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THE L-FUNCTIONS $L(s, \mathrm{Sym}^m(r), \pi)$

BY

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Dedicated to the memory of Robert Arnold Smith

ABSTRACT. The exact form for the gamma factor for the L-function corresponding to the m-th symmetric power of a cuspidal automorphic representation of PGL(2) is given. This information is used to obtain, via a theorem of Landau, bounds for the eigenvalues of Hecke operators.

0. **Introduction**. Let $\pi = \bigotimes_p \pi_p$ be a cuspidal automorphic representation of $\mathrm{PGL}(2, Q_{\mathbb{A}})$ and let $r = r_2$ be the standard 2-dimensional representation of $\mathrm{GL}(2, \mathbb{C})$. Let $L_p(s, \pi_p) = (1 - \alpha_p p^{-s})^{-1}(1 - \beta_p p^{-s})^{-1}$ be the local L-function corresponding to an unramified component π_p. For m a positive integer the Euler product

$$\prod_p \prod_{0 \leqq j \leqq m} (1 - \alpha_p^{m-j} \beta_p^j p^{-s})^{-1}$$

was first introduced by Serre [17], who also suggested, when π is associated to a holomorphic cusp form of integral weight, a companion gamma factor to go with the component π_∞. The importance of these functions has been recognized for some time. For $m = 2$ they first appeared in a disguised form in the classical work of Rankin [11] and Selberg [14] and were used to obtain partial information about the Petersson-Ramanujan Conjecture (PRC) concerning the size of the eigenvalues of the Hecke operators. For $m = 2$ they also appear in a fundamental way in the work of Gelbart and Jacquet [3] on the lifting of automorphic forms from GL(2) to GL(3); as an interesting consequence of their work they obtain bounds for the Fourier coefficients of cusp forms which are better than those obtained by Rankin's method. For $m = 3, 4, 5$ the functions $L(s, \mathrm{Sym}^m(r), \pi)$ have been the object of study of the second author ([19], [20]) who has established their meromorphic continuation and functional equation. For arbitrary m these functions were first studied by Serre [16] in connection with the Sato-Tate conjecture. They also appeared in a more general context in Langlands' article ([8], §8), where they were used to show that on the basis of the functoriality principle the representation $\pi = \bigotimes_p \pi_p$ satisfies a strong form of the PRC to the effect that the local components π_p are all tempered.

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Although the global properties of the functions $L(s, \mathrm{Sym}^m(r), \pi)$ for arbitrary m are not yet fully understood, the properties of the local factors corresponding to representations of $\mathrm{GL}(m+1, \mathbb{R})$ and $\mathrm{GL}(m+1, \mathbb{Q}_p)$ are fairly well known. It is the purpose of this note to indicate, following the suggestion of Serre in the holomorphic case [17] and of Gelbart and Jacquet when $m = 2$, how to construct the appropriate local L-factors at infinity. We also indicate how one can improve the results of Gelbart and Jacquet concerning the PRC for a cusp form on PGL(2).

1. **Local L-factors at infinity**. When $\pi = \bigotimes_p \pi_p$ is a cuspidal automorphic representation of $\mathrm{PGL}(2, \mathbb{Q}_{\mathbb{A}})$ and π_∞ is a discrete series representation Serre, ([17], p. 14), indicated a way of writing the local L-factor $L_\infty(s, \mathrm{Sym}^m(r), \pi_\infty)$. We extend here this construction to an arbitrary π_∞. Let us recall first the following special case of Langlands' functoriality principle (see Gelbart-Jacquet [3]). Let R be a semisimple finite dimensional complex representation of $\mathrm{GL}(m+1, \mathbb{C})$ and consider the map σ

$$\begin{array}{ccc} \mathrm{GL}(2,\mathbb{C}) & \xrightarrow{\quad\sigma\quad} & \mathrm{GL}(m+1,\mathbb{C}) \xrightarrow{R} \mathrm{GL}(V_{\mathbb{C}}) \\ \searrow & & \nearrow \\ & \mathrm{PGL}(2,\mathbb{C}) & \end{array}$$

where the diagonal arrow going up corresponds to the representtion of maximal weight m. To the map σ there is conjecturally associated a lift σ_* which sends automorphic representations π on $\mathrm{GL}(2, \mathbb{Q}_{\mathbb{A}})$ to automorphic representations in $\mathrm{GL}(m+1, \mathbb{Q}_{\mathbb{A}})$, with the property that the corresponding L-functions satisfy

$$L(s, R \cdot \sigma, \pi) = L(s, R, \sigma_*(\pi)),$$

with a similar equality holding for the local L-factors. From Langlands' classification scheme [8], and the fact that an admissible irreducible representation of $\mathrm{GL}(m+1, \mathbb{R})$ is uniquely determined by its L-function, we obtain the precise form of the local factor at infinity as follows. Recall that if π_∞ is an irreducible unitary representation of $\mathrm{PGL}(2, \mathbb{R})$, then it falls into one of the following three categories (Satake [12]),:

(i) a discrete series representation indexed by half an odd integer $(w-1)/2$

(ii) a supplementary series representation indexed by a real number ν, $0 \le \nu \le \frac{1}{2}$

(iii) a principal series representation indexed by a pure imaginary number $i\nu \in i\mathbb{R}$.

