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Variation of Mixed Hodge Structures Associated to an Equisingular One-dimensional Family of Calabi-Yau Threefolds

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Abstract. We study the variations of mixed Hodge structures (VMHS) associated with a pencil $\mathcal{X}$ of equisingular hypersurfaces of degree d in $\mathbb{P}^4$ with only ordinary double points as singularities, as well as the variations of Hodge structures (VHS) associated with the desingularization of this family $\widetilde{\mathcal{X}}$. The notion of a set of singular points being in *homologically good position* is introduced, and, by requiring that the subset of nodes in (algebraic) general position is also in homologically good position, we can extend Griffiths' description of the F^2-term of the Hodge filtration of the desingularization to this case, where we can also determine the possible limiting mixed Hodge structures (LMHS). The particular pencil $\mathcal{X}$ of quintic hypersurfaces with 100 singular double points with 86 of them in (algebraic) general position that served as the starting point for this paper is treated with particular attention.

1 Introduction

In 1941 W. V. D. Hodge proved that the complex de Rham cohomology $H^k(X,\mathbb{C})$ of every compact Kähler manifold splits as a direct sum of spaces $H^{p,q}(\cong H^q(X,\Omega^p_X))$, where $p+q=k$, currently called the Hodge decomposition of $H^k(X,\mathbb{C})$ (see [17]). The pair $(H^k(X,\mathbb{Z}),\{H^{p,q}\})$ is called a (pure) Hodge structure of weight k. All varieties will be considered algebraic and defined over the complex numbers $\mathbb{C}$. Unless otherwise stated, our notation will be consistent with Deligne's (see [9]).

Another way of looking at a Hodge structure is to consider the associated Hodge filtration

$$F^jH^k(X,\mathbb{C}) \overset{\text{def}}{=} \oplus_{p\geq j}H^{p,q}$$

and the pair $(H^k(X,\mathbb{Z}),\{F^jH^k(X,\mathbb{C})\})$.

If $X\subset\mathbb{P}^{n+1}$ is a hypersurface, then the only interesting cohomology group is $H^n(X,\mathbb{C})$ and because of Lefschetz' theorem, we only need to consider the so-called

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primitive cohomology $PH^n(X,\mathbb{C}) = \{\eta \in H^n(X,\mathbb{C}) \mid \eta \cdot H = 0\}$, where H is the class of a hyperplane section on the corresponding projective space.

Griffiths studied the (pure) Hodge structure of smooth projective hypersurfaces X and gave a description of it in terms of its Jacobian ring (see [11]). More precisely, let $X = V(f) \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree d and let

$$\mathscr{H}_k(X) \overset{\text{def}}{=} \left\{ \left[\frac{P\Omega}{f^k} \right] \in A_k^{n+1} \mod dA_{k-1}^n \mid \deg(P) = kd - (n+2) \right\}, \tag{1.1}$$

where A_k^j denotes the space of rational j-forms on $\mathbb{P}^{n+1}$ with a pole of order k along X and $\Omega = \sum_{i=0}^{n+1}(-1)^i x_i dx_0 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{n+1}$. Then we have a commutative diagram

(1.2)

$$\begin{array}{ccccccccc}
0 & \hookrightarrow & \mathscr{H}_1(X) & \hookrightarrow & \cdots & \hookrightarrow & \mathscr{H}_n(X) & \hookrightarrow & \mathscr{H}_{n+1}(X) \\
\downarrow & & \downarrow\cong & & & & \downarrow\cong & & \downarrow\cong \\
0 & \hookrightarrow & F^n PH^n(X,\mathbb{C}) & \hookrightarrow & \cdots & \hookrightarrow & F^1 PH^n(X,\mathbb{C}) & \hookrightarrow & F^0 PH^n(X,\mathbb{C}),
\end{array}$$

where the horizontal arrows in the first line correspond to the natural inclusion given by multiplication by f. Moreover, if $J(f)$ is the Jacobian ideal of f and $R_f \overset{\text{def}}{=} \mathbb{C}[X_0,\dots,X_n]/J(f)$ is the Jacobian ring of f, then the above identification induces isomorphisms between $(R_f)_{(k+1)d-n-2}$ and $PH^{n-k,k}(X,\mathbb{C})$.

For singular varieties, Deligne developed in 1971 the theory of mixed Hodge structures (see [9]), which involves in general the existence of a good desingularization due to Hironaka.

Griffiths and others have tried to give an alternative description for the mixed Hodge structure of a singular variety in some cases. The most important case for us is that of a singular projective hypersurface on the projective space with isolated singularities, the simplest of which is only nodes as singularities. For hypersurfaces of dimension less than or equal to 3, Griffiths [12] (who considers three dimensional hypersurfaces with one ordinary double point) and later Steenbrik [21] (who considers surfaces with isolated singularities) gave a description of the relevant cohomology group of its proper transform under normalization in terms of the Jacobian ring of the polynomial defining it. More precisely, let $X = V(f) \subset \mathbb{P}^{n+1}$ be a hypersurface of degree d, and assume its singular locus Σ consists of ordinary double points. Let $\widetilde{X}$ be its proper transform under normalization. If we define the $\mathscr{H}_k$ as before as well as the vector space

$$\mathscr{H}_2^1(X) \overset{\text{def}}{=} \left\{ \left[\frac{P\Omega}{f^2} \right] \in A_2^{n+1} \mod dA_1^n \mid \deg(P) = 2d - (n+2) \text{ and } P(Q) = 0 \,\forall Q \in \Sigma \right\},$$

given by the *first adjunction condition on* A_2^{n+1}, then if $|\Sigma| = 1$, we get a partial generalization of commutative diagramm (1.2) for the n-th cohomology of the primitive part of $\widetilde{X}$, namely,

$$\begin{array}{ccccc} 0 & \hookrightarrow & \mathscr{H}_1(X) & \hookrightarrow & \mathscr{H}_2^1(X) \\ \downarrow & & \downarrow\cong & & \downarrow\cong \\ 0 & \hookrightarrow & F^nPH^n(\widetilde{X},\mathbb{C}) & \hookrightarrow & F^{n-1}PH^n(\widetilde{X},\mathbb{C}), \end{array} \tag{1.3}$$

where the horizontal map in the arrow below corresponds to the natural map

$$\mathscr{H}_1(X) \longrightarrow \mathscr{H}_2^1(X)$$

given by $\frac{P\Omega}{f} \mapsto \frac{fP\Omega}{f^2}$. A direct generalisation of (1.3) for $|\Sigma| \geq 2$ is not as straightforward as it may seem, and we show in Theorem 3.2(ii) that to assume that all the points of Σ are in (algebraic) general position is not enough, so further imposing the condition that Σ is a hg set (see Definition 3.3) gives the expected generalization, as we have proved in Corollary 3.5 and in Corollary 3.6.

If we now consider a smooth family $\pi\colon \mathcal{X} \longrightarrow B \subset \mathbb{P}^1$, over a Zariski open set B, then on every fiber X_t, one has a Hodge structure $(H^n(X_t,\mathbb{Z}),\{F^pH^n(X_t,\mathbb{C})\})$, and the Hodge filtration extends to a global filtration $\mathcal{F}^p\mathcal{H}^n$, where $\mathcal{H}^n \overset{\text{def}}{=} R^n\pi_*\mathbb{C}\otimes\mathcal{O}_B$. It is well known that the monodromy of the family gives rise to a connection, called the Gauss–Manin connection (GM)

$$\nabla\colon \mathcal{H}^n \longrightarrow \mathcal{H}^n\otimes\Omega_B,$$

which is compatible with the Hodge filtration. More explicity, the GM-connection satisfies the Griffiths transversality condition (also called the horizontality condition)

$$\nabla\colon \mathcal{F}^p\mathcal{H}^n \longrightarrow \mathcal{F}^{p-1}\mathcal{H}^n\otimes\Omega_B.$$

Recall that any polarized VHS $\mathbb{H}$ of weight k on B induces a map from B to the classifying space D of polarized Hodge structures of weight k, which can be seen as a Zariski open set on a projective variety $\check{D}$ parametrizing flags $F^k\subset\cdots\subset F^1\subset V_{\mathbb{C}}$ of nationality $(f^k,\ldots,f^1)$ satisfying the first Riemann–Hodge bilinear relation, where $f^j \overset{\text{def}}{=} \dim F^j$. Since $\check{D}$ is projective, the map $B\to D$ induced by $\mathbb{H}$ can be extended to a map $\mathbb{P}^1\to\check{D}$, in particular to any point $p\in\mathbb{P}^1\backslash B$ we can associate a filtration $F^k(p)\subset\cdots\subset F^1(p)\subset V_{\mathbb{C}}$ satisfying the first Riemann-Hodge bilinear relation. There is no reason for this filtration to satisfy the second Riemann–Hodge bilinear relation, and in general, it will not, so *a priori*, there does not exist a polarized Hodge structure of weight k at p. However, in a small analytic neighborhood U of $p\in\mathbb{P}^1\backslash B$, the local monodromy $\pi_1(U,b)\cong\mathbb{Z}$ regardless of the choice of $b\in U$, and the generator of this group induces a linear transformation T on $\mathbb{H}_b$, compatible with the Hodge filtration, called the *monodromy operator*. T can be thought of as a linear transformation of $V_{\mathbb{C}}$, and as such, it will be compatible with the filtration $F^k(p)\subset\cdots\subset F^1(p)\subset V_{\mathbb{C}}$. If we write $T=T_s\circ T_u$, where T_s and T_u are the semisimple and the unipotent part of T respectively, it can be proved (see [20] Chapter 11 theorem 11.8 (monodromy theorem) and lemma-definition 11.9 or [18]—monodromy theorem §9.1) that $T_s^m=1$ for some integer m, and the least l such that $(T_u-\mathrm{Id})^l=0$ is less than or equal to $m+1$. The triple $(W_\bullet,F^\bullet,V)$ (or simply $(W_\bullet,F^\bullet)$ whenever V is clear from the context), defines a MHS on V, called the *limit MHS* at p (in the sense of Schmid), where $W_\bullet$

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is the monodromy weight filtration on V associated to $N = \log(T_u)$ (see [14, p. 255], [13, pp. 106–107], and Lemma 6.4). Finally, since the GM-connection associated with the VHS $\mathbb{H}$ satisfies Griffiths transversality, the nilpotent operator N induces linear maps $N_j\colon H^{j,k-j} \to H^{j-1,k-j+1}$, where $H^{j,k-j} \overset{\text{def}}{=} F^j(p)/F^{j+1}(p)$. The previous discussion is true for the case of $PR^k\pi_*\mathbb{C}$, the primitive part of the k-th higher direct image of $\mathbb{C}$.

From now on, unless explicitly stated otherwise, $X \subset \mathbb{P}^{n+1}$ will be a singular hypersurface with singular locus Σ; $\pi\colon \widehat{\mathbb{P}}^{n+1} \to \mathbb{P}^{n+1}$ will denote the blow up of $\mathbb{P}^{n+1}$ along Σ, and $\widehat{\Sigma}$ will be the exceptional divisor on $\widehat{\mathbb{P}}^{n+1}$; moreover, $\widetilde{X}$ will be the strict transform of X and $\widetilde{\Sigma} = \widehat{\Sigma} \cap \widetilde{X}$.

