contractible.

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Proof Start by picking a handle decomposition of W consisting of a single 0-handle, n 1-handles, m 2-handles, some number of 3-handles and no 4-handles. Let $W^{(d)}$ denote the union of all handles of dimension at most d. Let $\alpha_1, \ldots, \alpha_n \subseteq \partial W^{(1)}$ be curves dual to the belt spheres of the added 1-handles. The homology classes $[\alpha_1] \ldots [\alpha_n]$ give a basis for $H_1(W^{(1)}) \cong \mathbb{Z}^n$. Let $\beta_1, \ldots, \beta_m \subseteq \partial W^{(1)}$ be the attaching spheres of the added 2-handles. We express the homology classes $[\beta_1], \ldots, [\beta_m] \in H_1(W^{(1)})$ as vectors in terms of the basis $[\alpha_1], \ldots, [\alpha_n]$ in order to get an $(m \times n)$ presentation matrix, P, for $H_1(W) = 0$. See, for example, [10, Section 4.2].

As P presents the trivial group, a sequence of row-moves reduces P to an $(m \times n)$ matrix with 1's on the main diagonal and 0's elsewhere. These row-moves in turn can be realized by reordering and reversing some β_i's and by sliding some β_i's over other β_j's. See, for example, [10, Section 5.1]. Thus, after performing these moves, we can assume that in $H_1(W^{(1)})$, $[\beta_i] = [\alpha_i]$ for all $i \leq n$ and $[\beta_i] = 0$ for all $i > n$.

Let W' be the result of adding to $W^{(1)}$ 2-handles along $\beta_1, \ldots, \beta_n$. Since $[\beta_i] = [\alpha_i]$ in $H_1(W^{(1)})$ for $i = 1, \ldots n$, we see that $H_1(W') = H_2(W') = 0$. Because W' has neither 3- nor 4-handles, $H_3(W') = H_4(W') = 0$. Thus, W' is a homology ball. Let $Y' = \partial W'$ and V be given by starting with $Y' \times [0,1]$, adding 2-handles along $\beta_{n+1}, \ldots, \beta_m$ and finally adding all 3-handles. Then $W = W' \cup_{Y'} V$. Because W and W' are both homology balls, a Mayer–Veitoris argument reveals that V is a homology cobordism. By its very construction, (V, Y') has only 2- and 3-handles. This gives the desired result when W is a homology ball.

Now suppose that W is a contractible 4-manifold. Pick a basepoint $q \in \partial W^{(1)}$. After a choice of basing arcs, we see that $\pi_1(\partial W^{(1)}, q) \cong \pi_1(W^{(1)}, q)$ is the free group on the set of homotopy classes $[\alpha_1], \ldots, [\alpha_n]$. Moreover, as W is simply connected,

$$\pi_1(\partial W^{(1)}) = \ker\big(\pi_1(\partial W^{(1)}, q) \longrightarrow \pi_1(W, q) = \{0\}\big)$$

is normally generated by the homotopy classes of the attaching spheres of the 2-handles $[\beta_1], \ldots, [\beta_m]$, after a choice of basing arcs. Thus, each α_i is homotopic in $\partial W^{(1)}$ to a product of conjugates of the attaching regions:

$$[\alpha_i] = \prod_{k=1}^{\ell_i} [c_{i,k} \beta_{j_{i,k}}^{\epsilon_{i,k}} c_{i,k}^{-1}]$$

for some choices of arcs $c_{i,k}$ running from q to a point on $\beta_{j_{i,k}}$ and $\epsilon_{i,k} = \pm 1$. After a small homotopy, we can assume that the various $c_{i,k}$ are all embedded, have disjoint interiors, and have interiors disjoint from the various β_j. Similar to the proof of Proposition 2.1, we build a knot A_i in $\partial W^{(1)}$ representing the homotopy class $[\alpha_i]$ by starting with a small unknot near q and banding with framed pushoffs of the various $\beta_{j_{i,k}}$ (or their reverses if $\epsilon_{i,k} = -1$) along bands centered on the $c_{i,k}$. Let $A_1, \ldots, A_n$ be the resulting knots. By construction, $[A_i] = [\alpha_i]$ in $\pi_1(\partial W^{(1)}, q)$.