For these representations we know that the local L-factor $L_\infty(s, \pi)$ is given respectively by

(i) $G_{\mathbb{R}}(s + (w-1)/2) G_{\mathbb{R}}(s + (w+1)/2)$

(ii) $G_{\mathbb{R}}(s + \nu) G_{\mathbb{R}}(s - \nu)$

(iii) $G_{\mathbb{R}}(s + i\nu) G_{\mathbb{R}}(s - i\nu)$,

where $G_{\mathbb{R}}(s) = \pi^{-s/2} \Gamma(s/2)$.

DEFINITION. *Let π_∞ be an irreducible unitary representation of* $\mathrm{PGL}(2, \mathbb{R})$. *Let $L_\infty(s, \pi_\infty) = G_{\mathbb{R}}(s + \nu_1) G_{\mathbb{R}}(s + \nu_2)$ be the local L-function at infinity. The local L-function $L_\infty(s, \mathrm{Sym}^m(r), \pi_\infty)$ is given by the following formulas:*

I. *If* π_∞ *is a holomorphic discrete series with* $v_1 = (w-1)/2$ *and m is odd, then*

$$L_\infty(s, \mathrm{Sym}^m(r), \pi_\infty) = \prod_{0 \leqq j \leqq (m-1)/2} G_{\mathbb{C}}(s + (m-2j)v_1),$$

where $G_{\mathbb{C}}(s) = G_{\mathbb{R}}(s)G_{\mathbb{R}}(s+1)$; *if m is even, then*

$$L_\infty(s, \mathrm{Sym}^m(r), \pi_\infty) = G_{\mathbb{R}}(s+\kappa)L_\infty(s, \mathrm{Sym}^{m-1}(r), \pi_\infty),$$

where $\kappa = 0$ *if* $m/2$ *is even and* 1 *otherwise.*

II. *If* π_∞ *is in the complementary or in the principal series, then*

$$L_\infty(s, \mathrm{Sym}^m(r), \pi_\infty) = \prod_{0 \leqq j \leqq m} G_{\mathbb{R}}(s + (m-j)v_1 + jv_2).$$

Let us verify that definition I makes sense, the other case being similar. Under the Langlands' correspondence, the local factor $L_\infty(s, \pi_\infty) = G_{\mathbb{C}}(s + \frac{1}{2}(w-1))$ is the L-function $L_\infty(s, \chi)$ of the Hecke character $\chi\colon \mathbb{C}^\times \to \mathbb{C}^\times$, $\chi(z) = z^{-w}(z\bar{z})^{-\frac{1}{2}(w-1)}$ of the Weil group $W_{\mathbb{C}} = \mathbb{C}^\times$; hence the representation π_∞ corresponds to the 2-dimensional representation $\sigma(\chi) = \mathrm{Ind}(W_{\mathbb{C}} : W_{\mathbb{R},\chi})$ of the Weil group $W_{\mathbb{R}}$. Now, a simple combinatorial argument, using Mackey's tensor product theorem and the binomial theorem, gives the following decomposition for m odd

$$\mathrm{Sym}^m \sigma(\chi) = \bigoplus_{0 \leqq j \leqq \frac{1}{2}(m-1)} \sigma(\chi^{m-2j});$$

corresponding to the middle entry in a pascal triangle we get for m even

$$\mathrm{Sym}^m \sigma(\chi) = \mathrm{Sym}^{m-1}\sigma(\chi) \oplus (\mathrm{sgn})^{\frac{1}{2}m},$$

where $\mathrm{sgn} : \mathbb{R}^\times \to \{1, -1\}$ is the 1 dimensional representation $\mathrm{sgn}(x) = x^{-1}|x|$.