The paper is organized as follows. In Section 2, we generalize the classical definition of adjointness of $\mathscr{H}_2^1$ to isolated singularities of higher order, denoted as s-adjointness in Definition 2.2. The most important result in this section is given by Proposition 2.4, which is a sheaf theoretic formulation of the notion of s-adjointness with pole order conditions and a partial description of $\mathscr{H}^k(\widetilde{X})$ in terms of rational forms on $\mathbb{P}^4$, at least when the singular locus of X consist of ordinary double points. In Section 3, we define the notion of homologically good sets and study its relations to the notion of points in algebraic general position. The central results are Theorem 3.2, generalizing diagramm (1.3) above, together with Corollaries 3.5 and 3.6. In Section 4, recalling the definition of generalized Hodge numbers, the main result is given by Proposition 4.2, which computes the generalized Euler characteristic polynomial of X and $\widetilde{X}$ using the techniques introduced in [6]. In Section 5, using the technique of cubical hyperresolutions of [15], the main results are as follows. First, the computation of the Mixed Hodge structure of a nodal threefold (Proposition 5.1) applied in example 5.2, where we actually compute the number of points in algebraic general position. The other important result in this section is given in Proposition 5.4 and Remark 5.5 by finding an exact relation between the defect and the failure of Σ to impose linearly independent conditions on polynomials of degree $2d - 5$. In Section 6, we consider the VHS associated with the Lefschetz pencil of the desingularizations and compute the possible weight filtrations corresponding to the limit MHS in Theorem 6.5. Another important and natural result is that the VMHS associated with an equisingular pencil of nodal threefolds is indeed a geometric and admissible VMHS in the sense of [22], stated and proved in Proposition 6.6.

2 Generalized Adjointness Conditions

Let $\Omega^4_{\mathbb{P}}(kX)$ be the sheaf on $\mathbb{P}^4$ of four-rational forms with a pole of order k along the hypersurface X or shortly $\Omega^4_{\mathbb{P}}(k)$. Then it follows that $H^0(\mathbb{P}^4, \Omega^4_{\mathbb{P}}(k)) = A^4_k(X)$. For a polynomial F, we denote by $\mu_P(F)$ the multiplicity of F in P (see [16]). Given a subset $T \subset X$, let us denote by $\Omega^4_{\mathbb{P}}(kX, sT)$ the subsheaf of $\Omega^4_{\mathbb{P}}(k)$ of four-rational forms with a pole of order k on X and multiplicity at least s on every point $P \in T$.

Definition 2.1 Given $f \in \mathbb{C}[y_0, \dots, y_n]$, the s-adjoint condition on f relative to T is given by $\mu_p(f) \geq s$ for all $p \in T$. Note that $s = 1$ if and only if $T \subset V(f)$.

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Definition 2.2 The space of four-rational forms with poles of order k along X and s-adjoint to T is defined as follows:

$$A_k^4(X, sT) = \left\{\psi \in A_k^4(X) | \psi = \frac{h\Omega}{f^k},\ h \text{ is } s\text{-adjoint relative to } T\right\}.$$

In particular, if $T = \Sigma = \mathrm{Sing}(X)$, it follows that $H^0(\mathbb{P}, \Omega_{\mathbb{P}}^4(kX, s\Sigma)) = A_k^4(X_t, s\Sigma)$. Clearly, $s \le d = \deg(f)$. We have already defined the vector space $\mathscr{H}_2^1$ following Griffiths' [11] notation, and it is clear that $\mathscr{H}_2^1 = A_2^4(X_t, \Sigma)/dA_1^4(X)$. If Σ consists of ordinary double points, then trivially $dA_1^3(X) \subset A_2^4(X, \Sigma)$, but in general, it is not even possible to compare $dA_{k-1}^3(X)$ with $A_k^4(X, (k-1)\Sigma)$. Hence, we can define the following quotient:

$$\mathscr{H}_k^s = \frac{A_k^4(X, s\Sigma)}{dA_{k-1}^3 \cap A_k^4(X, s\Sigma)},$$

which is the vector space of top rational forms with poles of order k along X and satisfying the s-adjoint condition relative to Σ, naturally generalising one-adjointness relative to Σ given by [12].

Remark 2.3 In this sense given G, a finite subset of polynomials, one can generalize the adjointness condition relative to T if for all $h \in G$, the s-adjoint condition is satisfied on h.

Let us return to the sheaf theoretic version of forms with pole order and adjointness conditions.

Note. We will often write $\Omega_{\mathbb{P}}^4(k, s)$ as short hand notation for $\Omega_{\mathbb{P}^4}^4(kX, s\Sigma)$. Analogously, $\Omega_{\widetilde{\mathbb{P}}}^4(k)$ will stand for $\Omega_{\widetilde{\mathbb{P}}^4}^4(k\widetilde{X})$.

Proposition 2.4 *With notation as above, if Σ consists of ordinary double points, then for $N = 2k - 3$ positive and $s \ge N$, we have $\pi^*(\Omega_{\mathbb{P}}^4(k, s)) \subset \Omega_{\widetilde{\mathbb{P}}}^4(k)$.*

Proof This is a local straightforward computation. ∎

3 Elementary Results for Nodal Hypersurfaces on $\mathbb{P}^4$

Given a projective variety X, we will say that a finite set $T \subset X$ is a set of points in *algebraic general position* or *in general position*, for short, if they impose $|T|$ conditions on polynomials of degree d passing through all of them, for all $d \ge 1$.

Lemma 3.1 *For any scheme Y of dimension $n > 0$, any locally free sheaf H of finite rank and any non-singular subvariety $Z \in Y$, we have $H_Z^i(Y, H) = 0$ for all $i < codim_Y(Z)$.*

Proof It follows by excision, since for any $\mathrm{P} \in Y$ with smooth closure Z, the local ring $\mathcal{O}_{\mathrm{P}}$ is a regular local ring of depth $= \mathrm{codim}_Y(Z)$. ∎

The central result in this section is given by the following theorem.

Theorem 3.2

(i) $\mathscr{H}_1 \xrightarrow{\pi^*} H^{3,0}(\widetilde{X})$ *is an isomorphism, and*

(ii) $\mathscr{H}_2^1 \xrightarrow{\pi^*} F^2H^3(\widetilde{X},\mathbb{C})$ *is injective.*

Proof It is well known that $H^{3,0}(\widetilde{X}) \cong H^0(\widehat{\mathbb{P}}^4,\Omega^4_{\widehat{\mathbb{P}}^4}(\widetilde{X}))$, as already shown in [12, theorem 10.8], and the first assertion is equivalent to $\pi^*\colon H^0(\mathbb{P}^4,\Omega^4_{\mathbb{P}^4}(X)) \to H^0(\widehat{\mathbb{P}}^4,\Omega^4_{\widehat{\mathbb{P}}^4}(\widetilde{X}))$ being an isomorphism.

Let

$$U = \mathbb{P}^4\backslash\Sigma \overset{j}{\hookrightarrow} \mathbb{P}^4 \quad \text{and} \quad \widehat{U} = \widehat{\mathbb{P}}^4\backslash\widehat{\Sigma} \overset{\hat{\jmath}}{\hookrightarrow} \widehat{\mathbb{P}}^4 .$$

Then $\pi|_{\widehat{U}}\colon \widehat{U} \to U$ is an isomorphism, and, in particular, we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^0_\Sigma(\mathbb{P}^4,\Omega^4_{\mathbb{P}^4}(X)) & \longrightarrow & H^0(\mathbb{P}^4,\Omega^4_{\mathbb{P}^4}(X)) & \xrightarrow{j^*} & H^0(U,\Omega^4_{\mathbb{P}^4}(X)) & \longrightarrow & H^1_\Sigma(\mathbb{P}^4,\Omega^4_{\mathbb{P}^4}(X)) \\
 & & & & \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \pi^*|_U} & & \\
0 & \longrightarrow & H^0_{\widehat{\Sigma}}(\widehat{\mathbb{P}}^4,\Omega^4_{\widehat{\mathbb{P}}^4}(\widetilde{X})) & \longrightarrow & H^0(\widehat{\mathbb{P}}^4,\Omega^4_{\widehat{\mathbb{P}}^4}(\widetilde{X})) & \xrightarrow{\hat{\jmath}^{-1}} & H^0(\widehat{U},\Omega^4_{\widehat{\mathbb{P}}^4}(\widetilde{X})) & \longrightarrow & H^1_{\widehat{\Sigma}}(\widehat{\mathbb{P}}^4,\Omega^4_{\widehat{\mathbb{P}}^4}(\widetilde{X})),
\end{array}$$

where the map $\pi^*|_U$ is an isomorphism, and the rows are exact.

Since $\widehat{\mathbb{P}}^4$ is regular in codimension 1 and $\widehat{\Sigma}$ is a disjoint union of exceptional divisors E_P above the points $P \in \Sigma$, $H^0_{\widehat{\Sigma}}(\widehat{\mathbb{P}}^4,\Omega^4_{\widehat{\mathbb{P}}^4}(\widetilde{X})) \cong \oplus_{P\in\Sigma} H^0_{E_P}(\widehat{\mathbb{P}}^4,\Omega^4_{\widehat{\mathbb{P}}^4}(\widetilde{X})) = 0$.

On the other hand, since $\Omega^4_{\mathbb{P}^4}$ is locally free and every point in Σ is a regular point in $\mathbb{P}^4$, by Lemma 3.1, one also has that $H^1_\Sigma(\mathbb{P}^4,\Omega^4_{\mathbb{P}^4}(X)) = H^0_\Sigma(\mathbb{P}^4,\Omega^4_{\mathbb{P}^4}(X)) = 0$, so the diagram above becomes

$$\begin{array}{ccccccc}
0 & \longrightarrow & H^0(\mathbb{P}^4,\Omega^4_{\mathbb{P}^4}(X)) & \xrightarrow{j^*} & H^0(U,\Omega^4_{\mathbb{P}^4}(X)) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \pi^*|_U} & & \\
0 & \longrightarrow & H^0(\widehat{\mathbb{P}}^4,\Omega^4_{\widehat{\mathbb{P}}^4}(\widetilde{X})) & \xrightarrow{\hat{\jmath}^*} & H^0(\widehat{U},\Omega^4_{\widehat{\mathbb{P}}^4}(\widetilde{X})) & \longrightarrow & H^1_{\widehat{\Sigma}}(\widehat{\mathbb{P}}^4,\Omega^4_{\widehat{\mathbb{P}}^4}(\widetilde{X})).
\end{array}$$

Since $\pi^* \circ j^* = \hat{\jmath}^* \circ \pi^*$ is an isomorphism, π^* is injective and $\hat{\jmath}^*$ is surjective. But $\hat{\jmath}^*$ is injective; therefore, $\hat{\jmath}^*$ and π^* are isomorphisms.

As for (ii), if we denote by d the total differential, then we have the following commutative diagram:

$$\begin{array}{ccc}
A^3_1(X) & \overset{\pi_1^*}{\hookrightarrow} & A^3_1(\widetilde{X}) \\
\downarrow{\scriptstyle d} & & \downarrow{\scriptstyle \tilde{d}} \\
A^4_2(X,\Sigma) & \xrightarrow{\pi^*} & A^4_2(\widetilde{X}) \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\
\mathscr{H}^1_2(X) & \xrightarrow{[\pi^*]} & \mathscr{H}_2(\widetilde{X}),
\end{array} \tag{3.1}$$

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where we distinguish the pullback on 3-forms from the pullback on 4-forms through the subindex 1, and where p and q are the natural quotient maps. Observe that $\operatorname{Coker}(d) = \mathscr{H}_2^1$ and similarly $\operatorname{Coker}(\widetilde{d}) = \mathscr{H}_2(\widetilde{X})$ and the last horizontal arrow $[\pi^*]$ is induced by the universal property of the quotient.

Claim 1. π_1^* is an isomorphism.

