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# Parallel weak envelope solitons in multi-ion plasmas

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Heavy ions frequently appear as minor components in space plasmas, for example as charged helium in the solar wind and heavy ions in the vicinity of comets. Both the different components of ions and the associated plasma waves are observed by extraterrestrial in situ measurements. These plasma waves appear as large-amplitude magnetic field fluctuations in space plasmas. They must be described appropriately by means of multifluid equations. Because of the nonlinear nature of these waves, we here investigate nonlinear waves in multi-ion plasmas. Solitary waves that can only exist in a magnetized bi-ion plasma are presented. We employ a perturbation theory at the linear solution of a left-hand circularly polarized, low-frequency (below the proton gyrofrequency) plasma wave and take only the first nonlinear terms into account. Thus the multifluid equations are reduced to a single equation of the type of a nonlinear Schrödinger equation. The derived soliton solution is valid for magnetic field amplitudes lower than 10% of the ambient unperturbed magnetic field. The solutions are discussed for plasma parameters that are typical of the solar wind. A density enhancement can be observed within the soliton, where the helium ion density is more enhanced than the proton density.

## 1. Introduction

Plasmas composed of several particle species are observed by various spacecraft in the heliosphere. These plasmas consist mainly of electrons and protons, but also contain one or more non-negligible fractions of heavy ions, such as helium ions in the solar wind (see e.g. Marsch et al. 1982) or water group ions in the vicinity of comets (see e.g. Tsurutani and Smith 1986; Tsurutani et al. 1987a). These minor components of ions may change the plasma properties significantly. Large-amplitude magnetic field fluctuations are commonly observed by extraterrestrial in situ measurements in such plasmas (see e.g. Tsurutani et al. 1987b; Tsurutani 1991; Schwartz et al. 1992). They reveal the nonlinear nature of plasma waves in multi-ion plasmas. This motivates us to investigate the properties of nonlinear waves (especially solitons) in such plasmas.

A first impression of the effect caused by adding a third species to an electron–proton plasma is obtained by performing a linear mode analysis. For instance, in the low-frequency regime (below the proton cyclotron frequency $\Omega_p$), a new left-hand polarized mode emerges (Melrose 1986). In the case of propagation parallel to the magnetic field, the proton cyclotron mode splits into two modes, as shown

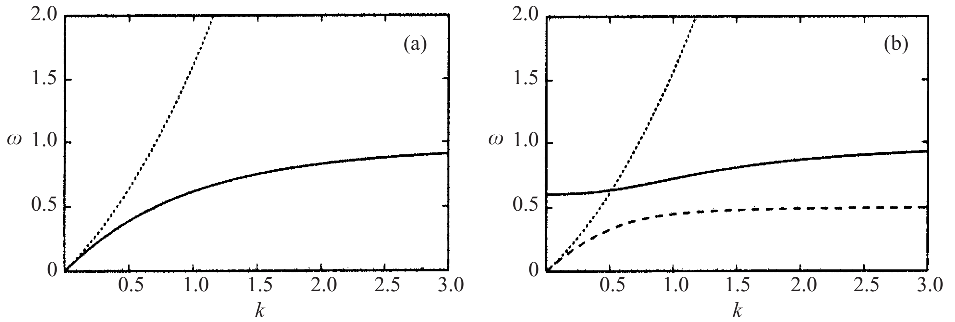


**Figure 1.** Dispersion relations $\omega(k)$ for propagation parallel to the ambient magnetic field in (a) a cold collisionless electron–proton plasma and (b) an electron–proton–helium plasma with 10% $^4\mathrm{He}^{2+}$ ($N_{0i} = 0.1N_{0e}$, $N_{0p} = 0.8N_{0e}$). The frequency $\omega$ is normalized to the proton cyclotron frequency and the wavenumber $k$ is normalized to the inverse ion inertial length of the electron–proton plasma. The dotted curve corresponds to the right-hand polarized whistler mode. The other curves represent left-hand polarized modes. We draw attention to the solid line in (b), which begins ($k$=0) at $\omega_I$ and ends ($k \to \infty$) at $\mathbf{\Omega}_p$.

in Fig. 1. The associated dispersion relation for a cold and collisionless plasma (see e.g. Mann et al. 1997) is given by

$$k^2 - \omega^2 \frac{\omega_I - \omega}{(\mathbf{\Omega}_p - \omega)(\mathbf{\Omega}_i - \omega)} = 0, \tag{1.1a}$$

with

$$\omega_I = Z_i \kappa_i \mathbf{\Omega}_i + Z_p \kappa_p \mathbf{\Omega}_p , \tag{1.1b}$$

where $\mathbf{\Omega}_i$ and $\mathbf{\Omega}_p$ are the cyclotron frequencies, $Z_i$ and $Z_p$ are the elementary charge numbers (i.e. $Z_p = 1$), and $\kappa_i$ and $\kappa_p$ are the undisturbed relative (with respect to the electron number density, i.e. $\kappa_\alpha = N_{0\alpha}/N_{0e}$) number densities of the ions and the protons respectively. We are interested in studying the nonlinear properties of the so-called *hybrid mode*. Its linear dispersion relation is represented by the solid curve in Fig. 1(b). It has frequencies between $\omega_I$ at $k = 0$ and $\mathbf{\Omega}_p$ at $k \to \infty$.

