Proceedings of the Royal Society of Edinburgh, **136A**, 889–907, *2006*

Radial solutions concentrating on spheres of nonlinear Schrödinger equations with vanishing potentials

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(MS received 1 June 2005; accepted 5 October 2005)

We prove the existence of radial solutions of

$$-\varepsilon^2 \Delta u + V(|x|)u = u^p, \quad x \in \mathbb{R}^n, \quad u \in W^{1,2}(\mathbb{R}^n), \quad u > 0,$$

concentrating on a sphere for potentials which might be zero and might decay to zero at infinity. The proofs use a perturbation technique in a variational setting, through a Lyapunov–Schmidt reduction.

1. Introduction

There is a great deal of literature on nonlinear Schrödinger (NLS) equations with potentials, such as

$$\left.\begin{aligned} -\varepsilon^2 \Delta u + V(x)u = u^p, \quad x \in \mathbb{R}^n,\\ u \in W^{1,2}(\mathbb{R}^n), \quad u > 0. \end{aligned}\right\} \tag{1.1}$$

Solutions of (1.1) for ε small are often called semiclassical states. The main feature of these semiclassical states, u_ε of (1.1), is that they concentrate, in the sense that outside the concentration set u_ε tend uniformly to zero as $\varepsilon \to 0$. Roughly, a classical result states that if V is smooth and satisfies

$$\exists V_0, V_1 > 0, \quad \text{such that } 0 < V_0 \leqslant V(x) \leqslant V_1, \tag{V_0}$$

for any *stable* isolated stationary point x_0 of V, there exists a solution of (1.1) concentrating at x_0 (see, for example, [2, 10]). Stable stationary points of V are maxima, minima and, more generally, points where the local degree of V' is non-zero. Solutions concentrating at a point are called spikes.

More recently, assumption (V_0) has been relaxed. The NLS equation with potentials which can be zero, but such that $\liminf_{|x|\to\infty} V(x) > 0$, has been studied in [7, 8]. Furthermore, a class of *positive* potentials which might decay to zero at

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https://doi.org/10.1017/S0308210500004789 Published online by Cambridge University Press

infinity has been considered (see [4, 5], and also [11, 13] for some previous partial results). However, in the aforementioned papers, only the existence of spikes is proved.

On the other hand, by dealing with radial potentials, namely with equations like

$$\left.\begin{aligned} -\varepsilon^2\Delta u + V(|x|)u = u^p, \quad x \in \mathbb{R}^n, \\ u \in W^{1,2}(\mathbb{R}^n), \quad u > 0, \end{aligned}\right\} \tag{1.2}$$

new classes of solutions concentrating on spheres have been found (see [3]). Let us define the auxiliary *weighted* potential M by setting

$$M(r) = r^{n-1}V^{\theta}(r), \quad \text{where } \theta = \frac{p+1}{p-1} - \frac{1}{2}.$$

Let $r^* > 0$ be a strict maximum or minimum of M and suppose that $p > 1$ and that V is smooth. Under the above assumptions, it has been proved in [3] that (1.2) possesses a radial solution which concentrates as $\varepsilon \to 0$ at the sphere $|x| = r^* > 0$.

In the present paper we consider radial potentials which might be zero and might decay to zero at infinity. By improving the preceding results, we show the existence of a semiclassical radial state concentrating on a sphere (see theorem 2.1 below).

2. Main result

We will assume that the potential $V \in C^1(\mathbb{R}_+, \mathbb{R})$ satisfies:

(V_1) $V(r) \geqslant 0$, $\forall r \in \mathbb{R}_+$ and there exist $r_0, a_1, a_2 > 0$ such that

$$\frac{a_1}{r^2} \leqslant V(r) \leqslant a_2, \quad \forall r > r_0; \tag{2.1}$$

(V_2) $\exists V_1' > 0$ such that $|V'(x)| \leqslant V_1'$, $\quad \forall r \in \mathbb{R}_+$.

The assumptions $V(r) \leqslant a_2$ and $(V2)$ are essential in the proof of lemmas 3.2 and 3.3 (see also [3]).

The main result of this paper is the following theorem.

Theorem 2.1. *Let $p > 1$ and suppose that (V_1) and (V_2) hold. Moreover, let us assume that there exists $r^* > r_0$ such that M has an isolated local maximum or minimum at $r = r^*$. Then for $\varepsilon \ll 1$, equation (1.2) has a solution that concentrates at the sphere $|x| = r^*$.*

Let us emphasize that, as in [3], any power $p > 1$ is allowed. It is also worth pointing out that, as a consequence of the fact that concentration arises where $V(r) > 0$, the preceding solutions behave like those found in [3] ad not as in [7,8] (see remark 5.1, below). It would be interesting to establish the existence of semiclassical states concentrating on a sphere of radius $\hat{r} > 0$, where $V(\hat{r}) = 0$. Note that such a $\hat{r}$ would be a minimum of M.

As a particular case in which the preceding theorem applies, let us consider a potential V such that

$$V(r) > 0 \quad \forall r > 0, \quad \text{and} \quad \exists \alpha \in (0, 2] : V(r) \sim r^{-\alpha}, \quad \text{as } r \to +\infty. \tag{2.2}$$

We find that $M(0) = 0$ and $M(r) \sim r^{n-1-\alpha\theta}$ as $r \to +\infty$. Thus, $M(r) \to 0$ as $r \to +\infty$, provided that $\alpha\theta > n-1$ and, hence, in such a case, M has a maximum. Therefore, an application of theorem 2.1 yields the following corollary.

COROLLARY 2.2. *Suppose that V satisfies (V_2) and (2.2). Moreover, let $\alpha\theta > n-1$. Then, for $\varepsilon \ll 1$, equation (1.2) has a solution that concentrates at a sphere.*

In order to prove theorem 2.1, we consider the equation

$$-\Delta u + V(\varepsilon|x|)u = u^p, \quad u \in W^{1,2}(\mathbb{R}^n), \quad u > 0. \tag{2.3}$$

If u is a solution of (2.3), then $u(x/\varepsilon)$ solves (1.2). This paper is devoted to proving the existence of solutions of (2.3), for ε small. Following the procedure used in [3,5], we first use a Lyapunov–Schmidt reduction in order to substitute (2.3) with an equivalent system consisting of an *auxiliary equation* and a *bifurcation equation* (see § 3).

A great deal of work is devoted to solve the auxiliary equation. First, it is shown that solutions of this equation can be found by searching the fixed points of a map S_ε in an appropriate set Γ_ε. Roughly, we look for solutions of the form $z + w$, where z is peaked near $r^* > r_0$ and w is small. This allows us to take advantage of the fact that, for r close to r^*, we have $V(r) \geqslant C > 0$ (see the discussion before lemma 3.2).

As for the set Γ_ε, it consists of functions with an appropriate decay at infinity and also near the region where V might vanish. The main point is to show that, for ε sufficiently small, S_ε is a contraction and maps Γ_ε into itself. The arguments with which to prove these facts are given in § 4 and require several new ingredients with respect to [3, 5], in order to handle the possible decay of V to zero (which is not allowed in [3]) as well as the fact that V may vanish in a compact set, which is not allowed in [3, 5]. In particular, the arguments needed to control the decay at infinity of $S_\varepsilon(w)$, $w \in \Gamma_\varepsilon$, are quite different from the ones employed in [5] (see remark 4.1 and the last part of the proof of lemma 4.5).

Once that the auxiliary equation is solved, study of the reduced finite-dimensional functional is carried out as in [3].

In order to keep the paper to a reasonable length, the arguments that are similar to those employed in [3, 5] have been sketched and details left to the interested reader.

2.1. Notation

$B_R(y) = \{x \in \mathbb{R}^n : |x-y| \leqslant R\}$; $B_R = B_R(0)$; $|B_R(y)|$ denotes the Lebesgue measure of $B_R(y)$.

$c_1, c_2, \dots$ and $C_1, C_2, \dots$ denote positive, possibly different, constants.