Proof of Claim 1. As before, we have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0_\Sigma(\mathbb{P}^4, \Omega^3_{\mathbb{P}^4}(X)) & \longrightarrow & H^0(\mathbb{P}^4, \Omega^3_{\mathbb{P}^4}(X)) & \xrightarrow{j^*} & H^0(U, \Omega^3_{\mathbb{P}^4}(X)) & \longrightarrow & H^1_\Sigma(\mathbb{P}^4, \Omega^3_{\mathbb{P}^4}(X)) \\ & & & & \downarrow \pi_1^* & & \downarrow \pi_1^*|_U & & \\ 0 & \longrightarrow & H^0_{\widetilde{\Sigma}}(\widehat{\mathbb{P}}^4, \Omega^3_{\widehat{\mathbb{P}}^4}(\widetilde{X})) & \longrightarrow & H^0(\widehat{\mathbb{P}}^4, \Omega^3_{\widehat{\mathbb{P}}^4}(\widetilde{X})) & \xrightarrow{\hat{j}^*} & H^0(\widehat{U}, \Omega^3_{\widehat{\mathbb{P}}^4}(\widetilde{X})) & \longrightarrow & H^1_{\widetilde{\Sigma}}(\widehat{\mathbb{P}}^4, \Omega^3_{\widehat{\mathbb{P}}^4}(\widetilde{X})), \end{array}$$

where the map $\pi_1^*|_U$ is an isomorphism, and, exactly as before, this diagram becomes

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(\mathbb{P}^4, \Omega^3_{\mathbb{P}^4}(X)) & \xrightarrow{j^*} & H^0(U, \Omega^3_{\mathbb{P}^4}(X)) & \longrightarrow & 0 \\ & & \downarrow \pi_1^* & & \downarrow \pi_1^*|_U & & \\ 0 & \longrightarrow & H^0(\widehat{\mathbb{P}}^4, \Omega^3_{\widehat{\mathbb{P}}^4}(\widetilde{X})) & \xrightarrow{\hat{j}^*} & H^0(\widehat{U}, \Omega^3_{\widehat{\mathbb{P}}^4}(\widetilde{X})) & \longrightarrow & H^1_{\widetilde{\Sigma}}(\widehat{\mathbb{P}}^4, \Omega^3_{\widehat{\mathbb{P}}^4}(\widetilde{X})), \end{array}$$

and the claim follows.

Claim 2. π^* is injective.

Proof of Claim 2. Clearly we have a a morphism of sheaves $\pi^*\colon \Omega^4_{\mathbb{P}^4}(2,1) \to \pi_* \Omega^4_{\widehat{\mathbb{P}}^4}(2)$, and it is enough to show injectivity on the stalk at every point.

If $\widehat{U}$ is an open set on $\widehat{\mathbb{P}}^4$ whose intersection with $\widehat{\Sigma}$ is empty, then π is an isomorphism from $\widehat{U}$ to $U = \pi(\widehat{U})$, and so $\pi^*\colon \Omega^4_{\mathbb{P}^4}(2,1)(U) \to \pi_*\Omega^4_{\widehat{\mathbb{P}}^4}(2)(U) = \Omega^4_{\widehat{\mathbb{P}}^4}(2)(\widehat{U})$ is an isomorphism. Now let us consider an open set $U \subset \mathbb{P}^4$ containing just the point $P \in \Sigma$; then any rational 4-form ω on $\mathbb{P}^4$ with poles of order 2 along X can be written in the form $\frac{F\,dz}{(z\cdot z)^2}$, where $z = (z_1, z_2, z_3, z_4)$ are local coordinates and $z \cdot z$ is a local equation defining X on U. The form ω satisfies the first adjoint condition relative to Σ in U if and only if $F(P) = 0$. But in this case,

$$\pi^*(\omega)(u,v) = \frac{u^3 F(uv,u)\,d\,u\,d\,v}{u^4(1+v\cdot v)^2} = \frac{F(uv,u)\,d\,u\,d\,v}{u(1+v\cdot v)^2}.$$

Since the zero set of a non-constant holomorphic function is a hypersurface on U, and π is a birational morphism, $\pi^*(\omega) = 0$ if and only if $\pi(\widehat{U}) \subset V(F) \cap U$ if and only if $F \equiv 0$.

Claim 3. $[\pi^*]$ is monomorphism.

Proof of Claim 3. Assume $\overline{\varphi} \in \mathscr{H}_2^1$ satisfies $[\pi^*](\overline{\varphi}) = 0$ and let $\varphi \in A_2^4(X, \Sigma)$ be any representative of $\overline{\varphi}$. Then $\pi^*(\varphi) = d\,\mathfrak{h}$ for some $\mathfrak{h} \in A_1^3(\widetilde{X})$, and by Claim 1, there exist some $\beta \in A_1^3(X)$ such that $\pi_1^*\beta = \mathfrak{h}$; therefore,

$$\pi^*(d\,\beta) = d\,(\pi_1^*\beta) = d\,\mathfrak{h} = \pi^*(\varphi),$$

and the injectivity of π^* implies that $\varphi = d\,\beta \in d\,A_1^3(X)$, *i.e.*, $\overline{\varphi} = 0 \in \mathscr{H}_2^1$.

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The theorem now follows from the fact that $\mathscr{H}_2(\widetilde{X}) \hookrightarrow F^2H^3(\widetilde{X},\mathbb{C})$ as proven *e.g.*, in [12, proposition 16.3, equation (16.10)]. ∎

Let $X \subset \mathbb{P}^4$ be a nodal hypersurface of degree d with m nodes and let P be a node on X. Then in an analytic neighborhood U of P in $\mathbb{P}^4$, we can write $X \cap U = V(z \cdot z)$, where $z = (z_1, z_2, z_3, z_4) \in \mathbb{C}^4$ and $z_i = x_i + \sqrt{-1} y_i$ for $i \in \{1,2,3,4\}$. With this notation, $z \cdot z = x \cdot x - y \cdot y + 2\sqrt{-1}(x \cdot y)$, and following the technique of continous deformation used in [12] (in particular the notation before equation (15.3)), we can consider a family X_ϵ of hypersurfaces with $m-1$ nodes that degenerate to X, that is to say that $X - U \cong X_\epsilon - U$. Note that the three-dimensional real spheres $\delta_\epsilon = \{x \cdot x = \epsilon, y = 0\}$ are contained in the $X_\epsilon \cap U = \{z \cdot z = \epsilon\}$, so the family of hypersurfaces $\{X_\epsilon\}$ degenerate to X and the latter is a singular hypersurface with m double points. Observe that there exists a 3-cell $\theta_\epsilon(P)$ on $U \cap X_\epsilon$ such that $\theta_\epsilon(P) \cdot \delta_\epsilon(P) = 1$ as shown in [12]. The sphere $\delta_0(P) \overset{\text{def}}{=} \lim_{\epsilon\to 0} \delta_\epsilon(P)$ is contractible to a point in X, while the 3-cell $\theta_0(P) \overset{\text{def}}{=} \lim_{\epsilon\to 0} \theta_\epsilon(P)$ gives a non-zero element of $H_3(X,\mathbb{Z}) \otimes \mathbb{Q}$. However, it can happen that $\theta_0(P)$ belongs to the subspace of $H_3(X,\mathbb{Z}) \otimes \mathbb{Q}$ generated by $\{\theta_0(Q) \mid Q \text{ is a node of X}, Q \neq P\}$. This motivates the following definition.

Definition 3.3 Given a nodal hypersurface $X \subset \mathbb{P}^4$, we will say that a set T of nodes on X is *homologically good* (hg) if the corresponding set of three-cells $\{\theta_0(P)\}_{P\in T}$, is a $\mathbb{Q}$-linearly independent set of elements in $H_3(X,\mathbb{Z}) \otimes \mathbb{Q}$ and T is maximal with this property. In particular, there will be a *vanishing cycle* $\delta_0(P)$ for every node in T.

Remark 3.4 Let X, U, and X_ϵ be as above, let T be a homologically good set of nodes on X, $P \in T$, and let $\widetilde{X}$ be the strict transform of X under the blow-up of X on P. The strict transform of $\theta_0(P)$ no longer represents an element in the homology of $\widetilde{X}$, since it is no longer a cycle in $\widetilde{X}$, and $\delta_0(P)$ is contractible to a point already in X; therefore,

$$\operatorname{rank} H_3(\widetilde{X},\mathbb{Z}) \leq \operatorname{rank} H_3(X_\epsilon,\mathbb{Z}) - 2.$$

In what follows, we will assume that $X \subset \mathbb{P}^4$ is a nodal hypersurface of degree d with m nodes, $l \leq m$ of which are in general position. Further, let $Y \subset \mathbb{P}^4$ be a smooth hypersurface of the same degree and let $a = \dim H^{0,3}(Y) = \dim H^{3,0}(Y)$ and $b = \dim H^{1,2}(Y) = \dim H^{2,1}(Y)$. Then more is true in Theorem 3.2(ii).

Corollary 3.5 *If X, $\widetilde{X}$ and Y are as before, then*

$$a + b - l = \frac{1}{2}\dim H^3(Y,\mathbb{C}) - l = \dim \mathscr{H}_2(Y) - l = \dim \mathscr{H}_2^1 \leq \dim F^2H^3(\widetilde{X},\mathbb{C}),$$

and therefore $\operatorname{rank} H_3(\widetilde{X},\mathbb{Z}) \geq 2a + 2b - 2l$.

Since for every point in a hg set T the rank of $H_3(\widetilde{X},\mathbb{Z})$ drops off by two with respect to the rank of $H_3(Y,\mathbb{Z})$, the inequality above imposes an upper bound for the number of nodes in any hg set. In particular, Lemma 3.2 shows that the number of vanishing cycles is at most l; i.e., *there cannot be more nodes forming an hg set on X than the number of nodes in general position.*

If the l nodes in general position form an hg set on X, we actually have

$$\dim \mathscr{H}_2^1 = \dim \mathscr{H}_2(Y) - l = \dim F^2H^3(Y,\mathbb{C}) - l = \frac{1}{2}\dim H^3(Y,\mathbb{C}) - l = a + b - l,$$

so that, in this case, Theorem 3.2 implies that the map $\mathscr{H}_2^1 \to F^2H^3(\widetilde{X},\mathbb{C})$ is in fact an isomorphism.

The discussion in Corollary 3.5 also implies the following corollary.

Corollary 3.6 *If $X \subset \mathbb{P}^4$ is a nodal hypersurface of degree d, where Σ consist of m nodes in general position and is an hg set on X, then $\dim H^{2,1}(\widetilde{X}) = b-m$, $\dim H^3(\widetilde{X},\mathbb{C}) = \operatorname{rank} H_3(\widetilde{X},\mathbb{Z}) = 2a+2b-2m$. In particular, $m \le h^{2,1}(Y)$, where $Y \subset \mathbb{P}^4$ is a smooth hypersurface of degree d.*

Remark 3.7 In particular, for a quintic hypersurface X on $\mathbb{P}^4$, we obtain the nice bound $m \le 101$ for the number of nodes in general position that also constitute an hg set. In this case (see [1,19,23]), this bound is almost sharp. Observe that the maximal number of nodes for a quintic hypersurface is expected to lie between 130 and 135, but they do not lie in general position (*i.e.*, they impose less than 130 conditions, illustrating one form of the Cayley–Bacarach theorem). If Σ is a finite set of nodes, how many independent conditions does Σ impose on homogeneous polynomials of degree d passing through Σ and how many of them form an hg set? This is not the original formulation of the Cayley–Bacharach theorem but a form of this type of theorem (see also [10, p. 297]). The exact relation will be given by the defect of X considered in Proposition 5.4.