Since A_i is constructed by starting with the unknot and banding it with the $\beta_{j_{i,k}}$, we can slide each A_i over the various $\beta_{j_{i,k}}$ to reduce the link $A_1 \cup \cdots \cup A_n$ to the unlink in $\partial W^{(2)}$. Thus, the various A_i bound disjoint disks in $\partial W^{(2)}$. Restrict a framing of these disks to a framing on A_i. Form a new 4-manifold by starting with $W^{(2)}$ and adding 2-handles along the knots $A_1, \ldots, A_n$. Because the A_i bound disjoint disks in $\partial W^{(2)}$ the boundary of this new 4-manifold is the connected sum of $\partial W^{(2)}$ with n

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copies of $S^1 \times S^2$. Add 3-handles along these new non-separating 2-spheres. The 2-handles added to A_i together with these 3-handles form cancelling pairs [10, Proposition 4.2.9] so that the 4-manifold resulting from adding these 2- and 3-handles to $W^{(2)}$ is diffeomorphic to $W^{(2)}$. Construct a 4-manifold diffeomorphic to W by adding the remaining 3-handles.

Thus, we can assume that W has a handle structure with 2-handles attached along framed curves $A_1, \dots, A_n, \beta_1, \dots, \beta_m \subseteq \partial W^{(1)}$ where $[A_1], \dots, [A_n]$ give a free basis for $\pi_1(W^{(1)})$. Let W' be given by adding to $W^{(1)}$ only the 2-handles attached along $A_1, \dots, A_n$. It now follows that W' is a simply connected homology ball. Therefore, W is contractible by a standard application of the Hurewicz homomorphism and the Whitehead theorem. See, for example, [6, Corollary 6.70]. Let $Y' = \partial W'$ and V be given by starting with $Y' \times [0,1]$, adding 2-handles along $\beta_1, \dots, \beta_m$ and then 3-handles. This gives the desired result. ∎

Finally, we prove Theorems 1.2 and 1.3.

***Theorem* 1.2** *Let Y be a homology sphere that bounds a smooth contractible 4-manifold. Let K be a knot in Y. There exists a knot K' in a homology sphere Y' such that (Y,K) is homology concordant to (Y',K') and K' is homotopic in Y' to a third knot K'' that bounds a smoothly embedded disk in a smooth contractible 4-manifold bounded by Y'.*

Proof Let Y be the boundary of a contractible smooth 4-manifold W. Let K be a knot in Y. Appeal to Proposition 3.1 to decompose W as $W' \cup_{Y'} V$ where W' is a contractible 4-manifold with boundary Y', V is a homology cobordism from Y to Y', (V,Y) has only relative 1- and 2-handles, and (W',Y') has no 1-handles. As (V,Y) has only 1- and 2-handles, we can realize V by starting with $Y \times [0,1]$ and adding 1- and 2-handles on $Y \times \{1\}$. We isotope K slightly to make it disjoint from the attaching regions for these handles. The image of $K \times [0,1]$ in $Y \times [0,1] \subseteq V$ gives a homology concordance from K to some knot K' in Y'. Now, $\pi_1(W')$ is trivial so that K' is nullhomotopic in W'. Since W' has no 3-handles, Theorem 1.5 applies and we see that K' is homotopic to some other knot K'' in Y' that bounds a smoothly embedded disk in W'. ∎

***Theorem* 1.3** *Let Y be a homology sphere that bounds a homology ball W. Let K be a knot in Y that is nullhomotopic in W. There exists a knot K' in a homology sphere Y' such that (Y,K) is homology concordant to (Y',K') and K' is homotopic in Y to a third knot K'' that bounds a smoothly embedded disk in a smooth homology ball bounded by Y'.*