3. Functional setting and finite-dimensional reduction

We will work in the Hilbert space

$$E = \left\{ u \in \mathcal{D}^{1,2}(\mathbb{R}^n) : u \text{ is radial}, \int_{\mathbb{R}^n} V(\varepsilon|x|)u^2(x)\,\mathrm{d}x < \infty \right\}.$$

https://doi.org/10.1017/S0308210500004789 Published online by Cambridge University Press

endowed with the scalar product and norm given, respectively, by

$$(u,v) = \int_{\mathbb{R}^n} [\nabla u \cdot \nabla v + V(\varepsilon|x|)uv]\,\mathrm{d}x, \quad \|u\|^2 = (u,u). \tag{3.1}$$

Since we are dealing with a nonlinearity like u^p where, possibly, $p \geqslant (n+2)/(n-2)$, a truncation is in order. Given $\bar{c} > 0$, $\hat{c} > 0$ and $\vartheta > 0$ such that $\vartheta(p-1) > n$, we choose $F_\varepsilon \in C^2(\mathbb{R}^n \times \mathbb{R})$ satisfying:

$$F_\varepsilon(|x|,u) = \begin{cases} \dfrac{1}{p+1}|u_+|^{p+1} & \text{if } |u| < \hat{c}(1+\varepsilon|x|)^{-\vartheta}, \\ \bar{c}(1+|\varepsilon x|)^{-\vartheta(p+1)} & \text{if } |u| > 2\hat{c}(1+\varepsilon|x|)^{-\vartheta}. \end{cases} \tag{3.2}$$

Specifically, let us define

$$F_\varepsilon(|x|,u) = \Upsilon(x,u)\frac{1}{p+1}|u_+|^{p+1} + (1-\Upsilon(x,u))\bar{c}(1+\varepsilon|x|)^{-\vartheta(p+1)},$$

where

$$\Upsilon(x,u) = \mu(\hat{c}^{-1}|u|(1+\varepsilon|x|)^{\vartheta}).$$

Here $\mu(s)$ is a real C^∞ function equal to one for $s < 1$ and equal to zero for $s > 2$.

We set

$$I_\varepsilon(u) = \tfrac{1}{2}\|u\|^2 - \int_{\mathbb{R}^n} F_\varepsilon(|x|,u(|x|))\,\mathrm{d}x, \quad u \in E. \tag{3.3}$$

A direct computation shows that I_ε is of class C^2 on E (the inequality $\vartheta(p-1) > n$ is needed).

Remark 3.1. Let $u_\varepsilon \in E$ be a critical point of I_ε. If there exist $\hat{c}$ and ϑ such that $|u_\varepsilon(r)| < \hat{c}(1+\varepsilon r)^{-\vartheta}$, then such a u_ε is a solution of (2.3).

To solve $I'_\varepsilon(u) = 0$, we will use a perturbation method based on a finite-dimensional reduction. For a broad exposition of the abstract framework as well as for further applications, we refer the reader to [1]. First, some preliminaries are in order. Observe that, under our conditions on V, $V(r^*) > 0$. For the sake of clarity, we will assume throughout the paper that $V(r^*) > 1$. Choose $\delta > 0$ such that $r^* - 4\delta > r_0$ and

$$V(r) \geqslant 1, \quad \forall r \in (r^* - 4\delta, r^* + 4\delta), \tag{3.4}$$

For any $\varepsilon > 0$ small, we will take ρ as a parameter satisfying

$$r^* - \delta \leqslant \varepsilon\rho \leqslant r^* + \delta, \tag{3.5}$$

and, for any $k > 0$, we denote by $U_k(r)$ the unique positive even function satisfying

$$-U'' + kU = U^p, \quad U \in W^{1,2}(\mathbb{R}).$$

It is well known that U_k decays exponentially to zero when $r \to \pm\infty$.

Define $\tau : \mathbb{R} \to \mathbb{R}$, a C^∞ function, such that $0 \leqslant \tau(r) \leqslant 1$, $\tau(r) = 1$ if $|r| \leqslant 1$ and $\tau(r) = 0$ if $|r| \geqslant 2$. We shall use the cut-off function $\phi_{\varepsilon,\rho}$ defined as

$$\phi_{\varepsilon,\rho}(r) = \phi_\varepsilon(r) = \tau\left(\frac{\varepsilon}{\delta}(r-\rho)\right).$$

On setting

$$R_d = \rho - \frac{d}{\varepsilon}, \qquad T_d = \rho + \frac{d}{\varepsilon},$$

ϕ_ε happens to satisfy

$$\phi_\varepsilon(r) = \begin{cases} 0 & \text{if } r \notin [R_{2\delta}, T_{2\delta}], \\ 1 & \text{if } r \in [R_\delta, T_\delta]. \end{cases}$$

Set $z_{\varepsilon,\rho}(r) = \phi_\varepsilon(r) U_{V(\varepsilon\rho)}(r-\rho)$ and

$$Z_\varepsilon = Z = \{z_{\varepsilon,\rho}(r) : \rho \text{ satisfying (3.5)}\}.$$

We shall look solutions of (2.3) in the form $u = z_{\varepsilon,\rho} + w$, with $z_{\varepsilon,\rho} \in Z_\varepsilon$ and $w \perp z_{\varepsilon,\rho}$. Letting P denote the orthogonal projection onto $(T_{z_{\varepsilon,\rho}} Z)^\perp$, the equation $I'_\varepsilon(z_{\varepsilon,\rho} + w) = 0$ is equivalent to the system

$$PI'_\varepsilon(z_{\varepsilon,\rho} + w) = 0 \quad \text{(auxiliary equation)}, \tag{3.6}$$

$$(Id - P) I'_\varepsilon(z_{\varepsilon,\rho} + w) = 0 \quad \text{(bifurcation equation)}. \tag{3.7}$$

Most of the paper is devoted to solve the auxiliary equation. In the last section we deal with the bifurcation equation.

Some preliminary estimates are in order. First of all, as in [3, lemma 3.1], we find that, for all ρ satisfying (3.5), the following expression holds:

$$\|z_{\varepsilon,\rho}\| \sim \varepsilon^{(1-n)/2}. \tag{3.8}$$

In the next two lemmas, we shall estimate $\|I'_\varepsilon(z_{\varepsilon,\rho})\|$ and $\|[PI''_\varepsilon(z_{\varepsilon,\rho})]^{-1}\|$. Let us point out that, since $z_{\varepsilon,\rho}$ has a uniform exponential decay at infinity, we can choose $\hat{c}$ and ϑ such that $|z_{\varepsilon,\rho}(x)| < \hat{c}(1+\varepsilon r)^{-\vartheta}$. Thus, in the following lemmas we may suppose that

$$I_\varepsilon(u) = \tfrac{1}{2}\|u\|^2 - \frac{1}{p+1}\int_{\mathbb{R}^n} |u_+|^{p+1}\,\mathrm{d}x.$$

Lemma 3.2. *For all ρ satisfying (3.5), the following expression holds:*

$$\|I'_\varepsilon(z_{\varepsilon,\rho})\| \sim \varepsilon\|z_{\varepsilon,\rho}\| \sim \varepsilon^{(3-n)/2}.$$

Proof. The same calculation as in [3, p. 437] yields

$$I'_\varepsilon(z_{\varepsilon,\rho})[v] = -(n-1)\int_0^\infty r^{n-2} z'_{\varepsilon,\rho} v \,\mathrm{d}r + \int_0^\infty r^{n-1}[-z''_{\varepsilon,\rho} + V(\varepsilon r) z_{\varepsilon,\rho} - z^p_{\varepsilon,\rho}] v \,\mathrm{d}r. \tag{3.9}$$

Let us evaluate the first integral. Denote by $K_{2\delta} = [R_{2\delta}, T_{2\delta}]$ the support of $z_{\varepsilon,\rho}$. Since ρ satisfies (3.5), we may infer that $r \in K_{2\delta}$ implies that $\varepsilon r \in [r^* - 3\delta, r^* + 3\delta]$, namely that $r \sim \varepsilon^{-1}$. Hence,

$$\int_0^\infty r^{n-2} z'_{\varepsilon,\rho} v \,\mathrm{d}r \sim \varepsilon \int_0^\infty r^{n-1} z'_{\varepsilon,\rho} v \,\mathrm{d}r = \varepsilon \int_{r \in K_{2\delta}} r^{n-1} z'_{\varepsilon,\rho} v \,\mathrm{d}r.$$

Moreover, the following inequalities hold:

$$\left|\int_{K_{2\delta}} r^{n-1} z'_{\varepsilon,\rho} v\,\mathrm{d}r\right| \leqslant \left[\int_{K_{2\delta}} r^{n-1} v^2\,\mathrm{d}r\right]^{1/2} \left[\int_{K_{2\delta}} r^{n-1} |z'_{\varepsilon,\rho}|^2\,\mathrm{d}r\right]^{1/2}$$
$$\leqslant C_1 \left[\int_{K_{2\delta}} r^{n-1} v^2\,\mathrm{d}x\right]^{1/2} \|z_{\varepsilon,\rho}\|.$$

From (3.4) and (3.5) it follows that $1 \leqslant V(\varepsilon r), \forall \varepsilon r \in K_{2\delta}$, and hence

$$\int_{K_{2\delta}} r^{n-1} v^2\,\mathrm{d}r \leqslant \int_{K_{2\delta}} r^{n-1} V(\varepsilon r) v^2\,\mathrm{d}r \leqslant \|v\|^2.$$

In conclusion, we find that

$$\int_0^\infty r^{n-2} z'_{\varepsilon,\rho} v\,\mathrm{d}r \sim \varepsilon \|v\| \cdot \|z_{\varepsilon,\rho}\|.$$

Similar arguments can be used to estimate the second integral in (3.9) and the proof can be completed as in [3]. □

The following uniform estimate is one of the keys of the proof of theorem 2.1.