Since it is well known that dispersion and nonlinear effects have to balance each other in solitons, we expect to find solitons at this hybrid mode: owing to the fact that this mode has an inflection point, we can adjust the 'sign' of the dispersion to our needs by simply choosing an appropriate $k$. In contrast to Alfvén solitons (Spangler and Sheerin 1982; Mjølhus and Wyller 1986, 1988; Mann 1988; Verheest 1990), where the inflection point of the associated Alfvén mode is at $k = 0$, $\omega = 0$, we expect here to have a finite $\omega$ that ranges somewhere between $\omega_I$ and $\mathbf{\Omega}_p$. Thus we do not find ordinary solitons; rather, we find envelope solitons, similar to the *whistler solitons* (Karpman and Washimi 1977; Spatschek et al. 1979) but at a much larger time scale. However, it is not the electrons but rather the two different ion species that determine the dynamics of the soliton.

We describe the three-component plasma by the multifluid equations. Since the frequency of the hybrid mode is very much smaller than the plasma frequency and the electron cyclotron frequency, the waves are predominantly carried by the motions of the ions, and the electrons merely act as a neutralizing fluid. This allows us to simplify the multifluid equations, by assuming quasineutrality and neglecting the displacement current and the electron mass (Kakutani et al. 1967). By eliminating the electron-related variables from the equations, we obtain the *bi-ion Hall equations*.

The ansatz of a wave packet is made, and thus the carrier wave, which obeys the linear dispersion relation (1.1), is separated from the envelope function. Since we know the properties of the carrier wave from linear theory, we are able to deduce the magnitude of the involved variables in terms of the modulus of the magnetic field. This enables us to employ a perturbation theory that is truncated after the third order (the lowest order with nonlinear terms). Through this procedure, a nonlinear Schrödinger equation is obtained.

Finally, we investigate our resulting equations by applying them to an actual cold bi-ion plasma. We realize that we must confine the soliton's magnetic field amplitude to 10 % of the unperturbed magnetic field. Furthermore, we detect a local density enhancement within the soliton, whereas the two ion species are treated differently: one of the species becomes more enhanced than the other one.

## 2. Bi-ion Hall equations

We describe the magnetized and collisionless plasma by means of bi-ion Hall equations, which can easily be derived from the multifluid equations that we have presented previously (Hackenberg et al. 1998) by eliminating the electron-related variables and the electric field.

In order to formulate the equations in a dimensionless manner, we normalize the time with the inverse of the proton cyclotron frequency $\mathbf{\Omega}_p = eB_0/m_pc$ (where $e$ is the (positive) elementary charge, $B_0$ is the magnitude of the undisturbed magnetic field, $m_p$ is the proton mass and $c$ is the velocity of light), and the spatial coordinates are given in terms of the reference length, $l_{\rm ref} = (m_pc^2/4\pi N_{0e}e^2)^{1/2}$. Then, the reference velocity is found to be $v_{\rm ref} = l_{\rm ref}\mathbf{\Omega}_p = B_0/(4\pi m_pN_{0e})^{1/2}$. The velocity $v_{\rm ref}$ corresponds to the Alfvén velocity in a pure electron–proton plasma with undisturbed electron number density $N_{0e}$. Note that this normalization is independent of the composition of the plasma. The magnetic field $\mathbf{B}$, the particle number densities $N_\alpha$ and all velocities are normalized by the undisturbed background magnetic field $B_{\rm ref} = B_0$, the unperturbed particle number densities $N_{{\rm ref}\alpha} = N_{0\alpha}$ and the reference velocity $v_{\rm ref}$ respectively. The unperturbed temperature $T_{0\alpha}$ and the unperturbed partial pressure $p_{0\alpha}$ are expressed by the plasma betas $\beta_\alpha = 8\pi p_{0\alpha}/B_0^2 = 8\pi N_{0\alpha}k_{\rm B}T_{0\alpha}/B_0^2$, which are already dimensionless quantities. From now on, all quantities are normalized, unless otherwise noted.

Since we only investigate waves propagating parallel to the $x$ axis, we restrict ourselves to the case where all varying quantities depend only on the time $t$ and the $x$ coordinate. Owing to quasineutrality, the electron number density is given by

$$N_e = \kappa_pZ_pN_p + \kappa_iZ_iN_i\ . \tag{2.1}$$

Despite our claim to eliminate all electron-related variables, we keep $N_e$ in our equations as a kind of abbreviation.

The multifluid equations consist of the continuity equations

$$\frac{\partial}{\partial t}N_\alpha + \frac{\partial}{\partial x}N_\alpha V_{\alpha x} = 0 \tag{2.2}$$