With the same techniques, one can prove a similar result for surfaces on $\mathbb{P}^3$ and curves on $\mathbb{P}^2$.

4 Generalized Hodge Numbers

Following Danilov and Khovanskiĭ (see [6, § 1], in particular Definition 1.5, Proposition 1.8, and Corollaries 1.9 and 1.10), we define the generalized Hodge numbers:

$$e^{p,q} = e^{p,q}(X) \stackrel{\text{def}}{=} \sum_k (-1)^k h^{p,q}(H_c^k(X))$$

as well as the generalized Euler characteristic polynomial

$$e(X;x,\overline{x}) \stackrel{\text{def}}{=} \sum_{p,q} e^{p,q}(X)x^p\overline{x}^q,$$

which in the sequel we will simply denote $e(X)$, and $\operatorname{coeff}_{e(X)}()$ is the coefficient of the term in parenthesis. We summarize some well known results about this polynomial (see [6]) in a single lemma.

Lemma 4.1

- *Suppose X is a disjoint union of a finite number of locally closed subvarieties X_i, $i \in I$. Then $e(X) = \sum_i e(X_i)$.*

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- *If $f\colon X \to Y$ is a bundle with fiber F that is locally trivial in the Zariski topology, then $e(X) = e(Y) \times e(F)$.*
- *If X is a point, then $e(X) = 1$.*
- $e(\mathbb{P}^1) = 1 + x\overline{x}$.
- $e(\mathbb{P}^n) = 1 + x\overline{x} + \cdots + (x\overline{x})^n$.
- *Let $\pi\colon \widehat{X} \to X$ be the blow up of X along a subvariety Y of codimension $r+1$ in X. Then*

$$e(\widehat{X}) = e(X) + e(Y)[x\overline{x} + \cdots + (x\overline{x})^r].$$

As an application of the above lemma, we will compute the generalized Euler polynomial of X for a projective hypersurface on $\mathbb{P}^4$ of degree d with precisely m nodes (l of which are in general position) as the singular locus Σ. To fix notation, let $\widehat{\mathbb{P}}^4$ be the blow up of $\mathbb{P}^4$ along Σ, let $\widehat{X}$ be the inverse image of X on $\widehat{\mathbb{P}}^4$, and let $\widetilde{X}$ be the strict transform of X and Y a non-singular hypersurface of degree d on $\mathbb{P}^4$. Further, let $\widehat{\Sigma}$ be the inverse image of Σ and $\widetilde{\Sigma} = \widehat{\Sigma} \cap \widetilde{X}$.

Outside the singular locus the blow up is an isomorphism; therefore, one has the following quasi-projective varieties:

$$X - \Sigma \overset{\text{def}}{=} W \cong \widehat{W} \overset{\text{def}}{=} \widehat{X} - \widehat{\Sigma} \cong \widetilde{X} - \widetilde{\Sigma} \overset{\text{def}}{=} \widetilde{W}.$$

Now, we recall Bott's theorem on the particular situation of $\mathbb{P}^n$ ([3, theorems IV and IV']):

$$H^p(\mathbb{P}^n, \Omega^q) = \begin{cases} 0 & \text{for } p \neq q, \\ \mathbb{C} & \text{for } p = q \leq n, \end{cases} \tag{4.1}$$

and in particular for $n = 4$: $e(\mathbb{P}^4) = 1 + x\overline{x} + x^2\overline{x}^2 + x^3\overline{x}^3 + x^4\overline{x}^4$. It follows inmediately that $\mathrm{Gr}_F^j H^n(\mathbb{P}^4) = H^{j,n-j}(\mathbb{P}^4)$ and the only non-zero graded part is when $\mathrm{coeff}_{e(\mathbb{P}^4)}(x^j\overline{x}^{n-j}) = 1$, hence

$$\mathrm{Gr}_F^2 H^4(\mathbb{P}^4) = H^{2,2} = \mathbb{C}. \tag{4.2}$$

Also,

$$e(\widehat{\mathbb{P}}^4) = e(\mathbb{P}^4) + e(\Sigma)(x\overline{x} + \cdots + (x\overline{x})^3)$$

using that $e(\Sigma) = m$ and substituting in the above formula:

$$h^{p,q}(\widehat{\mathbb{P}}^4) = \begin{cases} 0 & \text{if } p \neq q, \\ 1 & \text{if } p = q = 0, \\ m+1 & \text{if } 1 \leq p = q \leq 3, \\ 1 & \text{if } p = q = 4. \end{cases} \tag{4.3}$$

It follows that $h^{1,1}(\widehat{\mathbb{P}}^4) = h^{2,2}(\widehat{\mathbb{P}}^4) = h^{3,3}(\widehat{\mathbb{P}}^4) = m+1$. After these basic preliminaries, the main result in this section is the following proposition.

Proposition 4.2 *Let $X, \widetilde{X}$ and Y as above; then*

$$e(\widetilde{X}) = 1 + (m+1)x\overline{x} - ax^3 - (b-l)x^2\overline{x} - (b-l)x\overline{x}^2 - a\overline{x}^3 + (1+m)x^2\overline{x}^2 + x^3\overline{x}^3,$$

$$e(X) = 1 + (1-m)x\overline{x} - ax^3 - (b-l)x^2\overline{x} - (b-l)x\overline{x}^2 - a\overline{x}^3 + x^2\overline{x}^2 + x^3\overline{x}^3,$$

where $a = h^{3,0}(Y)$, $b = h^{2,1}(Y)$.

Proof Observe that $\widehat{\Sigma} = \cup_{x\in\Sigma} E_x$ and by cutting each E_x with $\widetilde{X}$, we obtain a quadric surface Q_x, hence $e(\widetilde{\Sigma}) = \Sigma_x e(Q_x)$, but each summand is equal to

$$e(\mathbb{P}^1 \times \mathbb{P}^1) = e(\mathbb{P}^1)^2 = 1 + 2x\overline{x} + x^2\overline{x}^2,$$

so

$$e(\widetilde{\Sigma}) = m(1 + 2x\overline{x} + x^2\overline{x}^2). \tag{4.4}$$

Moreover, $e^{p,q}(\widetilde{W}) = e^{p,q}(\widetilde{X}) - e^{p,q}(\widetilde{\Sigma})$ and

$$e^{p,q}(X) = e^{p,q}(W) + e^{p,q}(\Sigma) = e^{p,q}(\widetilde{W}) + e^{p,q}(\Sigma) = e^{p,q}(\widetilde{X}) - e^{p,q}(\widetilde{\Sigma}) + e^{p,q}(\Sigma).$$

Since $h^3(\widetilde{X}) = 2a + 2b - 2l$ (see Corollary 3.6), Lefschetz Hyperplane Theorem tell us that:

$$e(\widetilde{X}) = 1 + (m+1)x\overline{x} - ax^3 - (b-l)x^2\overline{x} - (b-l)x\overline{x}^2 - a\overline{x^3} + (1+m)x^2\overline{x}^2 + x^3\overline{x}^3. \tag{4.5}$$

Finally, $e(X) = e(\widetilde{X}) - (m + 2mx\overline{x} + mx^2\overline{x}^2) + m$. The result follows directly by substituting the value of $e(\widetilde{X})$ in equation (4.5). ∎

Using the Hodge numbers of the total transform $\widehat{\mathbb{P}}^4$ given by equation (4.3), we can conclude the following corollary.

Corollary 4.3 *In diagram (3.1), $d(A_1^3(X)) = \widetilde{d}(A_1^3(\widetilde{X})) = 0$; hence,* Coker$(d)$ = $\mathscr{H}_2^1 = A_2^4(X, \Sigma)$ *and* Coker$(\widetilde{d}) = \mathscr{H}_2(\widetilde{X}) = A_2^4(\widetilde{X})$.

Proof Since $\widetilde{X}$ is smooth, then the hodge numbers $e^{p,q}(\widetilde{X}) = (-1)^{p+q}h^{p,q}(\widetilde{X})$, in particular, $h^{2,0} = h^{0,2} = 0$ by the computation above. This implies that $H^0(\widetilde{X}, \widehat{\Omega}^2_{\widetilde{X}}) \subset H^0(\widetilde{X}, \Omega^2_{\widetilde{X}}) = 0$. Recall the exact sequence of residues in [12, Lemma 10.9(ii)]:

$$0 \longrightarrow \widehat{\Omega}^q_{\widetilde{\mathbb{P}}} \longrightarrow \widehat{\Omega}^q_{\widetilde{\mathbb{P}}}(1) \longrightarrow \widehat{\Omega}^{q-1}_{\widetilde{X}} \to 0$$

and its associated long sequence for $q = 3$:

$$0 \longrightarrow H^0(\widetilde{\mathbb{P}}, \widehat{\Omega}^3_{\widetilde{\mathbb{P}}}) \longrightarrow H^0\big(\widetilde{\mathbb{P}}, \widehat{\Omega}^3_{\widetilde{\mathbb{P}}}(1)\big) \longrightarrow H^0(\widetilde{X}, \widehat{\Omega}^2_{\widetilde{X}}) \longrightarrow \cdots.$$

Also, $H^0(\widetilde{\mathbb{P}}, \widehat{\Omega}^3_{\widetilde{\mathbb{P}}}) \subset H^0(\widetilde{\mathbb{P}}, \Omega^3_{\widetilde{\mathbb{P}}}) = 0$ (see equation (4.3)), since the last term for the above sequence is already zero so must be the middle term. In particular, $d(A_1^3(X)) \subset \widetilde{d}(A_1^3(\widetilde{X})) = 0$. ∎

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5 Mixed Hodge Structure of a Nodal 3-fold

Given a singular scheme X defined over $\mathbb{C}$, Guillen, Navarro, *et. al.* defined a *cubical hyperresolution* $X_\bullet$ of X (see[15, Exposé III, proposition 3.3]), which induces a spectral sequence

$$E_1^{p,q} = H^q(X_p, \mathbb{C}) \Longrightarrow H^{p+q}(X, \mathbb{C})$$

providing a natural Mixed Hodge Structure on $H^{p+q}(X, \mathbb{C})$ (we set $X_p \overset{\text{def}}{=} \sqcup_{|\alpha|=p+1} X_\alpha$).

In our situation, a cubical hyperresolution can be constructed from the following pullback diagram

$$\begin{array}{ccc} \Sigma \times_X \widetilde{X} & \longrightarrow & \widetilde{X} \\ \downarrow & & \downarrow \pi \\ \Sigma & \longrightarrow & X. \end{array}$$

Since $\Sigma \times_X \widetilde{X} \cong \widetilde{\Sigma}$, the projection to the first factor gets identified with $\pi_|$, the restriction of π to $\widetilde{\Sigma}$, while the projection to the second factor gets identified with the natural inclusion $i: \widetilde{\Sigma} \hookrightarrow \widetilde{X}$, yielding the cubical hyperresolution

$$X_1 \underset{\pi_|}{\overset{i}{\rightrightarrows}} X_0 \longrightarrow X,$$

where $X_1 = \widetilde{\Sigma}$ and $X_0 = \widetilde{X} \sqcup \Sigma$. Therefore, $E_1^{0,q} = H^q(X_0, \mathbb{C})$, $E_1^{1,q} = H^q(X_1, \mathbb{C})$ and $E_1^{p,q} = 0$ for all $p \geq 2$. Clearly this spectral sequence degenerates at E_2, so we have

$$0 \to E_2^{1,2} \longrightarrow H^3(X, \mathbb{C}) \longrightarrow E_2^{0,3} \to 0,$$

where

$$E_2^{0,3} = \operatorname{Ker}\big(H^3(X_0, \mathbb{C}) \xrightarrow{\pi_|^* - i^*} H^3(X_1, \mathbb{C})\big)$$

and

$$E_2^{1,2} = H^2(X_1, \mathbb{C}) / \Big(\operatorname{Im}(H^2(X_0, \mathbb{C}) \xrightarrow{\pi_|^* - i^*} H^2(X_1, \mathbb{C})\Big).$$

Since $H^3(X_0, \mathbb{C}) = H^3(\widetilde{X}, \mathbb{C})$ and $H^3(X_1, \mathbb{C}) = 0$, then $E_2^{0,3} = H^3(\widetilde{X}, \mathbb{C})$ is a pure Hodge structure of weight 3.