Lemma 3.3. *$PI''(z_{\varepsilon,\rho})$ is a compact perturbation of the identity. Moreover, $PI''_\varepsilon(z_{\varepsilon,\rho})$ is uniformly invertible on $(T_{z_{\varepsilon,\rho}} Z)^\perp$ for all $z_{\varepsilon,\rho} \in Z$, $|\varepsilon\xi| \leqslant 1$. Namely, there exists $C' > 0$ such that if ε is sufficiently small, then $\|[PI''_\varepsilon(z_{\varepsilon,\rho})]^{-1}\| \leqslant C'$.*

Proof. The fact that $PI''(z_{\varepsilon,\rho})$ is a compact perturbation of the identity map has been proved in [5, lemma 4]. We turn our attention to its uniform invertibility. By direct computation, we have

$$\begin{aligned} I''_\varepsilon(z_{\varepsilon,\rho})[z,z] &= \int_{\mathbb{R}^n} (|\nabla z|^2 + V(\varepsilon x) z^2 - p z^{p+1})\,\mathrm{d}x \\ &= \int_{\mathbb{R}^n} (|\nabla z|^2 + V(\varepsilon x) z^2 - z^{p+1})\,\mathrm{d}x + (1-p)\int_{\mathbb{R}^n} z^{p+1}\,\mathrm{d}x \\ &= I'_\varepsilon(z_{\varepsilon,\rho})[z] + (1-p)\int_{\mathbb{R}^n} z^{p+1}\,\mathrm{d}x. \end{aligned}$$

Recall that $I'_\varepsilon(z_{\varepsilon,\rho})[z] \leqslant \bar{C}\varepsilon\|z\|^2 \sim \bar{C}\varepsilon^{2-n}$, and observe that

$$\int z^{p+1} \sim \varepsilon^{1-n}.$$

Therefore, $I''_\varepsilon(z_{\varepsilon,\rho})[z,z] \leqslant -c\|z\|^2$ for ε small.

We will use the notation

$$\dot{z}_{\varepsilon,\rho} = \frac{\partial}{\partial\rho} z_{\varepsilon,\rho}(r) = \frac{\partial}{\partial\rho}[\phi_{\varepsilon,\rho}(r) U_{V(\varepsilon\rho)}(r)].$$

It is easy to verify that

$$\dot{z}_{\varepsilon,\rho} = \tau\left(\frac{\varepsilon}{\delta}(r-\rho)\right)\left(\varepsilon V'(\varepsilon r)\left(\frac{\partial}{\partial V} U_V(r-\rho)\right) - \frac{\partial}{\partial r} U_V(r-\rho)\right) - \frac{\varepsilon}{\delta}\tau'\left(\frac{\varepsilon}{\delta}(r-\rho)\right) U_V(r-\rho).$$

Taking into account the definition of U and τ, we can conclude that $\dot{z}_{\varepsilon,\rho} \sim -z'_{\varepsilon,\rho}$. This will be used later, at the end of the proof.

Define $X = \langle z_{\varepsilon,\rho}, \dot{z}_{\varepsilon,\rho}\rangle$. We will show that the following inequality holds:

$$I''_\varepsilon(z_{\varepsilon,\rho})[v,v] \geqslant c\|v\|^2, \quad \forall v \in X^\perp.$$

Let us fix $v \in X^\perp$, and suppose that $\|v\| = 1$. First, we claim that there exists $R \in (\varepsilon^{-1/4}, \varepsilon^{-1/2})$ such that

$$\int_{R<||x|-\rho|<R+1} [|\nabla v|^2 + v^2]\,\mathrm{d}x < 2\varepsilon^{1/2}\|v\|^2 = 2\varepsilon^{1/2}. \tag{3.10}$$

Recall that $V(\varepsilon x) \geqslant 1$ in $[R_{3\delta}, T_{3\delta}]$. Therefore, it follows that

$$\int_{||x|-\rho|<\varepsilon^{-1/2}+1} [|\nabla v|^2 + v^2]\,\mathrm{d}x \leqslant \int_{||x|-\rho|<\varepsilon^{-1/2}+1} [|\nabla v|^2 + V(\varepsilon x)v^2]\,\mathrm{d}x \leqslant 1.$$

Note that the sum

$$\sum_{R\in\mathbb{N}}^{\varepsilon^{-1/4}<R<\varepsilon^{-1/2}} \int_{R<||x|-\rho|<R+1} [|\nabla v|^2 + v^2]\,\mathrm{d}x \leqslant 1$$

has more than $\frac{1}{2}\varepsilon^{-1/2}$ summands (for ε small). Thus, it is always possible to choose $R \in \mathbb{N}$, $R \in (\varepsilon^{-1/4}, \varepsilon^{-1/2})$ so that (3.10) holds.

For such $R > 0$, define χ_R as a C^∞ radial function verifying $0 \leqslant \chi_R(r) \leqslant 1$, $|\nabla\chi_R(r)| \leqslant 2$, and

$$\chi_R(r) = \begin{cases} 1 & ||x| - \rho| < R, \\ 0 & ||x| - \rho| > R + 1. \end{cases}$$

We define $v_1 = \chi_R v$ and $v_2 = v - v_1$. First of all, note that the subsequent estimates of the norms of v_1 and v_2 hold:

$$\|v_1\|^2 - \int_{||x|-\rho|<R} [|\nabla v|^2 + V(\varepsilon x)v^2]\,\mathrm{d}x = O(\varepsilon^{1/2}),$$

$$\|v_2\|^2 - \int_{||x|-\rho|>R+1} [|\nabla v|^2 + V(\varepsilon x)v^2]\,\mathrm{d}x = O(\varepsilon^{1/2}).$$

Hence, $\|v_1\|^2 + \|v_2\|^2 - 1 = O(\varepsilon^{1/2})$, and this implies that $(v_1, v_2) = O(\varepsilon^{1/2})$. After these preliminaries, we decompose the above into

$$I''_\varepsilon(z_{\varepsilon,\rho})[v,v] = I''_\varepsilon(z_{\varepsilon,\rho})[v_1,v_1] + I''_\varepsilon(z_{\varepsilon,\rho})[v_2,v_2] + 2I''_\varepsilon(z_{\varepsilon,\rho})[v_1,v_2]. \tag{3.11}$$

The last term of the above equation can be estimated easily, as follows:

$$I''_\varepsilon(z_{\varepsilon,\rho})[v_1,v_2] = (v_1,v_2) - \int_{R<||x|-\rho|<R+1} pz^{p-1}v_1v_2 \sim \varepsilon^{1/2}.$$

As for the second term of (3.11), we have

$$I''_\varepsilon(z_{\varepsilon,\rho})[v_2,v_2] = \|v_2\|^2 - \int_{||x|-\rho|>R} pz^{p-1}v_2^2.$$

By using Hölder and Sobolev inequalities, and the exponential decay of z, we obtain

$$\int_{||x|-\rho|>R} pz^{p-1}v_2^2 \leqslant \left[\int_{||x|-\rho|>R} (pz^{p-1})^{N/2}\right]^{2/N} \left[\int_{||x|-\rho|>R} v^{2^*}\right]^{2/2^*} \leqslant o_\varepsilon(\|v_2\|)^2,$$

and, thus,

$$I''_\varepsilon(z_{\varepsilon,\rho})[v_2, v_2] \sim \varepsilon^{1/2}.$$

We now focus on the first term of (3.11). Observe that, since v_1 has compact support, it belongs to $W^{1,2}(\mathbb{R}^n)$. Actually, we have

$$\int_{\mathbb{R}^n} v_1^2 \,\mathrm{d}x = \int_{||x|-\rho|<R+1} v_1^2 \,\mathrm{d}x \leqslant \int_{||x|-\rho|<R+1} V(\varepsilon x)v_1^2 \,\mathrm{d}x \leqslant \|v_1\|^2.$$

We are concerned with the estimate of

$$\begin{aligned} I''_\varepsilon(z_{\varepsilon,\rho})[v_1, v_1] &= \int_{\mathbb{R}^n} [|\nabla v_1|^2 + [V(\varepsilon x) - pz_{\varepsilon,\rho}^{p-1}]v_1^2] \,\mathrm{d}x \\ &= \int_{||x|-\rho|<R+1} [|\nabla v_1|^2 + [V(\varepsilon\rho) - pz_{\varepsilon,\rho}^{p-1}]v_1^2] \,\mathrm{d}x \\ &\quad + \int_{||x|-\rho|<R+1} [V(\varepsilon x) - V(\varepsilon\xi)]v_1^2 \,\mathrm{d}x. \end{aligned}$$

