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# Saint-Venant's principle in blow-up for higher-order quasilinear parabolic equations

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We prove localization estimates for general $2m$th-order quasilinear parabolic equations with boundary data blowing up in finite time, as $t \to T^-$. The analysis is based on energy estimates obtained from a system of functional inequalities expressing a version of Saint-Venant's principle from the theory of elasticity. We consider a special class of parabolic operators including those having fixed orders of algebraic homogenuity $p > 0$. This class includes the second-order heat equation and linear $2m$th-order parabolic equations ($p = 1$), as well as many other higher-order quasilinear ones with $p \neq 1$. Such homogeneous equations can be invariant under a group of scaling transformations, but the corresponding least-localized regional blow-up regimes are not group invariant and exhibit typical exponential singularities $\sim \mathrm{e}^{(T-t)^{-\gamma}} \to \infty$ as $t \to T^-$, with the optimal constant $\gamma = 1/[m(p+1)-1] > 0$. For some particular equations, we study the asymptotic blow-up behaviour described by perturbed first-order Hamilton–Jacobi equations, which shows that general estimates of exponential type are sharp.

## 1. Introduction: statement of the problem, blow-up localization, Saint-Venant's principle and energy estimates

Saint-Venant's principle, formulated in the theory of linear elasticity in the middle of the nineteenth century, led to the concept of energy estimates, and in the second half of the twentieth century became a fundamental tool of the general theory of linear and nonlinear partial differential equations (PDEs). Using such energy estimates, several important problems on existence, uniqueness and asymptotic properties of solutions to different classes of PDEs were solved. The main feature of the method of Saint-Venant's principle consists in integral estimates on solutions over suitable families of subdomains in the space of independent variables. It is well known that sharp energy estimates are of principal importance in nonlinear equations with singularities and measurable coefficients admitting generalized


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or weak solutions only. In applications, choosing suitable continuous variables or, if necessary, discrete partitions, the PDE under consideration generates a system of differential-functional inequalities for energy functionals. The type of duality between PDEs and corresponding systems of functional inequalities then plays a key role.

In this paper we present an application of Saint-Venant's principle to localization blow-up phenomena for a class of quasilinear $2m$th-order parabolic equations. The study of such singular blow-up processes needs delicate estimates near finite blow-up time. This leads to new systems of functional inequalities, where special choices of infinite partitions in the independent variables are necessary.

We begin with the mathematical statement of the basic problem. A mechanical and physical basement of blow-up singularity formation problems and a survey of results on energy estimates representing Saint-Venant's principle will be presented next.

### 1.1. Statement of the problem

Without loss of generality, we formulate the basic example posed in a simple geometry. Let $\Omega = \{|x| > 1\}$ be the open complement of the unit ball $B = \{|x| < 1\}$ in $\mathbb{R}^N$. In the cylindrical domain $Q = \Omega \times (0,T)$, we consider the Cauchy–Dirichlet problem for a general quasilinear $2m$th-order parabolic equation (here, $m \geqslant 1$ is integer and $q > 0$ is a fixed exponent),

$$(|u|^{q-1}u)_t + \sum_{|\alpha|=m} (-1)^m D_x^\alpha a_\alpha(x,t,u,\dots,D_x^m u) = 0, \tag{1.1}$$

$$u(x,0) = u_0(x) \in L_{q+1}(\Omega) \quad \text{in } \Omega, \tag{1.2}$$

$$D_x^\alpha (u - f) = 0 \quad \text{on } \partial\Omega \times (0,T) \quad \text{for any } |\alpha| \leqslant m-1. \tag{1.3}$$

The functions $a_\alpha(x,t,\xi)$ are assumed to be continuous, and the elliptic operator on the left-hand side satisfies the following growth and coercivity conditions: there exist positive constants $p \geqslant q$ and $d_1$, $d_2$ such that

$$\sum_{|\alpha|=m} a_\alpha(x,t,\xi)\xi_\alpha \geqslant d_1 \biggl(\sum_{|\beta|=m} |\xi_\beta|\biggr)^{p+1}, \tag{1.4}$$

$$|a_\alpha(x,t,\xi)| \leqslant d_2 \biggl(\sum_{|\beta|=m} |\xi_\beta|\biggr)^{p} \quad \text{for all } (x,t,\xi) \in \bar{Q} \times \mathbb{R}^{n(m)}, \tag{1.5}$$

where $n(m)$ is the number of distinct multi-indices of the length not exceeding $m$. We use the usual notations

$$\alpha = (\alpha_1,\dots,\alpha_N),$$
$$|\alpha| = \alpha_1 + \cdots + \alpha_N,$$
$$D_x^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}},$$
$$D_x^m u = \{D_x^\alpha u,\ |\alpha| = m\}.$$

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The function $f(x,t)$ determining the boundary values is a suitable extension from the lateral boundary $\partial\Omega \times (0,T)$ into the domain $Q$ such that, for any $T_0 < T$, we have

$$f(\cdot,t) \in C([0,T_0]; L_{q+1}(\Omega)) \cap L_{p+1}(0,T_0; W^m_{p+1}(\Omega)), \tag{1.6}$$

$$f_t(\cdot,t) \in L_1(0,T_0; L_{q+1}(\Omega)) \cap L_r(0,T_0; L_r(\Omega)), \quad r = \frac{p+1}{p+1-q}. \tag{1.7}$$

The main feature of the problems under consideration is as follows. We assume that the boundary function $f(\cdot,t)$ **blows up** as $t \to T^-$. The rate of blow-up is characterized by the following function defined for $t \in (0,T)$:

$$\begin{aligned} F(t) = \sup_{0<\tau<t} \left( \int_\Omega |f(x,\tau)|^{q+1}\,\mathrm{d}x \right) + \int_0^t \int_\Omega |D^m_x f(x,\tau)|^{p+1}\,\mathrm{d}x\mathrm{d}\tau \\ + \left( \int_0^t \left( \int_\Omega |f_\tau(x,\tau)|^{q+1}\,\mathrm{d}x \right)^{1/(q+1)} \mathrm{d}\tau \right)^{q+1} \to \infty \quad \text{as } t \to T^-. \end{aligned} \tag{1.8}$$

Without loss of generality, we always suppose that the support of $f(\cdot,t)$ is uniformly bounded for $t \in (0,T)$. Given $S \subset \partial\Omega$, by $W^m_{p+1}(\Omega,S)$ we denote, as usual, the closure in the norm of the Sobolev space $W^m_{p+1}(\Omega)$ of the subset of functions from $C^\infty(\Omega)$ vanishing in a neighbourhood of $S$. We set $W^m_{p+1}(\Omega,\emptyset) \equiv W^m_{p+1}(\Omega)$ and by $\langle\cdot,\cdot\rangle$ denote the duality product between $(W^m_{p+1}(\Omega,\partial\Omega))^*$ and $W^m_{p+1}(\Omega,\partial\Omega)$.

DEFINITION 1.1. A function $u(x,t)$ is said to be a **generalized energy solution** of the problem (1.1)–(1.3) if, for any $T_0 < T$, we have

(i) $u - f \in L_{p+1}(0,T_0; W^m_{p+1}(\Omega,\partial\Omega)) \cap C([0,T_0]; L_{q+1}(\Omega))$;

(ii) $(|u|^{q-1}u)_t \in L_{(p+1)/p}(0,T_0; (W^m_{p+1}(\Omega,\partial\Omega))^*)$;

(iii) $u$ satisfies the initial condition (1.2) and the following integral identity,

$$\begin{aligned} \int_0^{T_0} \langle (|u|^{q-1}u)_t, \chi\rangle\,\mathrm{d}t \\ + \int_0^{T_0} \int_\Omega \sum_{|\alpha|=m} a_\alpha(x,t,u(x,t),\dots,D^m_x u(x,t)) D^\alpha_x \chi(x,t)\,\mathrm{d}x\mathrm{d}t = 0, \end{aligned} \tag{1.9}$$

where $\chi(x,t) \in L_{p+1}(0,T_0; W^m_{p+1}(\Omega,\partial\Omega))$ is an arbitrary function.

A well-known literature is devoted to questions of local existence and uniqueness of generalized (weak) solutions of the mixed problem for equations (1.1). In the case $q = 1$, $f = 0$, problem (1.1)–(1.3) was studied in detail in the 1960s (see [34] and the references therein). In the case $m = 1$ and bounded domains $\Omega$, existence of energy solutions follows from [2] for arbitrary $T_0 < T$ by the second inclusions in (1.6), (1.7). For $m > 1$ and $f = 0$, the solvability of the problem under consideration was established in [12] in the case where (1.1) has a variational structure, in particular, if $a_\alpha(x,t,\xi) \equiv |\xi_\alpha|^{p-1}\xi_\alpha$. The results admit a natural extension to more general higher-order equations and boundary conditions.

### 1.2. Localization of blow-up

The paper is devoted to the study of the localization of blow-up boundary regimes.

DEFINITION 1.2. The set of points $y \in \bar{\Omega}$ satisfying, for any small $\varepsilon > 0$,

$$\sup_{0<t<T}\left(\int_{\{|x-y|<\varepsilon\}\cap\Omega} |u(x,t)|^{q+1}\,\mathrm{d}x + \int_0^t \int_{\{|x-y|<\varepsilon\}\cap\Omega} |u(x,\tau)|^{p+1}\,\mathrm{d}x\mathrm{d}\tau\right) = \infty,$$

is said to be the *singularity* or the *blow-up subset*, denoted by $\Omega_{\mathrm{S}} \equiv \Omega_{\mathrm{S}}(u)$, of the energy solution $u(x,t)$ of problem (1.1)–(1.3). $\Omega_{\mathrm{R}} \equiv \bar{\Omega} \setminus \Omega_{\mathrm{S}}$ is the corresponding *non-singular subset*.

Actually, for sufficiently regular solutions, it is expected that $u(x_0,t)$ is uniformly bounded as $t \to T^-$ at any interior point $x_0 \in \Omega_{\mathrm{R}} = \Omega \setminus \Omega_{\mathrm{S}}$ and, vise versa, $u(x_0,t)$ is not uniformly bounded as $t \to T^-$ for any $x_0 \in \Omega_{\mathrm{S}}$. For such regular solutions, the blow-up set $\Omega_{\mathrm{S}}$ is known as the *localization domain* of the solution (or of the boundary blow-up regime). In the one-dimensional case, $L = \operatorname{meas} \Omega_{\mathrm{S}}$ is called the *localization length* of the blow-up regime (see [45, ch. 3]). We also use the following terminology [45]. Given bounded initial data, we say that the blow-up regime, prescribed by the function $f(x,t)$ on $\partial\Omega$, is *localized* (respectively, *non-localized*) if $\operatorname{meas} \Omega_{\mathrm{S}} < \infty$ (respectively, $\operatorname{meas} \Omega_{\mathrm{S}} = \infty$). Localized blow-up regime is called S-regime if $\operatorname{meas} \Omega_{\mathrm{S}} \in (0,\infty)$ (regional blow-up) and LS-regime if $\operatorname{meas} \Omega_{\mathrm{S}} = 0$ (this includes the case of single-point blow-up). A non-localized blow-up regime with $\operatorname{meas} \Omega_{\mathrm{S}} = \infty$ is also called HS-regime (global blow-up).

A general treatment of blow-up processes naturally occurred in the 1930–1950s in the context of Semenov's chain reaction theory (1930s), adiabatic explosion and combustion theory (we mention Frank-Kamenetskii's equation (1938) admitting blow-up in the non-stationary version and the first blow-up analysis by Todes (1933) (see [22, 57])). A strong influence was due to blow-up singularities in gas dynamics, the intense-explosion (focusing) problem with second-kind self-similar solutions considered by Bechert *et al*. in the 1940s (see [3, p. 127] and [55,56]). Another classical area of blow-up processes occurring in the 1960s was nonlinear optics. Here, the main model is the nonlinear (cubic) Schrödinger equation admitting blowing-up self-focusing solutions (see references in the book [53] and in surveys in [33, 35]).

A great interest to blow-up phenomena in the 1960s was also generated by highly non-stationary and nonlinear problems of controlled thermonuclear fusion (CTF) and in the inertial confinement fusion. Basov and Krohin [7] proposed the use of concentrated (focusing) laser beams for the heating of a deuterium-tritium (DT) pellet to create plasma of super-high temperatures. A remarkable feature of boundary blow-up regimes in CTF problems was revealed numerically by Nuckolls *et al*. in [36], showing a principle possibility of the laser blow-up-like compression of a DT drop to super-high densities *without shock waves* (these crucial results were first announced by Teller in Montreal in 1972). Such a compression was expected to initiate a thermonuclear burning (by the Lawson criterion, the rate of produced thermonuclear energy at a fixed ion temperature is proportional to the product of plasma density and confinement time).

The electron and ion conductivity of fully ionized plasma are known to depend essentially on the temperature. A simple mathematical model including the heat propagation only (gas-dynamics phenomena are known to be inertial and, under certain hypotheses, can be neglected at the purely heat stage) consists of a quasilinear second-order heat equation

$$u_t - (k(u)u_x)_x = 0 \quad \text{in } \mathbb{R}_+ \times (0,T), \tag{1.10}$$

with given bounded initial data (temperature) $u(x,0) = u_0(x)$, and a blow-up boundary regime at $x=0$: $u(0,t) = f(t) \to \infty$ as $t \to T^- < \infty$. Here, $k(u) > 0$ is the given heat conductivity coefficient satisfying $k(u) \to \infty$ as $u \to \infty$ ($k(u) \sim u^{5/2}$ for the fully ionized plasma) and $f(t)$ is a prescribed temperature on the surface of the DT target, which is generated by laser beams and increases in a blowing-up (peaking) regime.

A first detailed analysis of the striking effect of space **localization** of blow-up boundary regime (**S-regime of blow-up** generated by a blow-up **standing wave**) in the quasilinear diffusion equation (1.10) with the power law $k(u) = u^\sigma$, $\sigma > 0$, was performed by Samarskii and Sobol' in [43] (such explicit blow-up solutions were used before as parabolic barriers (see typical references in Kalashnikov's survey [24])). In the 1970s, the effect of space localization in $\Omega_{\mathrm{S}}$, where temperature $u(x,t)$ tends to infinity as $t \to T$, became a popular subject and posed a number of typical problems for quasilinear parabolic PDEs. In particular, the problem of localization of blow-up solutions in reaction–diffusion equations was first proposed by Kurdyumov [30] in 1974 (see ch. 3 and 4 in [45], devoted to localization analysis). It was shown that, in a quasilinear model, the heat and burning localization arises in plasma with the electron heat conductivity and heat source due to amalgamation of DT nuclei [58]. Such a model, as well as different aspects of localization effects, are discussed in [32]. An extensive list of references on localization of blow-up dissipative structures with the historical review can be found in Kurdyumov's survey paper [31]. Main mathematical results on existence and non-existence of localization of boundary blow-up and singularity formation for equation (1.10) with rather arbitrary monotone conductivity coefficients $k(u)$ are summarized in [45, ch. 3] (see also surveys in [15, 21]). A necessary and sufficient condition of localization for the porous medium equation (1.10) with $k(u) = u^\sigma$, $\sigma > 0$, was established in [23]. For general $N$-dimensional quasilinear second-order parabolic equations ($m = 1$), localization conditions were obtained in [49, 50] by means of the method of energy estimates we are going to use below for arbitrary higher-order equations.

In the present analysis, the blow-up singularity is generated on the boundary by means of a blow-up function. On the other hand, interior blow-up can be generated by extra source-type lower-order operators if, for instance, we consider the Cauchy problem for the quasilinear heat equation

$$u_t = \nabla \cdot (k(u)\nabla u) + Q(u), \tag{1.11}$$

where $Q(u) \geqslant 0$ has a superlinear growth as $u \to \infty$. Both types of blow-up (boundary and interior) phenomena are essentially related to each other, the localization terminology stays the same, though blow-up in (1.11) is more delicate (see the book by Bebernes and Eberly [8], Levine's survey [33] and ch. 4 in [45] for main results

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and related references). In general, localization problems of interior blow-up for such higher-order semilinear and quasilinear parabolic equations remain open.

### 1.3. Main results and plan of the paper

We prove localization of boundary blow-up for general $2m$th-order quasilinear parabolic equations. Higher-order semilinear and quasilinear diffusion operators occur in several applications, including thin film theory, nonlinear diffusion, lubrication theory, flame and wave propagation, phase transition at critical Lifschitz points and bistable systems (e.g. the Kuramoto–Sivashinskii equation and the extended Fisher–Kolmogorov equation) (see a number of models and a list of references in the book by Peletier and Troy [42]).

It turns out that, in the special case $p = q$, when the parabolic operators exhibit some kind of 'linear' properties due to coinciding algebraic homogenuities, the asymptotic behaviour of blow-up solutions is rather 'nonlinear' in the sense that, as it is seen from the theorem below, blow-up estimates do not exhibit any scaling invariance (see further comments below). The localized blow-up regimes admit the following description for $2m$th-order quasilinear parabolic equations.

THEOREM 1.3. *Let $p = q > 0$. Given a blow-up boundary regime $f$, let the corresponding function (1.8) satisfy, as $t \to T^-$,*

$$F(t) \leqslant c \exp\{A(T-t)^{-1/[m(p+1)-1+\xi_0]}\}, \tag{1.12}$$

*where $c$, $A$ are arbitrary ¯xed positive constants and the constant $\xi_0 > 0$ can be arbitrarily small. Then, for arbitrary constant $\delta > 0$, any energy solution $u(x,t)$ of the problem (1.1){(1.3) satis¯es*

$$\int_{\{|x|>1+\delta\}} |u(x,t)|^{p+1}\,\mathrm{d}x < C = C(\delta) < \infty \quad \text{for all } t \in (0,T). \tag{1.13}$$

This means that the singularity set $\Omega_{\mathrm{S}}$ is concentrated on the boundary $\partial\Omega$ and $f$ is a localized blow-up LS-regime. The unit ball $\Omega$ can be replaced by any bounded domain with sufficiently smooth boundary and the result says again that for such boundary blow-up, the singularity set is concentrated on the boundary. We will present examples (see theorem 3.1 in § 3) showing that such exponential estimates of localized blow-up regimes are sharp and cannot be improved in general. In a forthcoming paper [16], we prove a general estimate of the singularity blow-up subsets for the case of the regional blow-up occurring for $\xi_0 = 0$ in (1.12). We also establish the asymptotic estimates of *non-localized* blow-up regimes, in particular, corresponding to (1.12) with $\xi_0 \in (1 - m(p+1), 0)$.

Theorem 1.3 is proved in §§ 4–6 by a modification of the energy estimate method of the type [41]. We use a version of energy estimates for quasilinear $2m$th-order parabolic equations developed in [1, 46]. An important feature of our asymptotic analysis is that, in order to exhibit the localization of blow-up, we derive an infinite functional system of inequalities for a series of suitable energy functions by means of a specifically organized spatial-time partition of the $(x,t)$-domain concentrated near blow-up singularity.