Similarly, $H^2(X_0, \mathbb{C}) = H^2(\widetilde{X}, \mathbb{C}) \cong \mathbb{C}^m$ and $H^2(X_1, \mathbb{C}) = H^2(\widetilde{\Sigma}, \mathbb{C}) \cong \mathbb{C}^{2m}$, so $E_2^{1,2} \cong \mathbb{C}^m$ is a pure Hodge structure of weight 2, and we recover the Clemens–Schmidt exact sequence

$$0 \longrightarrow W_2 H^3(X, \mathbb{C}) \longrightarrow H^3(X, \mathbb{C}) \longrightarrow H^3(\widetilde{X}, C) \longrightarrow 0 \tag{5.1}$$

with $W_2 H^3(X, \mathbb{C}) = E_2^{1,2} \cong \mathbb{C}^m$, which is to be expected for a cubical hyperresolution, as pointed out in [20, Corollary 5.42].

Remember that, by virtue of Theorem 4.2, if Σ consists of m nodes, where precisely l of them are in general position (and assuming they are also in homologically good position), one has

(a) $\dim H^3(\widetilde{X}, \mathbb{C}) = 2a + 2b - 2l$

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(b) $H^3(\widetilde{X}, \mathbb{C}) \cong \oplus \widetilde{H}^{i,j}$, where $\dim \widetilde{H}^{0,3} = \dim \widetilde{H}^{3,0} = a$ and $\dim \widetilde{H}^{1,2} = \dim \widetilde{H}^{2,1} = b - l$.

(c) $\dim H^3(X, \mathbb{C}) = 2a + 2b - 2l + m$.

Moreover, in this situation we have the following proposition.

Proposition 5.1

$$\operatorname{Gr}_F{}^k H^3(X, \mathbb{C}) = \begin{cases} \mathbb{C}^a & \textit{if } k = 0, 3, \\ \mathbb{C}^{b-l+m} & \textit{if } k = 1, \\ \mathbb{C}^{b-l} & \textit{if } k = 2\ . \end{cases}$$

Observe that $m - l$ is precisely the failure of Σ to impose independent conditions on homogeneous polynomials of degree 5 (see Remark 3.7).

Example 5.2 Let X be the quintic threefold on $\mathbb{P}^5$ defined by the equations $p_1 = 0$ and $4p_5 - 15p_2p_3 = 0$, where $p_k = \sum_0^5 x_i^k$ is the k-th power symmetric function. Then the singular locus of X consists of precisely 100 nodes that are the orbits of $(1 : -1 : 1 : -1 : 1 : -1)$ and of $(1 : -1 : 1 : -1 : z : -z)$ under the symmetric group on six letters S_6, where $7z^2 + 16 = 0$. For this quintic threefold, using the kernel extension `PLURAL` of `SINGULAR` 2-0-6 (see [7]), we have written a program that allows us to conclude that the 100 nodes impose only 86 conditions on the space of quintics passing through them, so in this case, $l = 86 < 100 = m$. It is not difficult to see that this quintic threefold is actually a singular Calabi–Yau threefold on $H = V(p_1) \cong \mathbb{P}^4$.

As Candelas, de la Ossa, ét al. have shown in [4], $\dim H^3(Y, \mathbb{C}) = 204$ for a smooth quintic threefold on $\mathbb{P}^4$, and if, additionally, the nodes in general position form an hg set, then Corollary 5.1 can be written as:

$$\operatorname{Gr}_F{}^k H^3(X, \mathbb{C}) = \begin{cases} \mathbb{C} & \text{if } k = 0, 3, \\ \mathbb{C}^{115} & \text{if } k = 1, \\ \mathbb{C}^{15} & \text{if } k = 2. \end{cases}$$

and $\dim W_2 H^3(X, \mathbb{C}) = 100$.

Recall that we have a commutative diagram of long exact sequences with compact support:

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & H_c^i(U) & \longrightarrow & H^i(X) & \longrightarrow & H^i(\Sigma) & \longrightarrow & H_c^{i+1}(U) & \longrightarrow & \dots \\ & & \cong \big\downarrow \pi^* & & \big\downarrow \pi^* & & \big\downarrow \pi^* & & \cong \big\downarrow \pi^* & & \\ \dots & \longrightarrow & H_c^i(\widetilde{U}) & \longrightarrow & H^i(\widetilde{X}) & \longrightarrow & H^i(\widetilde{\Sigma}) & \longrightarrow & H_c^{i+1}(\widetilde{U}) & \longrightarrow & \dots\ . \end{array}$$

Since Σ is zero dimensional, $H^i(\Sigma) = 0$ for all $i > 0$. In particular,

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & H_c^4(U) & \xrightarrow{\cong} & H^4(X) & \longrightarrow & 0 & \longrightarrow & H_c^5(U) & \xrightarrow{\cong} & H^5(X) & \longrightarrow & 0 \\ & & \cong \big\downarrow \pi^* & & \big\downarrow \pi^* & & \big\downarrow \pi^* & & \cong \big\downarrow \pi^* & & \big\downarrow \pi^* & & \\ 0 & \longrightarrow & H_c^4(\widetilde{U}) & \longrightarrow & H^4(\widetilde{X}) & \longrightarrow & H^4(\widetilde{\Sigma}) & \longrightarrow & H_c^5(\widetilde{U}) & \longrightarrow & H^5(\widetilde{X}) & \longrightarrow & 0 \end{array}$$

is exact and commutative. From the generalized Hodge numbers, equations (4.4) and (4.5), we have $H^i(\widetilde{\Sigma}) = 0$ for $i > 4$, $H^3(\widetilde{\Sigma}) = 0$, $H^4(\widetilde{\Sigma}) \cong \mathbb{C}^m$, $H^4(\widetilde{X}) \cong \mathbb{C}^{m+1}$, and $H^5(\widetilde{X}) = 0$. Therefore, the second row of the above diagram simplifies to:

$$0 \longrightarrow \mathbb{C}^\beta \longrightarrow \mathbb{C}^{m+1} \longrightarrow \mathbb{C}^m \longrightarrow \mathbb{C}^r \to 0,$$

where β is the fourth Betti number of X. Applying the Euler characteristic to this exact sequence: $\beta-(m+1)+m-r = 0$ hence $r = \beta-1$. It follows that $H^5(X) \cong H^5_c(\widetilde{U}) \cong \mathbb{C}^{\beta-1}$ for some $\beta \leq m+1$.

On the other hand, Clemens [5] and later Werner [24] introduced the following Mayer–Vietoris type exact sequence:

$$0 \longrightarrow H_4(Y) \longrightarrow H_4(X) \xrightarrow{k} \mathcal{R} \xrightarrow{b} H_3(Y) \xrightarrow{\gamma} H_3(X) \longrightarrow 0,$$

where Y is a smooth threefold of the same degree as X and $\mathcal{R}$ is a free $\mathbb{Z}$-module of rank $m = |\Sigma|$. This allows us to compute the *defect* of X as $\delta \overset{\text{def}}{=} \operatorname{rank}(\operatorname{Im}(k))$. As a consequence of their definition, they show that $\beta_2(X) = 1$ and $\delta = \beta - 1$.

Corollary 5.3 *If the l double points are in general position (resp. form a hg set), then $\delta \geq 2(m-l)$ (resp. $\delta = 2(m-l)$). In particular, $l \geq \frac{m}{2}$.*

Proof If the l points are in general position, then $\dim H^3(\widetilde{X}) \geq 2(a+b-l)$, and by equation (5.1), $\dim H^3(X) = m + \dim H^3(\widetilde{X}) \geq 2(a+b-l)+m$, but $\operatorname{rank}(\operatorname{Im}(b)) = h^3(Y) - h^3(X) = 2(a+b) - h^3(X) \leq 2(a+b) - 2(a+b-l) - m = 2l - m$ (inequality is an equality if all the double points form a hg set). Hence, $\delta = \operatorname{rank}(\operatorname{Ker}(k)) = m - \operatorname{rank}(\operatorname{Im}(b)) \geq m - (2l-m) = 2(m-l)$ and $m+1 \geq \beta = \delta + 1 \geq 2(m-l)+1$. Therefore, $m \leq 2l$. ∎

In order to find an exact relation between δ and the failure of Σ to impose linearly independent conditions on polinomials of degree $2d-5$ (compare with Remark 3.7), we will use and prove the following proposition.

Proposition 5.4 *If the l double points are in (algebraic) general position and form an hg set, then*

$$\delta = m - l + a + b - \binom{2d-1}{4}.$$

Proof By [24, Satz Kap. IV p. 27], $\delta = m - \binom{2d-1}{4} + \dim(A_2^4(X,\Sigma))$. By Corollary 4.3, the last term $A_2^4(X,\Sigma) = \mathscr{H}_2^1$ and by the assumption on Σ, the dimension of the latter is equal to $a+b-l$. ∎

Remark 5.5 The significance of the last corollary is that the difference between the defect and the failure of Σ to impose conditions on polynomials of degree $2d-5$ is equal to $a+b-\binom{2d-1}{4}$, which depends only on the degree of X and the dimensions $h^{3,0}, h^{2,1}$ of a smooth Y of the same degree as X.

6 Equisingular Families

Let

$$\begin{array}{ccc} \overline{\mathcal{X}} & \hookrightarrow & \mathbb{P}^4 \times \mathbb{P}^1 \\ & {}_{f}\searrow & \downarrow \\ & & \mathbb{P}^1 \end{array}$$

be a Lefschetz pencil of hypersurfaces on $\mathbb{P}^4$, where the vertical arrow is the projection on the second factor, and assume that there is a maximal non-empty open subset $B \subset \mathbb{P}^1$ over which the family

$$\begin{array}{ccc} \mathcal{X} = f^{-1}(B) & \hookrightarrow & \overline{\mathcal{X}} \\ f\downarrow & & \downarrow \widetilde{f} \\ B & \hookrightarrow & \mathbb{P}^1 \end{array}$$

is real analytically trivial and such that the singular locus Σ_t of every fiber X_t consists of exactly m nodes. Then the higher direct image $\mathbb{H} = R^3 f_* \mathbb{C}$ is a local system, with fiber $H^3(X_t, \mathbb{C})$ admitting a MHS.