Proof Let Y be the boundary of a smooth homology ball W. The proof begins in the same manner as the proof of Theorem 1.2. Appeal to Proposition 3.1 to decompose W as $W' \cup_{Y'} V$ where W' is a homology ball with boundary Y', V is a homology cobordism from Y to Y', (V,Y) has only relative 1- and 2-handles, and (W',Y') has no relative 1-handles. As (V,Y) has only 1- and 2-handles, we realize V by starting with $Y \times [0,1]$ and adding 1- and 2-handles to $Y \times \{1\}$. We can isotope K slightly to get K disjoint from the attaching regions. The image of $K \times [0,1]$ in $Y \times [0,1] \subseteq V$ gives a concordance from K to some knot K_0' in Y'. From here the proof differs from that of Theorem 1.2. As W' is not simply connected, we cannot conclude that K_0' is nullhomotopic in W'.

Let K_+' be a pushoff of K_0'. The choice of pushoff will not be relevant to the proof. Pick a basepoint $q \in Y'$ that lies on K_+'. Since K is nullhomotopic in W by assumption, K_+' is nullhomotopic in W as well. Therefore, the homotopy class $[K_+']$ lies in $\ker(\pi_1(V,q) \to \pi_1(W,q))$. Recall that (W',Y') has only 2- and 3-handles. After making a choice of basing arcs, the homotopy classes of the attaching regions for these 2-handles, $\beta_1,\dots,\beta_n \subseteq Y'$, normally generate $\ker(\pi_1(V,q) \to \pi_1(W,q))$. Thus, $[K_+']$ is equal in $\pi_1(V,q)$ to a product of conjugates of these β_i. More precisely, in $\pi_1(V,q)$, $[K_+'] = \prod_{k=1}^m [c_k \beta_{i_k}^{\epsilon_k} c_k^{-1}]$ for some choices of embedded curves $c_k \subseteq V$ running from q to a point on β_{i_k}. Since (V,Y') has no relative 1-handles, $\pi_1(Y',q) \to \pi_1(V,q)$ is onto, and we can assume that each c_k is embedded in Y'. With this assumption, we have $\prod_{k=1}^m [c_k \beta_{i_k}^{\epsilon_k} c_k^{-1}] \in \pi_1(Y',q)$ is a product of conjugates of the attaching regions $\beta_1,\dots,\beta_n$. Thus, $\prod_{k=1}^m [c_k \beta_{i_k}^{\epsilon_k} c_k^{-1}]$ is in $\ker(\pi_1(Y',q) \to \pi_1(W',q))$. Theorem 1.5 concludes that $\prod_{k=1}^m [c_k \beta_{i_k}^{\epsilon_k} c_k^{-1}] \in \pi_1(Y',q)$ is represented by a knot K'' in Y' that bounds a smoothly embedded disk in W'.

Summarizing the proof so far, there is a smoothly embedded annulus C in V bounded by $K \subseteq Y$ and $K_0' \subseteq Y'$. In turn, K_0' is homotopic in V to another knot $K'' \subseteq Y'$ that bounds a smoothly embedded disk in W'. If this homotopy were in Y', then we would be done. It remains to alter the annulus C and the knot K_0' to arrange that the homotopy from K_0' to K'' lies in Y'.

Let $\nu(K) \cong K \times D^2$ be an open product neighborhood of K in Y, $\nu(C)$ be the image of $\nu(K) \times [0,1]$ in $Y \times [0,1] \subseteq V$, and $\nu(K_0') = \nu(C) \cap Y'$. Choosing these tubular neighborhoods small enough, K_+' is disjoint from $\nu(K_0')$. As $[K_+'] = [K'']$ in $\pi_1(V,q)$, it follows that the product $[K'_+]^{-1}[K'']$ is nullhomotopic in V. Thus, $[K'_+]^{-1}[K''] \in \ker(\pi_1(V - \nu(C),q) \to \pi_1(V,q))$, which is normally generated by the homotopy class of a meridian $m(K_0') \subseteq Y'$ of K_0' based at q. Thus, we conclude that there is a product of conjugates of meridians of K_0', $\delta = \prod_k [d_k m(K_0')^{\epsilon_k} d_k^{-1}]$ with each d_k an embedded arc in $V - \nu(C)$, such that $[K'_+]^{-1}[K'']\delta^{-1}$ is nullhomotopic in $V - \nu(C)$.