Before proceeding with proofs of the theorem, in order to anticipate typical features of boundary blow-up in the given class of equations, we study some special scaling invariant quasilinear and nonlinear second- and higher-order parabolic equations, which will be proved to admit non-invariant blow-up asymptotics. In §§ 2 (second-order quasilinear equations) and 3 ($2m$th-order linear parabolic equations), we give more detailed characteristics of such blow-up singularities, including the study of the geometric shape of localization domains and the asymptotic behaviour of blow-up solutions. In particular, we prove that in these two classes of problems, the exponential blow-up function on the boundary,

$$f(t) = \exp\{(T-t)^{n_*}\}, \quad \text{with the critical exponent } n_* = -\frac{1}{[m(p+1)-1]}, \tag{1.14}$$

always leads to a localized S-regime with the localization domain of bounded positive measure, $\operatorname{meas} \Omega_{\mathrm{S}} \in \mathbb{R}_+$. Therefore, theorem 1.3 establishes the optimal characterization of LS-regimes. Moreover, we show that an arbitrary negative perturbation of the critical exponent $n_*$ transforms this localized S-regime into the non-localized HS-regime with the boundary blow-up

$$f(t) = \exp\{(T-t)^{n}\}, \quad n < n_*, \tag{1.15}$$

where $\operatorname{meas} \Omega_{\mathrm{S}} = \infty$ and $u(x,t) \to \infty$ as $t \to T^-$ uniformly on compact subsets.

The class of quasilinear equations with $p = q$ is a special one. It follows from (1.12) that the main blowing-up regimes are not scaling invariant (though, in examples, the homogeneous parabolic operators are invariant under a group of scaling transformations). Some approximate scaling invariant asymptotics are obtained by an extra nonlinear logarithmic transformation $v = \ln u$. In the next two sections we show that the asymptotic blow-up behaviour in terms of $v(x,t)$ is described by a blow-up similarity solution of a nonlinear equation of Hamilton–Jacobi type, so that the higher-order parabolic terms form singular perturbations of first-order equations.

For the linear heat equation with $m = 1$ and for weakly quasilinear equations (1.10), $N = 1$, the above results on the blow-up singularity formation via Hamilton–Jacobi equations are well known (see first results in [44] and a detailed analysis in [45, ch. 3]).

It is worth mentioning that the case $p > q$ is simpler in the sense that the main blow-up asymptotics are given by scaling-invariant self-similar solutions of model equations [17]. On the other hand, this more nonlinear case generates some specific mathematical difficulties concerning application of our method.

### 1.4. Method: Saint-Venant's principle and energy estimates

Saint-Venant's principle and problem [4–6] formulated in the 1850s play a fundamental role in the linear theory of elastic equilibrium. First rigorous energy estimates on exponential decay of a parametrized strain energy and a strain inequality for a cylinder with a loaded cross-section on the end were established by Toupin [54] and by Knowles [25] (in these papers, principle earlier references can be found). In particular, a detailed analysis in [25] established second-order estimates on smooth solutions of the biharmonic (fourth-order elliptic) equation

$$\Delta^2\varphi = 0 \quad \text{in } R, \qquad \varphi = \varphi_x = \varphi_y = 0 \quad \text{on } \partial R, \tag{1.16}$$

where $R \subset \mathbb{R}^2$ is a bounded simply domain (the cross-section of the cylinder) with smooth boundary $\partial R$. It is proved that the strain energy

$$E(z) = \iint_{R_z} (\varphi_{xx}^2 + \varphi_{yy}^2 + 2\varphi_{xy}^2)\,\mathrm{d}x\mathrm{d}y$$

contained in the subdomain $R_z = R \cap \{x \geqslant z\}$ ($z \geqslant 0$ is a parameter) decays at least exponentially with $z$. Pointwise derivatives estimates were also proved by using mean-value theorems for biharmonic functions. Such energy estimates in a weak form applied to more general second-order elliptic equations [26] and to the heat (second-order parabolic) equation [27].

Estimates of behaviour of the energy integrals over the families of inner subdomains of diameter $r \to 0$ (i.e. those having the structure of Saint-Venant's principle; we do not mention other important approaches) play an important role in the study of regularity of generalized solutions of elliptic and parabolic equations. Such estimates for second-order equations were derived by Campanato [13]. Solonnikov [51, 52] proposed a method of obtaining such estimates for energy solutions of quasilinear divergent elliptic higher-order equations, which is based on the use of Hopf's cut-off functions. The behaviour of energy integrals for solutions of a general class of higher-order elliptic equations near the boundary was studied in [28], where references to earlier papers of the 1970s on polyharmonic and equations from elasticity theory can be found.

A systematic extension of ideas of different a priori estimates via introducing a parameter for classes of PDEs is due to Oleinik and collaborators (see [28, 37–41] and a full list of references in the survey in [41], where, in particular, the uniqueness in Tikhonov–Täcklind classes for linear second- and higher-order parabolic equations and systems was proved). These were advanced applications of Saint-Venant's principle to linear higher-order PDEs. Extra references and further comments on energy methods for quasilinear higher-order PDEs can be found in § 4 of this paper, where we also discuss and compare known energy approaches based on differential and discrete systems of functional inequalities.

## 2. Second-order parabolic equations

If $m = 1$, then the maximum principle applies, and for some typical equations we can study the blow-up behaviour in more detail, including the asymptotic behaviour and the geometry of localization domains. Let $\omega \subset \mathbb{R}^N$ be a bounded connected domain with the complement $E_\omega = \boldsymbol{R}^N \setminus \bar{\omega}$. In this section we consider the exterior problem with $m = 1$, $p > 0$, where, for convenience, we set $T = 1$ (and $T = 0$ later on),

$$(|u|^{p-1}u)_t = \Delta_p u \equiv \sum_{i=1}^{N} (|\nabla u|^{p-1} u_{x_i})_{x_i} \quad \text{in } Q = E_\omega \times (0,1), \tag{2.1}$$

$$u(x,t) = f(x,t) \quad \text{on } S = \partial\omega \times (0,1), \qquad u(x,0) = u_0(x) \quad \text{in } E_\omega. \tag{2.2}$$

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We assume that $f(x,t)$ blows up at $t=1^-$. Without loss of generality, we suppose that $f \geqslant 1$ and $u_0(x) \geqslant 1$, and hence $u \geqslant 1$ in $Q$ by the maximum principle. Such quasilinear parabolic equations of the second order are well known in the literature (see the classical book [34] by Lions, Kalashnikov's survey [24] and DiBenedetto's book [14]). By an energy solution we mean a weak one. Since the operator of the parabolic equation is not monotone, the uniqueness of such a weak solution is not straightforward. We then assume that $u(x,t)$ is a unique maximal solution, which is constructed by monotone smooth approximations. It satisfies the maximum principle and the usual comparison holds. Such a construction of extended limit semigroups of extreme (maximal or minimal) solutions is an essential and necessary feature for nonlinear parabolic equations admitting strong blow-up or extinction-like finite-time singularities (see [20, § 2] and the references therein).

In order to describe the asymptotic behaviour of main blow-up singularities for this equation, as in the linear case $p=1$ [44] (see details in [45, ch. 3]), we perform the change of the dependent variable $u=\mathrm{e}^v$ for $v \geqslant 0$, which leads to the quasilinear equation with a typical Hamilton–Jacobi operator and the diffusion $p$-Laplacian term

$$v_t = |\nabla v|^{p+1} + p^{-1}\Delta_p v, \quad u=\mathrm{e}^v. \tag{2.3}$$

The main idea is to prove that the asymptotic behaviour of this blow-up singularity is described by the corresponding Hamilton{Jacobi equation

$$V_t = |\nabla V|^{p+1}. \tag{2.4}$$

More precisely, we will show that the S-regime of boundary blow-up regional localization is given by its separate-variable solution

$$V_*(x,t) = (1-t)^{-1/p}\theta(x), \tag{2.5}$$

where $\theta \geqslant 0$ solves the stationary Hamilton–Jacobi equation

$$|\nabla\theta|^{p+1} - \frac{\theta}{p} = 0 \quad \text{in } E_\omega. \tag{2.6}$$

### 2.1. S-regime in one dimension

If $N=1$ and $E_\omega=\mathbb{R}_+$, equation (2.6) becomes a first-order ordinary differential equation (ODE) of the form

$$|\theta'|^{p+1} - \frac{\theta}{p} = 0, \tag{2.7}$$

which gives the following sufficiently smooth compactly supported solution,

$$\theta = \left(1-\frac{x}{x_0}\right)_+^{(p+1)/p}, \quad x\in\mathbb{R}_+, \tag{2.8}$$

where $x_0$ is the e®ective localization length,

$$x_0(p) = (p+1)p^{-p/(p+1)}, \tag{2.9}$$

of the boundary blow-up S-regime prescribed at the origin $x=0$,

$$v(0,t) = (1-t)^{-1/p}, \quad t\in(0,1). \tag{2.10}$$

On the other hand, it also satisfies the Neumann boundary condition $-v_x(0,t) = c_0(1-t)^{-1/p}$, where $c_0 = -\theta'(0) > 0$. In what follows, without loss of generality, we fix the blow-up Dirichlet boundary condition (2.10). The function $\theta$ is sufficiently smooth, $|\theta'|^{p+1} \in C^1$, and is Kruzhkov's entropy (viscosity) solution [29] of the Hamilton–Jacobi equation (2.4).

We prove that in the parabolic equation (2.3) the elliptic $p$-Laplacian operator is negligible on the asymptotic blow-up stage $t \to 1^-$ and forms a singular perturbation of the Hamilton–Jacobi equation. We introduce the rescaled function

$$g(x,\tau) = (1-t)^{1/p} v(x,t), \tag{2.11}$$

where $\tau = -\ln(1-t) \to \infty$ as $t \to 1^-$ is the new time variable, and arrive at the perturbed Hamilton–Jacobi equation

$$g_\tau = |\nabla g|^{p+1} - p^{-1} g + \mathrm{e}^{-\tau/p} \Delta_p g \quad \text{for } \tau > 0, \quad g(0,\tau) \equiv 1. \tag{2.12}$$

The initial function $g_0(x) \geqslant 0$ is assumed to be bounded.

Theorem 2.1.

(i) We have

$$g(x,\tau) = \theta(x) + o(1) \quad \text{as } \tau \to \infty \text{ uniformly.} \tag{2.13}$$

(ii) The e®ective localization length of $u(x,t)$ is bounded and positive,

$$L(u) = \operatorname{meas}\Big\{x > 0 : \limsup_{t\to 1} u(x,t) = \infty\Big\} = x_0(p). \tag{2.14}$$

In the proof, we derive an exact rate of convergence in (2.13) which is different in the parameter ranges $p > 1$, $p = 1$ and $p < 1$.

Proof. (i) The main difficulty is that the $p$-Laplacian is not bounded and is not well defined on the compactly supported function $\theta(x)$ at the interface $x = x_0$. Therefore, we first compare two solutions $\bar v$ and $\bar V_*$ of the problem posed in the domain $Q_1 = (0, 2x_0) \times (0,1)$ with the same boundary condition at $x = 0$ and an extra symmetric condition on the right-hand lateral boundary:

$$\bar v(2x_0, t) \equiv v(0,t) = (1-t)^{-1/p}.$$

The corresponding solution of the Hamilton–Jacobi equation

$$\bar V_*(x,t) = (1-t)^{-1/p} \bar\theta(x),$$

where $\bar\theta(x) = |1 - x/x_0|^{(p+1)/p}$, is indeed $V_*$ reflected relative to $x = x_0$. As we show, this reflected solution is sufficiently regular for comparison by the maximum principle, since $\Delta_p \bar V_*(x_0, t)$ makes sense.

If $p \leqslant 1$, then $\theta(x) \in C^2$, and we can use simpler straightforward computations. In any case, for any $p > 0$ and especially for $p > 1$, we assume that the solution $\bar v$ is sufficiently smooth using a standard approximation via a suitable regularization of the $p$-Laplacian

$$\Delta_p v \mapsto \Delta_{p,\varepsilon} v = \nabla \cdot [(|\nabla v|^2 + \varepsilon^2)^{(p-1)/2} \nabla v], \quad \varepsilon > 0.$$

Consider the difference $w = \bar{v} - \bar{V}_*$ satisfying the parabolic equation

$$w_t = |\bar{v}_x|^{p+1} - |(\bar{V}_*)_x|^{p+1} + p^{-1}\Delta_{p,\varepsilon}(\bar{V}_* + w).$$

The right-hand side can be written down in the form $w_t = \boldsymbol{B}w + p^{-1}\Delta_{p,\varepsilon}(\bar{V}_*)$, where, on regularized solutions, the operator $\boldsymbol{B}w = aw_{xx} + bw_x$ is a differential second-order one with sufficiently smooth coefficients. Passing to the limit $\varepsilon \to +0$ relative to the regularizing parameter, we construct a weak plane supersolution $\bar{w}(t)$, depending on the variable $t$ only, and satisfying the differential inequality

$$\bar{w}' = p^{-1}\sup_x \Delta_p \bar{V}_*(x,t) \equiv c_p(1-t)^{-1/p}, \qquad c_p = p^{-1}\sup_{x>0}\Delta_p\bar{\theta}(x) = [p(p+1)]^{-1},$$

where, as we have seen, the finiteness of the constant $c_p$ plays a key role in the regularization argument. Choosing $\bar{w}(0) = C_0 \gg 1$, by the maximum principle, we conclude that, for $p \neq 1$, $|\bar{v}(x,t) - \bar{V}_*(x,t)| \leqslant \bar{w}(t) = C_0 + \bar{c}_p(1-t)^{(p-1)/p}$ for any bounded initial data $\bar{v}(x,0)$. For $p = 1$, we get the logarithmic term $c_p|\ln(1-t)|$ in the right-hand side (cf. a general approach in [45, p. 379]). Assume now that $p \neq 1$. In particular, this implies that at the end point of the localization domain $x = x_0$ there holds $\bar{v}(x_0,t) \leqslant C_0 + \bar{c}_p(1-t)^{(p-1)/p}$.

Consider now the solution $v(x,t)$ of the original problem in the half-line $\{x > 0\}$. One can see that $v \leqslant \bar{v}$ in $Q_1$, so that $v \leqslant C_0 + \bar{c}_p(1-t)^{(p-1)/p}$ in $(0,x_0] \times (0,1)$. On the other hand, using standard monotonicity (in $x$) properties of large solutions of parabolic equations, we have that, by the maximum principle, for any small $\varepsilon > 0$ as $t \to 1^-$, $v(x,t) \leqslant v(x_0 - \varepsilon, t)$ for any $x \geqslant x_0$. Passing to the limit $\varepsilon \to 0$, we then obtain that, for $t \approx 1^-$, $v(x,t) \leqslant C_1 + \bar{c}_p(1-t)^{(p-1)/p}$ for $x \geqslant x_0$.

One can see that $\Delta_p\theta \geqslant 0$, so that the compactly supported function $V_*(x,t)$ is a subsolution of the parabolic equation and hence the following estimate from below holds: $v(x,t) \geqslant -C_2 + V_*(x,t)$. Summing up all the above estimates, we arrive at the rate of convergence in (2.13).

(ii) The calculations leading to the asymptotic expansion (2.13) mean that, as $t \to 1^-$,

$$u(x,t) = \mathrm{e}^{(1-t)^{-1/p}\theta(x)}\mathrm{e}^{O((1-t)^{(p-1)/p})}, \tag{2.15}$$

and hence

$$C_2\mathrm{e}^{(1-t)^{-1/p}\theta(x)} \leqslant u(x,t) \leqslant C_3\mathrm{e}^{O((1-t)^{(p-1)/p})}\mathrm{e}^{(1-t)^{-1/p}\theta(x)}.$$

For $p = 1$,

$$C_2\mathrm{e}^{(1-t)^{-1}\theta(x)} \leqslant u(x,t) \leqslant C_3(1-t)^{-c_1}\mathrm{e}^{(1-t)^{-1}\theta(x)}.$$

Obviously, this implies that $L \geqslant x_0(p)$. On the other hand, setting $x = x_0$, where $\theta(x_0) = 0$, we get $u(x_0,t) \leqslant C_3(1-t)^{-c_p}$ as $t \to 1^-$. Any power-like blow-up on the boundary $u(0,t) = (1-t)^n$, with $n < 0$, is known to correspond to a blow-up LS-regime with the zero effective localization length. This follows from the general theorem 1.3 stated in § 1, to be proved later on. On the other hand, it can be proved by the standard comparison with the corresponding localized self-similar solution of the form

$$u_*(x,t) = (1-t)^n f(\eta), \qquad \eta = \frac{x}{(1-t)^{1/(p+1)}},$$

where $f > 0$ solves the ODE

$$(|f'|^{p-1}f')' - (p+1)^{-1}(f^p)'\eta + npf^p = 0, \quad \eta > 0, \quad f(0) = 1.$$

The self-similar strictly monotone decreasing $C^\infty$-profile $f(\eta) > 0$ for all $\eta > 0$ satisfies the following asymptotic behaviour: $f(\eta) = C_*\eta^{n(p+1)}(1 + o(1)) \to 0$ as $\eta \to \infty$ ($C_* > 0$). Then the solution $u_*$ is effectively localized at $x = 0$ in the sense that, for any $x > 0$, as $t \to 1^-$,

$$u_*(x,t) \to u_*(x,1^-) = C_*x^{n(p+1)} < \infty.$$

By the usual comparison of two solutions $u(x,t)$ and $u_*(x - x_0, t)$, which are assumed to be smooth enough by monotone regularization, this implies that $u(x,t)$ is also uniformly bounded for any $x > x_0$, so that $L = x_0$. □

### 2.2. Non-localized HS-regime in one dimension

We prove that there exist blow-up functions from the same exponential class staying above the S-regime leading to non-localized HS-regimes (i.e. blow-up regimes that are 'higher than S'). We thus consider the blow-up boundary condition

$$u(0,t) = \mathrm{e}^{(1-t)^n}, \quad \text{with } n < -\frac{1}{p}. \tag{2.16}$$

THEOREM 2.2. *Let (2.16) hold. Then $u(x,t)$ blows up as $t \to 1^-$ uniformly on the inde¯nitely expanding subset*

$$\{0 \leqslant x \leqslant c_0(1-t)^\beta\}, \qquad \beta = \frac{np+1}{p+1} < 0, \tag{2.17}$$

*with a constant $c_0 = c_0(p,n) > 0$.*

*Proof.* The new function $v = \ln u$ satisfies the boundary condition $v(0,t) = (1-t)^n$. The proof is based on comparison from below of the given solution $v(x,t)$ with the self-similar solution of the Hamilton–Jacobi equation (2.4)

$$V_*(x,t) = (1-t)^n h(\eta), \qquad \eta = \frac{x}{(1-t)^\beta}, \tag{2.18}$$

where $h \geqslant 0$ solves the first-order ODE $|g'|^{p+1} - \beta g'\eta + ng = 0$ for $\eta > 0$ with $g(0) = 1$. This ODE admits a scaling invariance and reduces to an autonomous ODE, which is integrated in quadratures (see a similar analysis of the case $p = 1$ in [45, p. 171]). As the result, we obtain that, for $n < -1/p$, such a monotone profile $g$ exists and vanishes at a finite point $\eta_0 = \operatorname{meas}\operatorname{supp} g < \infty$. We then set $g(\eta) \equiv 0$ for all $\eta \geqslant \eta_0$, so that $g(\eta)$ becomes a continuous (but non-smooth) compactly supported rescaled viscosity solution of the Hamilton–Jacobi equation. The comparison of $v$ and $V_*$ in the domain $\{x < \eta_0(1-t)^n\} \times (0,1)$, where both solutions are sufficiently smooth, is performed as in the case of the S-regime above (see also [45, p. 382]) and gives uniform blow-up on any unbounded subset (2.17) with arbitrary $c_0 < \eta_0$. □

A general stability approach to the asymptotic degeneracy of parabolic flows into the Hamilton–Jacobi ones was developed in [18, 19]. Using these results, it is not difficult to show that the self-similar solution (2.18) of the Hamilton–Jacobi equation (2.4) actually describes the asymptotic behaviour of the HS-regime for the parabolic equation (2.3). This is similar to the behaviour in the linear case $p = 1$ [45, p. 381].