For a fixed $t \in B$, let $\widehat{\mathbb{P}^4}$ be the blow up of $\mathbb{P}^4$ along Σ_t and $\widetilde{X}_t$ be the strict transform of X_t. Further, let $\widehat{\Sigma}_t$ be the inverse image of Σ_t (*i.e.*, the disjoint union of the exceptional divisors along the m nodes) and $\widetilde{\Sigma}_t = \widehat{\Sigma}_t \cap \widetilde{X}_t$. Since the multiplicity of every point in Σ_t is 2, $\widetilde{X}_t$ is a projective, non-singular variety, and we have a diagram

$$\begin{array}{ccccc} \Sigma_t & \hookrightarrow & X_t & \hookrightarrow & \mathbb{P}_4 \\ \uparrow & & \uparrow \pi & & \pi \uparrow \\ \widetilde{\Sigma}_t & \hookrightarrow & \widetilde{X}_t & \hookrightarrow & \widehat{\mathbb{P}}_4 \end{array}$$

Let $\widetilde{\mathcal{X}} \overset{\widetilde{f}}{\to} B$ be the smooth family formed by the union of $\widetilde{X}_t$ along B (also a Lefschetz pencil). Then the higher direct image $\widetilde{\mathbb{H}} = R^3 \widetilde{f}_* \mathbb{C}$ is a VHS on B, in particular we have a GM-connection

$$\widetilde{\nabla}^{GM} \colon \widetilde{\mathcal{H}}^3 \longrightarrow \widetilde{\mathcal{H}}^3 \otimes \Omega^1_B,$$

where $\widetilde{\mathcal{H}}^3 \overset{\text{def}}{=} R^3 \widetilde{f}_* \mathbb{C} \otimes \mathcal{O}_B$. As seen in the introduction, at every point $p \in \mathbb{P}^1 \backslash B$, there exists a limit MHS $(W_\bullet, F\bullet)$, as well as an extension of the monodromy operator T, inducing the weight filtration $W_\bullet$, and such that the corresponding nilpotent operator $N = \log(T_u)$ has nilpotence degree ≤ 4.

Example 6.1 It is not difficult to see, using the above description and the notation of Proposition 5.1, that for a smooth quintic threefold $X \subset \mathbb{P}^4$, one has $a = 1$ and $b = 101$ (see [4]). Moreover, if X_t is the smooth family of quintic threefolds in $\mathbb{P}^4$ given by $x^5 + y^5 + z^5 + w^5 + u^5 - 5txyzwu$, it has been shown by Candelas ét. al. that the GM-connection induces a maximal unipotent map on $H^3(X_t, \mathbb{C})$, for any t, whose nilpotent part N satisfies $N(H^{p,3-p}) \subset H^{p-1,3-p+1}$ for $0 \leq p \leq 3$ with $N^3 \neq 0$

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but $N^4 = 0$. In particular, one has a splitting of the Hodge structure:

$$H^3(X_t, \mathbb{C}) = J \oplus_{i=1}^{100} V_i(-1),$$

where J is a Hodge structure of weight 3 and type $(1,1,1,1)$, and each $V_i(-1)$ is a Hodge structure of weight 3 and type $(0,1,1,0)$, associated with a Hodge structure V_i of weight one and type $(1,1)$ (see also [4] for the quintic family of threefolds in connection with mirror symmetry). Here, as usual, $V_i(-1) = V_i \otimes \mathbb{Z}(-1)$ and $\mathbb{Z}(-1)$ is the Tate-Hodge structure of weight 2.

Example 6.2 More generally, for a pencil of Calabi–Yau threefolds on $\mathbb{P}^4$, we have

$$\dim(\widetilde{H}^{0,3}) = \dim(\widetilde{H}^{3,0}) = 1 \quad \text{and} \quad k = \dim(\widetilde{H}^{1,2}) = \dim(\widetilde{H}^{2,1}),$$

hence $\widetilde{H} \cong \mathbb{C}^{2k+2}$.

In the same spirit as the example given in [4] and Example 5.2, keeping the notation there, for the case $n = 5$, consider the pencil of quintic hypersurfaces in $H = \mathbb{P}^4$ defined by

$$f_{(\alpha,\beta)} = \alpha p_5 - \frac{5(\alpha+\beta)}{6} p_2 p_3.$$

Let $\mathcal{M} \subset H \times \mathbb{P}^1$ be the corresponding incidence family. Clearly, for each $(\alpha : \beta)$, we have a quintic $\mathcal{M}_{(\alpha:\beta)} \subset \mathbb{P}^4$. This family has already been introduced and studied by Van Straten [23]. In *loc.cit* (see Theorem 2), he shows that $\mathcal{M}_{(\alpha:\beta)}$ is a singular variety for a general value of $(\alpha : \beta) = (\frac{\alpha}{\beta} : 1)$, except for the quintics associated with

$$q_1 = 25, q_2 = 1, q_3 = -3, q_4 = 0, q_5 = -2, q_6 = \infty.$$

For $t \in \mathbb{P}^1 - \{q_1, \ldots, q_6\}$, the singular locus, $\Sigma_t = \operatorname{Sing}(\mathcal{M}_t)$ consist of 100 nodes (compare with the bound $m \leq 101$ computed in Remark 3.7). In Example 5.2, we have seen that only 86 of these nodes are in general position; therefore, for a general member of this family, we have $\dim H^3(X_t) = 132$ and

$$Gr_F^k H^3(X_t, \mathbb{C}) \cong \begin{cases} \mathbb{C} & \text{if } k = 0, 3, \\ \mathbb{C}^{115} & \text{if } k = 1, \\ \mathbb{C}^{15} & \text{if } k = 2, \end{cases}$$

while $\dim H^3(\widetilde{X}_t) = 32$ and

$$\operatorname{rank} \widetilde{H}^{k,3-k} = \begin{cases} 1 & \text{if } k = 0, 3, \\ 15 & \text{if } k = 1, 2. \end{cases}$$

Before we study the LMHS of the VPHS given by $\widetilde{\mathcal{X}} \to B$, we introduce a very well known inductive method to calculate the monodromy weight filtration and advise the reader interested in the main result to skip to Proposition 6.5. For that, let m be an integer, $H_{\mathbb{Q}}$ be a $\mathbb{Q}$-vector space and $N\colon H_{\mathbb{Q}} \to H_{\mathbb{Q}}$ be a nilpotent endomorphism such that $N^{m+1} = 0$ but $N^m \neq 0$. Following Donagi (see [13, remark, p. 69]), we can introduce the following $\mathbb{Q}$-spaces for r, s positive integers satisfying $r \leq m, s \leq m+1$: $M_{r,s} = \operatorname{Ker} N^{m-r} \cap \operatorname{Im} N^s$. These spaces satisfy the following relations: $M_{0,s} \supset M_{1,s} \supset \cdots \supset M_{m,s} = 0$, and similarly, $M_{r,0} \supset M_{r,1} \supset \cdots \supset M_{r,m+1} = 0$. Observe that the nilpotent operator N admits a natural extension to $N\colon H_{\mathbb{C}} \to H_{\mathbb{C}}$.

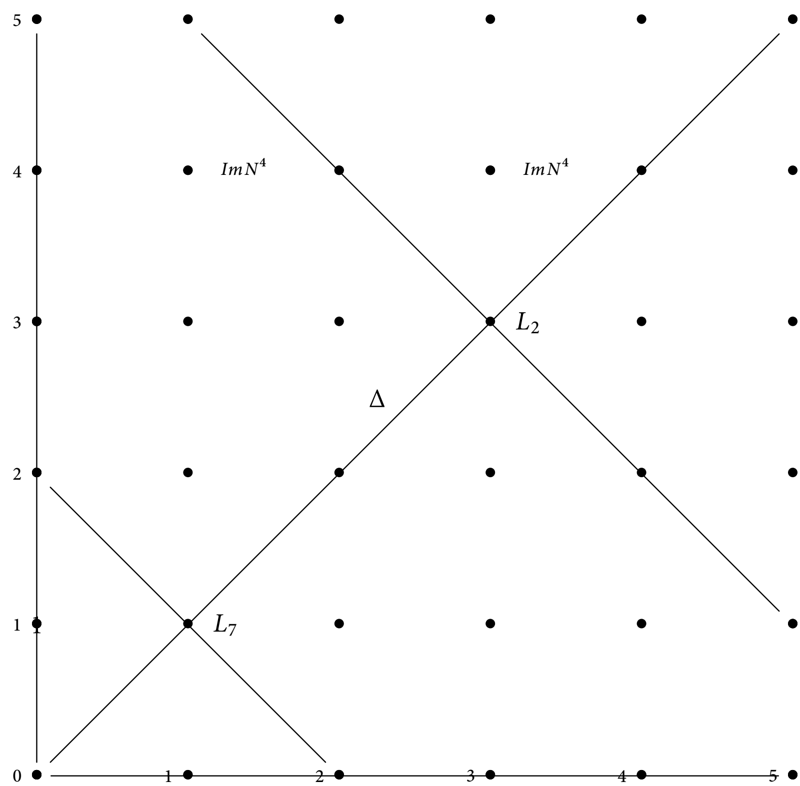
Consider an increasing filtration on $H_{\mathbb{Q}}$ given by the $\mathbb{Q}$-vector spaces:

$$W_q \overset{\text{def}}{=} < \Sigma_{2m-q-1=r+s} M_{r,s} >,$$

for $0 \leq q \leq 2m-1$, while $W_{2m} \overset{\text{def}}{=} H_{\mathbb{Q}}$.

Example 6.3 Since the sum in the formula is internal, one need not compute all terms in the formula above, and many redundant terms occur. If one represents the lattice of subspaces $M_{r,s}$ as integral points in the plane, observe first that $M_{m,j} = 0$ and $M_{j,m+1} = 0$ for all j, since $N^0 = id$ (the identity) and $N^{m+1} = 0$; therefore, the relevant terms lie in the integral points of a finite array of $(m-1) \times m$. Moreover, if $s - r \geq 1$, then $N^{m-r+s} = 0$, *i.e.*, $\operatorname{Im} N^s \subset \operatorname{Ker} N^{m-r}$ and $M_{r,s} = \operatorname{Im} N^s$.

The subspace W_q is the sum of the subspaces represented by the integral points lying on the line $L_q \overset{\text{def}}{=} \{(r,s) | r + s = 2m - q - 1\}$, and we want to know which of the corresponding subspaces $M_{r,s}$ actually contribute to the sum:



If $q = 2b$ for some integer b, then the intersection of the diagonal $\Delta \overset{\text{def}}{=} \{(r,s) \mid r = s\}$, and L_q is not an integral point, but $(m-b-1, m-b) \in L_q$. As observed above, since $m - b - (m-b-1) = 1$, $M_{m-b-1,m-b} = \operatorname{Im} N^{m-b}$. Moreover, all points $(r,s) \in L_q$ lying above the diagonal Δ satisfy $s \geq m - b > m - b - 1 \geq r$; therefore,

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$s - r \geq 1$ and $M_{r,s} = \operatorname{Im} N^s \subset \operatorname{Im} N^{m-b} = M_{m-b-1,m-b}$ for all such points $(r,s) \in L_q$; in particular, the corresponding $M_{r,s}$ do not contribute anything new to W_q.

If q is odd, then $2m - q - 1$ is even and L_q intersects the diagonal Δ at the integral point $(m - (q+1)/2, m - (q+1)/2)$. In this case, all points $(r,s) \in L_q$ above the diagonal Δ satisfy $s > m - (q+1)/2 > r$ and again $s - r \geq 1$, so $M_{r,s} = \operatorname{Im} N^s \subset \operatorname{Im} N^{m-(q+1)/2}$. Since $s + (q+1)/2 > m$ for all such points, one has $N^{(q+1)/2+s} = 0$, *i.e.*, $\operatorname{Im} N^s \subset \ker N^{(q+1)/2}$ as well; therefore, $M_{r,s} \subset M_{m-(q+1)/2, m-(q+1)/2}$, and these $M_{r,s}$ do not contribute anything new to W_q.