When p is a finite prime, and π_p is unramified, i.e., $L_p(s, \pi_p) = (1 - \alpha_p p^{-s})^{-1}(1 - \beta_p p^{-s})^{-1}$, then we have (Serre [16])

$$L_p(s, \mathrm{Sym}^m(r), \pi_p) = \prod_{0 \leq j \leq m} (1 - \alpha_p^{m-j}\beta_p^j p^{-s})^{-1}.$$

When π_p is ramified, Jacquet's construction of the principal L-functions [4] also leads to a description of $L_p(s, \mathrm{Sym}^m(r), \pi_p)$ as well as that of the $\in$-factors $\in_p(s, \mathrm{Sym}^m(r), \pi_p, \psi)$. In the following we write

$$L(s, \mathrm{Sym}^m(r), \pi) = L_\infty(s, \mathrm{Sym}^n(r), \pi_\infty)L_f(s, \mathrm{Sym}^m(r), \pi),$$

where

$$L_f(s, \mathrm{Sym}^m(r), \pi) = \prod_{p<\infty} L_p(s, \mathrm{Sym}^m(r), \pi_p),$$

and r is the standard 2-dimensional representation of $\mathrm{GL}(2, \mathbb{C})$.

2. **Bounds for the eigenvalues of Hecke operators**. The PRC for a cuspidal automorphic representation $\pi = \bigotimes_p \pi_p$ of $\mathrm{PGL}(2, Q_A)$ is the statement that the local

representations at the finite primes are tempered. This is equivalent to the assertion that the local representations are cuspidal or belong to the principal series parametrized by parameters α_p and β_p of absolute value 1. A similar statement is supposed to hold for the local component at infinity; namely it should be a unitary principal series or a discrete series. When the latter holds Deligne has shown that the PRC is true at all the finite primes. In the following we discuss an analytic method for obtaining information about π_p which depends on the knowledge we have of $L(s, \mathrm{Sym}^m(r), \pi)$.

Suppose that under the Langlands' functoriality principle $L(s, \mathrm{Sym}^m(r), \pi)$ is not only entire, but also corresponds to the principal L-function of an automorphic representation π^* on $\mathrm{GL}(m+1)$. By the results of Jacquet, P. Shapiro, and Shalika [5], the Rankin-Selberg convolution

$$L(s, \pi^* \times \pi^*)$$

has a meromorphic continuation to the whole s-plane with simple poles at $s = 1, 0$. If we write

$$L(s, \pi^* \times \pi^*) = L_\infty(s, \pi^*_\infty \times \pi^*_\infty) L_f(s, \pi^* \times \pi^*)$$

and let

$$L_f(s, \pi^* \times \pi^*) = \sum_{n=1}^{\infty} b^{[m]}(n) n^{-s},$$

then, using the exact form of the gamma factor in the functional equation, we obtain from Landau's well known theorem on the mean values of the coefficients of Dirichlet series ([6], [13])

$$\sum_{n \leq x} b^{[m]}(n) = c(\pi) x + O(x^{\mu + \epsilon}),$$

where $\mu = (\nu - 1)/(\nu + 1)$ and $\nu = (m+1)^2$ is the number of factors $G_{\mathbb{R}}(s + v)$ which appear in the functional equation for $L(s, \pi^* \chi \pi^*)$; if we now observe that $b^{[m]}(p)$ is essentially $\max(|\alpha_p|, |\beta_p|)^{2m}$, we obtain the following result.

THEOREM. *With notations as above, suppose* $L(s, \mathrm{Sym}^m(r), \pi)$ *is the principal* L-*function of a cuspidal automorphic representation* π^* *of* $\mathrm{GL}(m+1)$. *Then we have*

$$\max(|\alpha_p|, |\beta_p|) \ll p^{(m+2)/2(m^2+2m+2)+\epsilon}.$$

When $m = 2$, in which case we can apply the Gelbart-Jacquet lift from automorphic cuspidal representations of GL(2) to GL(3), followed by the Rankin-Selberg convolution on GL(3), we obtain the bound

$$\max(|\alpha_p|, |\beta_p|) \ll p^{1/5+\epsilon}.$$

This is an improvement over the bound $p^{1/4+\epsilon}$ which is obtained by Gelbart-Jacquet ([3], Theorem 9.3). The estimate with the exponent $1/5$ also has been obtained independently by Serre [18] and by Ram Murty [23].

The estimate $a(n) \ll n^{\frac{1}{4}+\epsilon}$ for the coefficients of Maass wave forms on $\Gamma_0(N)$ seems to have been obtained by Proskurin [10] by using a special type of Selberg trace formula. This estimate is included as a special case of a very general result proved by Gelbart and Jacquet for arbitrary cuspidal automorphic forms of $\mathrm{GL}(2, K_{\mathbb{A}})$ and K any number field ([3], Theorem 9.3). In view of the recent work of Flicker on liftings from GL(2) to GL(3), it may be worthwhile to reconsider the arguments of Proskurin and Gelbart-Jacquet.

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