Using the boundedness of V', we may infer that $|V(\varepsilon r) - V(\varepsilon\rho)| \leqslant M\varepsilon|r-\rho| \leqslant 2M\varepsilon^{1/2}$. Let

$$(u, v)_\rho = \int_{\mathbb{R}} [u'(r)\cdot v'(r) + V(\varepsilon\rho)u(r)v(r)] \,\mathrm{d}r$$

denote a scalar product in $W^{1,2}(\mathbb{R}^n)$, and let $\|u\|_\rho^2 = (u, u)_\rho$ denote the associated norm. Observe that $\rho^{(n-1)/2}\|v_1\|_\rho \sim \|v_1\|$. We now write $v_1 = \phi + w$, where $\phi \in X$ and $w \perp_\rho X$, where $\perp_\rho$ stands for orthogonality in the $(\cdot,\cdot)_\rho$ sense. Let us show that ϕ is small compared with w, so v_1 turns out to be close to w. Note that ϕ is given by

$$\phi = (v_1, z_{\varepsilon,\rho})_\rho z_{\varepsilon,\rho}\|z_{\varepsilon,\rho}\|_\rho^{-2} + (v_1, \dot z_{\varepsilon,\rho})_\rho \dot z_{\varepsilon,\rho}\|\dot z_{\varepsilon,\rho}\|_\rho^{-2},$$

We first show that $|(v_1, z_{\varepsilon,\rho}) - \rho^{n-1}(v_1, z_{\varepsilon,\rho})_\rho| = o_\varepsilon(1)\rho^{(n-1)/2}$. Indeed,

$$\begin{aligned} |(v_1, z) - \rho^{n-1}(v_1, z)_\rho| &= \left| \int_{|r-\rho|<R+1} (r^{n-1} - \rho^{n-1})v_1'(r)z'_{\varepsilon,\rho}(r) \right. \\ &\quad \left. + \int_{|r-\rho|<R+1} [r^{n-1}V(\varepsilon r) - \rho^{n-1}V(\varepsilon\rho)]v_1 z_{\varepsilon,\rho} \,\mathrm{d}x \right| \\ &= o_\varepsilon(1) \int_{|r-\rho|<R+1} r^{n-1}|v_1'(r)z'_{\varepsilon,\rho}(r)| + r^{n-1}v_1(r)z_{\varepsilon,\rho}(r) \,\mathrm{d}r \\ &= o_\varepsilon(1)\rho^{(n-1)/2}, \end{aligned}$$

since $\|v_1\| \leqslant 1 + C\sqrt{\varepsilon}$ and $\|z\| \sim \rho^{(n-1)/2}$. Taking into account $v = v_1 + v_2$ and $v \perp X$ with respect to the scalar product in E, we deduce that $|(v_1, z_{\varepsilon,\rho})| = |(v_2, z_{\varepsilon,\rho})|$. We

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then find that

$$\begin{aligned}|(v_1, z_{\varepsilon,\rho})| &= |(v_2, z_{\varepsilon,\rho})|\\ &\leqslant \int_{||x|-\rho|>R} [|\nabla v_2 \cdot \nabla z_{\varepsilon,\rho}| + V(\varepsilon x)|v_2| z_{\varepsilon,\rho}]\,\mathrm{d}x\\ &\leqslant \left(\int_{\mathbb{R}^n} |\nabla v_2|^2\,\mathrm{d}x\right)^{1/2} \left(\int_{||x|-\rho|>R} |\nabla z_{\varepsilon,\rho}|^2\,\mathrm{d}x\right)^{1/2}\\ &\qquad + C_2 \left(\int_{\mathbb{R}^n} V(\varepsilon x)|v_2|^2\,\mathrm{d}x\right)^{1/2} \left(\int_{||x|-\rho|>R} z_{\varepsilon,\rho}^2\,\mathrm{d}x\right)^{1/2}.\end{aligned}$$

Recall that $R > \varepsilon^{-1/4}$. Then, because of the uniform exponential decay of $z_{\varepsilon,\rho}$ and its derivatives, we find that $|(v_1, z_{\varepsilon,\rho})| = o_\varepsilon(1)$. We then conclude that $|(v_1, z_{\varepsilon,\rho})_\rho| = o_\varepsilon(1)\rho^{(1-n)/2}$. In the same way, we can estimate $(v_1, \dot{z}_{\varepsilon,\rho})_\rho$. From this we obtain $\|\phi\| = o_\varepsilon(1)$, $\|\phi\|_\rho = o_\varepsilon(1)\rho^{(1-n)/2}$.

We turn our attention to w. Note that $\|w\|_\rho \leqslant C\rho^{(1-n)/2}$. From the expression of $\dot{z}_{\varepsilon,\rho}$, we can easily conclude that $\|z'_{\varepsilon,\rho} + \dot{z}_{\varepsilon,\rho}\|_\rho = o_\varepsilon(1)$. The non-degeneracy result of [12] then implies that

$$\int_{\mathbb{R}_+} [w'(r)^2 + (V(\varepsilon\rho) - p z_{\varepsilon,\rho}^{p-1}) w(r)^2]\,\mathrm{d}r \geqslant c_1 \|w\|_\rho^2 - o_\varepsilon(\rho^{1-n}) \tag{3.12}$$

By reasoning as above, we can conclude that

$$\int_{\mathbb{R}^n} [|\nabla w|^2 + (V(\varepsilon\xi) - p z_{\varepsilon,\rho}^{p-1})^2] w^2\,\mathrm{d}x \geqslant c_2 \rho^{n-1} \|w\|_\rho^2 - o_\varepsilon(1) \geqslant c_3 \|w\|^2 - o_\varepsilon(1).$$

From the previous computations, we deduce that

$$I''_\varepsilon(z_{\varepsilon,\rho})[v, v] \geqslant c_4 \|v_1\|^2 + c_5 \|v_2\|^2 - o_\varepsilon(1).$$

Since $\|v_1\|^2 + \|v_2\|^2 = 1 + O(\varepsilon^{1/2})$, we may infer that

$$I''_\varepsilon(z_{\varepsilon,\rho})[v, v] \geqslant c_6,$$

and the proof is completed. □

The preceding lemma allows us to transform the auxiliary equation into a fixed-point problem. Actually, on fixing $z_{\varepsilon,\rho}$, and letting

$$S_\varepsilon = S_{\varepsilon,\rho} : (T_{z_{\varepsilon,\rho}} Z)^\perp \mapsto (T_{z_{\varepsilon,\rho}} Z)^\perp$$

be defined by

$$S_\varepsilon(w) = w - [P I''_\varepsilon(z_{\varepsilon,\rho})]^{-1}(P I'_\varepsilon(z_{\varepsilon,\rho} + w)),$$

equation (3.6) is equivalent to $S_\varepsilon(w) = w$.

4. Fixed points of S_ε

Fixed points of S_ε are found in an appropriate subset of E. For $m = a_1/\varepsilon^2$, let us set

$$u_1(r) = r^{-\sigma}, \quad \sigma = \frac{n - 2 + \sqrt{(n-2)^2 + 4m}}{2}, \tag{4.1}$$

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and, given $c_1 > 0$, let W_ε denote the set of $w \in E$ such that the following pointwise estimates hold:

$$|w(r)| \leqslant 2c_1\sqrt{\varepsilon}\exp\{-\delta/\varepsilon\}, \qquad \forall r \in [0, R_{3\delta}], \tag{4.2}$$

$$|w(r)| \leqslant c_1\sqrt{\varepsilon}\exp\{-|r - R_{2\delta}|\}, \quad \forall r \in [R_{3\delta}, R_{2\delta}], \tag{4.3}$$

$$|w(r)| \leqslant c_1\sqrt{\varepsilon}\frac{u_1(r)}{u_1(T_{2\delta})}, \qquad \forall r \geqslant T_{2\delta}. \tag{4.4}$$

Next, let us also define

$$\Gamma_\varepsilon = \{w \in W_\varepsilon, w \perp z_{\varepsilon,\rho} \mid \|w\| \leqslant c_0\varepsilon\|z_{\varepsilon,\rho}\|\}.$$

In the above definitions, c_0 and c_1 are positive constants to be defined (see equations (4.9), (4.13)).

We will show that there exist $c_0, c_1 > 0$ such that S_ε maps Γ_ε into itself and is a contraction there (see proposition 4.3, below).

Remark 4.1. The function u_1 is a fundamental solution of

$$-\Delta u + \frac{m}{|x|^2}u = 0,$$

to which equation (2.3) will be compared. Let us also point out that, in contrast with [5], here it does not suffice to take m sufficiently large. Rather, we need to choose $m \sim \varepsilon^{-2}$; it is this choice that allows us to prove that $S_\varepsilon(w)$, $w \in \Gamma_\varepsilon$, has the decay at infinity required in (4.4).