As $\nu(C)$ is the image of $\nu(K) \times [0,1]$ in $Y \times [0,1] \subseteq V$ and (V,Y) has only 1- and 2-handles, it follows that $(V - \nu(C), Y - \nu(K))$ has only 1- and 2-handles. Turning this handle structure upside-down, $(V - \nu(C), Y' - \nu(K_0'))$ has only relative 2- and 3-handles. In particular, $\pi_1(Y - \nu(K_0')) \to \pi_1(V - \nu(C))$ is onto. Thus, we can assume that each d_k lies in $Y' - \nu(K_0')$. As a consequence, $\delta \in \pi_1(Y' - \nu(K_0'), q)$.

We now have $[K'_+]^{-1}[K'']\delta^{-1} \in \ker(\pi_1(Y' - \nu(K_0')) \to \pi_1(V - \nu(C)))$. Again making use of the fact that $(V - \nu(C), Y' - \nu(K_0'))$ has no 1-handles, Proposition 2.1 implies that the homotopy class $[K'_+]^{-1}[K'']\delta^{-1} \in \pi_1(Y' - \nu(K_0'))$ contains a knot J that bounds a smoothly embedded disk D_J in $V - \nu(C)$. The disk $D_J \subseteq V$ is disjoint from the annulus C. We band C to D_J along an embedded arc in Y' and see a homology concordance from K to a new knot K' with $[K'] = [K_+'][J]$ in $\pi_1(Y',q)$. Expanding in $\pi_1(Y',q)$,

$$[K'] = [K_+'][J] = [K_+']\big([K'_+]^{-1}[K'']\delta^{-1}\big) = [K'']\delta^{-1}.$$

As δ is a product of conjugates of meridians of K_0', δ is nullhomotopic in Y'. Thus, in $\pi_1(Y',q)$, $[K'] = [K'']$.

Summarizing, there is a homology cobordism V from Y to Y' in which the knots K and K' cobound a smoothly embedded annulus. The knot $K' \subseteq Y'$ is homotopic in Y' to a third knot K'', which bounds a smoothly embedded disk in the homology ball W'. This completes the proof. ∎

4 The Equivalence Relation Generated by Homotopy and Homology Concordance

Next we set our eyes on Theorem 1.6, which concludes that two knots in possibly different homology spheres cobound an immersed annulus in a homology cobordism if and only if they are related by a sequence of homology concordances and homotopies. Recall the following notation from the introduction:

- $(Y,K) \simeq_h^3 (Y,J)$ means K and J cobound an immersed annulus in the homology sphere Y.
- $(Y,K) \simeq_c (X,J)$ means there exists a homology cobordism from Y to X in which K and J cobound a smoothly embedded annulus.
- $(Y,K) \simeq_h^4 (X,J)$ means there exists a homology cobordism from Y to X in which K and J cobound an immersed annulus.

First we check that the operation of connected sum of pairs is well defined under each of these equivalence relations. Recall that the connected sum $(Y,K)\#(X,J) = (Y\#X, K\#J)$ is given by picking points $p \in K$ and $q \in J$, removing small neighborhoods $\nu(p) \subseteq Y$ and $\nu(q) \subseteq X$, of p and q, so that $K - \nu(p)$ and $J - \nu(q)$ are properly embedded arcs from a point p_+ to a point p_- and from q_+ to q_-. We form the connected sum $Y\#X$ by gluing these two spherical boundaries together along an orientation reversing diffeomorphism that sends p_+ to q_- and p_- to q_+. The connected sum $K\#J$ is given by gluing together $K - \nu(p)$ and $J - \nu(q)$.