### 2.3. S-regime in the $N$-dimensional geometry

By $\omega \subset \mathbb{R}^N$ we again denote a bounded smooth domain with complement $E_\omega$. Consider the following blow-up condition:

$$u(x,t) = \mathrm{e}^{(1-t)^{-1/p}} \quad \Rightarrow \quad v(x,t) = (1-t)^{-1/p} \quad \text{on } \partial\omega, t \in (0,1).$$

The rescaled function $g(x,\tau)$ satisfies (2.12) with the boundary condition $g(x,\tau) = 1$ on $\partial\omega \times \mathbb{R}_+$. The corresponding Hamilton–Jacobi stationary profile solves the stationary exterior problem

$$|\nabla\theta|^{p+1} - \frac{\theta}{p} = 0 \quad \text{in } E_\omega, \qquad \theta(x) = 1 \quad \text{on } \partial\omega.$$

One can expect that this solution describes the asymptotic behaviour of the parabolic equation. Then $\operatorname{supp}\theta = \{x \in E_\omega : \theta(x) > 0\}$ is the effective localization domain of the solution $V_*(x,t)$. We now compare this localization domain for the Hamilton–Jacobi equation (2.4) and the localization domain

$$\omega_{\mathrm{S}}(v) = \Big\{x \in E_\omega : \limsup_{t\to 1} v(x,t) = \infty\Big\}$$

of solutions $v(x,t)$ of (2.3) with bounded initial data $v_0$.

THEOREM 2.3. *The following hold.*

(i) $\omega_{\mathrm{S}}(v)$ *is bounded.*

(ii) *The $x_0$-neighbourhood of $\omega$,*

$$N_{x_0}(\omega) = \{y \in E_\omega : \operatorname{dist}(y, \partial\omega) \leqslant x_0(p)\},$$

*is contained in $\omega_{\mathrm{S}}(v)$.*

*Proof.* (i) Let $\omega_*$ be any convex smooth domain such that $\omega \subseteq \omega_*$. We use comparison with one-dimensional solutions. Fix a point $y_0 \in \partial\omega_*$ and let $l(y_0)$ be the hyperplane tangent to $\partial\Omega_*$ at $y = y_0$. Denote by $z$ the distance to $l(y_0)$. Consider the one-dimensional solution $v_1(z,t)$ of (2.3) with the blow-up function $v_1(0,t) = (1-t)^{-1/p}$ prescribed on the hyperplane, with sufficiently large initial data $v_1(z,0)$. By theorem 2.3, $v_1(z,t)$ is effectively localized in the infinite strip of the width $x_0(p)$ with the outer plane lateral boundary parallel to $l(y_0)$. Since $v \leqslant v_1$ in the outer half-space $\{z > 0\}$, the solution $v(x,t)$ is uniformly bounded at any $x$ beyond the localization strip. Thus $\omega_{\mathrm{S}}(v)$ is contained in the intersection of all such tangent strips constructed for all points $y_0 \in \partial\omega_*$, which implies boundedness of the localization domain and some extra estimates to be improved below.

(ii) In order to prove the second result, we will compare the solutions with the radially symmetric self-similar solutions $V_*(r,t) = (1-t)^{-1/p}\theta(r)$, $r = |x|$, of the Hamilton–Jacobi equation, where $\theta$ solves the same stationary equation (2.6), which, in the radial geometry, becomes the same ODE (2.7) with a prime denoting differentiation with respect to $r$. Then we obtain the same solution (2.8). Fix a point $y_0 \in \omega$ and a small $\varepsilon > 0$ such that the ball $B_\varepsilon(y_0) \subset \omega$. Consider the radial solution $v_\varepsilon(r,t)$, $r = |x-y_0|$, of the parabolic equation in the cylinder $B_\varepsilon(y_0)\times(0,1)$, $v_\varepsilon = (1-t)^{-1/p}$ on $\partial B_\varepsilon$, with bounded radial initial data. Let $V_*(r,t)$ be the corresponding approximate self-similar solution described above. The comparison from below establishes that the domain of localization $\{\varepsilon < r < \varepsilon + x_0\}$ of $V_*$ is contained in the domain of localization of $v_\varepsilon$.

As the last step, we compare $v_\varepsilon$ with the solution $v : v \geqslant v_\varepsilon$ in $Q$ by the maximum principle. Since $y_0 \in \omega$ and $\varepsilon > 0$ are arbitrary, we obtain the result. □

In the case of the convex domain $\omega$, this simple comparison from above with plane (one-dimensional) self-similar solutions $V_*$ and the comparison from below with the radial ones establish the equality $\omega_{\mathrm{S}}(v) = N_{x_0}(\omega)$. It is not difficult to see that the equality holds for more general domains $\omega$. It is curious that though $\partial\omega$ can be arbitrarily smooth, the boundary of the blow-up set $\omega_{\mathrm{S}}(v)$ is not necessarily $C^1$-smooth (see [45, p. 167]).

## 3. Blow-up localization in linear $2m$th-order equations

The linear $2m$th-order parabolic equation in one dimension,

$$u_t = (-1)^{m+1} D_x^{2m} u \quad \text{in } Q = \mathbb{R}_+ \times (0,1), \tag{3.1}$$

belongs to the case $p = q = 1$ of equations with Hamilton–Jacobi blow-up singularities. These problems can be studied using the convolution representation of the general solutions. Without loss of generality, we consider the following blow-up boundary condition,

$$\left.\begin{aligned} u(0,t) = f(t) \to \infty, \quad t \to 1^-, \\ u_x = \cdots = D_x^{m-1} u = 0, \quad x = 0, \quad t \in (0,1), \end{aligned}\right\} \tag{3.2}$$

and take zero initial data. Let $b(x,t)$ be the fundamental self-similar solution of (3.1),

$$b(x,t) = t^{-1/2m} F(\xi), \qquad \xi = \frac{x}{t^{1/2m}}, \tag{3.3}$$

where $F(\xi)$ is a unique symmetric solution of the linear ODE

$$(-1)^{m+1} F^{(2m)} + \frac{1}{2m}(F\xi)' = 0 \quad \text{in } \mathbb{R}, \qquad \int_{-\infty}^{\infty} F(\xi)\,\mathrm{d}\xi = 1. \tag{3.4}$$

By the cosine Fourier transform, we have that

$$F(\xi) = \frac{1}{\pi}\int_0^\infty \mathrm{e}^{-s^{2m}} \cos(s\xi)\,\mathrm{d}s.$$

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The solution of (3.1), (3.2) is given by the convolution potential

$$u(x,t) = 2(-1)^m \int_0^t f(\tau) D_x^{2m-1} b(x, t-\tau)\,\mathrm{d}\tau$$
$$\equiv 2(-1)^m \int_0^t f(\tau) F^{(2m-1)}\left[\frac{x}{(t-\tau)^{1/2m}}\right](t-\tau)^{-1}\,\mathrm{d}\tau. \tag{3.5}$$

For the heat equation, $m=1$, classification of boundary blow-up via the potential is easy (see [45, ch. 3, 6]).

### 3.1. S-regime of localization, HS-regimes

It follows from (3.5) that, as $t \to 1^-$, the asymptotic behaviour of the blow-up boundary regime $f(t)$ essentially depends on the asymptotic behaviour of the rescaled kernel $F(\xi)$ as $\xi \to \infty$. It is well known (and can be easily seen from the ODE (3.4)) that the rescaled kernel $F(\xi)$ is oscillatory as $\xi \to \infty$ for $m > 1$. A standard asymptotic analysis shows that (3.4) admits the solutions with the asymptotic behaviour

$$F(\xi) \sim \mathrm{e}^{-a|\xi|^\alpha} \quad \text{as } \xi \to \infty, \quad \alpha = \frac{2m}{2m-1} \in (1,2), \tag{3.6}$$

where $a \in \mathbb{C}$ satisfies the algebraic equation $(-1)^m(\alpha a)^{2m-1} + 1/2m = 0$. It has $2m-1$ different roots $\{a_k\}$, and $F(\xi)$ for $\xi \gg 1$ exhibits the behaviour corresponding to (3.6) with the maximal $\operatorname{Re} a_k < 0$. The leading asymptotic terms of $F$ are given by two linearly independent expressions of the type

$$\left.\begin{aligned} &\sim \mathrm{e}^{-a_m|\xi|^\alpha}\cos(b_m|\xi|^\alpha) \quad (\text{or } \sin(b_m|\xi|^\alpha)) \quad \text{as } \xi \to \infty,\\ &a_m = (2m-1)(2m)^{-\alpha}\cos\nu_m, \qquad b_m = (2m-1)(2m)^{-\alpha}\sin\nu_m, \end{aligned}\right\} \tag{3.7}$$

where $\nu_m = \pi(m-1)/(2m-1)$. These asymptotic estimates are enough to establish the following S- and HS-regimes of blow-up for the linear parabolic equations.

THEOREM 3.1.

(i) *For any $m > 1$, the blow-up S-regime occurs for the boundary function*

$$f(t) = \mathrm{e}^{(1-t)^{-1/(2m-1)}} \quad \text{as } t \to 1^-, \tag{3.8}$$

*and the effective localization length is given by*

$$L(u) = a_m^{-1/\alpha} > l_m = 2m(2m-1)^{-1/\alpha}. \tag{3.9}$$

(ii) *The HS-regime with any exponent $n < -1/(2m-1)$,*

$$f(t) = \mathrm{e}^{(1-t)^n} \quad \text{as } t \to 1^-, \tag{3.10}$$

*is not localized and $\omega_{\mathrm{S}}(u) = \mathbb{R}_+$.*

For $m = 1$, we have the equality $L(u) = l_1 = 2$ (see [45, p. 132]).

Proof. (i) The localization result follows from potential (3.5), where we use the above asymptotics of the rescaled kernel. We present an analysis proving (3.9) and also showing the asymptotic behaviour of the solutions as $t \to 1^-$ to be compared with the Hamilton–Jacobi limits given below. By shifting $t \mapsto t+1$, the blow-up moment becomes $T=-0$ and now $t<0$. Using (3.6) in potential (3.5), we obtain that, for $t \approx -0$, the main asymptotic expansion terms of solution $u(x,t)$ are composed from oscillatory integrals of the form

$$\int^{t} \mathrm{e}^{(-\tau)^{-\gamma} - a_m x^\alpha (t-\tau)^{-\gamma}} \cos[b_m x^\alpha (t-\tau)^{-\gamma}] \,\mathrm{d}\tau, \quad \gamma = \frac{1}{2m-1}. \tag{3.11}$$

Here we omit lower-order terms, which do not affect convergence or divergence of the integral as $t \to -0$, provided that the main prescribed exponential part stays dominant. Setting $\tau = \mu t$, $\mu > 1$, we arrive at the integrals

$$\int^{(-t)^{-1}} \mathrm{e}^{(-t)^{-\gamma} G(\mu)} \cos[b_m x^\alpha (-t)^{-\gamma} (\mu-1)^{-\gamma}] \,\mathrm{d}\mu, \tag{3.12}$$

where the function $G(\mu) = \mu^{-\gamma} - a_m x^\alpha (\mu-1)^{-\gamma}$, $\mu > 1$, has a typical bell-shaped form with a single maximum at $\mu_* = (1 - a_m^{1/\alpha} x)^{-1}$ for $x < L \equiv a_m^{-1/\alpha}$, where $G(\mu_*) = g(x) \equiv (1 - x/L)^\alpha$. Thus, if $x < L$, by the Cauchy criterion, integrals of the type (3.11) diverge as $t \to -0$, in view of the strong dominance of the singular multiplier with fast exponential growth. Using Taylor's expansion of $G(\mu)$ in a neighbourhood of $\mu = \mu_*$, we conclude that, in the first approximation, the corresponding non-oscillatory part (the envelope of the oscillating functional family) can be asymptotically estimated by the exponential function

$$\mathrm{e}^{(-t)^{-\gamma} G(\mu_*)} \equiv \mathrm{e}^{(-t)^{-\gamma} g(x)} \to \infty \tag{3.13}$$

as $t \to -0$ for any $x < L$. One can see from (3.12) that, for any fixed $x > L$, as $t \to -0$, integrals like (3.11) converge absolutely. The behaviour in the localization domain $x \in (0, L)$ is oscillatory, so that $\limsup u(x,t) = \infty$ and $\liminf u(x,t) = -\infty$ as $t \to -0$ (cf. Hamilton–Jacobi limits explained below).

(ii) The proof is similar. The divergence of the integral for any $x > 0$ is straightforward for any $n < -1/(2m-1)$. A more precise estimate can also be obtained and it corresponds to the similarity solutions of the Hamilton–Jacobi equation. □

### 3.2. The Hamilton–Jacobi structure of blow-up singularities

Let us show that the above exponential blow-up regimes have a Hamilton–Jacobi structure. This can be seen directly from the potential. On the other hand, we can use the same nonlinear transformation $u = \mathrm{e}^v$, which leads to the perturbed complex Hamilton–Jacobi equation for the function $v : Q \to \mathbb{C}$,

$$v_t = (-1)^{m+1} (v_x)^{2m} + \boldsymbol{P}_m(v), \tag{3.14}$$

where $\boldsymbol{P}_m$ is a quasilinear polynomial $2m$th-order operator, which can be computed. For instance,

$$\boldsymbol{P}_2(v) = -v_{xxxx} - 4 v_x v_{xxx} - 6 (v_x)^2 v_{xx} - 3 (v_{xx})^2.$$

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The corresponding Hamilton–Jacobi equation composed from the leading operator of the maximal algebraic homogenuity order,

$$v_t = (-1)^{m+1}(v_x)^{2m}, \tag{3.15}$$

admits the solution in separate variables

$$V_*(x,t) = (1-t)^{-1/(2m-1)}\theta(x), \tag{3.16}$$

where $\theta : \mathbb{R} \to \mathbb{C}$ solves the complex ODE

$$(-1)^{m+1}(\theta')^{2m} - \frac{\theta}{2m-1} = 0 \quad \text{for } x > 0, \quad \theta(0) = 1. \tag{3.17}$$

For odd $m = 1, 3, \dots$, this ODE admits a non-negative compactly supported solution

$$\tilde{\theta}(x) = \left(1 - \frac{x}{l_m}\right)_+^{\alpha}, \qquad l_m = 2m(2m-1)^{-1/\alpha}, \qquad \alpha = \frac{2m}{2m-1}. \tag{3.18}$$

The constant $l_m$ is the same as that calculated via the potential. For $m = 1$, where $l_1 = 2$, function (3.18) actually occurs in the asymptotic behaviour as $t \to -0$ (see [45, ch. 3]). Though, for $m > 1$, as we have seen above, this non-oscillatory profile does not describe the asymptotic behaviour, $\tilde{\theta}$ correctly explains the behaviour of the envelope to a family of oscillatory rescaled blow-up profiles. In general, the ODE (3.17) admits complex-valued solutions with compactly supported real part describing the oscillatory blow-up behaviour. For any $m > 1$, problem (3.17) has the complex solutions $\theta(x) = (1 - bx)^{\alpha}$, where $b \in \mathbb{C}$ solves the algebraic equation $(-1)^{m+1}(\alpha b)^{2m} = 1/(2m-1)$, so that $b = \mathrm{e}^{\mathrm{i}\varphi_k}/l_m$, $\varphi_k = \pi(2k+1)/2m$ for even $m$, and $b = \mathrm{e}^{\mathrm{i}\varphi_k}/l_m$, $\varphi_k = \pi k/m$ for odd $m$, $k = 0, 1, \dots, 2m-1$. Then $\theta(x) = \rho^{\alpha}\mathrm{e}^{\mathrm{i}\alpha\psi}$, where $\rho^2(x) = 1 + x^2/l_m^2 - 2x\cos\varphi_k/l_m$ and $\tan\psi(x) = -l_m x \sin\varphi_k/(l_m - x\cos\varphi_k)$. This gives a typical oscillating behaviour of the blowing-up solution

$$u(x,t) \sim \operatorname{Re} \mathrm{e}^{V_*(x,t)} \equiv \mathrm{e}^{(1-t)^{-\gamma}\rho^{\alpha}(x)\cos(\alpha\psi(x))}\cos[(1-t)^{-\gamma}\rho^{\alpha}(x)\sin(\alpha\psi(x))]. \tag{3.19}$$

We again observe the exponential envelope to the oscillating family (3.19),

$$\exp\{(1-t)^{-\gamma}\rho^{\alpha}(x)\cos(\alpha\psi(x))\} \quad \text{as } t \to 1^-.$$

In a similar way, we can construct blow-up HS-regimes with $v(0,t) = (1-t)^n$, where $n < -1/(2m-1)$. Approximate self-similar solutions satisfying the Hamilton–Jacobi equation are

$$V_*(x,t) = (1-t)^n h(\eta), \qquad \eta = \frac{x}{(1-t)^{\beta}}, \qquad \beta = \frac{n(2m-1)+1}{2m} < 0,$$

where $h : \mathbb{R} \to \mathbb{C}$ solves the complex ODE $(-1)^{m+1}(h')^{2m} - \beta h'\eta + nh = 0$. Complex-valued profiles $h$ describe the oscillating character of global blow-up.

## 4. Energy estimates and functional systems

We return to the general problem (1.1)–(1.3) and present our basic system of functional inequalities, which will be later on compared with other possible energy estimates of Saint-Venant's principle types.