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for the protons ($\alpha$=$p$) and the ions ($\alpha$=$i$). The three spatial components of the momentum equation for the protons are

$$\frac{N_e}{\Omega_p}\frac{d_p}{dt}V_{px} = \rho_i[(V_{py}-V_{iy})B_z-(V_{pz}-V_{iz})B_y]-\frac{1}{2}\frac{\partial}{\partial x}B^2-\frac{\beta_p N_e}{2\rho_p}\frac{\partial}{\partial x}N_p^{\gamma_p}-\frac{\beta_e}{2}\frac{\partial}{\partial x}N_e^{\gamma_e}, \tag{2.3a}$$

$$\frac{N_e}{\Omega_p}\frac{d_p}{dt}V_{py} = \rho_i[(V_{pz}-V_{iz})B_x-(V_{px}-V_{ix})B_z]+B_x\frac{\partial}{\partial x}B_y\ , \tag{2.3b}$$

$$\frac{N_e}{\Omega_p}\frac{d_p}{dt}V_{pz} = \rho_i[(V_{px}-V_{ix})B_y-(V_{py}-V_{iy})B_x]+B_x\frac{\partial}{\partial x}B_z\ , \tag{2.3c}$$

with the comoving derivative

$$\frac{d_\alpha}{dt}=\frac{\partial}{\partial t}+V_{\alpha x}\frac{\partial}{\partial x},$$

the charge density $\rho_\alpha = \kappa_\alpha Z_\alpha N_\alpha$ and the polytropic index $\gamma_\alpha$. The momentum equation for the ions is obtained from the proton momentum equation (2.3a–c) by simply exchanging the indices $p$ and $i$. The induction equation closes our set of equations:

$$\frac{\partial}{\partial t}B_y = \frac{\partial}{\partial x}\frac{1}{N_e}\left[B_x\frac{\partial}{\partial x}B_z-\rho_p(V_{px}B_y-V_{py}B_x)-\rho_i(V_{ix}B_y-V_{iy}B_x)\right], \tag{2.4a}$$

$$\frac{\partial}{\partial t}B_z = -\frac{\partial}{\partial x}\frac{1}{N_e}\left[B_x\frac{\partial}{\partial x}B_y-\rho_p(V_{pz}B_x-V_{px}B_z)-\rho_i(V_{iz}B_x-V_{ix}B_z)\right]. \tag{2.4b}$$

We further simplify our equations by assuming that the waves are propagating parallel to the ambient magnetic field, i.e. the magnetic field is aligned with the $x$ axis. Thus its constant $x$ component becomes $B_x = 1$.

Since there is a high symmetry in the transverse equations (2.3b,c) and (2.4a,b), it is convenient to define the complex variables

$$b = i\,B_y + B_z\ , \qquad v_p = i\,V_{py}+V_{pz}\ , \qquad v_i = i\,V_{iy}+V_{iz}\ . \tag{2.5}$$

We are able to do this, because the original variables are real variables. They can be recovered by taking the real and imaginary part of the new variables.

## 3. Derivation of a nonlinear Schrödinger equation

According to our goal of searching for envelope solitons (see Sec. 1), we make the ansatz of a wave packet:

$$b^*(x,t)=\hat{b}(x,t)\,e^{i(kx-\omega t)}\ , \qquad v_\alpha^*(x,t)=\hat{v}_\alpha(x,t)\,e^{i(kx-\omega t)}\ . \tag{3.1}$$

We have taken $b^*$ and $v_\alpha^*$ instead of $b$ and $v_\alpha$ to ensure that the carrier wave is left-hand polarized for positive $\omega$ and $k$. The asterisk denotes the complex conjugate. The dispersion relation (1.1) (see Fig. 1) defines the relationship $\omega(k)$. The envelope functions $\hat{b}$ and $\hat{v}_\alpha$ should vary far less rapidly than the carrier, i.e. the inequalities

$$\left|\frac{\partial\hat{b}/\partial x}{\hat{b}}\right| \ll k\ , \qquad \left|\frac{\partial\hat{b}/\partial t}{\hat{b}}\right| \ll \omega, \tag{3.2a}$$

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and

$$\left|\frac{\partial \hat{v}_\alpha/\partial x}{\hat{v}_\alpha}\right| \ll k\,, \qquad \left|\frac{\partial \hat{v}_\alpha/\partial t}{\hat{v}_\alpha}\right| \ll \omega \tag{3.2b}$$

should hold.

By inserting the ansatz (3.1) into the momentum equations, we get

$$\frac{\rho_p}{\Omega_p}\frac{d_p}{dt}V_{px} + \frac{\rho_i}{\Omega_i}\frac{d_i}{dt}v_{ix} = -\frac{1}{2}\left(\frac{\partial}{\partial x}|\hat{b}|^2 + \beta_e\frac{\partial}{\partial x}N_e^{\gamma_e} + \beta_p\frac{\partial}{\partial x}N_p^{\gamma_p} + \beta_i\frac{\partial}{\partial x}N_i^{\gamma_i}\right), \tag{3.3}$$

$$\frac{1}{\Omega_p}\frac{d_p}{dt}V_{px} - \frac{1}{\Omega_i}\frac{d_i}{dt}v_{ix} = -\tfrac{1}{2}i[(\hat{v}_p^* - \hat{v}_i^*)\hat{b} - \text{c.c.}] - \frac{\beta_p}{2\rho_p}\frac{\partial}{\partial x}N_p^{\gamma_p} + \frac{\beta_i}{2\rho_i}\frac{\partial}{\partial x}N_i^{\gamma_i} \tag{3.4}$$

for the longitudinal components and

$$\frac{\rho_p}{\Omega_p}\frac{d_p}{dt}\hat{v}_p + \frac{\rho_i}{\Omega_i}\frac{d_i}{dt}\hat{v}_i = \frac{\partial}{\partial x}\hat{b} - \frac{i\rho_p\hat{v}_p}{\Omega_p}(kV_{px}-\omega) - \frac{i\rho_i\hat{v}_i}{\Omega_i}(kV_{ix}-\omega) + ik\hat{b}, \tag{3.5}$$