Proposition 4.1 *Suppose that (Y,K), (Y',K'), and (X,J) are knots in homology spheres. For any $\simeq \in \{\simeq_h^3, \simeq_c, \simeq_h^4\}$, if $(Y,K) \simeq (Y',K')$, then $(Y,K)\#(X,J) \simeq (Y',K')\#(X,J)$. (In the case of $\simeq_h^3$, we assume $Y = Y'$.)*

Proof The proofs in the cases of $\simeq_c$ and $\simeq_h^4$ are nearly identical. Suppose that (Y,K), (Y',K'), and (X,J) are knots in homology spheres. Let W be a homology cobordism from Y to Y' in which A is an embedded (or immersed in the case of $\simeq_h^4$) annulus bounded by K and K'. Let α be an embedded curve in A running from a point on K to a point on K'. If A is not embedded, then α is chosen to be disjoint from all self-intersection points of A. Let p be a point on J. Glue together $W - \nu(\alpha)$ and $(X - \nu(p)) \times [0,1]$ to get a homology cobordism V from $Y\#X$ to $Y'\#X$. Let A' be the result of gluing $A - \nu(\alpha)$ to $(J - \nu(p)) \times [0,1]$ in this homology cobordism. Then A' is an embedded (or immersed if A is not embedded) annulus bounded by $K\#J$ and $K'\#J$. Thus, if $(Y,K) \simeq_c (Y',K')$, then $(Y\#X, K\#J) \simeq_c (Y'\#X, K'\#J)$, and if $(Y,K) \simeq_h^4 (Y',K')$, then $(Y\#X, K\#J) \simeq_h^4 (Y'\#X, K'\#J)$.

Suppose that $(Y,K) \simeq_h^3 (Y',K')$ so that $Y = Y'$ and K is homotopic to K' in Y. We can assume that the homotopy is constant on a small neighborhood of some point p of K. Using this as the point we use in the connected sum construction and using the constant homotopy on $J \subseteq Y$, we see that $K\#J$ and $K'\#J$ are homotopic in $Y\#X$, completing the proof. ■

***Theorem* 1.6** *Let Y and X be homology spheres. Let K and J be knots in Y and X, respectively. If there exists a smooth homology cobordism from Y to X in which K is homotopic to J, then there exist knots K' and J' in some homology sphere Z such that*

$$(Y,K) \simeq_c (Z,K') \simeq_h^3 (Z,J') \simeq_c (X,J).$$

Proof of Theorem 1.6 Suppose that (Y,K) and (X,J) are knots in homology spheres and that $(Y,K) \simeq_h^4 (X,J)$.

By Proposition 4.1, $(Y\#-X, K\#-J) \simeq_h^4 (X\#-X, J\#-J)$. Here, $-X$ indicates the orientation reverse of X and $-J$ the orientation reverse of J. Let V be a homology cobordism from $Y\#-X$ to $X\#-X$ in which $K\#-J$ and $J\#-J$ cobound an immersed annulus C. Just as for knots in S^3, $J\#-J$ bounds a smoothly embedded disk, Δ, in a homology ball, W, bounded by $X\#-X$. Glue together V and W along their common $X\#-X$ boundary. This gives a homology ball bounded by $Y\#-X$ in which $K\#-J$ bounds an immersed disk, $C \cup \Delta$. Thus, $K\#-J$ is nullhomotopic in a homology ball. Theorem 1.3 now applies and concludes that there exists another homology sphere Z_0 and knots L and L' in Z_0 such that

$$(Y\#-X, K\#-J) \simeq_c (Z_0,L) \simeq_h^3 (Z_0,L') \simeq_c (S^3,U),$$

where U is the unknot in S^3. By Proposition 4.1, we can take the connected sum of each of these terms with (X,J) to get

$$\big((Y\#-X)\#X, (K\#-J)\#J\big) \simeq_c (Z_0\#X, L\#J) \simeq_h^3 (Z_0\#X, L'\#J) \simeq_c (X,J).$$

Together with the observation that $(Y,K) \simeq_c (Y\#(-X\#X), K\#(-J\#J))$, this completes the proof. ■

Acknowledgments The author would like to thank MinHoon Kim, Arunima Ray, JungHwan Park, Paolo Aceto, Kent Orr, Aliakbar Daemi, and Adam Levine for helpful conversations at various stages of the development of the ideas leading to this paper.

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Department of Mathematics, University of Wisconsin–Eau Claire, USA
e-mail: daviscw@uwec.edu
URL: www.uwec.edu/daviscw