### 4.1. Energy estimates

Let us introduce families of the subdomains of $Q$. For any fixed $s > 1$ and $\delta > 0$, we define

$$\Omega(s) = \{|x| > s\}, \qquad \Omega(s,\delta) = \Omega(s) \setminus \Omega(s+\delta), \qquad \Omega = \Omega(1),$$
$$Q_a^b(s,\delta) = \Omega(s,\delta) \times (a,b), \quad 0 \leqslant a \leqslant b \leqslant T, \qquad Q_a^b(s) \equiv Q_a^b(s,\infty).$$

We will use the interpolation Gagliardo–Nirenberg inequalities for the domains $\Omega(s,\delta)$. For any $0 \leqslant j < m$, we have

$$\|D_x^j v\|_{r,\Omega(s,\delta)} \leqslant k_1 \delta^{-j-n(r-g)/rg} \|v\|_{g,\Omega(s,\delta)} + k_2 \|D_x^m v\|^{\theta}_{p+1,\Omega(s,\delta)} \|v\|^{1-\theta}_{g,\Omega(s,\delta)}, \quad (4.1)$$

where $v(x)$ is an arbitrary function from the Sobolev space

$$W^m_{p+1}(\Omega(s,\delta)) \cap L_g(\Omega(s,\delta)),$$

$\|v\|^h_{h,\Omega} = \int_\Omega |v|^h \,\mathrm{d}x$ and $g > 0$, $r > 1$ are fixed exponents. The constant $\theta \in [j/m, 1]$ is determined by

$$\frac{1}{r} - \frac{j}{n} = \theta\left(\frac{1}{p+1} - \frac{m}{n}\right) + \frac{1-\theta}{g}. \quad (4.2)$$

Positive constants $k_1$ and $k_2$ do not depend on $v(x)$, $s$ and $\delta$. Let $\eta_0(h) \geqslant 0$ be a $C^m$-smooth cut-off function such that

$$\left.\begin{aligned} \eta_0(h) &= 0 \quad \text{for } h \leqslant 0, \\ \eta_0(h) &= 1 \quad \text{for } h \geqslant 1, \\ 0 \leqslant \eta_0(h) &\leqslant 1 \quad \text{for } h \in \mathbb{R}. \end{aligned}\right\} \quad (4.3)$$

Lemma 4.1. *Let $u(x,t)$ be an arbitrary energy solution of the problem (1.1)–(1.3). Then, for any $0 \leqslant a < b < T$, $s > 1$, $\delta > 0$ and any $\varepsilon > 0$, we have*

$$\int_{\Omega(s+\delta)} |u(x,b)|^{q+1} \,\mathrm{d}x + \int_a^b \int_{\Omega(s+\delta)} |D_x^m u(x,t)|^{p+1} \,\mathrm{d}x\mathrm{d}t$$
$$\leqslant (1+\varepsilon) \int_{\Omega(s)} |u(x,a)|^{q+1} \,\mathrm{d}x + \delta^{-m(p+1)} c(\varepsilon) \int_a^b \int_{\Omega(s,\delta)} |u(x,t)|^{p+1} \,\mathrm{d}x\mathrm{d}t. \quad (4.4)$$

*The positive constant $c(\varepsilon) \to \infty$ as $\varepsilon \to 0$ depends on parameters of the problem and is independent of $u(x,t)$, $s$ and $\delta$.*

*Proof.* Let us fix an $s_1 \in (1,s)$, $s > 1$, $\delta > 0$, and introduce the cut-off functions

$$\eta(x) = \eta_0\left(\frac{|x|-s}{\delta}\right), \quad \eta_1(x) = \eta_0\left(\frac{|x|-s_1}{\delta_1}\right), \quad \text{where } \delta_1 = s - s_1 > 0.$$

Then $\eta_1^r(x)\eta(x) \equiv \eta(x)$ for every exponent $r > 0$ and, for any $t \in (0,T)$,

$$\langle b(u)_t, \eta u\rangle = \langle b(u)_t, \eta_1^q \eta u\rangle = \langle b(\eta_1 u)_t, \eta u\rangle = \langle b(\eta_1 u)_t, \eta\eta_1 u\rangle = \langle b(v)_t, \eta v\rangle,$$

where $b(u) = |u|^{q-1}u$ and $v = \eta_1 u$. As a consequence, similar to proposition 3.2 in [11], we obtain the formula of integration by parts,

$$(q+1)q^{-1}\int_a^b \langle (|u|^{q-1}u)_t, u\eta\rangle \, \mathrm{d}t = \int_\Omega |u(x,b)|^{q+1}\eta(x)\,\mathrm{d}x - \int_\Omega |u(x,a)|^{q+1}\eta(x)\,\mathrm{d}x. \tag{4.5}$$

We substitute the test function $u(x,t)\eta(x)$ into the integral identity (1.9), where $u$ is the given energy solution. Using (4.5) and the structural conditions (1.4), (1.5), standard transformations yield

$$\begin{aligned}\int_{\Omega(s)} |u(x,b)|^{q+1}\eta\,\mathrm{d}x &+ \int_a^b\int_{\Omega(s)} |D_x^m u|^{p+1}\eta\,\mathrm{d}x\mathrm{d}t \\ &\leqslant \int_{\Omega(s)} |u(x,a)|^{q+1}\eta\,\mathrm{d}x \\ &\quad + c\biggl(\int_{Q_a^b(s,\delta)} |D_x^m u|^{p+1}\,\mathrm{d}x\mathrm{d}t\biggr)^{p/(p+1)} \\ &\quad\quad \times\biggl(\sum_{i=0}^{m-1}\int_{Q_a^b(s,\delta)} |D_x^i u|^{p+1}|D_x^{m-i}\eta|^{p+1}\,\mathrm{d}x\mathrm{d}t\biggr)^{1/(p+1)}.\end{aligned} \tag{4.6}$$

Let us estimate the second term on the right-hand side by using the interpolation inequality (4.1) with $j = 0,1,2,\dots,m-1$, $r = p+1$ and $g = q+1$. We also use the following obvious property of the cut-off function $\eta(x)$: $|D_x^i\eta| \leqslant c\delta^{-i}$ for $i \geqslant 0$. As a result, for $i = 0,1,2,\dots,m-1$, we have

$$\begin{aligned}&\int_{\Omega(s,\delta)} |D_x^i u(x,t)|^{p+1}|D_x^{m-i}\eta|^{p+1}\,\mathrm{d}x \\ &\quad\leqslant c\delta^{-(m-i)(p+1)}\int_{\Omega(s,\delta)} |D_x^i u(x,t)|^{p+1}\,\mathrm{d}x \\ &\quad\leqslant c\delta^{-(m-i)(p+1)}\biggl[k_1^{p+1}\delta^{-i(p+1)}\int_{\Omega(s,\delta)} |u|^{p+1}\,\mathrm{d}x \\ &\qquad\qquad + k_2^{p+1}\biggl(\int_{\Omega(s,\delta)} |D_x^m u|^{p+1}\,\mathrm{d}x\biggr)^{i/m}\biggl(\int_{\Omega(s,\delta)} |u|^{p+1}\,\mathrm{d}x\biggr)^{(m-i)/m}\biggr] \\ &\quad\leqslant c\delta^{-m(p+1)}k_1^{p+1}\int_{\Omega(s,\delta)} |u|^{p+1}\,\mathrm{d}x \\ &\qquad + ck_2^{p+1}\biggl(\int_{\Omega(s,\delta)} |D_x^m u|^{p+1}\,\mathrm{d}x\biggr)^{i/m}\biggl(\delta^{-m(p+1)}\int_{\Omega(s,\delta)} |u|^{p+1}\,\mathrm{d}x\biggr)^{(m-i)/m} \\ &\quad\leqslant \varepsilon\int_{\Omega(s,\delta)} |D_x^m u|^{p+1}\,\mathrm{d}x + c(\varepsilon)\delta^{-m(p+1)}\int_{\Omega(s,\delta)} |u|^{p+1}\,\mathrm{d}x,\end{aligned}$$

where, at the last step, we apply Young's inequality with a parameter $\varepsilon > 0$. Substituting these estimates into (4.6) and using Young's inequality where necessary, we obtain

$$\int_{\Omega(s)} |u(x,b)|^{q+1}\eta\,\mathrm{d}x + \int_a^b \int_{\Omega(s)} |D_x^m u(x,t)|^{p+1}\eta\,\mathrm{d}x\mathrm{d}t \leqslant \int_{\Omega(s)} |u(x,a)|^{q+1}\eta\,\mathrm{d}x + \varepsilon \int_{Q_a^b(s,\delta)} |D_x^m u(x,t)|^{p+1}\,\mathrm{d}x\mathrm{d}t + c(\varepsilon)\delta^{-m(p+1)} \int_{Q_a^b(s,\delta)} |u(x,t)|^{p+1}\,\mathrm{d}x\mathrm{d}t, \tag{4.7}$$

where $\varepsilon > 0$ can be arbitrarily small and $c(\varepsilon) \to \infty$ as $\varepsilon \to 0$.

Consider (4.7) as the relation between the following functions of argument $s$:

$$A_b(s) = \int_{\Omega(s)} |u(x,b)|^{q+1}\,\mathrm{d}x, \qquad A_a(s) = \int_{\Omega(s)} |u(x,a)|^{q+1}\,\mathrm{d}x,$$
$$B(s) = \int_{Q_a^b(s)} |u(x,t)|^{p+1}\,\mathrm{d}x\mathrm{d}t, \qquad H(s) = \int_{Q_a^b(s)} |D_x^m u(x,t)|^{p+1}\,\mathrm{d}x\mathrm{d}t.$$

It then follows from (4.7) that

$$A_b(s+\delta) + H(s+\delta) \leqslant \varepsilon H(s) + c(\varepsilon)\delta^{-m(p+1)}[B(s) - B(s-\delta)] + A_a(s). \tag{4.8}$$

Since $s > 1$ and $\delta > 0$ are arbitrary continuous arguments in (4.8), the following inequalities hold for any $j = 1, 2, \dots,$

$$A_b\left(s + \frac{\delta}{2^{j-1}}\right) + H\left(s + \frac{\delta}{2^{j-1}}\right) \leqslant \varepsilon H\left(s + \frac{\delta}{2^j}\right) + c(\varepsilon)\left(\frac{2^j}{\delta}\right)^{m(p+1)}\left[B\left(s + \frac{\delta}{2^j}\right) - B\left(s + \frac{\delta}{2^{j-1}}\right)\right] + A_a\left(s + \frac{\delta}{2^j}\right), \tag{4.9}$$

which are derived from (4.8), replacing $s \mapsto s + \delta/2^j$ and $\delta \mapsto \delta/2^j$.

Let us construct a process of successive estimates using (4.9). It is similar to the procedure used in [52]. To this end, consider it for $j = 1$ and next estimate $H(s + \frac{1}{2}\delta)$ on the right-hand side by means of the same inequality for $j = 2$. Continue this process cyclically for $j = 3, 4, \dots$ Combining all these estimates, we deduce that

$$\begin{aligned} &A_b(s+\delta) + H(s+\delta) \\ &\leqslant \varepsilon H\left(s + \frac{\delta}{2}\right) + c(\varepsilon)\left(\frac{2}{\delta}\right)^{m(p+1)}\left[B\left(s + \frac{\delta}{2}\right) - B(s+\delta)\right] + A_a\left(s + \frac{\delta}{2}\right) \\ &\leqslant \varepsilon^2 H\left(s + \frac{\delta}{4}\right) + c(\varepsilon)\left[\left(\frac{2}{\delta}\right)^{m(p+1)}\left[B\left(s + \frac{\delta}{2^2}\right) - B(s+\delta)\right]\right. \\ &\quad \left. + \varepsilon\left(\frac{2^2}{\delta}\right)^{m(p+1)}\left[B\left(s + \frac{\delta}{2^2}\right) - B\left(s + \frac{\delta}{2}\right)\right]\right] + A_a\left(s + \frac{\delta}{2}\right) + \varepsilon A_a\left(s + \frac{\delta}{2^2}\right) \\ &\leqslant \cdots \end{aligned}$$

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$$
\begin{aligned}
&\leqslant \varepsilon^j H\Big(s+\frac{\delta}{2^j}\Big) \\
&\quad + c(\varepsilon)\Big(\frac{2}{\delta}\Big)^{m(p+1)} \sum_{k=0}^{j-1}(2^{m(p+1)}\varepsilon)^k\Big[B\Big(s+\frac{\delta}{2^{k+1}}\Big)-B\Big(s+\frac{\delta}{2^k}\Big)\Big] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \sum_{k=0}^{j-1}\varepsilon^k A_a\Big(s+\frac{\delta}{2^{k+1}}\Big) \\
&\leqslant \varepsilon^j H\Big(s+\frac{\delta}{2^j}\Big) + A_a\Big(s+\frac{\delta}{2^j}\Big)\sum_{k=0}^{j-1}\varepsilon^k \\
&\quad + c(\varepsilon)\Big(\frac{2}{\delta}\Big)^{m(p+1)}\Big[B\Big(s+\frac{\delta}{2^j}\Big)-B(s+\delta)\Big]\sum_{k=0}^{j-1}(2^{m(p+1)}\varepsilon)^k. \qquad (4.10)
\end{aligned}
$$

Setting in this inequality $\varepsilon \leqslant \varepsilon_0 = 2^{-m(p+1)-1}$ and passing to the limit $j \to \infty$ yields

$$
A_b(s+\delta) + H(s+\delta) \leqslant c(\varepsilon)2^{m(p+1)+1}\frac{[B(s)-B(s+\delta)]}{\delta^{m(p+1)}} + \frac{A_a(s)}{1-\varepsilon}. \qquad (4.11)
$$

Here we have used the continuity property of the functions $A_a(s)$ and $B(s)$. Inequality (4.11) is equivalent to (4.4). This completes the proof of lemma 4.1. □

## 4.2. System of functional inequalities

We introduce an arbitrary strictly monotone sequence

$$
\{t_j\} \to T^-, \quad \Delta_j = t_j - t_{j-1} > 0 \quad \text{for } j = 1,2,\dots, t_0 \in (0,T), \qquad (4.12)
$$

and define two families of energy functions of the energy solution $u(x,t)$ of problem (1.1)–(1.3) for $s > 1$, $j \geqslant 1$,

$$
\left.
\begin{aligned}
I_j(s) &= \int_{t_{j-1}}^{t_j}\int_{\Omega(s)} |u(x,t)|^{p+1}\,\mathrm{d}x\mathrm{d}t, \\
h_j(s) &= \sup_{t\in(t_{j-1},t_j)}\int_{\Omega(s)} |u(x,t)|^{q+1}\,\mathrm{d}x, \\
I_0(s) &= \int_0^{t_0}\int_{\Omega(s)} |u(x,t)|^{p+1}\,\mathrm{d}x\mathrm{d}t, \\
h_0(s) &= \int_{\Omega(s)} |u(x,t_0)|^{q+1}\,\mathrm{d}x.
\end{aligned}
\right\} \qquad (4.13)
$$

LEMMA 4.2. Given an arbitrary energy solution of (1.1){(1.3), for any $s > 1$ and $\delta > 0$, functions (4.13) satisfy the following system of functional inequalities,

$$
h_j(s+\delta) \leqslant (1+\varepsilon_j)h_{j-1}(s) + c(\varepsilon_j)\frac{\Delta I_j(s)}{\delta^{m(p+1)}}, \quad \Delta I_j(s) \equiv I_j(s) - I_j(s+\delta), \quad (4.14)
$$

$$
I_j(s+\delta) \leqslant c_1\Delta_j^{1-\theta_1}\Big[(1+\varepsilon_j)h_{j-1}(s) + c(\varepsilon_j)\frac{\Delta I_j(s)}{\delta^{m(p+1)}}\Big]^{1+\nu}, \quad j = 1,2,\dots, \quad (4.15)
$$

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where $c_1 > 0$ is a constant,

$$\theta_1 = \frac{n(p-q)}{n(p-q)+m(p+1)(q+1)}, \qquad \nu = \frac{(p-q)(1-\theta_1)}{q+1}, \tag{4.16}$$

$\{\varepsilon_j > 0\}$ is an arbitrary sequence and positive constants $c(\varepsilon_j) \to \infty$ as $\varepsilon_j \to 0$.

Proof. Setting in (4.4) $a = t_{j-1}$ and $b = \tilde{t}$, where $\tilde{t} \in (t_{j-1}, t_j]$ is arbitrary, we obtain (4.14). In order to prove (4.15), we write down the interpolation inequality (4.1) with $\delta = \infty$, $j = 0$, $r = p+1$, $g = q+1$ and, replacing $s \mapsto s+\delta$,

$$\int_{\Omega(s+\delta)} |u|^{p+1}\,\mathrm{d}x \leqslant k_2^{p+1}\biggl(\int_{\Omega(s+\delta)} |D_x^m u|^{p+1}\,\mathrm{d}x\biggr)^{\theta_1}\biggl(\int_{\Omega(s+\delta)} |u|^{q+1}\,\mathrm{d}x\biggr)^{(1-\theta_1)(p+1)/(q+1)},$$

where $\theta_1$ is taken from (4.16). Integrating this inequality in $t$ and using estimates (4.14) and (4.4), we get

$$\begin{aligned} I_j(s+\delta) &\leqslant c_2\biggl(\int_{t_{j-1}}^{t_j}\int_{\Omega(s+\delta)} |D_x^m u|^{p+1}\,\mathrm{d}x\mathrm{d}t\biggr)^{\theta_1} \\ &\qquad \times\biggl[\int_{t_{j-1}}^{t_j}\biggl(\int_{\Omega(s+\delta)} |u|^{q+1}\,\mathrm{d}x\biggr)^{(p+1)/(q+1)}\mathrm{d}t\biggr]^{1-\theta_1} \\ &\leqslant c_1[R_j(s,\delta)]^{\theta_1}\Delta_j^{1-\theta_1}[R_j(s,\delta)]^{(p+1)(1-\theta_1)/(q+1)} \\ &\equiv c_1\Delta_j^{1-\theta_1}[R_j(s,\delta)]^{1+(p-q)(1-\theta_1)/(q+1)}, \end{aligned} \tag{4.17}$$

where

$$R_j(s,\delta) \equiv (1+\varepsilon_j)h_{j-1}(s) + c(\varepsilon_j)\frac{\Delta I_j(s)}{\delta^{m(p+1)}}.$$

Inequality (4.17) gives (4.15). □

### 4.3. From differential to functional systems of inequalities

There exist various types of energy approaches to nonlinear higher-order parabolic PDEs, developed in the last three decades, which are based on distinct (sometimes simpler than (4.14), (4.15)) differential or discrete systems of functional inequalities.

The method of introducing parameters [41] dealt with fully discretized (in both $s$ and $t$ variables) systems of functional inequalities and allowed sharp descriptions of solutions to linear higher-order parabolic PDEs to be obtained in the case of an initial singularity posed at $t = 0$ (for unbounded initial data from Tikhonov–Täcklind classes). This initial singularity demanded an infinite-time partition near $t = 0$. Extensions of this approach to divergent quasilinear parabolic equations are given in [1, 46]. In [41], an abstract scheme of the method and a survey on such methods of introducing a parameter can be found.

Later on, in applications to localization and finite propagation to quasilinear higher-order elliptic and parabolic equations, the energy methods leading to elegant

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differential inequalities with respect to the spatial variable $s$ were proposed in [9, 11]. In [47, 48], the phenomenon of finite speed of propagation was studied via differential-functional inequalities in $s$. In these problems, the time-behaviour of energy solutions was not highly non-stationary and any multi-step time-discretization was not necessary (in fact, a one-time-step approximation was sufficient).

Localization blow-up phenomena present another type of *eventual singularity* occurring at finite blow-up time $t = T^-$, essentially depending on both nonlinear properties of the equations and the blow-up boundary regime. We face new types of evolution singularity in nonlinear dynamical systems. In the analysis of the evolution singularities, the type of partitions is not known *a priori*, and, in fact, should be chosen according to unknown asymptotic properties of the unbounded blow-up solutions.