In the remainder of the paper, we will make use of the following result.

Lemma 4.2. *For all ρ satisfying (3.5), all $w \in \Gamma_\varepsilon$ and all $0 \leqslant s \leqslant 1$, the equality*

$$\|I''_\varepsilon(z_{\varepsilon,\rho} + sw) - I''_\varepsilon(z_{\varepsilon,\rho})\| = o(\varepsilon^{q/2}), \quad q = 1 \wedge (p-1),$$

holds.

Proof. We can choose $\hat{c}$ and ϑ in (3.2) such that $|z_{\varepsilon,\rho} + w| < \hat{c}(1 + \varepsilon r)^{-\vartheta}$. We can then assume that

$$I_\varepsilon(u) = \tfrac{1}{2}\|u\|^2 - \frac{1}{p+1}\int_{\mathbb{R}^n} |u_+|^{p+1}$$

and, using the notation $q = 1 \wedge (p-1)$, we have

$$\begin{aligned}|I''_\varepsilon(z_{\varepsilon,\rho} + sw)[v,v] - I''_\varepsilon(z_{\varepsilon,\rho})[v,v]| &\leqslant C_1\int_{\mathbb{R}^n} |(|z_{\varepsilon,\rho}| + |w|)^{p-1} - z_{\varepsilon,\rho}^{p-1}|v^2\,\mathrm{d}x \\ &\leqslant C_2\int_{\mathbb{R}^n} |w|^q v^2\,\mathrm{d}x.\end{aligned}$$

In order to estimate the last integral, we write

$$\int_{\mathbb{R}^n} |w|^q v^2\,\mathrm{d}x = \int_{|x|\leqslant R_{3\delta}} |w|^q v^2\,\mathrm{d}x + \int_{R_{3\delta}\leqslant|x|\leqslant T_{3\delta}} |w|^q v^2\,\mathrm{d}x + \int_{|x|\geqslant T_{3\delta}} w^q v^2\,\mathrm{d}x. \tag{4.5}$$

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Let us evaluate the first integral on the right-hand side. We have

$$\left|\int_{|x|\leqslant R_{3\delta}} |w|^q v^2\,\mathrm{d}x\right| \leqslant \left[\int_{|x|\leqslant R_{3\delta}} v^{2^*}\,\mathrm{d}x\right]^{2/2*} \left[\int_{|x|\leqslant R_{3\delta}} |w|^{qn/2}\,\mathrm{d}x\right]^{2/n}$$
$$\leqslant C_1\|v\|^2 \left[\int_{|x|\leqslant R_{3\delta}} |w|^{qn/2}\,\mathrm{d}x\right]^{2/n}.$$

Using (4.2) we find that

$$\int_{|x|\leqslant R_{3\delta}} |w|^{qn/2}\,\mathrm{d}x \leqslant C_2 \exp\left\{\frac{-C_3}{\varepsilon}\right\}|B_{R_{3\delta}}| \leqslant C_4\varepsilon^{-n}\exp\left\{\frac{-C_3}{\varepsilon}\right\},$$

and hence

$$\int_{|x|\leqslant R_{3\delta}} |w|^q v^2\,\mathrm{d}x \leqslant C_5\varepsilon^{-n}\exp\left\{\frac{-C_3}{\varepsilon}\right\}\|v\|^2. \tag{4.6}$$

Consider now the second integral on the right-hand side of (4.5). Using (4.12), we get

$$\int_{R_{3\delta}\leqslant|x|\leqslant T_{3\delta}} |w|^q v^2\,\mathrm{d}x \leqslant c\varepsilon^{q/2}\int_{R_{3\delta}\leqslant|x|\leqslant T_{3\delta}} v^2\,\mathrm{d}x.$$

For x in the annulus $R_{3\delta}\leqslant|x|\leqslant T_{3\delta}$ we obtain $r_0 < r^* - 4\delta \leqslant \varepsilon r \leqslant r^* + 4\delta$. As in the proof of lemma 3.2, $V(\varepsilon r) \geqslant 1$ for $\varepsilon r \in [r^* - 4\delta, r^* + 4\delta]$, and hence

$$\int_{R_{3\delta}\leqslant|x|\leqslant T_{3\delta}} v^2\,\mathrm{d}x \leqslant C_6\|v\|^2,$$

which yields

$$\int_{R_{3\delta}\leqslant|x|\leqslant T_{3\delta}} |w|^q v^2\,\mathrm{d}x \leqslant C_7\varepsilon^{q/2}\|v\|^2. \tag{4.7}$$

Finally, for

$$r \geqslant T_{3\delta} = \rho + \frac{3\delta}{\varepsilon}$$

we obtain

$$|w(r)| \leqslant c_1\sqrt{\varepsilon}\frac{u_1(r)}{u_1(T_{2\delta})}.$$

Then

$$\left|\int_{|x|\geqslant T_{3\delta}} |w|^q v^2\,\mathrm{d}x\right| \leqslant C_8\varepsilon^{q/2}\int_{|x|\geqslant T_{3\delta}}\left(\frac{u_1(|x|)}{u_1(T_{2\delta})}\right)^q v^2\,\mathrm{d}x$$
$$\leqslant C_8\varepsilon^{q/2}\left[\int_{\mathbb{R}^n} v^{2^*}\,\mathrm{d}x\right]^{2/2*}\left[\int_{|x|\geqslant T_{3\delta}}\left(\frac{u_1(|x|)}{u_1(T_{2\delta})}\right)^{qn/2}\mathrm{d}x\right]^{2/n}$$
$$\leqslant C_8\varepsilon^{q/2}\|v\|^2\left[\int_{|x|\geqslant T_{3\delta}}\left(\frac{u_1(|x|)}{u_1(T_{2\delta})}\right)^{qn/2}\mathrm{d}x\right]^{2/n}.$$

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The last integral can be estimated as follows:

$$\int_{|x|\geqslant T_{3\delta}} \left(\frac{u_1(|x|)}{u_1(T_{2\delta})}\right)^{qn/2} \mathrm{d}x = T_{2\delta}^{\sigma qn/2} \int_{r\geqslant T_{3\delta}} r^{n-1} r^{-\sigma qn/2}\,\mathrm{d}r$$

$$= T_{2\delta}^{\sigma qn/2} \frac{T_{3\delta}^{n-\sigma qn/2}}{\sigma qn/2 - n} = \frac{T_{3\delta}^{n}}{\sigma qn/2 - n}\left(\frac{T_{2\delta}}{T_{3\delta}}\right)^{\sigma qn/2}.$$

Since

$$\frac{T_{2\delta}}{T_{3\delta}} \leqslant \frac{r^* + 3\delta}{r^* + 4\delta} < 1 \quad \text{and} \quad \sigma \sim \varepsilon^{-1},$$

the above expression tends to zero.

On substituting this inequality, (4.6) and (4.7) into (4.5), we obtain

$$\left|\int_{\mathbb{R}^n} w^q v^2\,\mathrm{d}x\right| \leqslant C\varepsilon^{q/2}\|v\|^2, \quad q = 1 \wedge (p-1), \tag{4.8}$$

for some $C > 0$, and the lemma follows. □

Proposition 4.3. *$S_\varepsilon(\Gamma_\varepsilon) \subset \Gamma_\varepsilon$, and is a contraction, provided that ε is sufficiently small.*

Proposition 4.3 is an immediate consequence of the following two lemmas.

Lemma 4.4. *There exists $c_0 > 0$ such that, for ε sufficiently small, we have*

$$\|S_\varepsilon(w)\| \leqslant c_0\varepsilon \quad \textit{for all } w \in \Gamma_\varepsilon.$$

Moreover, S_ε is a contraction in Γ_ε.

Lemma 4.5. *For all ε sufficiently small, we have $S_\varepsilon(\Gamma_\varepsilon) \subset \mathcal{W}_\varepsilon$, for every $w \in \Gamma_\varepsilon$.*

Proof of lemma 4.4. Let $\bar{C} > 0$ such that $\|I'_\varepsilon(z_{\varepsilon,\rho})\| \leqslant \bar{C}\varepsilon\|z_{\varepsilon,\rho}\|$. Observe that, by lemma 3.3, $\|[PI''_\varepsilon(z_{\varepsilon,\rho})]^{-1}\| \leqslant C'$ for some $C' > 0$. Choose

$$c_0 = 2C'\bar{C}, \tag{4.9}$$

in the definition of Γ_ε. We first compute $S'_\varepsilon(w)$ for some $w \in \Gamma_\varepsilon$. The equality

$$S'_\varepsilon(w)[v] = v - [PI''_\varepsilon(z_{\varepsilon,\rho})]^{-1}(PI''_\varepsilon(z_{\varepsilon,\rho} + w)[v])$$

holds. We apply $PI''_\varepsilon(z_{\varepsilon,\xi})$, and obtain

$$\|PI''_\varepsilon(z_{\varepsilon,\rho})[S'_\varepsilon(w)[v]]\| = \|PI''_\varepsilon(z_{\varepsilon,\rho})[v] - PI''_\varepsilon(z_{\varepsilon,\rho} + w)[v]\| \leqslant C\varepsilon^{(1\wedge(p-1))/2}\|v\|$$

where we have used lemma 4.2. Then, for any w_1, $w_2 \in \Gamma_\varepsilon$, we use the mean value theorem to get

$$S_\varepsilon(w_1) - S_\varepsilon(w_2) = S'_\varepsilon[sw_1 + (1-s)w_2](w_1 - w_2),$$

for some $s \in (0,1)$. Since the $sw_1 + (1-s)w_2$ belong to Γ_ε, we find that

$$\|S_\varepsilon(w_1) - S_\varepsilon(w_2)\| = o_\varepsilon(1)\|w_1 - w_2\|. \tag{4.10}$$

Equation (4.10) yields the contraction property for S_ε.