$$\frac{1}{\Omega_p}\frac{d_p}{dt}\hat{v}_p - \frac{1}{\Omega_i}\frac{d_i}{dt}\hat{v}_i = -i[(\hat{v}_p - \hat{v}_i) - (V_{px} - V_{ix})\hat{b}] - \frac{i\hat{v}_p}{\Omega_p}(kV_{px}-\omega) + \frac{i\hat{v}_i}{\Omega_i}(kV_{ix}-\omega) \tag{3.6}$$

for the transverse components. The induction equation yields

$$\frac{\partial}{\partial t}\hat{b} - i\omega\hat{b} = \left\{\frac{\partial}{\partial x}\frac{1}{N_e} + \frac{ik}{N_e}\right\}\left[-i\left(\frac{\partial}{\partial x}\hat{b} + ik\hat{b}\right) + \rho_p(\hat{v}_p - V_{px}\hat{b}) + \rho_i(\hat{v}_i - V_{ix}\hat{b})\right]. \tag{3.7}$$

The term enclosed by curly braces in (3.7) is an operator, i.e. the differentiation has to be applied to the terms within the square brackets.

Before we can apply the perturbation theory, we have to know the order of magnitude of our variables. We compare each variable with the (small, positive) quantity $\epsilon$.

We begin with the amplitude of the magnetic field $\hat{b}$, which is (by definition) of order $\epsilon$, i.e. $\hat{b} = O(\epsilon)$. In contrast, all constants (including $\omega$ and $k$) are of order one, i.e. $O(1)$. For very small $\epsilon$, we already know the solution of our equations, since this is the result of the linear theory: owing to the presence of the left-hand polarized wave, there are transverse velocities, which are of the same order as the magnetic field (see (3.5) and (3.6)), i.e. $\hat{v}_\alpha = O(\epsilon)$; but there are no density fluctuations $n_\alpha = N_\alpha - 1$ and no longitudinal velocities $V_{\alpha x}$ in the linear theory. They will be induced by the nonlinear terms $\hat{b}\hat{b}^*$, $\hat{v}_p^*\hat{b}$ and $\hat{v}_i^*\hat{b}$ in (3.3) and (3.4), which are of order $\epsilon^2$, i.e. $n_\alpha = O(\epsilon^2)$ and $V_{\alpha x} = O(\epsilon^2)$.

In the first two orders of the perturbation theory, only the transverse equations (3.5)–(3.7) have to be taken into account, since the longitudinal equations (3.3) and (3.4) first contribute to the third order. From that, we obtain

$$\frac{\partial}{\partial x}\hat{b} + \frac{1}{v_g}\frac{\partial}{\partial t}\hat{b} + O(\epsilon^3) = 0, \tag{3.8}$$

with the group velocity

$$v_g = \frac{\partial\omega}{\partial k} = \frac{2\omega k\,(\Omega_p - \omega)(\Omega_i - \omega)}{k^2\,(2\Omega_p\Omega_i - \omega\Omega_p - \omega\Omega_i) - \omega^3}\,. \tag{3.9}$$

This means that the peak of our wave packet has to move with a velocity close to the group velocity of the carrier wave. For the sake of completeness, the transverse

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velocities are given by

$$\hat{v}_p = -\frac{\Omega_p}{\Omega_p - \omega}\,\frac{\omega}{k}\left[\hat{b} - \frac{i}{2}\,\frac{\omega^2 - k^2(\Omega_p - \Omega_i)}{(\Omega_p - \omega)(\Omega_i - \omega)}\,\frac{v_g}{k^2}\,\frac{\partial}{\partial x}\,\hat{b}\right] + O(\epsilon^3), \tag{3.10}$$

$$\hat{v}_i = -\frac{\Omega_i}{\Omega_i - \omega}\,\frac{\omega}{k}\left[\hat{b} - \frac{i}{2}\,\frac{\omega^2 + k^2(\Omega_p - \Omega_i)}{(\Omega_p - \omega)(\Omega_p - \omega)}\,\frac{v_g}{k^2}\,\frac{\partial}{\partial x}\,\hat{b}\right] + O(\epsilon^3)\,. \tag{3.11}$$

In fact, the first term on the right-hand side of (3.4),

$$\tfrac{1}{2}i[(\hat{v}_p^* - \hat{v}_i^*)\hat{b} - \text{c.c.}] = -\frac{\Omega_p - \Omega_i}{\omega_I - \omega}\left(1 + \frac{kv_g}{\omega_I - \omega}\right)\frac{\partial}{\partial x}\,|\hat{b}|^2 + O(\epsilon^4) = O(\epsilon^3) \tag{3.12}$$

does not contribute to the second order of the perturbation theory.