We now discuss our basic system of functional inequalities given in lemma 4.2. In order to explain key properties of solutions of system (4.14), (4.15) to be studied, we present a derivation of a hierarchy of simpler differential-discrete systems originating the system under consideration.

Consider the simplest *second-order* model problem (2.1), (2.2). For any energy solution $u(x,t)$, we introduce the corresponding two energy functions,

$$H(s,t) = \int_{\Omega(s)} |u(x,t)|^{p+1}\,\mathrm{d}x \quad \text{and} \quad E(s,t) = \int_{\Omega(s)} |D_x u(x,t)|^{p+1}\,\mathrm{d}x. \tag{4.18}$$

Assuming some additional regularity of $u(x,t)$, by straightforward computations, multiplying equation (2.1) by $u$, integrating over $\Omega(s)$, using integration by parts and the corresponding interpolation inequalities, one obtains the following partial differential inequality (PDI),

$$H_t + E \leqslant c(E_s)^{(p+1)/(p+2)} H^{1/(p+2)} \quad \text{in } Q^* = \{s \geqslant 1,\ t \in (0,T)\}, \tag{4.19}$$

where the functions are estimated from above on the boundary $s = 1$

$$H(1,t) \leqslant cF_0(t) \quad \text{and} \quad E(1,t) \leqslant cF_0(t) \quad \text{for } t \in (0,T)$$

by the corresponding blow-up data,

$$F_0(t) = \int_{\{s>1\}} (|f(x,t)|^{p+1} + |D_x f(x,t)|^{p+1})\,\mathrm{d}x,$$

which is directly related to (1.8). It is important that the PDI (4.19) contains two unknown functions (4.18), and therefore one needs an extra relation between them. We do not know if there exists a natural way to study the asymptotics of this 'undetermined' PDI of the first order.

On the other hand, introducing suitable partitions in $t$, integrating (4.19) over each time-interval and using imbeddings as a relation between $H$ and $E$ makes it possible to obtain an infinite differential (in $s$) discrete (in $t$) functional system, which can be studied (see [49]).

It is not difficult to see that, for any $2m$ equations with $m > 1$, using such a procedure, it is impossible to derive a PDI like (4.19). More precisely, the spatial derivatives in $s$ can be obtained by using a special weighted energy functional [10, 11], but the time-derivative in $t$, which is of crucial importance for essentially non-stationary solutions, cannot be preserved. Moreover, as far as we know, partitions in both independent variables $\{s, t\}$ leading to systems in lemma 4.2 is the only possible way to derive a functional relation for a single energy; we mean the iterative procedure that led from the inequality (4.6) for two energy functionals to (4.11), where the second higher-order energy $H$ is not available on the right-hand side.

Let us focus on an extra advantage of the fully discretized functional system of the type (4.14), (4.15) concerning the precise asymptotic blow-up behaviour. As it was shown in §§ 2 and 3, the asymptotic behaviour of exponential blow-up regimes like (3.8) (cf. (1.12)) is described by similarity solutions of the associated first-order Hamilton–Jacobi equations. Theorem 1.3 establishes sharp estimates of such similarity types with the optimal limit exponent $n = -1/[m(p+1) - 1]$, so that our system of functional inequalities, at least, partly contains this delicate Hamilton–Jacobi asymptotics. It is worth mentioning, returning to the PDI (4.19) for $m = 1$, that we did not succeed in a formal derivation of such asymptotics. For instance, making a natural assumption that

$$E(s,t) \approx H(s,t) \quad \text{as } t \to T^-,$$

we formally arrive at the Hamilton–Jacobi inequality:

$$H_t + H \leqslant c(H_s)^{(p+1)/(p+2)} H^{1/(p+2)}.$$

The corresponding first-order equation does not admit any approximate similarity solutions with necessary correct asymptotic blow-up properties.

### 4.4. Preliminary estimates

Corollary 4.3. *For equation (1.1) with $p = q$, the main functional system (4.14), (4.15) reads as follows. For any $s > 1$ and $\delta > 0$,*

$$h_j(s+\delta) \leqslant (1+\varepsilon_j) h_{j-1}(s) + c(\varepsilon_j) \frac{\Delta I_j(s)}{\delta^{m(p+1)}}, \tag{4.20}$$

$$I_j(s+\delta) \leqslant c_1 \Delta_j (1+\varepsilon_j) h_{j-1}(s) + c_1 c(\varepsilon_j) \Delta_j \frac{\Delta I_j(s)}{\delta^{m(p+1)}}, \quad j = 1, 2, \dots \tag{4.21}$$

Fixing an arbitrary strictly monotone decreasing sequence $\{\alpha_i > 0\}$, we define the normalized energy functions $H_j(s) = \alpha_j h_j(s)$ and $J_j(s) = \alpha_j I_j(s)$. In terms of them, system (4.20), (4.21) takes the form

$$H_j(s+\delta) \leqslant \beta_j H_{j-1}(s) + c(\varepsilon_j) \frac{\Delta J_j(s)}{\delta^{m(p+1)}}, \quad \beta_j \equiv \frac{(1+\varepsilon_j)\alpha_j}{\alpha_{j-1}}, \quad H_j = \alpha_j h_j, \tag{4.22}$$

$$J_j(s+\delta) \leqslant c_1 \beta_j \Delta_j H_{j-1}(s) + c_1 c(\varepsilon_j) \Delta_j \frac{\Delta J_j(s)}{\delta^{m(p+1)}}, \quad J_j = \alpha_j I_j. \tag{4.23}$$

The main reason for the introduction of weights $\{\alpha_j\}$ is that we can now guarantee that the iteration coefficients $\{\beta_j\}$ on the right-hand sides is strictly less than 1.

We now iterate this system. Namely, we estimate $H_{j-1}(s)$ on the right-hand side of (4.23) by means of (4.22) replacing $j \mapsto j-1$. Assuming now that $\delta = \delta_j$ in (4.22) and (4.23) depends on $j$ and denoting, for convenience, $d_j = \delta_j^{-m(p+1)}$, we obtain

$$\begin{aligned}J_j(s+\delta_j+\delta_{j-1}) &\leqslant c_1\beta_j\Delta_j H_{j-1}(s+\delta_{j-1}) + c_1 c(\varepsilon_j)\Delta_j \Delta J_j(s+\delta_{j-1})d_j \\ &\leqslant c_1\beta_j\beta_{j-1}\Delta_j H_{j-2}(s) \\ &\qquad + c_1[c(\varepsilon_j)\Delta_j d_j \Delta J_j(s+\delta_{j-1}) + c(\varepsilon_{j-1})\Delta_j\beta_j d_{j-1}\Delta J_{j-1}(s)].\end{aligned}$$

We again estimate $H_{j-2}(s)$ on the right-hand side by means of (4.22) with $j \mapsto j-2$,

$$\begin{aligned}&J_j(s+\delta_j+\delta_{j-1}+\delta_{j-2}) \\ &\quad\leqslant c_1\beta_j\beta_{j-1}\beta_{j-2}\Delta_j H_{j-3}(s) \\ &\qquad + c_1[c(\varepsilon_j)\Delta_j d_j \Delta J_j(s+\delta_{j-1}+\delta_{j-2}) \\ &\qquad\qquad + c(\varepsilon_{j-1})\Delta_j\beta_j d_{j-1}\Delta J_{j-1}(s+\delta_{j-2}) \\ &\qquad\qquad\quad + c(\varepsilon_{j-2})\Delta_j\beta_j\beta_{j-1} d_{j-2}\Delta J_{j-2}(s)].\end{aligned}$$

Repeating the same procedure, we arrive at

$$\begin{aligned}J_j\biggl(s+\sum_{i=0}^{j}\delta_i\biggr) &\leqslant c_1\beta_j\cdots\beta_1\Delta_j H_0(s+\delta_0) \\ &\qquad + c_1\sum_{k=1}^{j} c(\varepsilon_k)\Delta_j\beta_j\cdots\beta_{k+1}d_k\Delta J_k\biggl(s+\sum_{i=0}^{k-1}\delta_i\biggr),\end{aligned}$$

or, replacing $s+\sum_{i=0}^{j}\delta_i$ by $s$,

$$\begin{aligned}J_j(s) &\leqslant c_1\beta_j\cdots\beta_1\Delta_j H_0\biggl(s-\sum_{i=1}^{j}\delta_i\biggr) + c_1 c(\varepsilon_j)\Delta_j d_j[J_j(s-\delta_j)-J_j(s)] \\ &\qquad + c_1\sum_{k=1}^{j-1} c(\varepsilon_k)\Delta_j\beta_j\cdots\beta_{k+1}d_k J_k\biggl(s-\sum_{i=k}^{j}\delta_i\biggr),\end{aligned}$$

where $d_j = \delta_j^{-m(p+1)}$. Collecting like terms together, we derive the following result.

**Lemma 4.4.** *For $p = q$, normalized energy functions $J_j(s)$, $j = 1, 2, \dots$, given by (4.13), (4.23), satisfy the following functional system:*

$$\begin{aligned}J_j(s) &\leqslant \frac{\gamma_j}{1+\gamma_j}J_j(s-\delta_j) \\ &\qquad + \frac{c_1}{1+\gamma_j}\sum_{k=1}^{j-1} c(\varepsilon_k)\Delta_j\beta_j\cdots\beta_{k+1}J_k\biggl(s-\sum_{i=k}^{j}\delta_i\biggr)\delta_k^{-m(p+1)} \\ &\qquad + \frac{c_1}{1+\gamma_j}\beta_j\cdots\beta_1\Delta_j H_0\biggl(s-\sum_{i=1}^{j}\delta_i\biggr),\end{aligned} \tag{4.24}$$

*with*

$$s \geqslant 1+\sum_{i=1}^{j}\delta_i, \qquad \gamma_j = c_1 c(\varepsilon_j)\Delta_j\delta_j^{-m(p+1)}.$$

We now establish a global bound on $u(x,t)$ via the boundary function $F$.

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LEMMA 4.5. For any energy solution $u(x,t)$ of the problem (1.1){(1.3) with $p=q$, the following global a priori estimate holds:

$$\int_\Omega |u|^{p+1}\,\mathrm{d}x+\int_0^t\int_\Omega |D_x^m u|^{p+1}\,\mathrm{d}x\mathrm{d}\tau \leqslant C(\|u_0\|^{p+1}_{L_{p+1}(\Omega)}+F(t)),\quad t\in(0,T). \tag{4.25}$$

**Proof.** Using conditions (1.6), (1.7) and the definition of solution $u(x,t)$, it is easy to extend the proof of the integration-by-parts formula [2] to arbitrary $m>1$. Then

$$\begin{aligned}\int_0^t \langle(|u|^{p-1}u)_\tau, u-f\rangle\,\mathrm{d}\tau &= \frac{p}{p+1}\int_\Omega(|u(x,t)|^{p+1}-|u(x,0)|^{p+1})\,\mathrm{d}x\\ &\quad+\int_0^t\int_\Omega(|u(x,\tau)|^{p-1}u(x,\tau)-|u(x,0)|^{p-1}u(x,0))f_\tau(x,\tau)\,\mathrm{d}x\mathrm{d}\tau\\ &\quad-\int_\Omega(|u(x,t)|^{p-1}u(x,t)-|u(x,0)|^{p-1}u(x,0))f(x,t)\,\mathrm{d}x.\end{aligned}\tag{4.26}$$

Substituting the test function $\chi(x,t)=u(x,t)-f(x,t)$ into the integral identity (1.9) and using (4.26) and the structural conditions (1.4), (1.5), we conclude that

$$\begin{aligned}\frac{p}{p+1}\int_\Omega |u(x,t)|^{p+1}\,\mathrm{d}x+d_1\int_0^t\int_\Omega|D_x^m u|^{p+1}\,\mathrm{d}x\mathrm{d}\tau &\leqslant d_2\int_0^t\int_\Omega |D_x^m u(x,\tau)|^p|D_x^m f(x,\tau)|\,\mathrm{d}x\mathrm{d}\tau\\ &\quad+\int_\Omega(|u(x,t)|^{p-1}u(x,t)-|u(x,0)|^{p-1}u(x,0))f(x,t)\,\mathrm{d}x\\ &\quad-\int_0^t\int_\Omega[|u(x,\tau)|^{p-1}u(x,\tau)-|u(x,0)|^{p-1}u(x,0)]f_\tau(x,\tau)\,\mathrm{d}x\mathrm{d}\tau\\ &\quad+\frac{p}{p+1}\int_\Omega |u(x,0)|^{p+1}\,\mathrm{d}x,\end{aligned}\tag{4.27}$$

where $d_1$ is taken from (1.4). By Young's inequality, we derive the following straightforward estimates of the terms given on the right-hand side of (4.27):

$$\begin{aligned}\int_0^t\int_\Omega |D_x^m u(x,\tau)|^p|D_x^m f(x,\tau)|\,\mathrm{d}x\mathrm{d}\tau &\leqslant \varepsilon_1\int_0^t\int_\Omega|D_x^m u(x,\tau)|^{p+1}\,\mathrm{d}x\mathrm{d}\tau+c(\varepsilon_1)\int_0^t\int_\Omega|D_x^m f(x,\tau)|^{p+1}\,\mathrm{d}x\mathrm{d}\tau,\end{aligned}\tag{4.28}$$

$$\begin{aligned}\int_\Omega ||u(x,t)|^{p-1}u(x,t)-|u(x,0)|^{p-1}u(x,0)||f(x,t)|\,\mathrm{d}x &\leqslant \varepsilon_1[\|u(\cdot,t)\|^{p+1}_{L_{p+1}(\Omega)}+\|u(\cdot,0)\|^{p+1}_{L_{p+1}(\Omega)}]+c_2(\varepsilon_1)\|f(\cdot,t)\|^{p+1}_{L_{p+1}(\Omega)},\end{aligned}\tag{4.29}$$

$$\int_0^t \int_\Omega |u(x,0)|^p |f_\tau(x,\tau)| \,\mathrm{d}x\mathrm{d}\tau \leqslant \varepsilon_1 \|u_0\|^{p+1}_{L_{p+1}(\Omega)} + c_3(\varepsilon_1)\|f_\tau(\cdot,\tau)\|^{p+1}_{L_1(0,t;L_{p+1}(\Omega))}, \tag{4.30}$$

$$\begin{aligned}\int_0^t \int_\Omega |u(x,\tau)|^p |f_\tau(x,\tau)| \,\mathrm{d}x\mathrm{d}\tau &\leqslant \sup_{\tau\in(0,t)} \|u(\cdot,\tau)\|^p_{L_{p+1}(\Omega)} \int_0^t \biggl(\int_\Omega |f_\tau(x,\tau)|^{p+1}\,\mathrm{d}x\biggr)^{1/(p+1)} \mathrm{d}\tau \\ &\leqslant \varepsilon_1 \sup_{\tau\in(0,t)} \|u(\cdot,\tau)\|^{p+1}_{L_{p+1}(\Omega)} + c_4(\varepsilon_1)\|f_\tau(\cdot,\tau)\|^{p+1}_{L_1(0,t;L_{p+1}(\Omega))}. \end{aligned} \tag{4.31}$$

Combining (4.28)–(4.31) and substituting them into (4.27), we arrive at (4.25). □

## 5. First step: proof of localization of 'flat' LS-regime

The proof of theorem 1.3 consists of two steps. In this section we perform the first step and consider the case of 'flat' (lower than the optimal one in (1.12)) boundary blow-up regime, meaning that function $F$ in (1.8) satisfies

$$F(t) \leqslant c\exp\{A(T-t)^{-1/[m(p+1)+\xi_0]}\}, \quad t \in (0,T), \tag{5.1}$$

where $c$ and $A$ are fixed positive constants. The constant $\xi_0 > 0$ can be arbitrarily small.

Localization estimate (1.13) is derived obtaining a uniform boundedness with respect to $j$ of the energy $\sum_{i=1}^{j} I_i(s_0)$ for any fixed $s_0 > 1$. In the present case of the flat boundary regime (5.1), this uses the functional system (4.20), (4.21) with the following choice of its free parameters.

(i) Fix a small $\delta > 0$ ($\delta \to 0$ later on), and define a sequence of displacements (spatial partition) $\delta_i = 2^{-i}\delta$ for $i \geqslant 1$, $\delta_0 = 0$.

(ii) Choose unit normalized multipliers $\alpha_i = 1$ for $i \geqslant 1$.

(iii) All the parameters $\varepsilon_i$, $i \geqslant 1$, in the system (4.20), (4.21) and in (4.24) are constants,

$$0 < \varepsilon_i \equiv \varepsilon, \quad \text{such that } (1+\varepsilon)r_0 \equiv (1+\varepsilon)2^{-m(p+1)} = r_1 < 1. \tag{5.2}$$

(iv) The time displacements for $j = 1, 2, \dots$ (time partition) are

$$\Delta_j = \frac{\delta_j^{m(p+1)}}{c_1 c(\varepsilon)} = r_0^{j-1}\Delta_1, \qquad \Delta_1 = (c_1 c(\varepsilon))^{-1}(\tfrac{1}{2}\delta)^{m(p+1)}, \tag{5.3}$$

where $c_1$ and $c(\varepsilon)$ are the constants from (4.24). Then $\gamma_j = 1$ for all $j \geqslant 1$ and the functional system (4.24) takes the form

$$\begin{aligned} I_j(s) \leqslant \tfrac{1}{2} I_j(s-\delta_j) + \frac{1}{2}\sum_{k=1}^{j-1} r_1^{j-k} I_k\biggl(s - \sum_{i=k}^{j}\delta_j\biggr) \\ + \tfrac{1}{2}c_1 r_1^{j-1}\Delta_1 h_0\biggl(s - \sum_{i=1}^{j}\delta_i\biggr), \quad j \geqslant 1, \quad s > 1 + \sum_{i=1}^{j}\delta_i. \end{aligned} \tag{5.4}$$

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From (5.3), we calculate the value $t_0$ corresponding to displacements $\{\Delta_i,\ i \geqslant 1\}$,

$$T - t_0 = \sum_{i=1}^{\infty} \Delta_i = \Delta_1 \sum_{i=0}^{\infty} r_0^i = \frac{\Delta_1}{1 - r_0} = \frac{\delta^{m(p+1)}}{c_1 c(\varepsilon) 2^{m(p+1)}(1 - r_0)},$$

so that $t_0 = T - \delta^{m(p+1)}/c_1 c(\varepsilon)2^{m(p+1)}(1 - r_0) \to T^-$ as $\delta \to 0$ or $\varepsilon \to 0$.