We can summmarize these calculations saying that the only subspaces $W_{r,s}$ that contribute something new to W_q are those subspaces for which $(r,s) \in L_q$ lies below Δ, on $L_q \cap \Delta$ (if q is odd) or inmediatly above Δ (if q is even). For instance, for $m = 3$, one has

$$W_0 = \langle N^{3,0} \rangle, \qquad W_1 = \langle N^{2,2} \rangle, \qquad W_2 = \langle N^{1,2} + N^{2,1} \rangle,$$
$$W_3 = \langle N^{0.2} + N^{1,1} \rangle, \qquad W_4 = \langle N^{1,0} + N^{0,1} \rangle, \qquad W_5 = \langle N^{0,0} \rangle.$$

As one can see, at most 9 different summands contribute to W_q's in this case. More generally, denoting by N_m the maximal number of *different* summands contributing to the W_q's, an elementary counting argument shows that for arbitrary m, this number equals the number of points I_m in the isosceles triangle of height $m - 1$ and base $m - 1$ bounded below by the diagonal, plus the number of odd integers in the set $\{1, 2, \ldots, 2m - 3\}$ plus one, giving $N_m = \frac{m(m+3)}{2}$. In particular, for $m \geq 1$,

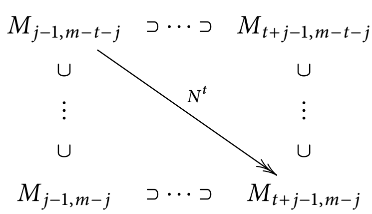
$$I_m = \frac{(m-1)^2}{2} \leq N_m \leq m^2.$$

This is already true for $m = 3$, as seen above. Also, $N_4 = 14$, $N_5 = 20$.

Lemma 6.4 *The filtration defined above satisfies Morrison's characterization of the weight filtration on* $H_{\mathbb{Q}}$ *associated with* N.[1] *(see [13, pp. 106–107]):*

(i) $N(W_k) \subset W_{k-2}$,
(ii) $W_{m-t}/W_{m-t-1} = \operatorname{Im}(N^t|_{W_{m+t}/W_{m-t-1}})$,
(iii) $W_{m+t-1}/W_{m-t-1} = \operatorname{Ker}(N^t|_{W_{m+t}/W_{m-t-1}})$.

Proof It is helpful to visualize the action of N^t on the Lattice formed by the $M_{r,s}$ as follows:



(i) Obviously $N(M_{a,b}) = N(\operatorname{Ker} N^{m-a} \cap \operatorname{Im} N^b) \subset \operatorname{Ker} N^{m-(a+1)} \cap \operatorname{Im} N^{b+1} = M_{a+1,b+1}$.

[1] The formula for W_k given in [13, p. 69] is incomplete. The procedure there described is correct; however, the formula has a misprint. Here, we include a more accurate formula in both cases for lack of another suitable reference.

(ii) Observe that $W_{m-t} = M_{t-1,m} + \cdots + M_{m,t-1}$, since $N^{m+1} \equiv 0$ and $\operatorname{Ker} N^0 = 0$. We claim that $N^t(M_{r,s}) = M_{r+t,s+t}$. Indeed, the first inclusion is the content of the proof above for (i). For the equality, let $x \in \operatorname{Ker} N^{m-r-t}$ and $x = \operatorname{Im}(N^{s+t}(y))$. Let $z = N^s(y)$; therefore, $N^t(z) = N^{s+t}(y) = x$ and $0 = N^{m-r-t}(x) = N^{m-r-t}(N^{s+t}(y)) = N^{m-r}(N^s(y)) = N^{m-r}(z)$, *i.e.*, $z \in M_{r,s}$. It follows that $N^t(W_{m+t}) = W_{m-t}$.

(iii) From (ii) it follows that $W_{m+t-1}/W_{m-t-1} \subset \operatorname{Ker}(N^t|_{W_{m+t}/W_{m-t-1}})$. For the other inclusion, it is enough to prove that $(N^t)^{-1}(W_{m-t-1}) \cap W_{m+t} \subset W_{m+t-1}$.

Indeed, since $W_{m-t-1} = \sum_a M_{m+t-a,a}$ with $0 \le m+t-a \le m$, it is enough to prove that $(N^t)^{-1}(M_{m+t-a,a}) \cap W_{m+t} \subset W_{m+t-1}$. Observe that $0 \le m+t-a \le m$ if and only if $a-m \le t \le a$.

If $z \in (N^t)^{-1}(M_{m+t-a,a}) \cap W_{m+t}$, then

$$N^t(z) \in \operatorname{Ker} N^{m-(m+t-a)} \cap \operatorname{Im} N^a = \operatorname{Ker} N^{a-t} \cap \operatorname{Im} N^a.$$

Hence, $N^t(z) = N^a(w)$ or $N^t(z - N^{a-t}(w)) = 0$ for some w, *i.e.*,

$$z \in \operatorname{Ker} N^t + \operatorname{Im} N^{a-t}.$$

But $N^t(z) \in \operatorname{Ker} N^{a-t}$; then $0 = N^{a-t}(N^t(z)) = N^a(z)$, *i.e.*, $z \in \operatorname{Ker} N^a$. Therefore,

$$z \in (\operatorname{Ker} N^t + \operatorname{Im} N^{a-t}) \cap \operatorname{Ker} N^a = \operatorname{Ker} N^t + \operatorname{Ker} N^a \cap \operatorname{Im} N^{a-t};$$

hence, $z \in M_{m-t,0} + M_{m-a,a-t} \subset W_{m+t-1}$. ∎

In order to compute the limit Hodge structure for the VPHS $\widetilde{\mathcal{H}}^3$ at a point $p \in \mathbb{P}^1\backslash B$, we apply the formula obtained in Example 6.3 and the fact that $N^r : \mathrm{Gr}^W_{n+r} \simeq \mathrm{Gr}^W_{n-r}$ for all r for the weight filtration centered at n (we say it is *symmetric* at n). Let us define $n_i \overset{\text{def}}{=} \dim(\operatorname{Im} N_i)$ and $m_i \overset{\text{def}}{=} \dim(\operatorname{Ker} N_i)$ for $i \in \{1,2,3\}$. To simplify the notation of the proof of the following proposition, we omit the tildes in the components $H^{i,j}$ of the local system $\widetilde{\mathcal{H}}^3$.

Proposition 6.5 *The limit Hodge filtration $(W_\cdot, F^\cdot_\infty)$ for the family $\widetilde{\mathcal{X}}$ can be described as follows, where $N_{i,j} \overset{\text{def}}{=} N_i \circ N_j$ and $o \overset{\text{def}}{=} \dim(M_{2,1})$:*

(i) *$N = 0$ and it is pure of weight three.*
(ii) *$N \neq 0, N^2 = 0$ there are two cases:*
  (a) *$N_1 \neq 0, N_3 \neq 0$ such that $N_{2,1} = N_{3,2} = 0$,*
  (b) *$N_1 = N_3 = 0$ with $N_2 \neq 0$.*

For these cases, the weight filtration centered at three is:

$$\mathrm{Gr}^W_i(H_{\mathbb{Q}}) = \begin{cases} 0 & i = 0,1,5,6 \\ \mathbb{C}^{n_2+2} & i = 2,4 \quad \text{(a)}, \\ \mathbb{C}^{n_2} & i = 2,4 \quad \text{(b)}, \\ \mathbb{C}^{2(m_2-1)} & i = 3 \quad \text{(a)}, \\ \mathbb{C}^{2(m_2+1)} & i = 3 \quad \text{(b)}. \end{cases}$$

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(iii) $N^2 \neq 0, N^3 = 0.$

$$\mathrm{Gr}_i^W(H_{\mathbb{Q}}) = \begin{cases} 0 & i = 0,6 \\ \mathbb{C}^2 & i = 1,5 \\ \mathbb{C}^{o-2} & i = 2,4 \\ \mathbb{C}^{2(k+1-o)} & i = 3. \end{cases}$$

Proof (i) $N = 0$; the weight filtration centered at three is: $W_i = 0$ for $i \in \{0,1,2\}$, and otherwise, $W_j = H_{\mathbb{Q}}$. Therefore, $\mathrm{Gr}_3(H_{\mathbb{Q}}) = H_{\mathbb{Q}}$.

(ii) Assume that $N \neq 0, N^2 = 0$. We have the following general decompositions for $\{\mathrm{Ker}\, N^i, \mathrm{Im}\, N^i\}_{i=1,2}$:

$$\mathrm{Ker}\, N = \oplus_{i=1}^3 \mathrm{Ker}\, N_i \oplus H^{0,3}, \qquad \mathrm{Im}\, N = \oplus_{i=1}^3 \mathrm{Im}\, N_i,$$
$$\mathrm{Ker}\, N^2 = \mathrm{Ker}\, N_{2,1} \oplus \mathrm{Ker}\, N_{3,2} \oplus H^{1,2} \oplus H^{0,3}, \qquad \mathrm{Im}\, N^2 = \mathrm{Im}\, N_{2,1} \oplus \mathrm{Im}\, N_{3,2},$$

and in both cases the weight filtration is given as

$$W_0 = W_1 = 0, \quad W_2 = \mathrm{Im}\, N, \quad W_3 = \mathrm{Ker}\, N,$$
$$W_4 = \mathrm{Ker}\, N^2 = W_5 = W_6 = H_{\mathbb{Q}}.$$

In particular we get $\mathrm{Gr}_i^W H_{\mathbb{Q}} = 0$ for $i \in \{0,1,5,6\}$ in both cases, as stated.

For the other graded groups, we have

(a) $N_1 \neq 0$ (and hence $N_3 \neq 0$ since the polarization is non-degenerate and the GM-connection is compatible with the metric induced by it) with $N_{2,1} = N_{3,2} = 0$. Hence the weight filtration simplifies further to:

$$0 = W_0 = W_1 \subset \mathrm{Im}\, N \subset \mathrm{Ker}\, N \subset W_4 = W_5 = W_6 = H_{\mathbb{Q}}$$

(b) $N_1 = N_3 = 0$ with $N_2 \neq 0$. In this case:

$$\mathrm{Im}\, N_2 = \mathrm{Im}\, N \subset \mathrm{Ker}\, N = H^{3,0} \oplus H^{0,3} \oplus H^{1,2} \oplus \mathrm{Ker}\, N_2.$$

Then for both cases above,

$$W_3 = \mathrm{Ker}\, N = \begin{cases} \mathrm{Ker}\, N_2 \oplus \mathrm{Ker}\, N_3 \oplus H^{0,3} & \text{(a)}, \\ H^{3,0} \oplus H^{0,3} \oplus H^{1,2} \oplus \mathrm{Ker}\, N_2 & \text{(b)}. \end{cases}$$

$$W_2 = \mathrm{Gr}_2^W H_{\mathbb{Q}} = \mathrm{Im}\, N = \begin{cases} \oplus_{i=1}^3 \mathrm{Im}\, N_i = \mathbb{C} \oplus \mathbb{C}^{n_2} \oplus \mathbb{C} & \text{(a)}, \\ \mathrm{Im}\, N_2 = \mathbb{C}^{n_2} & \text{(b)}. \end{cases}$$

Trivially,

$$\mathrm{Ker}\, N_2/\mathrm{Im}\, N_1 \simeq \mathbb{C}^{m_2-1} \simeq \mathrm{Ker}\, N_3/\mathrm{Im}\, N_2, \quad H^{1,2}/\mathrm{Im}\, N_2 \simeq \mathbb{C}^{m_2}.$$