In the third (and last) order of the perturbation theory, the induction equation (3.7) becomes

$$\begin{aligned} 2ik&\left\{\frac{\partial}{\partial x}\,\hat{b} + \frac{1}{v_g}\,\frac{\partial}{\partial t}\,\hat{b}\right\} + k\,\frac{v_g'}{v_g}\,\frac{\partial^2}{\partial x^2}\,\hat{b} \\ &-\frac{k^2(\Omega_i - \omega) - \omega^2}{\Omega_p - \Omega_i}\left(n_p - \frac{2\Omega_p - \omega}{\Omega_p - \omega}\,\frac{k}{\omega}\,V_{px}\right)\hat{b} \\ &+\frac{k^2(\Omega_p - \omega) - \omega^2}{\Omega_p - \Omega_i}\left(n_i - \frac{2\Omega_i - \omega}{\Omega_i - \omega}\,\frac{k}{\omega}\,V_{ix}\right)\hat{b} + O(\epsilon^4) = 0, \end{aligned} \tag{3.13}$$

with the group dispersion $v_g' = \partial v_g/\partial k$. Owing to the nonlinear terms, the densities $n_\alpha$ and the longitudinal velocities $V_{\alpha x}$ are also involved.

To proceed further, we have to express $n_\alpha$ and $V_{\alpha x}$ in terms of $\hat{b}$. In fact, owing to (3.3) and (3.12), they are connected to $|\hat{b}|^2$ (the squared modulus of the envelope). No phase information of $\hat{b}$ is involved. In order to integrate the continuity equation (2.2) and the longitudinal components of the momentum equation (3.3) and (3.4), we go into a comoving system (with velocity $U$), in which we assume $|\hat{b}|^2$, $n_\alpha$ and $V_{\alpha x}$ to be stationary. The envelope function $\hat{b}$ will not be stationary in this system. The coordinate transformation into this new system is given by

$$(x,t) \mapsto (\xi,\tau), \qquad \text{with} \quad \xi = x - Ut \quad \text{and} \quad \tau = t\,. \tag{3.14}$$

In particular, the substitutions

$$x \mapsto \xi + U\tau, \qquad \frac{\partial}{\partial x} \mapsto \frac{\partial}{\partial \xi}, \tag{3.15a}$$

$$t \mapsto \tau, \qquad \frac{\partial}{\partial t} \mapsto -U\,\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \tau} \tag{3.15b}$$

are to be applied to the above-mentioned equations. Since our variables are stationary in the comoving frame of reference, the $\tau$-derivatives vanish. Thus we obtain ordinary differential equations with respect to $\xi$.

At early times ($t\to-\infty$ or $\xi\to\infty$), the wave packet is far away and the plasma is still undisturbed. This provides boundary conditions,

$$\left.\begin{aligned} n_p(\xi\to\infty) &= 0\,, & V_{px}(\xi\to\infty) &= 0\,, \\ n_i(\xi\to\infty) &= 0\,, & V_{ix}(\xi\to\infty) &= 0\,, \end{aligned}\right\} \quad \text{and} \quad |\hat{b}|^2(\xi\to\infty) = 0\,, \tag{3.16}$$

which enables us to fix constants of integration.

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From (3.8), which can be written as

$$\frac{\partial}{\partial x}|\hat{b}|^2 + \frac{1}{v_g}\frac{\partial}{\partial t}|\hat{b}|^2 + O(\epsilon^3) = 0\,, \tag{3.17}$$

we obtain by integration

$$U = v_g + O(\epsilon)\,. \tag{3.18}$$

Therefore the speed $U$ of the comoving system must be close to the group velocity $v_g$, and we are able (if precision permits it) to replace $U$ by $v_g$. Integration of the continuity equation (2.2) yields

$$V_{px} = v_g\, n_p + O(\epsilon^3), \qquad V_{ix} = v_g\, n_i + O(\epsilon^3)\,, \tag{3.19}$$

and from the momentum equations (3.3) and (3.4) we get

$$\frac{\kappa_p Z_p}{\Omega_p}(C_{sp}^2 + \Omega_p c_{se}^2 - v_g^2)n_p + \frac{\kappa_i Z_i}{\Omega_i}(C_{si}^2 + \Omega_i c_{se}^2 - v_g^2)n_i + \frac{1}{2}|\hat{b}|^2 + O(\epsilon^3) = 0\,, \tag{3.20}$$

$$\frac{1}{\Omega_p}(C_{sp}^2 - v_g^2)n_p - \frac{1}{\Omega_i}(C_{si}^2 - v_g^2)n_i - \frac{1}{2}\frac{\Omega_p - \Omega_i}{\omega_I - \omega}\left(1 + \frac{k v_g}{\omega_I - \omega}\right)|\hat{b}|^2 + O(\epsilon^3) = 0\,. \tag{3.21}$$

The quantities

$$c_{se}^2 = \tfrac{1}{2}\beta_e\gamma_e\,, \qquad C_{sp}^2 = \Omega_p\frac{\beta_p\gamma_p}{2\kappa_p Z_p}\,, \qquad C_{si}^2 = \Omega_i\frac{\beta_i\gamma_i}{2\kappa_i Z_i}\,, \tag{3.22}$$

are closely related to the plasma sound speeds $C_{s1}$ and $C_{s2}$, which are given by the roots of a biquadratic polynomial (see Mann et al. 1997):

$$\begin{aligned}(x^2 - C_{s1}^2)(x^2 - C_{s2}^2) = x^4 - x^2\,[C_{sp}^2 + C_{si}^2 + (\kappa_p Z_p \Omega_p + \kappa_i Z_i \Omega_i)\, c_{se}^2] \\ + C_{sp}^2 C_{si}^2 + (\kappa_p Z_p \Omega_p C_{si}^2 + \kappa_i Z_i \Omega_i C_{sp}^2)\, c_{se}^2\end{aligned} \tag{3.23}$$

Now, all the necessary ingredients exist, and we can assemble the final equations. Since we have finished the perturbation theory, we refrain from writing down the orders $O(\epsilon)$.