Assume that $s$ satisfies the additional restriction $s - \delta_j - \sum_{i=1}^{j} \delta_i \geqslant 1$. Then the first term on the right-hand side of (5.4) can be estimated from above by means of the same inequality with the shifted argument $s \mapsto s - \delta_j$. This yields

$$\begin{aligned} I_j(s) &\leqslant 2^{-2} I_j(s - 2\delta_j) \\ &\quad + 2^{-2} \sum_{k=1}^{j-1} r_1^{j-k} I_k\biggl(s - \delta_j - \sum_{i=k}^{j} \delta_j\biggr) + 2^{-1} \sum_{k=1}^{j-1} r_1^{j-k} I_k\biggl(s - \sum_{i=k}^{j} \delta_j\biggr) \\ &\quad + c_1 r_1^{j-1} \Delta_1 \biggl[2^{-1} h_0\biggl(s - \sum_{i=1}^{j} \delta_i\biggr) + 2^{-2} h_0\biggl(s - \delta_j - \sum_{i=1}^{j} \delta_i\biggr)\biggr]. \end{aligned}$$

If we now assume that $s$ satisfies $s - 2\delta_j - \sum_{i=1}^{j} \delta_i \geqslant 1$, then the first term on the right-hand side is estimated by (5.4) with $s \mapsto s - 2\delta_j$. After similar $l$ steps, we obtain, for $j \geqslant 1$,

$$\begin{aligned} I_j(s) &\leqslant 2^{-l} I_j(s - l\delta_j) \\ &\quad + \sum_{k=1}^{j-1} r_1^{j-k} \sum_{i=1}^{l} 2^{-i} I_k(s - i\delta_j - \delta_{j-1} - \cdots - \delta_k) \\ &\quad + c_1 r_1^{j-1} \Delta_1 \sum_{i=1}^{l} 2^{-i} h_0(s - i\delta_j - \delta_{j-1} - \cdots - \delta_1), \quad s - l\delta_j - \sum_{i=1}^{j} \delta_i \geqslant 1. \end{aligned} \tag{5.5}$$

Observe that the restriction on $s$ in (5.5) holds provided that $s = s_0 \equiv 1 + 2\delta$ and $l = l_1 \equiv 2^j$. Indeed, for any $j \geqslant 1$,

$$s_0 - l_1\delta_j - \sum_{i=1}^{j-1} \delta_i = 1 + 2\delta - 2^j \delta_j - \delta \sum_{i=1}^{j-1} 2^{-i} = 1 + 2^{-j+1}\delta > 1.$$

From (5.5), we then derive the first main inequality

$$\begin{aligned} I_j(s_0) &\leqslant 2^{-l_1} I_j(s_0 - l_1\delta_j) \\ &\qquad + \sum_{k_1=1}^{j-1} r_1^{j-k} \sum_{i_1=1}^{l_1} 2^{-i_1} I_{k_1}(s_0 - i_1\delta_j - \delta_{j-1} - \cdots - \delta_{k_1}) \\ &\qquad + c_1 r_1^{j-1} \Delta_1 \sum_{i_1=1}^{l_1} 2^{-i_1} h_0(s_0 - i_1\delta_j - \delta_{j-1} - \cdots - \delta_1) \\ &\equiv A_1^{(1)} + A_1^{(2)} + A_1^{(3)}. \end{aligned} \tag{5.6}$$

Let us transform $A_1^{(2)}$ into the form

$$\begin{aligned}A_1^{(2)} &= r_1 \sum_{i_1=1}^{l_1} 2^{-i_1} I_{j-1}(s_0 - i_1\delta_j - \delta_{j-1}) \\ &\qquad + \sum_{k_1=1}^{j-2} r_1^{j-k_1} \times \sum_{i_1=1}^{l_1} 2^{-i_1} I_{k_1}(s_0 - i_1\delta_j - \delta_{j-1} - \cdots - \delta_{k_1}) \\ &\equiv A_{1,1}^{(2)} + A_{1,2}^{(2)}.\end{aligned}$$

In order to estimate the terms of $A_{1,1}^{(2)}$, we use (5.5) with $j \mapsto j-1$,

$$\begin{aligned}&I_{j-1}(s_0 - i_1\delta_j - \delta_{j-1}) \\ &\quad\leqslant 2^{-l_2} I_{j-1}(s_0 - i_1\delta_j - \delta_{j-1} - l_2\delta_{j-1}) \\ &\qquad + \sum_{k_2=1}^{j-2} r_1^{j-1-k_2} \sum_{i_2=1}^{l_2} 2^{-i_2} I_{k_2}(s_0 - i_1\delta_j - \delta_{j-1} - i_2\delta_{j-1} - \delta_{j-2} - \cdots - \delta_{k_2}) \\ &\qquad + c_1 r_1^{j-2} \varDelta_1 \sum_{i_2=1}^{l_2} 2^{-i_2} h_0(s_0 - i_1\delta_j - \delta_{j-1} - i_2\delta_{j-1} - \delta_{j-2} - \cdots - \delta_1).\end{aligned} \tag{5.7}$$

The restriction from above for $l_2$ now takes the form

$$s_0 - i_1\delta_j - \delta_{j-1} - l_2\delta_{j-1} - \sum_{i=1}^{j-2} \delta_i \geqslant 1,$$

and hence

$$l_2\delta_{j-1} \leqslant s_0 - 1 - i_1\delta_j - \sum_{i=1}^{j-1} \delta_i \equiv \delta - i_1\delta_j + \delta_{j-1}.$$

Substituting estimate (5.7) into the expression for $A_{1,1}^{(2)}$, after some manipulations in the right-hand side of (5.6), we arrive at

$$I_j(s_0) \leqslant A_2^{(1)} + A_2^{(2)} + A_2^{(3)},$$

where

$$A_2^{(1)} = A_1^{(1)} + r_1 \sum_{i_1=1}^{l_1} 2^{-(i_1+l_2)} I_{j-1}(s_0 - i_1\delta_j - \delta_{j-1} - l_2\delta_{j-1}),$$

$$\begin{aligned}A_2^{(2)} &= A_{1,2}^{(2)} + \sum_{i_1=1}^{l_1} 2^{-i_1} r_1 \sum_{k_2=1}^{j-2} r_1^{j-1-k_2} \sum_{i_2=1}^{l_2} 2^{-i_2} \\ &\qquad\qquad \times I_{k_2}(s_0 - i_1\delta_j - \delta_{j-1} - i_2\delta_{j-1} - \delta_{j-2} - \cdots - \delta_{k_2}) \\ &\equiv \sum_{k_2=1}^{j-2} \sum_{i_1=1}^{l_1} \sum_{i_2=0}^{l_2} r_1^{j-k_2} 2^{-(i_1+i_2)} \\ &\qquad\qquad \times I_{k_2}(s_0 - i_1\delta_j - i_2\delta_{j-1} - \delta_{j-1} - \delta_{j-2} - \cdots - \delta_{k_2})\end{aligned}$$

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and

$$A_2^{(3)} = A_1^{(3)} + c_1 r_1^{j-1} \Delta_1 \sum_{i_1=1}^{l_1} \sum_{i_2=1}^{l_2} 2^{-(i_1+i_2)} \times h_0(s_0 - i_1\delta_j - i_2\delta_{j-1} - \delta_{j-1} - \delta_{j-2} - \cdots - \delta_1)$$
$$\equiv c_1 r_1^{j-1} \Delta_1 \sum_{i_1=1}^{l_1} \sum_{i_2=0}^{l_2} 2^{-(i_1+i_2)} h_0(s_0 - i_1\delta_j - i_2\delta_{j-1} - \delta_{j-1} - \delta_{j-2} - \cdots - \delta_1).$$

We separate the terms in $A_2^{(2)}$ with $k_2 = j-2$ in the first term and write down the representation

$$A_2^{(2)} = A_{2,1}^{(2)} + A_{2,2}^{(2)}.$$

We estimate terms from $A_{2,1}^{(2)}$ by means of (5.5) with

$$j \mapsto j-2 \quad \text{and} \quad s \mapsto s_0 - i_1\delta_j - i_2\delta_{j-1} - \delta_{j-1} - \delta_{j-2}.$$

By manipulations similar to those at the previous step, we obtain a new form of estimate (5.6),

$$I_j(s_0) \leqslant A_3^{(1)} + A_3^{(2)} + A_3^{(3)}.$$

Performing these cyclic estimates $j-1$ times, we arrive at

$$I_j(s_0) \leqslant A_{j-1}^{(1)} + A_{j-1}^{(2)} + A_{j-1}^{(3)}, \tag{5.8}$$

where

$$\begin{aligned} A_{j-1}^{(1)} = {} & 2^{-l_1} I_j(s_0 - l_1\delta_j) \\ & + r_1 \sum_{i_1=1}^{l_1} 2^{-(i_1+l_2)} I_{j-1}(s_0 - i_1\delta_j - \delta_{j-1} - l_2\delta_{j-1}) \\ & + \cdots \\ & + r_1^h \sum_{i_1=1}^{l_1} \sum_{i_2=0}^{l_2} \cdots \sum_{i_h=0}^{l_h} 2^{-(i_1+\cdots+i_h+l_{h+1})} \\ & \qquad \times I_{j-h}\biggl(s_0 - i_1\delta_j - \cdots - i_h\delta_{j-h+1} - l_{h+1}\delta_{j-h} - \sum_{i=j-h}^{j-1} \delta_i\biggr) \\ & + \cdots \\ & + r_1^{j-2} \sum_{i_1=1}^{l_1} \sum_{i_2=0}^{l_2} \cdots \sum_{i_{j-2}=0}^{l_{j-2}} 2^{-(i_1+\cdots+i_{j-2}+l_{j-1})} \\ & \qquad \times I_2\biggl(s_0 - i_1\delta_j - \cdots - i_{j-2}\delta_3 - l_{j-1}\delta_2 - \sum_{i=2}^{j-1} \delta_i\biggr), \end{aligned}$$

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$$A^{(2)}_{j-1} = r_1^{j-1} \sum_{i_1=1}^{l_1} \sum_{i_2=0}^{l_2} \cdots \sum_{i_{j-1}=0}^{l_{j-1}} 2^{-(i_1+\cdots+i_{j-1})} \times I_1\Big(s_0 - i_1\delta_j - \cdots - i_{j-1}\delta_2 - \sum_{i=1}^{j-1}\delta_i\Big),$$

$$A^{(3)}_{j-1} = c_1 r_1^{j-1} \Delta_1 \sum_{i_1=1}^{l_1} \sum_{i_2=0}^{l_2} \cdots \sum_{i_{j-1}=0}^{l_{j-1}} 2^{-(i_1+\cdots+i_{j-1})} \times h_0\Big(s_0 - i_1\delta_j - \cdots - i_{j-1}\delta_2 - \sum_{i=1}^{j-1}\delta_i\Big).$$

The values $l_k$ satisfy the following restrictions from above:

$$\begin{aligned} s_0 - i_1\delta_j - i_2\delta_{j-1} - \cdots - i_{k-1}\delta_{j-k+2} - l_k\delta_{j-k+1} - \sum_{i=1}^{j-1}\delta_i \geqslant 1 \\ \Rightarrow \quad i_1\delta_j + i_2\delta_{j-1} + \cdots + i_{k-1}\delta_{j-k+2} + l_k\delta_{j-k+1} & \\ \leqslant 1 + 2\delta - 1 - \sum_{i=1}^{j-1}\delta_i & \\ \equiv 2\delta - \delta(1-2^{-j+1}) & \\ = \delta + \delta_{j-1}, \quad k = 1,2,\dots,j-1. & \end{aligned} \tag{5.9}$$

Our next goal is to establish appropriate estimates from above on $A^{(1)}_{j-1}$, $A^{(2)}_{j-1}$ and $A^{(3)}_{j-1}$. We begin with $A^{(1)}_{j-1}$. Let us write it down in the form

$$A^{(1)}_{j-1} = \sum_{h=0}^{j-2} r_1^h P_h, \tag{5.10}$$

where

$$P_0 = 2^{-l_1} I_j(s - l_1\delta_j),$$

$$P_1 = \sum_{i_1=1}^{l_1} 2^{-(i_1+l_2)} I_{j-1}(s_0 - i_1\delta_j - l_2\delta_{j-2} - \delta_{j-1})$$

and, for $h \geqslant 2$,

$$P_h = \sum_{i_1=1}^{l_1} \sum_{i_2=0}^{l_2} \cdots \sum_{i_h=0}^{l_h} 2^{-(i_1+\cdots+i_h+l_{h+1})} \times I_{j-h}\Big(s_0 - i_1\delta_j - \cdots - i_h\delta_{j-h+1} - l_{h+1}\delta_{j-h} - \sum_{i=j-h}^{j-1}\delta_i\Big),$$

By condition (5.9) on the number of iteration, the following estimate on $P_h$ holds:

$$P_h \leqslant I_{j-h}(1) \sum_{i_1=1}^{l_1} \sum_{i_2=0}^{l_2} \cdots \sum_{i_h=0}^{l_h} 2^{-(i_1+\cdots+i_h+l_{h+1})}, \quad h \leqslant j-2. \tag{5.11}$$

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According to (5.9), we also conclude that $l_{h+1}$ satisfies

$$i_1\delta_j + i_2\delta_{j-1} + \cdots + i_h\delta_{j-h+1} + l_{h+1}\delta_{j-h} \leqslant \delta + \delta_{j-1}.$$

Hence, by the definition of sequence $\{\delta_j\}$, we have

$$\delta_j(i_1 + 2i_2 + 2^2 i_3 + \cdots + 2^{h-1}i_h + 2^h l_{h+1}) \leqslant \delta_j(2^j + 2).$$

Therefore,

$$l_{h+1} \leqslant 2^{j-h} - 2^{-h}(i_1 + 2i_2 + 2^2 i_3 + \cdots + 2^{h-1}i_h) + 2^{1-h},$$

so that we can set

$$l_{h+1} = [2^{j-h} - 2^{-h}(i_1 + 2i_2 + \cdots + 2^{h-1}i_h) + 2^{1-h}],$$

where $[\cdot]$ denotes the integer part. Obviously, we have

$$l_{h+1} \geqslant 2^{j-h} - 2^{-h}(i_1 + 2i_2 + \cdots + 2^{h-1}i_h) - 1.$$

Extending estimate (5.11), we obtain

$$P_h \leqslant 2I_{j-h}(1) \sum_{i_1=1}^{\infty} \sum_{i_2=0}^{\infty} \cdots \sum_{i_h=0}^{\infty} 2^{-R(i_1,i_2,\dots,i_h)}, \tag{5.12}$$

where

$$R(i_1, i_2, \dots, i_h) = 2^{j-h} + i_1(1 - 2^{-h}) + i_2(1 - 2^{-(h-1)}) + \cdots + i_h(1 - 2^{-1}).$$

It then follows from (5.12) that

$$\begin{aligned} P_h &\leqslant 2I_{j-h}(1) \sum_{i_1=1}^{\infty} 2^{-(1-2^{-h})i_1} \prod_{l=2}^{h} \biggl(\sum_{i_l=0}^{\infty} 2^{-(1-2^{-(h-l+1)})i_l}\biggr) 2^{-2^{j-h}} \\ &\leqslant 2^{2^{-h}} 2^{-2^{j-h}} I_{j-h}(1) \prod_{l=1}^{h} b_i, \end{aligned}$$

where $b_i = (1 - 2^{-(1-2^{-i})})^{-1}$. In order to estimate constants $b_i$, we observe that, for arbitrarily small $\mu > 0$, there exists a number $k_0 = k_0(\mu)$ such that

$$b_i \leqslant 2 + \mu \quad \text{for any } i > k_0. \tag{5.13}$$

Hence we have

$$\prod_{i=1}^{h} b_i \leqslant (2 + \mu)^{h-k_0} g_0^{k_0} \quad \text{for } h > 1,$$

where $g_0 = \sqrt{2}/(\sqrt{2} - 1)$. Extending the above estimate on $P_h$, we obtain

$$P_h \leqslant 2^{2^{-h}} I_{j-h}(1) 2^{-2^{j-h}} (2 + \mu)^{h-k_0} g_0^{k_0}. \tag{5.14}$$

We now estimate $I_{j-h}(1)$ from above. By the *a priori* estimate (4.25), from lemma 4.5, it follows by assumption (5.1) that

$$I_{j-h}(1) \leqslant cF(t_{j-h}) = c_3 \exp\{A(T - t_{j-h})^{-\beta}\}, \qquad \beta = \frac{1}{m(p+1) + \xi_0}. \tag{5.15}$$

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By definition (5.3) of $\{\Delta_j\}$, we have

$$T - t_{j-h} = \sum_{i=j-h+1}^{\infty} \Delta_j = \sum_{i=j-h+1}^{\infty} r_0^{i-1}\Delta_1 = \frac{\Delta_1 r_0^{j-h}}{1-r_0} \equiv \frac{\Delta_1 2^{-m(p+1)(j-h)}}{1-r_0}.$$

Hence inequality (5.15) provides us with the estimate

$$I_{j-h}(1) \leqslant c_3 \exp\biggl\{A\biggl[\frac{1-r_0}{\Delta_1}\biggr]^{\beta} 2^{(j-h)(1-\xi_0\beta)}\biggr\}.$$

Substituting this estimate into (5.14), we get

$$P_h \leqslant c_3 2^{2^{-h}}(2+\mu)^{h-k_0} g_0^{k_0} \exp\{-\,2^{j-h} M(j-h,\xi_0,\delta)\}, \tag{5.16}$$

where the function $M(v,\xi_0,\delta)$ has the form

$$\begin{aligned} M(v,\xi_0,\delta) &= \ln 2 - A\biggl(\frac{1-r_0}{\Delta_1 2^{v\xi_0}}\biggr)^{\beta} \\ &\equiv \ln 2 - A\biggl[\frac{(1-r_0)c_1 c(\varepsilon)2^{m(p+1)}}{\delta^{m(p+1)}2^{v\xi_0}}\biggr]^{\beta}. \end{aligned} \tag{5.17}$$

Here, the constants $c_1$ and $c(\varepsilon)$ are given in (4.24), (5.3), $\varepsilon > 0$ satisfies (5.2) and $\beta$ is from (5.15). We need some properties of the functions $M(v,\xi_0,\delta)$. For arbitrarily small values of $\delta > 0$ and $\xi_0 > 0$, there exists finite $v_0 = v_0(\xi_0,\delta) > 0$ such that $M(v,\xi_0,\delta) \geqslant \frac{1}{2}\ln 2$ for any $v \geqslant v_0$. Hence estimate (5.16) can be extended as follows:

$$P_h \leqslant \begin{cases} c_3 2^{2^{-h}}(2+\mu)^{h-k_0} g_0^{k_0} \exp\{-\,2^{j-h}\frac{1}{2}\ln 2\} & \text{if } h \leqslant j - v_0, \\ c_3 2^{2^{-h}}(2+\mu)^{h-k_0} g_0^{k_0} \exp\{2^{1+v_0} B\delta^{-m(p+1)\beta}\} & \text{if } h > j - v_0. \end{cases} \tag{5.18}$$

The constant $B = A[(1-r_0)c_1 c(\varepsilon)]^{\beta}$ is independent of $h$, $j$ and other constants in (5.18). We have $k_0 \equiv k_0(\mu) \to \infty$ as $\mu \to 0$ and $v_0(\xi_0,\delta) \to \infty$ as $\xi_0 \to 0$ or $\delta \to 0$.