Hence,

$$\mathrm{Gr}_3^W H_{\mathbb{Q}} = \begin{cases} \mathrm{Ker}\, N_2/\mathrm{Im}\, N_1 \oplus \mathrm{Ker}\, N_3/\mathrm{Im}\, N_2 \simeq \mathbb{C}^{2(m_2-1)} & \text{(a)}, \\ H^{3,0} \oplus H^{0,3} \oplus H^{1,2}/\mathrm{Im}\, N_2 \oplus \mathrm{Ker}\, N_2 \simeq \mathbb{C}^{2(m_2+1)} & \text{(b)}. \end{cases}$$

(iii) $N^2 \neq 0, N^3 = 0$ with $N_{2,1} \neq 0, N_{3,2} \neq 0$. The weight filtration is explicitly:

$$\begin{aligned}
W_0 &= 0,\\
W_1 &= \operatorname{Im} N_{2,1} + H^{0,3} \simeq \mathbb{C}^2,\\
W_2 &= \operatorname{Im} N \cap \operatorname{Ker} N \simeq \mathbb{C}^o \quad (\text{note}: M_{1,2} \subset M_{2,1}),\\
W_3 &= \operatorname{Im} N + \operatorname{Ker} N \simeq \mathbb{C}^{2k+2-o},\\
W_4 &= \operatorname{Ker} N^2 = \operatorname{Ker} N_{3,2} + H^{1,2} + H^{3,0} = \mathbb{C}^{k-1} \oplus \mathbb{C}^k \oplus \mathbb{C}^1 = \mathbb{C}^{2k},\\
W_5 &= W_6 = H_{\mathbb{Q}}.
\end{aligned}$$

From which

$$\operatorname{Gr}_k^W H_{\mathbb{Q}} = \begin{cases} 0 & \text{for } k = 0,\\ \mathbb{C}^2 & \text{for } k = 1,\\ \mathbb{C}^{o-2} & \text{for } k = 2,\\ \mathbb{C}^{2(k+1-o)} & \text{for } k = 3. \end{cases}$$ ∎

We return to the study of the VMHS for the family $\mathcal{X}$ over B considered in the introduction.

By assumption, the family

$$\mathcal{X} \xrightarrow{f} B \subset \mathbb{P}^1$$

is real analytically trivial, *i.e.*, the sheaf $R^3 f_* \mathbb{C}$ is a local system on B. Additionally, by the RH-correspondence, there exists a GM-connection $\mathcal{H}^3 \xrightarrow{\nabla^{GM}} \mathcal{H}^3 \otimes \Omega_B^1$. Moreover, the weight filtration on the fibers fits together to form a subbundle $\mathcal{W}_2 R^3 f_* \mathbb{Q} \subset R^3 f_* \mathbb{Q}$, and we have a short exact sequence (see also equation (5.1)):

$$0 \longrightarrow \mathcal{W}_2 R^3 f_* \mathbb{Q} \longrightarrow R^3 f_* \mathbb{Q} \xrightarrow{\pi^*} R^3 \widetilde{f}_* \mathbb{Q} \longrightarrow 0.$$

A trivialization for $R^3 f_* \mathbb{C}$ induces a trivialization for $\mathcal{W}_2 R^3 f_* \mathbb{C}$, and so the action of the monodromy on $R^3 f_* \mathbb{C}$ is compatible with the action of the monodromy on $\mathcal{W}_2 R^3 f_* \mathbb{C}$; in particular, the GM-connection on $\mathcal{W}_2 R^3 f_* \mathbb{C} \otimes \mathcal{O}_B = \mathcal{W}_2 \mathcal{H}^3$ is just the restriction of ∇^{GM} on $\mathcal{H}^3$ to $\mathcal{W}_2 \mathcal{H}^3$, and by passing to the quotient, the short exact sequence above induces a connection $\overline{\nabla}^{GM}$ on $\widetilde{\mathcal{H}}^3$ with flat sections $R^3 \widetilde{f}_* \mathbb{C}$. By the uniqueness of the GM-connection (see [8, proposition 2.16]), this connection is none other than $\widetilde{\nabla}$ on $\widetilde{\mathcal{H}}^3$, *i.e.*, we have a short exact sequence that is compatible with the GM-connection:

(6.1)
$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{W}_2 \mathcal{H}^3 & \longrightarrow & \mathcal{H}^3 & \xrightarrow{\widetilde{\pi}^*} & \widetilde{\mathcal{H}}^3 & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle \nabla^{GM}} & & \downarrow{\scriptstyle \nabla^{GM}} & & \downarrow{\scriptstyle \widetilde{\nabla}} & & \\
0 & \longrightarrow & \mathcal{W}_2 \mathcal{H}^3 \otimes \Omega_B^1 & \longrightarrow & \mathcal{H}^3 \otimes \Omega_B^1 & \xrightarrow{\widetilde{\pi}^* \otimes id} & \widetilde{\mathcal{H}}^3 \otimes \Omega_B^1 & \longrightarrow & 0
\end{array}$$

Proposition 6.6 $(\mathcal{H}^3, \nabla^{GM})$ *is a VMHS.*

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Proof Observe that the Hodge filtrations are compatible, so we have a commutative diagram with exact arrows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & \mathcal{F}^3\mathcal{H}^3 & \xrightarrow{\widetilde{\pi}^*} & \mathcal{F}^3\widetilde{\mathcal{H}}^3 & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & \mathcal{F}^2\mathcal{H}^3 & \xrightarrow{\widetilde{\pi}^*} & \mathcal{F}^2\widetilde{\mathcal{H}}^3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F}^1 W_2\mathcal{H}^3 & \longrightarrow & \mathcal{F}^1\mathcal{H}^3 & \xrightarrow{\widetilde{\pi}^*} & \mathcal{F}^1\widetilde{\mathcal{H}}^3 & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & W_2\mathcal{H}^3 & \longrightarrow & \mathcal{H}^3 & \xrightarrow{\widetilde{\pi}^*} & \widetilde{\mathcal{H}}^3 & \longrightarrow & 0,
\end{array}$$

where all the rows are exact. In particular, $\nabla^{GM}(\mathcal{F}^p\mathcal{H}^3) \subset \mathcal{F}^{p-1}\mathcal{H}^3 \otimes \Omega^1_B$ by the commutativity of diagram (6.1); hence, it becomes a VMHS. ∎

Since f is quasi-projective, this VMHS is in fact graded-polarizable; indeed, this is a *geometric variation of mixed Hodge structure.*

We claim the following corollary.

Corollary 6.7 *We have that f is a geometric VMHS and an admissible variation of Hodge structure in the sense of Steenbrink–Zucker (see [20, Theorem 14.51] and [22]).*

A desingularization of the family produces a VPHS $\widetilde{\mathcal{H}}^3$ whose limit MHS can be described as in Proposition 6.5.

Denote by $\mathbb{M}_l(\mathbb{C})$ the set of l by l matrices over $\mathbb{C}$ and denote by $J(l) \in \mathbb{M}_l(\mathbb{C})$ the Jordan matrix with entries

$$J(l)_{r,s} = \begin{cases} 1 & \text{for } r = s+1, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\operatorname{rank}\operatorname{Ker}(J(l)^i) = i$ for $i \le l$, $\operatorname{rank}\operatorname{Ker}(J(l)^{i+1}) - \operatorname{rank}\operatorname{Ker}(J(l)^i) = 1\ \forall i < l$. The Jordan form $\mathbf{J}(A)$ of a nilpotent matrix $A \in M_n(\mathbb{C})$ is written as a direct sum of the corresponding Jordan block matrices. We call such a direct sum of Jordan block matrices simply a *Jordan matrix* of a Jordan form. If a Jordan block matrix $J(m)$ appears with multiplicity r, we denote it by $J(m)^r$.

Assume we have a VHS of type $(1, k, k, 1)$. Keeping the notation above for T, T_u and N, if $N \neq 0$ and $k \ge 2$, then the Jordan matrix of N is one of the following types:

- Type (1): $J(4) \oplus J(2)^s$ with $s \le k-1$,
- Type (2): $J(3)^2 \oplus J(2)^s$ with $s \le k-2$,
- Type (3): $J(2)^s$ where $s \le k+1$,
- Type (4): $J(3) \oplus J(2)^s$ with $s \le [\frac{2k-1}{2}]$.

Proposition 6.8 *The Jordan canonical form of N is of type* (1), (2), *or* (3).

Proof (1) A type (3) Jordan matrix decomposition implies that there are at most $k+1$ two by two blocks. This implies that $N^2 = 0$.

(2) A type (1) Jordan matrix decomposition corresponds to the maximal unipotent case, which is known to occur for instance for the family of [4].

(3) If $N^3 = 0$ but $N^2 \neq 0$, we know from linear algebra that all Jordan blocks are of size 3, 2, or 1, which correspond to either type (2) or type (4).

A type (4) Jordan Matrix decomposition is not possible. For that, recall the abstract situation of Example 6.2, namely, we have the following lemma.

Lemma 6.9 *Recall the notation of Proposition 6.5:*

$$H^{3,0} \xrightarrow{N_1} H^{2,1} \xrightarrow{N_2} H^{1,2} \xrightarrow{N_3} H^{0,3},$$

where $H^{3,0} \simeq H^{0,3} \simeq \mathbb{C}, H^{2,1} \simeq H^{1,2} \simeq \mathbb{C}^k$; then N_1 is one-to-one $\Leftrightarrow$ N_3 is surjective.

Proof of the lemma. The polarization Q is flat with respect to the connection N. ■

(4) Assume $N^3 \equiv 0$ but there exist a three-dimensional N-cyclic space

$$W_0 = < w, N(w), N^2(w) > .$$

Without loss of generality, assume either $w \in H^{3,0}$ or $w \in H^{2,1}$. Indeed, write $w = v_0 + v_1 + v_2 + v_3$ with $v_j \in H^{k-j,j}$. Then

$$N(w) = N(v_0) + N(v_1) + N(v_2),$$
$$N^2(w) = N^2(v_0) + N^2(v_1),$$

since $N(v_3) = N^2(v_2) = 0$.

If $N^2(v_0) \neq 0$, then $< v_0, N(v_0), N^2(v_0) >$ is a three-dimensional N-cyclic space. On the other hand, if $N^2(v_1) \neq 0$, $< v_1, N(v_1), N^2(v_1) >$ is a three-dimensional N-cyclic space.

(a) If $w = v_0 \in H^{3,0}$, since Q is non-degenerate there exist a $u \in H^{2,1}\setminus\{0\}$ such that $Q(u, N^2(w)) = 1$ and because of Q-flatness of the VHS with respect to ∇^{GM}:

$$Q(N(u), N(w)) + Q(u, N^2(w)) = 0;$$

therefore, $Q(N(u), N(w)) = -Q(u, N^2(w)) = -1$ thus $N(u) \neq 0$.
Similarly, $Q(N^2(u), w) = -Q(N(u), N(w)) = 1$ and $N^2(u) \neq 0$ either; therefore, $W_1 :=< u, N(u), N^2(u) >$ is a another three-dimensional N-cyclic space and $W_0 \cap W_1 = 0$.

(b) If $w = v_1 \in H^{2,1}$, then $N^2(w) \in H^{0,3}\setminus\{0\}$, and because of the non-singularity of Q, there exist a $u \in H^{3,0}\setminus\{0\}$ such that $Q(u, N^2 w) = 1$. As before, we will have

$$Q(N(u), N(w)) + Q(u, N^2(w)) = 0;$$

therefore, $Q(N(u), N(w)) = -Q(u, N^2(w)) = -1$ thus $N(u) \neq 0$ and again, $Q(N^2(u), w) = -Q(N(u), N(w)) = 1$ and $N^2(u) \neq 0$ either; therefore, $W_1 := < u, N(u), N^2(u) >$ is a another three-dimensional N-cyclic space and $W_0 \cap W_1 = 0$. ■

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