From (3.20) and (3.21), we get for the number densities

$$\begin{aligned} n_p = {} & \frac{\Omega_p\,|\hat{b}|^2}{2(v_g^2 - C_{s1}^2)(v_g^2 - C_{s2}^2)} \\ & \times\left[\left(\frac{\Omega_i - \omega}{\omega_I - \omega} - 1\right)\left(1 + \frac{k v_g}{\omega_I - \omega}\right)[C_{si}^2 + \Omega_i c_{se}^2 - v_g^2] + [C_{si}^2 - v_g^2]\right], \end{aligned} \tag{3.24a}$$

$$\begin{aligned} n_i = {} & \frac{\Omega_i\,|\hat{b}|^2}{2(v_g^2 - C_{s1}^2)(v_g^2 - C_{s2}^2)} \\ & \times\left[\left(\frac{\Omega_p - \omega}{\omega_I - \omega} - 1\right)\left(1 + \frac{k v_g}{\omega_I - \omega}\right)(C_{sp}^2 + \Omega_p c_{se}^2 - v_g^2) + (C_{sp}^2 - v_g^2)\right], \end{aligned} \tag{3.24b}$$

which are proportional to $|\hat{b}|^2$, and with (3.19) we also know the longitudinal velocities. Thus we can merge most of the third term in (3.13) into a constant

$$\begin{aligned} C_N = {} & \frac{k^2(\Omega_i - \omega) - \omega^2}{\Omega_p - \Omega_i}\left(1 - \frac{2\Omega_p - \omega}{\Omega_p - \omega}\frac{k}{\omega}v_g\right)\frac{n_p}{|\hat{b}|^2} \\ & - \frac{k^2(\Omega_p - \omega) - \omega^2}{\Omega_p - \Omega_i}\left(1 - \frac{2\Omega_i - \omega}{\Omega_i - \omega}\frac{k}{\omega}v_g\right)\frac{n_i}{|\hat{b}|^2}, \end{aligned} \tag{3.25}$$

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and rewrite (3.13) as

$$\left\{ i \left[ \frac{\partial}{\partial \tau} + (v_g - U) \frac{\partial}{\partial \xi} \right] + \frac{v_g'}{2} \frac{\partial^2}{\partial \xi^2} + C_N |\hat{b}|^2 \right\} \hat{b} = 0 \, , \tag{3.26}$$

which is of the form of a nonlinear Schrödinger equation. Note that $|\hat{b}|$ must be stationary in the comoving system.

The nonlinear Schrödinger equation (3.26) may be solved by the ansatz

$$\hat{b}(\xi, \tau) = \hat{b}(\xi) \, e^{-i\Omega\tau} \, , \tag{3.27}$$

which satisfies the stationarity of $|\hat{b}|$. Thus it is reduced to an ordinary differential equation

$$\left\{ \Omega + i(v_g - U) \frac{\partial}{\partial \xi} + \frac{v_g'}{2} \frac{\partial^2}{\partial \xi^2} + C_N |\hat{b}|^2 \right\} \hat{b}(\xi) = 0 \, , \tag{3.28}$$

whose general solution is given in the Appendix. For solitary boundary conditions, the solution is

$$\hat{b}(\xi) = b_0 \, \mathrm{sech}\left[ \left( \frac{C_N}{v_g'} \right)^{1/2} b_0 \, (\xi - \xi_0) \right] e^{i[(U - v_g)\xi / v_g' - \phi_0]}, \tag{3.29}$$

with the maximum amplitude

$$b_0 = \left[ -\frac{2\Omega}{C_N} - \frac{(U - v_g)^2}{v_g' C_N} \right]^{1/2} . \tag{3.30}$$

Note that the quantities within the square roots in (3.29) and (3.30) must be positive; in particular, $v_g' \, C_N > 0$ must hold.

We find

$$\mathbf{B} = \mathrm{Re}[b_0 \, (\mathbf{e}_z + i\mathbf{e}_y) \, e^{i\phi_0} \, e^{i[(U - v_g)/v_g' \, (x - Ut)]} \, e^{i[kx - (\omega + \Omega)t]}] \times \mathrm{sech}\left[ \left( \frac{C_N}{v_g'} \right)^{1/2} b_0 \, (x - Ut - \xi_0) \right] + \mathbf{e}_x \tag{3.31}$$

as a final solution in still-normalized variables. In the last exponential, the term $\omega + \Omega$ appears. It denotes a frequency shift of the carrier wave. The shift increases quadratically with the maximum amplitude $b_0$ (see (3.30) with $U = v_g$), and $\Omega$ is therefore called the *nonlinear frequency shift* (Whitham 1974).

## 4. Discussion

In order to illustrate the results of the previous section, an electron–proton–helium plasma with 10 % $^4He^{2+}$, i.e. $N_{0i} = 0.1 N_{0e}$ and $N_{0p} = 0.8 N_{0e}$, is adopted for example. Such a plasma can be found in the solar wind near the Sun. Other helium densities, as long as the densities do not become very small or large, will only change the quantitative but not the qualitative results of this discussion.