We now estimate $A_{j-1}^{(1)}$. In view of (5.18), it follows from (5.10) that

$$\begin{aligned} A_{j-1}^{(1)} \leqslant{}& \sum_{h=0}^{j-v_0} c_3 2^{2^{-h}} g_0^{k_0}[r_1(2+\mu)]^h (2+\mu)^{-k_0} \exp\{-\,2^{j-h}\tfrac{1}{2}\ln 2\} \\ &+ \sum_{h=j-v_0+1}^{j-2} c_3 2^{2^{-h}} g_0^{k_0}[r_1(2+\mu)]^h (2+\mu)^{-k_0} \exp\{2^{1+v_0} B\delta^{-m(p+1)\beta}\} \\ \equiv{}& A_{j-1,1}^{(1)} + A_{j-1,2}^{(1)}. \end{aligned}$$

We recall that, by (5.2), $r_1 = (1+\varepsilon)r_0 = (1+\varepsilon)2^{-m(p+1)}$. Therefore, for any $m \geqslant 1$ and $p > 0$, we can choose small fixed $\varepsilon > 0$ and the $\mu > 0$ from (5.13) such that

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$(2+\mu)r_1 = (2+\mu)(1+\varepsilon)/2^{m(p+1)} = \lambda_1 < 1$. Then we deduce the estimates

$$\begin{aligned} A^{(1)}_{j-1,1} &\leqslant 2c_3 g_0^{k_0}(2+\mu)^{-k_0} \sum_{h=0}^{j-v_0} \lambda_1^h \exp\{-2^{j-h}\tfrac{1}{2}\ln 2\} \\ &\leqslant c_4\lambda_1^j \sum_{h=0}^{j-v_0} \exp\{-(\tfrac{1}{2}\ln 2)2^{j-h} + (j-h)\ln\lambda_1^{-1}\} \\ &< c_4\lambda_1^j B_1, \end{aligned} \tag{5.19}$$

$$\begin{aligned} B_1 &= B_1(\lambda_1) \\ &= \sum_{i=1}^{\infty} \exp\{-(\tfrac{1}{2}\ln 2)2^i + i\ln\lambda_1^{-1}\} \\ &= \text{const.}, \end{aligned} \tag{5.20}$$

$$c_4 = 2c_3 g_0^{k_0}(2+\mu)^{-k_0}, \tag{5.21}$$

$$\begin{aligned} A^{(1)}_{j-1,2} &\leqslant B_2 \sum_{h=j-v_0+1}^{j-2} \lambda_1^h \\ &\leqslant B_2\lambda_1^{j-v_0+1}(1-\lambda_1)^{-1}, \end{aligned} \tag{5.22}$$

$$B_2 = c_4\exp\{2^{1+v_0}B\delta^{-m(p+1)\beta}\}. \tag{5.23}$$

Let us estimate the term $A^{(2)}_{j-1}$. We have

$$\begin{aligned} A^{(2)}_{j-1} &\leqslant r_1^{j-1} \sum_{i_1=1}^{\infty}\sum_{i_2=0}^{\infty}\cdots\sum_{i_{j-1}=0}^{\infty} 2^{-(i_1+\cdots+i_{j-1})} I_1(1) \\ &= r_1^{j-1} I_1(1) 2^{j-2} \\ &= \tfrac{1}{2}(2r_1)^{j-1} I_1(1). \end{aligned}$$

It follows that $2r_1 < \lambda_1 < 1$, and hence

$$A^{(2)}_{j-1} \leqslant 2^{-1}\lambda_1^{j-1} I_1(1). \tag{5.24}$$

Finally, consider $A^{(3)}_{j-1}$,

$$\begin{aligned} A^{(3)}_{j-1} &\leqslant c_1 r_1^{j-1} h_0(1)\varDelta_1 \sum_{i_1=1}^{\infty}\sum_{i_2=0}^{\infty}\cdots\sum_{i_{j-1}=0}^{\infty} 2^{-(i_1+i_2+\cdots+i_{j-1})} \\ &\leqslant \tfrac{1}{2}c_1(2r_1)^{j-1}\varDelta_1 h_0(1) \\ &\leqslant \tfrac{1}{2}c_1\lambda_1^{j-1}\varDelta_1 h_0(1). \end{aligned} \tag{5.25}$$

Thus we obtain from (5.8), due to (5.19)–(5.25), that $I_j(s_0) = I_j(1+2\delta) \leqslant \tilde{c}\lambda_1^j$ for any $j = 1, 2, \dots$, where $\tilde{c} = \tilde{c}(\delta)$ is independent of $j$ and $\delta > 0$ can be chosen arbitrarily small, and hence

$$I^{(t_j)}(s_0) \equiv \sum_{i=1}^{j} I_i(1+2\delta) \leqslant \tilde{c}\sum_{i=0}^{j-1}\lambda_1^i \leqslant \frac{\tilde{c}}{1-\lambda_1}, \quad j = 1, 2, \dots$$

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This uniform-in-$j$ estimate completes the proof of theorem 1.3 under the additional restriction (5.1).

## 6. End of the proof of theorem 1.3: 'steep' LS-regime

We now consider optimal blow-up LS-regimes. Namely, according to theorem 1.3, the boundary blow-up function satisfies, for $t \in (0, T)$,

$$F(t) \leqslant c \exp\{A(T-t)^{-\phi}\}, \quad \text{where } \phi = \frac{1}{[m(p+1)-1+\xi_0]} \quad \text{and} \quad \xi_0 > 0. \tag{6.1}$$

We begin with the functional system (4.24), but unlike the previous section, we use a different choice of free parameters in partitions. In the case of 'steep' boundary regimes (6.1), the optimal choice of the partition $\{\Delta_i\}$ has to depend on the spatial variable $s$; more precisely, on the *a priori* unknown profile of solution $u(x,t)$ under consideration,

$$I^{(t)}(s) = \int_{\Omega(s)\times(0,t)} |u(x,t)|^{p+1}\,\mathrm{d}x\mathrm{d}t \quad \text{for } t \approx T, \quad s > 1.$$

Therefore, we need to arrange an iteration procedure. In the first step, we perform estimates of functions $I_j(s)$ corresponding to the partition $\{\Delta_i\}$ connected with the initial boundary regime (6.1). In the next steps, the corresponding partitions will be chosen by means of 'fictitious, more flat' regimes obtained from energy estimates of solution in the previous step. Let us perform such a construction.

Step 1. Fix $\varepsilon_i = \varepsilon_0 = \text{const.}$, $i \geqslant 1$, where $\varepsilon_0$ is sufficiently small.

Step 2. Fix a positive constant $r_3 < 1$ and define the sequence $\{\alpha_j\}$ of normalized multipliers in (4.22), (4.23) as follows:

$$\alpha_j = \alpha_{j-1} r_2 \equiv \frac{\alpha_{j-1} r_3}{1+\varepsilon_0} \quad \Rightarrow \quad \beta_j = r_3 < 1. \tag{6.2}$$

Step 3. Let us fix a large constant $K > 0$ (in what follows, we pass to the limit $K \to \infty$) and define the partitions

$$\delta_i = (K+i)^{-1-\xi_1/m(p+1)}, \quad \Delta_i = (K+i)^{-m(p+1)-\xi_1}, \quad i \geqslant 1 \tag{6.3}$$

(we set $\delta_0 = 0$), where the constant $\xi_1 = \xi_1(\xi_0) > 0$ will be determined below. Under the above definitions, the functional system (4.24) takes the form

$$J_j(s) \leqslant \lambda_1 J_j(s-\delta_j) + \lambda_1 \sum_{k=1}^{j-1} \frac{\Delta_j}{\Delta_k} r_3^{j-k} J_k\biggl(s - \sum_{i=k}^{j} \delta_i\biggr) + c_2 r_3^j \Delta_j H_0\biggl(s - \sum_{i=1}^{j} \delta_i\biggr), \tag{6.4}$$

where

$$\lambda_1 = \frac{c_1 c(\varepsilon_0)}{1+c_1 c(\varepsilon_0)} < 1 \quad \text{and} \quad c_2 = \frac{c_1}{1+c_1 c(\varepsilon_0)}.$$

Recall that

$$J_j(s) = I_j(s) r_2^j = I_j(s) \exp\{-\ln r_2^{-1} j\}.$$

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System (6.4) looks similar to (5.4) for the flat LS-regime and we apply the same iteration procedure, which beforehand led (5.4) to (5.5). Then we obtain, for $j \geqslant 1$,

$$J_j(s) \leqslant \lambda_1^l J_j(s - l\delta_j) + \sum_{k=1}^{j-1} \frac{\Delta_j}{\Delta_k} r_3^{j-k} \sum_{i=1}^{l} \lambda_1^i J_k\biggl(s - i\delta_j - \sum_{h=k}^{j-1} \delta_h\biggr) + c_2 r_3^j \Delta_j \sum_{i=1}^{l} \lambda_1^{i-1} H_0\biggl(s - i\delta_j - \sum_{h=1}^{j-1} \delta_h\biggr) \equiv \tilde{A}_1^{(1)}(l) + \tilde{A}_1^{(2)}(l) + \tilde{A}_1^{(3)}(l). \tag{6.5}$$

Let us comment on a possible number of iterations $l$ in (6.5). Denote

$$\delta = \delta(K, \xi_1) = \sum_{i=1}^{\infty} \delta_j \equiv \sum_{i=1}^{\infty} (K + i)^{-1-\xi_1/m(p+1)}. \tag{6.6}$$

We have

$$\int_{K+1}^{\infty} x^{-1-\xi_1/m(p+1)}\,\mathrm{d}x \leqslant \delta \leqslant \int_{K}^{\infty} x^{-1-\xi_1/m(p+1)}\,\mathrm{d}x,$$

and consequently

$$\xi_1^{-1} m(p+1)(K+1)^{-\xi_1/m(p+1)} \leqslant \delta(K, \xi_1) \leqslant \xi_1^{-1} m(p+1) K^{-\xi_1/m(p+1)}, \tag{6.7}$$

so that, for any fixed $\xi_1 > 0$, $\delta(K, \xi_1) \to 0$ as $K \to \infty$. Denote

$$s_0 = 1 + 2\delta(K, \xi_1) \tag{6.8}$$

and determine the following number of iterations at the fist step, $l_1 = l_1(j) = [\delta/\delta_j]$ for $j \geqslant 1$. Then

$$s_0 - l_1\delta_j - \sum_{i=1}^{j} \delta_i = 1 + 2\delta - l_1\delta_j - \sum_{i=1}^{j} \delta_i \geqslant 1 + 2\delta - \delta - \sum_{i=1}^{j} \delta_i = 1 + \sum_{i=j+1}^{\infty} \delta_i > 1.$$

Hence (6.5) holds with $l = l_1$. We again decompose $\tilde{A}_1^{(2)}$ in a manner similar to that in the previous section to get

$$\tilde{A}_1^{(2)} = \frac{\Delta_j}{\Delta_{j-1}} r_3 \sum_{i_1=1}^{l_1} \lambda_1^{i_1} J_{j-1}(s_0 - i_1\delta_j - \delta_{j-1}) + \sum_{k_1=1}^{j-2} \frac{\Delta_j}{\Delta_{k_1}} r_3^{j-k_1} \sum_{i_1=1}^{l_1} \lambda_1^{i_1} J_{k_1}\biggl(s_0 - i_1\delta_j - \sum_{h=k_1}^{j-1} \delta_h\biggr) \equiv \tilde{A}_{1,1}^{(2)} + \tilde{A}_{1,2}^{(2)}. \tag{6.9}$$

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We estimate the terms in $\tilde{A}^{(2)}_{1,1}$ by using (6.5) similarly to (5.7),

$$\begin{aligned}
&J_{j-1}(s_0 - i_1\delta_j - \delta_{j-1})\\
&\quad\leqslant \lambda_1^{l_2} J_{j-1}(s_0 - i_1\delta_j - \delta_{j-1} - l_2\delta_{j-1})\\
&\qquad + \sum_{k_2=1}^{j-2} \frac{\Delta_{j-1}}{\Delta_{k_2}} r_3^{j-1-k_2} \sum_{i_2=1}^{l_2} \lambda_1^{i_2} J_{k_2}\biggl(s_0 - i_1\delta_j - \delta_{j-1} - i_2\delta_{j-1} - \sum_{h=k_2}^{j-2}\delta_h\biggr)\\
&\qquad + c_2 r_3^{j-1}\Delta_{j-1}\sum_{i_2=1}^{l_2}\lambda_1^{i_2-1} H_0\biggl(s_0 - i_1\delta_j - \delta_{j-1} - i_2\delta_{j-1} - \sum_{h=1}^{j-2}\delta_h\biggr). \qquad (6.10)
\end{aligned}$$

Substituting estimate (6.10) into the expression for $\tilde{A}^{(2)}_{1,1}$, by some obvious manipulations with the terms on the right-hand side in (6.5), we obtain

$$J_j(s_0) \leqslant \tilde{A}^{(1)}_2 + \tilde{A}^{(2)}_2 + \tilde{A}^{(3)}_2,$$

with $\tilde{A}^{(1)}_2$, $\tilde{A}^{(2)}_2$ and $\tilde{A}^{(3)}_2$ corresponding to $A^{(1)}_2$, $A^{(2)}_2$ and $A^{(3)}_2$ introduced in § 5. After $j-1$ steps of such calculations, similar to the previous section, we deduce that

$$J_j(s_0) \leqslant \tilde{A}^{(1)}_{j-1} + \tilde{A}^{(2)}_{j-1} + \tilde{A}^{(3)}_{j-1}, \quad j \geqslant 1, \qquad (6.11)$$

$$\begin{aligned}
\tilde{A}^{(1)}_{j-1} &= \lambda_1^{l_1} J_j(s_0 - l_1\delta_j)\\
&\qquad + \frac{\Delta_j}{\Delta_{j-1}} r_3 \sum_{i_1=1}^{l_1} \lambda_1^{i_1+l_2} J_{j-1}(s_0 - i_1\delta_j - l_2\delta_{j-1} - \delta_{j-1})\\
&\qquad + \frac{\Delta_j}{\Delta_{j-2}} r_3^2 \sum_{i_1=1}^{l_1}\sum_{i_2=0}^{l_2} \lambda_1^{i_1+i_2+l_3}\\
&\qquad\qquad \times J_{j-2}(s_0 - i_1\delta_j - i_2\delta_{j-1} - l_3\delta_{j-2} - \delta_{j-1} - \delta_{j-2})\\
&\qquad + \cdots\\
&\qquad + \frac{\Delta_j}{\Delta_2} r_3^{j-1} \sum_{i_1=1}^{l_1}\sum_{i_2=0}^{l_2}\cdots\sum_{i_{j-2}=0}^{l_{j-2}} \lambda_1^{i_1+i_2+\cdots+i_{j-2}+l_{j-1}}\\
&\qquad\qquad \times J_2\biggl(s_0 - i_1\delta_j - \cdots - i_{j-2}\delta_3 - l_{j-1}\delta_2 - \sum_{h=2}^{j-1}\delta_h\biggr),\\
\tilde{A}^{(2)}_{j-1} &= \frac{\Delta_j}{\Delta_1} r_3^{j-1} \sum_{i_1=1}^{l_1}\sum_{i_2=0}^{l_2}\cdots\sum_{i_{j-1}=0}^{l_{j-1}} \lambda_1^{i_1+i_2+\cdots+i_{j-1}}\\
&\qquad\qquad \times J_1\biggl(s_0 - i_1\delta_j - \cdots - i_{j-1}\delta_2 - \sum_{h=1}^{j-1}\delta_h\biggr),\\
\tilde{A}^{(3)}_{j-1} &= c_2 r_3^j \Delta_j \lambda_1^{-1} \sum_{i_1=1}^{l_1}\sum_{i_2=0}^{l_2}\cdots\sum_{i_{j-1}=0}^{l_{j-1}} \lambda_1^{i_1+i_2+\cdots+i_{j-1}}\\
&\qquad\qquad \times H_0\biggl(s_0 - i_1\delta_j - \cdots - i_{j-1}\delta_2 - \sum_{h=1}^{j-1}\delta_h\biggr).
\end{aligned}$$

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Note that the values of $l_k = l_k(i_1, i_2, \dots, i_{k-1})$, $k = 1, 2, \dots, j-1$, are determined from (5.9), where now $s_0$ is given in (6.8) and $\{\delta_i\}$ are defined in (6.3). Thus

$$\tilde{A}^{(1)}_{j-1} = \sum_{h=0}^{j-2} \frac{\Delta_j}{\Delta_{j-h}} r_3^h \tilde{P}_h, \tag{6.12}$$

where the $\tilde{P}_h$ are similar to $P_h$ from (5.10). In particular, the following estimate holds:

$$\tilde{P}_h \leqslant J_{j-h}(1) \sum_{i_1=1}^{l_1} \sum_{i_2=0}^{l_2} \cdots \sum_{i_h=0}^{l_h} \lambda_1^{i_1+i_2+\cdots+i_h+l_{h+1}}, \quad h \geqslant 0. \tag{6.13}$$

Using (5.9) again, we derive the condition determining possible values of $l_{h+1}$,

$$l_{h+1} \leqslant \frac{\delta - i_1\delta_j - i_2\delta_{j-2} - \cdots - i_h\delta_{j-h+1}}{\delta_{j-h}} \equiv a.$$

Hence we can set $l_{h+1} = [a]$ and then

$$\begin{aligned} i_1 + i_2 + \cdots + i_h + l_{h+1} \\ \geqslant \frac{\delta}{\delta_{j-h}} + i_1\left(1 - \frac{\delta_j}{\delta_{j-h}}\right) + i_2\left(1 - \frac{\delta_{j-1}}{\delta_{j-h}}\right) + \cdots + i_h\left(1 - \frac{\delta_{j-h+1}}{\delta_{j-h}}\right) - 1. \end{aligned}$$

Therefore, the following estimates on $\tilde{P}_h$ hold,

$$\begin{aligned} \tilde{P}_h &\leqslant \lambda_1^{-1} J_{j-h}(1) \exp\left\{-\ln \lambda_1^{-1} \frac{\delta}{\delta_{j-h}}\right\} \\ &\quad \times \sum_{i_1=1}^{\infty} \lambda_1^{(1-\delta_j/\delta_{j-h})i_1} \sum_{i_2=0}^{\infty} \lambda_1^{(1-\delta_{j-1}/\delta_{j-h})i_2} \cdots \sum_{i_h=0}^{\infty} \lambda_1^{(1-\delta_{j-h+1}/\delta_{j-h})i_h} \\ &\leqslant \lambda_1^{-1} J_{j-h}(1) \exp\left\{-\ln \lambda_1^{-1} \frac{\delta}{\delta_{j-h}}\right\} \Pi_h, \end{aligned} \tag{6.14}$$

where

$$\begin{aligned} \Pi_h = \lambda_1^{-(1-\delta_{j-1}/\delta_{j-h})} \cdots \lambda_1^{-(1-\delta_{j-h+1}/\delta_{j-h})} (\lambda_1^{-(1-\delta_j/\delta_{j-h})} - 1)^{-1} \\ \cdots (\lambda_1^{-(1-\delta_{j-h+1}/\delta_{j-h})} - 1)^{-1}. \end{aligned} \tag{6.15}$$