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Next, we show that $\|S_\varepsilon(w)\| \leqslant c_0\varepsilon$ for any $w \in \Gamma_\varepsilon$. Using (4.10) with $w_1 = w$ and $w_2 = 0$, we obtain

$$\|S_\varepsilon(w) - S_\varepsilon(0)\| = o_\varepsilon(1)\|w\|.$$

On the other hand,

$$\|S_\varepsilon(0)\| = \|[PI''_\varepsilon(z_{\varepsilon,\rho})]^{-1}(PI'_\varepsilon(z_{\varepsilon,\rho}))\| \leqslant C'\|PI'_\varepsilon(z_{\varepsilon,\rho})\| \leqslant C'\bar{C}\varepsilon\|z_{\varepsilon,\rho}\|.$$

Hence, we finally deduce

$$\begin{aligned}\|S_\varepsilon(w)\| &\leqslant \|S_\varepsilon(w) - S_\varepsilon(0)\| + \|S_\varepsilon(0)\| \\ &\leqslant o_\varepsilon(1)\|w\| + C'\bar{C}\varepsilon\|z_{\varepsilon,\rho}\| \leqslant (o_\varepsilon(1)c_0 + C'\bar{C})\|\varepsilon z_{\varepsilon,\rho}\| \\ &\leqslant c_0\varepsilon\|z_{\varepsilon,\rho}\|.\end{aligned}$$

The proof of lemma 4.4 is thus completed. □

Proof of lemma 4.5. Take $w \in \Gamma_\varepsilon$, and define $\tilde{w} = S_\varepsilon(w)$. We shall prove that $\tilde{w}$ satisfies (4.2)–(4.4).

By recalling the definition of S_ε, the equation $\tilde{w} = S_\varepsilon(w)$ can be rewritten as

$$\begin{aligned}&-\Delta\tilde{w} + V(\varepsilon x)\tilde{w} - pz_{\varepsilon,\rho}^{p-1}\tilde{w} \\ &\qquad = \Delta z_{\varepsilon,\rho} - V(\varepsilon x)z_{\varepsilon,\rho} + [(z_{\varepsilon,\rho} + w)_+]^p - pz_{\varepsilon,\rho}^{p-1}w + \eta(\Delta\dot{z}_{\varepsilon,\rho} + V(\varepsilon x)\dot{z}_{\varepsilon,\rho}),\end{aligned}$$

where $\eta = \|\dot{z}_{\varepsilon,\xi}\|^{-2}(I''_\varepsilon(z_{\varepsilon,\rho})[\tilde{w} - w] + I'_\varepsilon(z_{\varepsilon,\rho} + w), \dot{z}_{\varepsilon,\xi})$. Recall that $z_{\varepsilon,\rho} = 0$ for any $r \notin [R_{2\delta}, T_{2\delta}]$. Therefore, $\tilde{w}$ satisfies the equation

$$-\Delta\tilde{w}(x) + V(\varepsilon x)\tilde{w}(x) = [w_+(x)]^p, \quad \forall|x| \notin [R_{2\delta}, T_{2\delta}]. \tag{4.11}$$

By using the radial lemma [6, lemma A:III], we get the following inequality for any $r \geqslant R_{3\delta}$:

$$|\tilde{w}(r)| \leqslant r^{(2-n)/2}\|\tilde{w}\| \leqslant c_0\varepsilon r^{(2-n)/2}\|z_{\varepsilon,\rho}\| \leqslant C_0\sqrt{\varepsilon}. \tag{4.12}$$

Given such a constant C_0, we choose c_1 in the definition of Γ_e as

$$c_1 = 2C_0. \tag{4.13}$$

We will first prove inequalities (4.2) and (4.3). In order to do that, we define

$$\hat{V}(r) = \begin{cases} 0, & r \in [0, R_{3\delta}], \\ 1, & r \in (R_{3\delta}, R_{2\delta}]. \end{cases}$$

Because of the assumptions on V, it holds that $\hat{V}(r) \leqslant V(\varepsilon r)$. Hence, the maximum principle implies that $0 \leqslant \tilde{w} \leqslant \hat{w}$ in $B(0, R_{2\delta})$, where $\hat{w}$ is the solution of

$$\left.\begin{aligned}\Delta\hat{w} + \bar{V}(|x|)\hat{w} = [w_+(x)]^p, &\quad x \in B(0, R_{2\delta}), \\ \hat{w}(x) = \tilde{w}(x) \leqslant C_0\sqrt{\varepsilon}, &\quad |x| = R_{2\delta}.\end{aligned}\right\} \tag{4.14}$$

Therefore, it suffices to prove inequalities (4.2) and (4.3) for $\hat{w}$. Since w satisfies (4.2), $\Delta\hat{w} \leqslant \zeta$, where

$$\zeta = \left(2c_1\sqrt{\varepsilon}\exp\left\{\frac{-\delta}{\varepsilon}\right\}\right)^p.$$

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We then find that $\hat{w}$ satisfies

$$\hat{w}(R_{3\delta}) \leqslant \hat{w}(r) \leqslant \hat{w}(R_{3\delta}) + \zeta \frac{R_{3\delta}^2 - r^2}{2n}, \tag{4.15}$$

and this implies that

$$0 \geqslant \hat{w}'(R_{3\delta}) \geqslant -\frac{R_{3\delta}}{n}\left(2c_1\sqrt{\varepsilon}\exp\left\{\frac{-\delta}{\varepsilon}\right\}\right)^p.$$

This estimate will be very useful in what follows.

We now study equation (4.14) in the annulus $R_{3\delta} < r < R_{2\delta}$; in such a domain, $\hat{w}$ satisfies

$$-\Delta\hat{w} + \hat{w} = w_+(r)^p, \quad R_{3\delta} < r < R_{2\delta},$$

with the following conditions on the boundary:

$$\hat{w}(R_{2\delta}) < C_0\sqrt{\varepsilon}, \qquad 0 \geqslant \hat{w}'(R_{3\delta}) \geqslant -\frac{R_{3\delta}}{n}\zeta^p.$$

Let φ_1 and φ_2 be the fundamental solutions of the problem $-\Delta u + u = 0$, that is

$$\varphi_1(r) = r^{1-n/2}B_K\left(\frac{n-2}{2}, r\right), \quad \varphi_2(r) = r^{1-n/2}B_I\left(\frac{n-2}{2}, r\right),$$

where B_I and B_K are the modified Bessel functions of the first and second kind, respectively. From the basic properties of the Bessel functions (see [9, §§ 5.7 and 5.16.4]), we have

$$\begin{gathered}
\varphi_1(r) \sim \frac{1}{\sqrt{2}}r^{(1-n)/2}\mathrm{e}^{-r}, \quad \varphi_2(r) \sim \frac{1}{\sqrt{2}}r^{(1-n)/2}\mathrm{e}^{r}, \quad r \to +\infty, \\
\frac{\varphi_1'(r)}{\varphi_1(r)} \sim -1, \qquad \frac{\varphi_2'(r)}{\varphi_2(r)} \sim 1, \qquad r \to +\infty, \\
\varphi_1(r)\varphi_2'(r) - \varphi_1'(r)\varphi_2(r) = r^{1-n}, \quad \forall r > 0.
\end{gathered}$$

Using the preceding estimates, we can write $\hat{w}$ by means of the variation of constants:

$$\begin{aligned}
\hat{w}(r) = \varphi_1(r)\int_{R_{3\delta}}^r & s^{n-1}\varphi_2(s)[w_+(s)]^p\,\mathrm{d}s \\
& - \varphi_2(r)\int_{R_{3\delta}}^r s^{n-1}\varphi_1(s)[w_+(s)]^p\,\mathrm{d}s + a\varphi_1(r) + b\varphi_2(r),
\end{aligned}$$

for any $r \in [R_{3\delta}, R_{2\delta}]$ and some constants $a = a_\varepsilon$, $b = b_\varepsilon$. From these and the above estimate of $\hat{w}'(R_{3\delta})$, we get

$$|\hat{w}'(R_{3\delta})| = |a\varphi_1'(R_{3\delta}) + b\varphi_2'(R_{3\delta})| \leqslant \frac{R_{3\delta}}{n}\zeta^p. \tag{4.16}$$