For simplicity, we assume a cold plasma ($c_{se} = 0$, $C_{sp} = 0$, $C_{si} = 0$). The number densities (3.24) simplify to

$$n_p = \frac{\Omega_p}{\Omega_p - \omega} \left[ 1 + \frac{\omega^2 - k^2(\Omega_p - \omega)}{(\Omega_p - \omega)(\Omega_i - \omega)} \, \frac{v_g}{k} \right] \frac{\omega^2}{2 v_g^2 k^2} \, |\hat{b}|^2 \, , \tag{4.1a}$$

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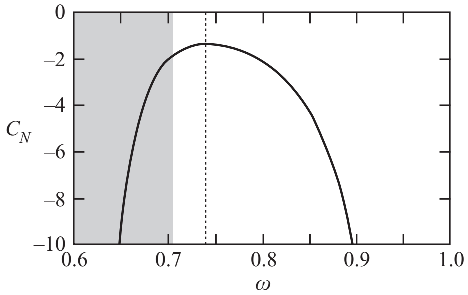


**Figure 2.** Plot of $C_N$ as a function of $\omega$. In the grey-shaded region, $v_g' > 0$. Hence no solitary wave can exist there. The dotted line at $\omega = 0.74\,\Omega_p$ marks the location of our calculation.

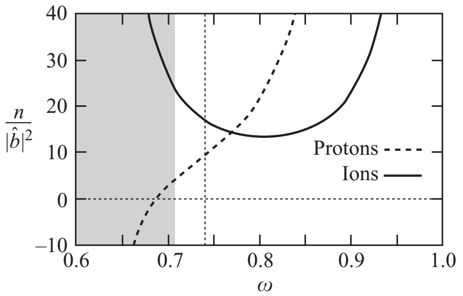


**Figure 3.** Ratio of the number density $n$ and the envelope amplitude $|\hat{b}|^2$ as a function of $\omega$; see (4.1). See Fig. 2 for further explanations.

$$n_i = \frac{\Omega_i}{\Omega_i - \omega}\left[1 + \frac{\omega^2 - k^2(\Omega_i - \omega)}{(\Omega_p - \omega)(\Omega_i - \omega)}\,\frac{v_g}{k}\right]\frac{\omega^2}{2v_g^2k^2}\,|\hat{b}|^2\,, \tag{4.1b}$$

and thus the constant $C_N$ is given by

$$C_N = \frac{\omega^3 - k^2(\Omega_p\Omega_i - \omega^2)}{(\Omega_p - \omega)(\Omega_i - \omega)}\;\frac{\omega^2}{4v_gk^3} \times\left[1 + \frac{\omega^2 - k^2(\Omega_p - \omega)}{(\Omega_p - \omega)(\Omega_i - \omega)}\,\frac{\omega^2 - k^2(\Omega_i - \omega)}{(\Omega_p - \omega)(\Omega_i - \omega)}\,\frac{(\Omega_p - \omega)(\Omega_i - \omega) + 2\Omega_p\Omega_i}{\omega^3 - k^2(\Omega_p\Omega_i - \omega^2)}v_g^2\right]. \tag{4.2}$$

First, we determine the region in which the soliton could exist, i.e. where $v_g' C_N > 0$ is fulfilled. The inflection point of the hybrid mode ($\omega_I \leqslant \omega < \Omega_p$) is located at $\omega \approx 0.705$ (see Fig. 1). To its left, the group dispersion $v_g'$ is positive, while to its right, group dispersion is negative. Thus the constant $C_N$ decides in which of these two regions we can find a solitary wave. In Fig. 2, $C_N$ is plotted as a function of $\omega$. Since $C_N$ is overall negative, the solitary wave can only exist at $\omega \gtrsim 0.705$.

In order to be beyond the boundary of existence ($\omega \gtrsim 0.705$), we choose $\omega = 0.74$ for example. At this point, we have $k = 1.11$, $v_g = 0.19$, $v_g' = -0.09$ and $C_N = -1.34$.

We conclude from (4.1), which is plotted in Fig. 3, that the coefficient of proportionality between $n_\alpha$ and $|\hat{b}|^2$ is of the order of ten. Since, owing to our perturbation theory, the number densities $n_\alpha$ must not exceed $|\hat{b}|^2$, the upper limit for $b_0$ is at 0.1, i.e. the maximum magnetic field amplitude should be limited to 10% of the unperturbed magnetic field.

It is very important that this inspection is done quantitatively with the present

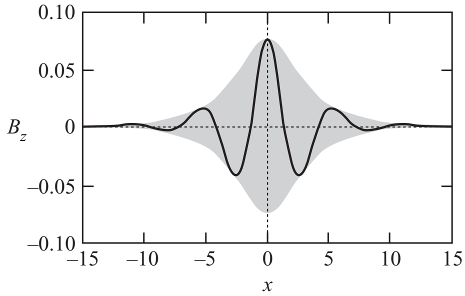


**Figure 4.** Snapshot ($t$ = 0) of the wave packet's magnetic field $B_z$, whose maximum is located at $x = 0$. The envelope $|\hat{b}|$ is depicted by the grey-shaded region.