To estimate $\Pi_h$ from above we need bounds on the last factors. By Lagrange's formula,

$$\lambda_1^{-(1-\delta_{j-k}/\delta_{j-h})} - 1 = \exp\left\{\ln \lambda_1^{-1}\left(1 - \frac{\delta_{j-k}}{\delta_{j-h}}\right)\right\} - 1 \geqslant \ln \lambda_1^{-1}\left(1 - \frac{\delta_{j-k}}{\delta_{j-h}}\right).$$

Therefore, equation (6.15) implies that, for any $h \geqslant 1$,

$$\Pi_h \leqslant \lambda_1^{-(h-1-(\delta_{j-1}+\cdots+\delta_{j-h+1})/\delta_{j-h})} \left[(\ln \lambda_1^{-1})^h \prod_{k=0}^{h-1} \left(1 - \frac{\delta_{j-k}}{\delta_{j-h}}\right)\right]^{-1}. \tag{6.16}$$

It follows from (6.3) that

$$1-\frac{\delta_{j-k}}{\delta_{j-h}}=1-\left(1-\frac{h-k}{K+j-k}\right)^{1+\xi_1/m(p+1)}\geqslant\frac{h-k}{K+j-k}$$

and

$$\prod_{k=0}^{h-1}\left(1-\frac{\delta_{j-k}}{\delta_{j-h}}\right)\geqslant\prod_{k=0}^{h-1}\frac{h-k}{K+j-k}=\frac{h!(K+j-h)!}{(K+j)!}\equiv(C_{K+j}^{h})^{-1},$$

where the binomial coefficient satisfies $(C_{K+j}^{h})^{-1}\geqslant 2^{-(K+j)}$. Therefore, we have

$$\varPi_h\leqslant\mu_1^h\lambda_1^{(\delta_{j-1}+\delta_{j-2}+\cdots+\delta_{j-h})/\delta_{j-h}}C_{K+j}^{h},\qquad \mu_1=(\lambda_1\ln\lambda_1^{-1})^{-1}.$$

Substituting this into (6.14) yields

$$\tilde{P}_h\leqslant\lambda_1^{(\delta_{j-1}+\delta_{j-2}+\cdots+\delta_{j-h+1})/\delta_{j-h}}\mu_1^hC_{K+j}^{h}J_{j-h}(1)\exp\left\{-\ln\lambda_1^{-1}\frac{\delta}{\delta_{j-h}}\right\}.\tag{6.17}$$

Let us estimate the last two factors on the right-hand side. By (6.7) and by the definition of $\delta_i$ in (6.3), we have

$$\frac{\delta}{\delta_{j-h}}\geqslant\frac{m(p+1)(K+j-h)^{1+\xi_1/m(p+1)}}{\xi_1(K+1)^{\xi_1/m(p+1)}}\tag{6.18a}$$

and

$$\exp\left\{-\ln\lambda_1^{-1}\frac{\delta}{\delta_{j-h}}\right\}\leqslant\exp\left\{-\frac{\ln\lambda_1^{-1}m(p+1)(K+j-h)^{1+\xi_1/m(p+1)}}{\xi_1(K+1)^{\xi_1/m(p+1)}}\right\}.\tag{6.18b}$$

We now estimate $J_{j-h}(1)$. By the *a priori* estimate (4.25), it follows from lemma 4.5, condition (6.1) on the boundary blow-up regime, definition (4.23) of $J_j(s)$ and from (6.2) that

$$J_{j-h}(1)\equiv r_2^{j-h}I_{j-h}(1)\leqslant r_2^{j-h}CF(t_{j-h})\leqslant Cr_2^{j-h}\exp\{A(T-t_{j-h})^{-\phi}\},\tag{6.19}$$

where $r_2=r_3/(1+\varepsilon_0)$. Let us derive an estimate on $T-t_i$ for any $i\geqslant 1$,

$$T-t_i=\sum_{k=i+1}^{\infty}\varDelta_k=\sum_{k=K+i+1}^{\infty}k^{-m(p+1)-\xi_1}.$$

Hence

$$\int_{K+i+1}^{\infty}x^{-[m(p+1)+\xi_1]}\,\mathrm{d}x\leqslant T-t_i\leqslant\int_{K+i}^{\infty}x^{-[m(p+1)+\xi_1]}\,\mathrm{d}x,$$

so that, for $i\geqslant 1$,

$$\phi_1(K+i+1)^{-1/\phi_1}\leqslant(T-t_i)\leqslant\phi_1(K+i)^{-1/\phi_1},\qquad\phi_1=\frac{1}{m(p+1)-1+\xi_1}.\tag{6.20}$$

Therefore, equation (6.19) implies that

$$J_{j-h}(1)\leqslant Cr_2^{j-h}\exp\{A_1(K+j-h+1)^{\phi/\phi_1}\},\tag{6.21}$$

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where $A_1 = A\phi_1^{-\phi}$. We now choose $\xi_1 = \xi_1(\xi_0) > 0$ such that

$$\frac{\phi}{\phi_1} \equiv \frac{m(p+1)-1+\xi_1}{m(p+1)-1+\xi_0} = 1 + \frac{\xi_1}{2m(p+1)} \quad \Rightarrow \quad \xi_1 = \xi_0 \frac{2m(p+1)}{m(p+1)+1-\xi_0}, \tag{6.22}$$

and, in particular, for any $\xi_0 > 0$,

$$\xi_1 > \gamma_0 \xi_0, \quad \gamma_0 = \frac{2m(p+1)}{m(p+1)+1} > 1. \tag{6.23}$$

Then estimate (6.21) for $h \geqslant 1$ takes the form

$$J_{j-h}(1) \leqslant C r_2^{j-h} \exp\{A_1(K+j-h+1)^{\nu_1}\}, \quad \nu_1 = 1 + \frac{\xi_1}{2m(p+1)} > 1. \tag{6.24}$$

Substituting estimates (6.18) and (6.24) into (6.17), we obtain

$$\tilde{P}_h \leqslant C\mu_1^h 2^{K+j} r_2^{j-h} \exp\{-A_2(K+j-h)^{2\nu_1-1} + A_1(K+j-h+1)^{\nu_1}\}, \tag{6.25}$$

where $A_2 = m(p+1)\ln\lambda_1^{-1}/\xi_1(K+1)^{2(\nu_1-1)}$. We need some extra simple manipulations. Let us write down the exponential term in the right-hand side of (6.25) in the form

$$\Gamma_1 \equiv -(K+j-h)^{\nu_1}\{A_2(K+j-h)^{\nu_1-1} - A_1(1+(K+j-h)^{-1})^{\nu_1}\}.$$

Consider the auxiliary function

$$\Gamma_2(v) = A_2(K+v)^{\nu_1-1} - A_1[1+(K+v)^{-1}]^{\nu_1}.$$

One can see that there exists $v_0 = v_0(A_2, A_1, K, \xi_1)$ such that $\Gamma_2(v) > 1$ for all $v > v_0$. Denote

$$\Gamma_2^{(0)} \equiv \Gamma_2^{(0)}(A_2, A_1, K, \xi_1) = \max_{2 \leqslant v \leqslant v_0} |\Gamma_2(v)|.$$

By the definition of $\Gamma_2(v)$, we have $\Gamma_1 \leqslant -(K+j-h)^{\nu_1}$ if $j-h \geqslant v_0$ and $\Gamma_1 \leqslant \Gamma_2^{(0)}(K+v_0)^{\nu_1}$ if $j-h < v_0$. Thus, from (6.25), we derive the following estimate:

$$\tilde{P}_h \leqslant \begin{cases} C\mu_1^h 2^{K+j} r_2^{j-h} \exp\{-(K+j-h)^{\nu_1}\}, & h \leqslant j - v_0, \\ C\mu_1^h 2^{K+j} r_2^{j-h} \exp\{\Gamma_2^{(0)}(K+v_0)^{\nu_1}\}, & h > j - v_0. \end{cases} \tag{6.26}$$

Let us come back to the estimate on $\tilde{A}_{j-1}^{(1)}$ from (6.11), (6.12). Hence, by (6.26), we have

$$\begin{aligned} \tilde{A}_{j-1}^{(1)} \leqslant C 2^{K+j} \sum_{h=0}^{j-v_0} \frac{\Delta_j}{\Delta_{j-h}} r_3^h r_2^{j-h} \mu_1^h \exp\{-(K+j-h)^{\nu_1}\} \\ + C 2^{K+j} \sum_{h=j-v_0+1}^{j-2} \frac{\Delta_j}{\Delta_{j-h}} r_3^h r_2^{j-h} \mu_1^h \exp\{\Gamma_2^{(0)}(K+v_0)^{\nu_1}\} \end{aligned}$$

$$
\begin{aligned}
&= C 2^{K+j} r_2^j \sum_{h=0}^{j-v_0} \frac{\Delta_j}{\Delta_{j-h}} (1+\varepsilon_0)^h \mu_1^h \exp\{-(K+j-h)^{\nu_1}\} \\
&\quad + C 2^{K+j} r_2^j \sum_{h=j-v_0+1}^{j-2} \frac{\Delta_j}{\Delta_{j-h}} [(1+\varepsilon_0)\mu_1]^h \exp\{\Gamma_2^{(0)}(K+v_0)^{\nu_1}\} \\
&\leqslant C 2^{K+j} r_2^j \sum_{h=0}^{j-v_0} \Lambda_1^h + c K_1 2^{K+j} r_2^j \sum_{h=j-v_0+1}^{j-2} \Lambda_1^h, \qquad (6.27)
\end{aligned}
$$

where $C$ is as given in (4.25), $K_1 = \exp\{\Gamma_2^{(0)}(K+v_0)^{\nu_1}\}$, $\Lambda_1 = (1+\varepsilon_0)\mu_1$ and $\mu_1 = (\lambda_1 \ln \lambda_1^{-1})^{-1}$. As a natural continuation of (6.27), we finally obtain

$$
\tilde{A}_{j-1}^{(1)} \leqslant C 2^{K+j} r_2^j K_1 \Lambda_1^{j-1} (\Lambda_1 - 1)^{-1} \equiv C_1 (2 r_2 \Lambda_1)^j, \quad j \geqslant 1. \qquad (6.28)
$$

Here and later on, constants $C_1, C_2, \dots$ depend on the known parameters and, in particular, are independent of $j$. Next, we have

$$
\begin{aligned}
\tilde{A}_{j-1}^{(2)} &\leqslant \frac{\Delta_j}{\Delta_1} r_3^{j-1} J_1(1) \sum_{i_1=1}^{\infty} \lambda_1^{i_1} \sum_{i_2=0}^{\infty} \lambda_1^{i_2} \cdots \sum_{i_{j-1}=0}^{\infty} \lambda_1^{i_{j-1}} \\
&\leqslant r_2^{j-1} (1+\varepsilon_0)^{j-1} K^{m(p+1)+\xi_1} (K+j)^{-m(p+1)-\xi_1} J_1(1) \lambda_1 (1-\lambda_1)^{-(j-1)} \\
&\leqslant C_2 [r_2 (1+\varepsilon_0)(1-\lambda_1)^{-1}]^j. \qquad (6.29)
\end{aligned}
$$

Hence

$$
\begin{aligned}
\tilde{A}_{j-1}^{(3)} &\leqslant c_2 r_3^j \Delta_j H_0(1) \sum_{i_1=1}^{\infty} \lambda_1^{i_1} \sum_{i_2=0}^{\infty} \lambda_1^{i_2} \cdots \sum_{i_{j-1}=0}^{\infty} \lambda_1^{i_{j-1}} \\
&\leqslant c_2 r_2^j (1+\varepsilon_0)^j (K+j)^{-m(p+1)-\xi_1} H_0(1) \lambda_1 (1-\lambda_1)^{-(j-1)} \\
&\leqslant C_3 [(1+\varepsilon_0) r_2 (1-\lambda_1)^{-1}]^j. \qquad (6.30)
\end{aligned}
$$

By (6.11) and (6.28)–(6.30), we have

$$
J_j(s_0) \leqslant C_4 r_2^j [2\Lambda_1 + 2(1+\varepsilon_0)(1-\lambda_1)^{-1}]^j \equiv C_4 r_2^j \Lambda_2^j.
$$

Recalling the definition of the normalized functions $J_i(s)$ in (4.23) and (6.2), we obtain $I_j(s_0) \leqslant C_4 \Lambda_2^j$, and hence, for any $j \geqslant 1$,

$$
I^{(t_j)}(s_0) = \sum_{i=1}^{j} I_i(s_0) \leqslant C_4 \sum_{i=1}^{j} \Lambda_2^i \leqslant C_5 \Lambda_2^j = C_5 \exp\{j \ln \Lambda_2\}. \qquad (6.31)
$$

By estimate (6.20),

$$
K + i \leqslant \phi_1^{\phi_1} (T - t_i)^{-\phi_1} \quad \text{for } i \geqslant 1.
$$

Substituting into (6.31), we get

$$
I^{(t_j)}(s_0) \leqslant C_5 \exp\{\Lambda_3 (T - t_j)^{-\phi_1}\},
$$

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where $\Lambda_3 = \phi_1^{\phi_1} \ln \Lambda_2$. Since $(T - t_j)/(T - t_{j-1}) \to 1$ as $j \to \infty$ monotone, this 'discrete' estimate implies the 'continuous' one for any $t > t_0$,

$$I^{(t)}(s_0) \leqslant C_6 \exp\{\Lambda_4 (T-t)^{-\phi_1}\}, \qquad \Lambda_4 = \Lambda_3 \left[\frac{T-t_1}{T-t_2}\right]^{\phi_1}. \tag{6.32}$$

Let us note that, by the *a priori* estimate (4.25) and condition (6.1), we have the 'initial estimate'

$$I^{(t)}(1) \leqslant CF(t) \leqslant C \exp\{A(T-t)^{-\phi}\}, \quad t \in (t_0, T), \quad \phi = \frac{1}{m(p+1) - 1 + \xi_0}. \tag{6.33}$$

Comparing estimates (6.32) and (6.33) shows that we essentially reduce the singularity order at $s = s_0$ relative to that at $s = 1$, since, by the construction, $\xi_1 > \gamma_0 \xi_0$, where $\gamma_0 > 1$.

We now start the next computational cycle by taking estimate (6.32) as the initial one instead of (6.33). Namely, for $s > s_0$, we introduce a sequence of new energy functions $I_j(s)$, keeping for convenience all the previous notations, so that, for $i \geqslant 1$,

$$\Delta_i = \Delta_i(\xi_2) \equiv (K+i)^{-m(p+1)-\xi_2} \omega_0, \qquad \delta_i = \delta_i(\xi_2) \equiv (K+i)^{-1-\xi_2/m(p+1)},$$

where $\delta_0 = 0$ by implication and $\xi_2$ is determined from the equality similar to (6.22),

$$\frac{m(p+1) - 1 + \xi_2}{m(p+1) - 1 + \xi_1} = 1 + \frac{\xi_2}{m(p+1)} \quad \Rightarrow \quad \xi_2 = \xi_1 \frac{2m(p+1)}{m(p+1) + 1 - \xi_1} > \gamma_0 \xi_1, \tag{6.34}$$

where $\gamma_0 > 1$ is as in (6.23). The new energy functions are

$$I_j(s) = \int_{t_{j-1}}^{t_j} \int_{\Omega(s)} |u(x,t)|^{p+1} \, dx dt, \quad s \geqslant s_0, \quad j \geqslant 1.$$

Here, $\{t_j\}$ is the new time partition of the interval $(0, T)$ corresponding to $\{\Delta_i\}$ defined above. Instead of $\delta = \delta(K, \xi_1)$ in (6.6), we set

$$\delta = \delta(K, \xi_2) = \sum_{i=1}^{\infty} \delta_i(\xi_2),$$

so that, similar to (6.7), we have

$$m(p+1)\xi_2^{-1}(K+1)^{-\xi_2/m(p+1)} \leqslant \delta(K, \xi_2) \leqslant m(p+1)\xi_2^{-1} K^{-\xi_2/m(p+1)}. \tag{6.35}$$

We also define the new value $s_1 = s_0 + 2\delta = 1 + 2\delta(K, \xi_1) + 2\delta(K, \xi_2)$. We perform calculations similar to those that led us from the initial estimate (6.33) to (6.32), but now with (6.32) as the initial one. We then obtain, for any $t < T$,

$$I^{(t)}(s_1) \leqslant C_7 \exp\{\Lambda_5 (T-t)^{-\phi_2}\}, \qquad \phi_2 = \frac{1}{m(p+1) - 1 + \xi_2}. \tag{6.36}$$

Using (6.36) as the initial estimate for the next cycle of calculations, etc., after a number $l$ of such cycles, we arrive at the estimate

$$I^{(t)}(s_{l-1}) \leqslant C_8 \exp\{\Lambda_6(T-t)^{-\phi_l}\}, \qquad \phi_l = \frac{1}{m(p+1)-1+\xi_l}, \qquad \Lambda_6 = \Lambda_6(l), \tag{6.37}$$

where $\xi_l \geqslant \gamma_0^l \xi_0$, $\gamma_0 > 1$, $s_{l-1} = 1 + 2\delta(K,\xi_1) + \cdots + 2\delta(K,\xi_l)$ and $\delta(K,\xi)$ satisfies (6.35). Choosing $l = l_0$ such that $\xi_{l_0} = 1+\nu$, $\nu > 0$, then

$$l_0 = 1 + \left[\frac{(\ln(1+\nu) + \ln \xi_0^{-1})}{\ln \gamma_0}\right].$$

As a consequence, we deduce that

$$\begin{aligned} s_{l_0-1} &\leqslant 1 + 2\sum_{i=1}^{l_0} \delta(K,\xi_i) \leqslant 1 + 2m(p+1)\sum_{i=1}^{l_0} \xi_0^{-1}\gamma_0^{-i} K^{-\xi_0\gamma_0^i/m(p+1)} \\ &\leqslant 1 + 2m(p+1)\xi_0^{-1}(\gamma_0-1)^{-1}K^{-\xi_0\gamma_0/m(p+1)}. \end{aligned} \tag{6.38}$$

Thus, performing $l_0$ cycles, we obtain the estimate

$$I^{(t)}(s_{l_0-1}) \leqslant C_9 \exp\{\Lambda_7(T-t)^{-1/[m(p+1)+\nu]}\}, \quad t < T, \quad \nu > 0. \tag{6.39}$$

We now choose the value of $K$. Given an arbitrarily small number $\mu > 0$, let $K = K(\mu)$ be such that $2m(p+1)\xi_0^{-1}(\gamma_0-1)^{-1}K^{-\xi_0\gamma_0/m(p+1)} = \mu$. Then, in view of (6.38), estimate (6.39) yields

$$I^{(t)}(1+\mu) \leqslant C_9 \exp\{\Lambda_7(T-t)^{-1/[m(p+1)+\nu]}\} \quad \text{for } t \in (0,T).$$

The constants $C_9$ and $\Lambda_7$ do not depend on $t$. Therefore, we arrive at a flat blow-up LS-regime posed on the lateral boundary of the domain $\{|x| > 1+\mu\} \times (0,T)$. By the results from § 5, the blow-up set is contained in $\{|x| \leqslant 1+\mu\}$. Since $\mu > 0$ can be chosen arbitrarily small, theorem 1.3 follows.

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