Next, let us compute $\hat{w}(R_{2\delta})$, which must be smaller than $C_0\sqrt{\varepsilon}$:

$$\begin{aligned}\hat{w}(R_{2\delta}) &= \varphi_1(R_{2\delta})\int_{R_{3\delta}}^{R_{2\delta}} s^{n-1}\varphi_2(s)[w_+(s)]^p\,\mathrm{d}s \\ &\quad - \varphi_2(R_{2\delta})\int_{R_{3\delta}}^{R_{2\delta}} s^{n-1}\varphi_1(s)[w_+(s)]^p\,\mathrm{d}s + a\varphi_1(R_{2\delta}) + b\varphi_2(R_{2\delta}) \\ &= \varphi_1(R_{2\delta})\varphi_2(R_{2\delta})\biggl[\int_{R_{3\delta}}^{R_{2\delta}} s^{n-1}\frac{\varphi_2(s)}{\varphi_2(R_{2\delta})}[w_+(s)]^p\,\mathrm{d}s \\ &\qquad - \int_{R_{3\delta}}^{R_{2\delta}} s^{n-1}\frac{\varphi_1(s)}{\varphi_1(R_{2\delta})}[w_+(s)]^p\,\mathrm{d}s\biggr] \\ &\quad + a\varphi_1(R_{2\delta}) + b\varphi_2(R_{2\delta}) \\ &\sim \frac{1}{2R_{2\delta}^{n-1}}\biggl[\int_{R_{3\delta}}^{R_{2\delta}} s^{n-1}\frac{\varphi_2(s)}{\varphi_2(R_{2\delta})}[w_+(s)]^p\,\mathrm{d}s - \int_{R_{3\delta}}^{R_{2\delta}} s^{n-1}\frac{\varphi_1(s)}{\varphi_1(R_{2\delta})}[w_+(s)]^p\,\mathrm{d}s\biggr] \\ &\quad + a\varphi_1(R_{2\delta}) + b\varphi_2(R_{2\delta})\end{aligned}$$

We observe that

$$\frac{\varphi_1(s)}{\varphi_1(R_{2\delta})} \sim \mathrm{e}^{-s+R_{2\delta}}, \qquad \frac{\varphi_2(s)}{\varphi_2(R_{2\delta})} \sim \mathrm{e}^{s-R_{2\delta}}, \qquad \forall s \in (R_{3\delta}, R_{2\delta}).$$

We now remark that, since w satisfies (4.3), the integral expressions above both have order $\sqrt{\varepsilon}^p$. By taking into account the above expression and the estimate (4.16), we obtain

$$|b| = |b_\varepsilon| \leqslant \sqrt{\varepsilon}C_0\mathrm{e}^{-R_{2\delta}}, \qquad |a| = |a_\varepsilon| \leqslant \frac{R_{3\delta}}{n}\zeta^p\mathrm{e}^{R_{3\delta}}.$$

Reasoning as above, we can now conclude that, for any $R \in [R_{3\delta}, R_{2\delta}]$,

$$\begin{aligned}\hat{w}(R) &= \varphi_1(R)\int_{R_{3\delta}}^{R} s^{n-1}\varphi_2(s)[w_+(s)]^p\,\mathrm{d}s \\ &\quad - \varphi_2(R)\int_{R_{3\delta}}^{R} s^{n-1}\varphi_1(s)[w_+(s)]^p\,\mathrm{d}s + a\varphi_1(R) + b\varphi_2(R) \\ &\sim \frac{1}{2R^{n-1}}\int_{R_{3\delta}}^{R} s^{n-1}\biggl[\frac{\varphi_2(s)}{\varphi_2(R)} - \frac{\varphi_1(s)}{\varphi_1(R)}\biggr][w_+(s)]^p\,\mathrm{d}s + a\varphi_1(R) + b\varphi_2(R).\end{aligned}$$

Recall that w satisfies (4.3), which implies that

$$[w_+(r)]^p \leqslant c_1\varepsilon^{p/2}\mathrm{e}^{p(r-R_{2\delta})} = c_1\varepsilon^{p/2}\mathrm{e}^{p(r-R)}\mathrm{e}^{p(R-R_{2\delta})}.$$

Therefore, reasoning as above, we see that the integral expressions above are of an order smaller than $\varepsilon^{p/2}\mathrm{e}^{p(R-R_{2\delta})}$. Finally, the term $a\varphi_1(R) + b\varphi_2(R)$ is of order

$$\begin{aligned}a\varphi_1(R) + b\varphi_2(R) &\leqslant \frac{R_{3\delta}}{n}\zeta^p\mathrm{e}^{R_{3\delta}}\mathrm{e}^{-R} + \sqrt{\varepsilon}C_0\mathrm{e}^{-R_{2\delta}}\mathrm{e}^{R} \\ &\leqslant \frac{R_{3\delta}}{n}\varepsilon^{p/2}\mathrm{e}^{(R_{3\delta}-R_{2\delta})p}\mathrm{e}^{R_{3\delta}-R} + C_0\sqrt{\varepsilon}\mathrm{e}^{R-R_{2\delta}},\end{aligned}$$

which implies that $\hat{w}$ satisfies (4.3).

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In particular, we obtain $\hat{w}(R_{3\delta}) \leqslant c_1\sqrt{\varepsilon}\mathrm{e}^{-\delta/\varepsilon}$. By inserting this into equation (4.15), we obtain

$$\hat{w}(r) \leqslant c_1\sqrt{\varepsilon}\mathrm{e}^{-\delta/\varepsilon} + \zeta\frac{R_{3\delta}^2}{2n} \leqslant 2c_1\sqrt{\varepsilon}\mathrm{e}^{-\delta/\varepsilon}, \quad \forall r < R_{3\delta},$$

that is, $\hat{w}$ satisfies (4.2).

Finally, let us prove inequality (4.4). It is easy to check (see [5]) that the fundamental solutions of

$$-\Delta u + \frac{m}{|x|^2}u = 0$$

are

$$u_1(r) = r^{-\sigma}, \quad u_2(r) = r^{2-n+\sigma}, \quad \sigma = \frac{n-2+\sqrt{(n-2)^2+4m}}{2}.$$

Observe that if $r > T_{2\delta}$, $\tilde{w}(r) \leqslant \breve{w}(r)$, where $\breve{w}$ is defined by

$$\begin{aligned} \Delta\breve{w} + \frac{m}{|x|^2}\breve{w} &= f(r)^p, \quad |x| > T_{2\delta}, \\ \breve{w}(x) &= C_0\sqrt{\varepsilon}, \quad |x| = T_{2\delta}, \\ \breve{w}(x) &\to 0 \qquad x \to \infty, \end{aligned}$$

where $m = a_1/\varepsilon^2$ (a_1 is given by (2.1)) and

$$f(r) = \left[c_1\sqrt{\varepsilon}\frac{u_1(r)}{u_1(T_{2\delta})}\right].$$

Note that, with this choice of m, we have $\sigma \sim \varepsilon^{-1}$.

Then, $\breve{w}$ is given by the variation of constants:

$$\begin{aligned} \breve{w}(r) = \frac{1}{d}u_1(r)\int_{T_{2\delta}}^{r} s^{n-1}u_2(s)f^p(s)\,\mathrm{d}s \\ - \frac{1}{d}u_2(r)\int_{T_{2\delta}}^{r} s^{n-1}u_1(s)f^p(s)\,\mathrm{d}s + au_1(r) + bu_2(r), \end{aligned} \tag{4.17}$$

where

$$d = d_\varepsilon = \sqrt{(n-2)^2+4m} \sim \varepsilon^{-1}$$

and $a = a_\varepsilon$, $b = b_\varepsilon$ are constants. Observe that, since w satisfies (4.2), both the integrals above are uniformly bounded for any r. Moreover, the condition $\breve{w}(r) \to 0$, $r \to +\infty$, implies that

$$b = \int_{T_{2\delta}}^{+\infty} u_1(s)f^p(s)s^{n-1}\,\mathrm{d}s.$$

Therefore, $\breve{w}$ can be rewritten as

$$\breve{w}(r) = \frac{1}{d}u_2(r)\int_{T_{2\delta}}^{r} s^{n-1}u_1(s)f^p(s)\,\mathrm{d}s + \frac{1}{d}u_1(r)\left[\int_{r}^{+\infty} s^{n-1}u_2(s)f^p(s)\,\mathrm{d}s + au_1(r)\right].$$