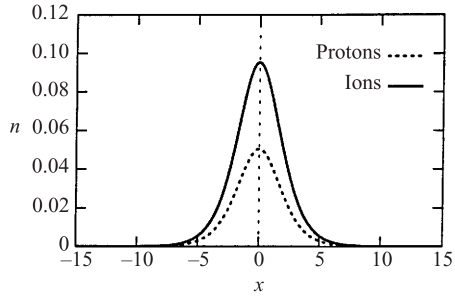


**Figure 5.** Enrichment of helium ions and protons within the wave packet. The maximum relative enrichments are 9.5% for the helium ions and 4.9% for the protons.

plasma. Otherwise there is the risk that the restrictions imposed in the derivation of the the nonlinear Schrödinger equation are violated. The nonlinear Schrödinger equation itself does not contain any hint as to which amplitudes are valid and which are not. This information can only be obtained from the applied perturbation theory.

We stay away from the maximum amplitude and choose $b_0 = 0.075$, resulting in a nonlinear frequency shift $\mathbf{\Omega} = 0.0038$, which is (being 0.5% of $\omega$) negligibly small. For simplicity, we choose $U = v_g$. We have not found any remarkable effect that would justify a more complicated choice.

Now all parameters are fixed, and we can have a look at the result. The $z$ component of the magnetic field is shown in Fig. 4 ($\xi_0 = 0$, $\phi_0 = 0$, $t = 0$). All-in-all, we get the following picture. The wave packet (depicted by the grey-shaded region in Fig. 4) moves with velocity $U = 0.19$ from left to right. The left-hand polarized carrier wave (represented by $B_z$), trapped within the wave packet, moves faster from left to right, with a phase velocity $v_{\mathrm{ph}} = \omega/k = 0.67$. Simultaneously, protons and helium ions are enriched within the solitary wave (see Fig. 5).

The helium ions are more enriched than the protons within the solitary wave. The fact that the helium ions are nearly twice as enriched as the protons is due to the chosen parameters, as shown in Fig. 3. The characteristics of the enrichment can be chosen through $\omega$. Note, $\omega$ or $k$ must not be too large, otherwise strong damping by kinetic effects occur in the real (not cold) plasma. Kinetic effects are neglected in multifluid descriptions.

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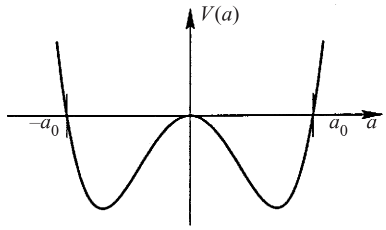


**Figure A.1.** Sketch of the potential $V(a)$. The mass starts infinitely slowly at $a = 0$, reaches its maximum amplitude at $a_0$, and then approaches $a = 0$ again, where it stops at infinite later time.

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## Appendix

With (3.28), we have derived an ordinary differential equation of the form

$$\left\{A + iB\,\frac{\partial}{\partial x} + C\,\frac{\partial^2}{\partial x^2} + |\psi(x)|^2\right\}\psi(x) = 0, \qquad \text{with} \quad A, B, C \in \mathbb{R}\,. \tag{A 1}$$

We shall show how this equation is solved (see Whitham 1974; Karpman 1975).

We denote differentiation with respect to $x$ with a prime. The ansatz

$$\psi(x) = a(x)e^{i\phi(x)}, \qquad \text{with} \quad a, \phi \in \mathbb{R}, \tag{A 2}$$

allows us to split (A 1) into its real part

$$A\,a - B\,a\,\phi' + C\,(a'' - a\,\phi'^2) + a^3 = 0 \tag{A 3}$$

and its imaginary part

$$Ba' + C\,(2\,a'\,\phi' + a\,\phi'') = 0\,. \tag{A 4}$$

By integrating (A 4), inserting the resulting formula for $\phi'$ in (A 3) and integrating again, one obtains

$$\left(A + \frac{B^2}{4C}\right)a^2 + \frac{1}{2}a^4 + \frac{D^2}{4C}\,\frac{1}{a^2} + C\,a'^2 = E\,, \tag{A 5}$$

with the two real constants of integration $D$ and $E$. Solitary boundary conditions, i.e. $a(x \to \infty) = 0$, $a'(x \to \infty) = 0$ and $\phi'(x \to \infty) = O(1)$, render the two constants zero ($D = E = 0$). Hence we can view (A 5) as an equation

$$V(a) + a'^2 = 0, \qquad \text{with} \quad V(a) = \left(\frac{A}{C} + \frac{B^2}{4C^2}\right)a^2 + \frac{1}{2C}a^4\,, \tag{A 6}$$

that describes the motion of a mass in a potential $V(a)$. Since solitary solutions correspond to non-cyclical solutions, the only possible shape of $V$ is as depicted in Fig. A.1. Thus the signs of the coefficients in (A 6) are given by

$$A + \frac{B^2}{4C} < 0, \qquad C > 0\,, \tag{A 7}$$

and we can integrate (A 6).

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Finally, the solution of (A 1) with the maximum amplitude $a_0$ located at $x_0$ is

$$\psi(x) = a_0 \operatorname{sech}\left[\frac{a_0}{(2C)^{1/2}}(x - x_0)\right] e^{-i[(B/2C)x + \phi_0]} \tag{A 8a}$$

with

$$a_0 = \left(-2A - \frac{B^2}{2C}\right)^{1/2}. \tag{A 8b}$$

Note that (A 1) can also be solved in a very similar manner for periodic solutions and so-called inverse solitons. One just has to choose other boundary conditions (and other signs for the potential coefficients).

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