Let us compute b explicitly:

$$b = c_1^p\varepsilon^{p/2}\int_{T_{2\delta}}^{+\infty} s^{-\sigma}s^{-p\sigma}T_{2\delta}^{p\sigma}s^{n-1}\,\mathrm{d}s = c_1^p\varepsilon^{p/2}\frac{T_{2\delta}^{-\sigma+n}}{(p+1)\sigma-n}.$$

Next, to obtain a, we use the boundary condition $\breve{w}(T_{2\delta}) = C_0\sqrt{\varepsilon}$. Precisely, from (4.17) we get

$$\breve{w}(T_{2\delta}) = C_0\sqrt{\varepsilon} = \frac{1}{d}(aT_{2\delta}^{-\sigma} + bT_{2\delta}^{2-n+\sigma}),$$

which yields

$$a = dC_0\sqrt{\varepsilon}T_{2\delta}^{\sigma} - bT_{2\delta}^{2-n+2\sigma}. \tag{4.18}$$

Finally, in order to prove that $\breve{w}$ satisfies (4.4), we study the expression

$$\frac{\breve{w}(r)}{u_1(r)}u_1(T_{2\delta})$$

as follows:

$$\frac{\breve{w}(r)}{u_1(r)}u_1(T_{2\delta}) = T_{2\delta}^{-\sigma}\frac{1}{d}\int_{T_{2\delta}}^{r} s^{n-1}u_2(s)f^p(s)\,\mathrm{d}s + T_{2\delta}^{-\sigma}\frac{1}{d}\left[\frac{u_2(r)}{u_1(r)}\int_r^{+\infty} s^{n-1}\frac{u_1(s)}{u_2(s)}u_2(s)f^p(s)\,\mathrm{d}s + a\right].$$

Taking into account the fact that $u_1(s)/u_2(s)$ is decreasing, we get

$$\frac{\breve{w}(r)}{u_1(r)}u_1(T_{2\delta}) \leqslant T_{2\delta}^{-\sigma}\frac{1}{d}\int_{T_{2\delta}}^{r} s^{n-1}u_2(s)f^p(s)\,\mathrm{d}s + T_{2\delta}^{-\sigma}\frac{1}{d}\left[\int_r^{+\infty} s^{n-1}u_2(s)f^p(s)\,\mathrm{d}s + a\right],$$

and, thus,

$$\frac{\breve{w}(r)}{u_1(r)}u_1(T_{2\delta}) \leqslant T_{2\delta}^{-\sigma}\frac{1}{d}\left[c_1^p\sqrt{\varepsilon}^pT_{2\delta}^{p\sigma}\int_{T_{2\delta}}^{+\infty} s^{n-1}s^{2-n+\sigma}s^{-p\sigma}\,\mathrm{d}s + a\right].$$

We use (4.18) to estimate the expression

$$\begin{aligned}
T_{2\delta}^{-\sigma}\frac{1}{d}a &= T_{2\delta}^{-\sigma}\left[C_0\sqrt{\varepsilon}T_{2\delta}^{\sigma} - \frac{1}{d}bT_{2\delta}^{2-n+2\sigma}\right] \\
&= C_0\sqrt{\varepsilon} - T_{2\delta}^{-\sigma}\frac{1}{d}c_1^p\varepsilon^{p/2}\frac{T_{2\delta}^{-\sigma+n}}{(p+1)\sigma-n}T_{2\delta}^{2-n+2\sigma} \\
&= C_0\sqrt{\varepsilon} - c_1^p\varepsilon^{p/2}\frac{1}{d}T_{2\delta}^2\frac{1}{(p+1)\sigma-n} \\
&\leqslant C_0\sqrt{\varepsilon}.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
T_{2\delta}^{-\sigma}\frac{1}{d}c_1^p\varepsilon^{p/2}T_{2\delta}^{p\sigma}\int_{T_{2\delta}}^{+\infty} s^{1-(p-1)\sigma}\,\mathrm{d}s &= \frac{1}{d}c_1^p\varepsilon^{p/2}T_{2\delta}^{(p-1)\sigma}\frac{T_{2\delta}^{2-(p-1)\sigma}}{(p-1)\sigma-2} \\
&= c_1^p\varepsilon^{p/2}\frac{1}{d}\frac{1}{(p-1)\sigma-2}T_{2\delta}^2.
\end{aligned}$$

Since $T_{2\delta} \sim \varepsilon^{-1}$, $\sigma \sim \varepsilon^{-1}$ and $d \sim \varepsilon^{-1}$, the above expression is also smaller than $C_0\sqrt{\varepsilon}$, provided ε is small enough. In conclusion we have

$$\breve{w}(r) \leqslant 2C_0\sqrt{\varepsilon}\frac{u_1(r)}{u_1(T_{2\delta})},$$

which proves (4.4). □

5. Completion of the proof of theorem 2.1

According to proposition 4.3, for every $z_{\varepsilon,\rho} \in Z$, the auxiliary equation (3.6) has a locally unique solution $w_{\varepsilon,\rho} \in \Gamma_\varepsilon$. It remains to solve the bifurcation equation, $(Id - P)I'_\varepsilon(z_{\varepsilon,\rho} + w_{\varepsilon,\rho}) = 0$, obtained by inserting $w_{\varepsilon,\rho}$ into (3.7). It is known (see [1, theorem 2.12]) that, in order to find a solution of the bifurcation equation, it suffices to find a stationary point of the *reduced functional*

$$\Phi_\varepsilon(\rho) = I_\varepsilon(z_{\varepsilon,\rho} + w_{\varepsilon,\rho}).$$

We follow [3] closely. The equality

$$\Phi_\varepsilon(\rho) = I_\varepsilon(z_{\varepsilon,\rho}) + I'_\varepsilon(z_{\varepsilon,\rho})[w_{\varepsilon,\rho}] + O(\|w_{\varepsilon,\rho}\|^2)$$

holds. Using lemma 3.2 and the fact that $w_{\varepsilon,\rho} \sim \varepsilon\|z_{\varepsilon,\rho}\| \sim \varepsilon^{(3-n)/2}$ (see the definition of Γ_ε and (3.8)), we deduce that $I'_\varepsilon(z_{\varepsilon,\rho})[w_{\varepsilon,\rho}] + O(\|w_{\varepsilon,\rho}\|^2) = O(\varepsilon^{3-n})$. Then, the same calculation as in [3, § 5,p. 449] yields

$$\Phi_\varepsilon(\rho) = C_0\rho^{n-1}V^\theta(\varepsilon\rho) + O(\varepsilon^{3-n}), \quad C_0 = \frac{1}{p+1}\int U^{p+1}.$$

Thus,

$$\varepsilon^{n-1}\Phi_\varepsilon(\rho) = C_0 M(\varepsilon\rho) + O(\varepsilon^2),$$

and the maximum (or minimum) r^* of M gives rise to a maximum (or minimum) $\rho_\varepsilon \sim r^*/\varepsilon$ of Φ_ε. It follows that $u_\varepsilon = z_{\rho_\varepsilon,\varepsilon} + w_{\rho_\varepsilon,\varepsilon}$ is a critical point of I_ε, defined in (3.3). Since $z_{\varepsilon,\rho}$ has a uniform exponential decay at infinity and $w_{\varepsilon,\rho}$ belongs to Γ_ε, there exist $\hat{c}$ and $\vartheta > 0$ such that $|u_\varepsilon(r)| \leqslant \hat{c}(1+\varepsilon|x|)^{-\vartheta}$ and, thus, according to remark 3.1, u_ε is a radial solution of (2.3). Consequently, $u_\varepsilon(r/\varepsilon)$ is a radial solution of (1.2) and, from $u_\varepsilon(r) \sim U(r - r^*/\varepsilon)$, it follows that $u_\varepsilon(r/\varepsilon) \sim U((r - r^*)/\varepsilon)$ concentrates near the sphere $r = r^*$ (here U stands for $U_{V(r^*)}$). This completes the proof of theorem 2.1.

REMARK 5.1. The fact that $u_\varepsilon(r/\varepsilon) \sim U((r - r^*)/\varepsilon)$ makes it clear that the solutions found in theorem 2.1 have the same behaviour as those in [3]. On the other hand, the peaks of the solutions found in [7, 8] become small as $\varepsilon \to 0$. As anticipated after the statement of theorem 2.1, this difference is due to the fact that in the present case the concentration arises in the region where $V(r) > 0$.

Acknowledgments

This research has been supported by MURST within the PRIN 2004 program 'Variational methods and nonlinear differential equations'.

D.R. has also been partly supported by the Ministry of Science and Technology (Spain), under Grant no. BFM2002-02649 and by J. Andalucia (FQM 116). He also thanks the SISSA (Trieste) for their hospitality.

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(*Issued 6 October 2006*)