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VARIATIONS ON Δ^1_1 DETERMINACY AND $\aleph_{\omega_1}$

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Abstract. We consider a seemingly weaker form of Δ^1_1 Turing determinacy.

Let $2 \leqslant \rho < \omega_1^{\mathsf{CK}}$, Weak-Turing-Det$_\rho(\Delta^1_1)$ is the statement:

Every Δ^1_1 set of reals cofinal in the Turing degrees contains two Turing distinct, Δ^0_ρ-equivalent reals.

We show in ZF^-:

Weak-Turing-Det$_\rho(\Delta^1_1)$ implies: for every $\nu < \omega_1^{\mathsf{CK}}$ there is a transitive model $M \models \mathsf{ZF}^- +$ "$\aleph_\nu$ exists".

As a corollary:

If every cofinal Δ^1_1 set of Turing degrees contains both a degree and its jump, then for every $\nu < \omega_1^{\mathsf{CK}}$, there is a transitive model: $M \models \mathsf{ZF}^- +$ "$\aleph_\nu$ exists".

- With a simple proof, this improves upon a well-known result of Harvey Friedman on the strength of Borel determinacy (though not assessed level-by-level).
- Invoking Tony Martin's proof of Borel determinacy, Weak-Turing-Det$_\rho(\Delta^1_1)$ implies Δ^1_1 determinacy.
- We show further that, assuming Δ^1_1 Turing determinacy, or Borel Turing determinacy, as needed:
  - Every cofinal Σ^1_1 set of Turing degrees contains a "hyp-Turing cone": $\{x \in \mathcal{D} \mid d_0 \leqslant_T x \leqslant_h d_0\}$.
  - For a sequence $(A_k)_{k<\omega}$ of analytic sets of Turing degrees, cofinal in $\mathcal{D}$, $\bigcap_k A_k$ is cofinal in $\mathcal{D}$.

Introduction. A most important result in the study of infinite games is Harvey Friedman's [3], where it is shown that a proof of determinacy, for Borel games, would require $\aleph_1$ iterations of the power set operation—and this is precisely what Tony Martin used in his landmark proof [7].

Our focus here is on the Turing determinacy results of [3], concentrating instead on the theory ZF^-, rather than Zermelo's Z. In the Δ^1_1 realm, Friedman essentially shows that the determinacy of Turing closed Δ^1_1 games—henceforth, Turing-Det(Δ^1_1)—implies the consistency of the theories $\mathsf{ZF}^- +$ "$\aleph_\nu$ exists", for all $\nu < \omega_1^{\mathsf{CK}}$. He does produce a level-by-level analysis entailing, e.g., that the determinacy of Turing closed Σ^0_{n+6} games implies the consistency of $\mathsf{ZF}^- +$ "$\aleph_n$ exists".[1,2]

Importantly, it was further observed by Friedman (unpublished) that these results extend to produce transitive models, rather than just consistency statements. See Martin's forthcoming book [9] for details, see also Van Wesep's [13].

We forgo in this paper the level-by-level analysis to provide, in §3, a simple proof of the existence of transitive models of ZF^- with uncountable cardinals, from Turing-Det(Δ^1_1). In so doing, we show that the full force of Turing determinacy isn't

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[1] Improved by Martin to Σ^0_{n+5}.

[2] In [10] Montalbán and Shore considerably refine the analysis of the proof theoretic strength of Det(Γ), for classes Γ, where $\Pi^0_3 \subseteq \Gamma \subseteq \Delta^0_4$.

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needed. The main result is Theorem 3.1, with a simply stated corollary. For context, by Martin's Lemma (see 1.2), Turing-Det(Δ^1_1) is equivalent to:

- *Every cofinal* Δ^1_1 *set of Turing degrees contains a cone of degrees*—i.e., a set $\{x \in \mathcal{D} \mid d_0 \leqslant_T x\}$.

Theorem (3.1). *Let* $2 \leqslant \rho < \omega_1^{\mathsf{CK}}$, *and assume every* Δ^1_1 *set of reals, cofinal in the Turing degrees, contains two Turing distinct,* Δ^0_ρ*-equivalent reals. For every* $\nu < \omega_1^{\mathsf{CK}}$, *there is a transitive model*: $M \models \mathsf{ZF}^- +$ "$\aleph_\nu$ exists".

Corollary (3.2). *If every cofinal* Δ^1_1 *set of Turing degrees contains both a degree and its jump, then for every* $\nu < \omega_1^{\mathsf{CK}}$, *there is a transitive model*: $M \models \mathsf{ZF}^- +$ "$\aleph_\nu$ exists".

In §4 several results are derived, showing that Turing-Det(Δ^1_1) imparts weak determinacy properties to the class Σ^1_1, such as [4.4]:

- *Every cofinal* Σ^1_1 *set of degrees includes a set* $\{x \in \mathcal{D} \mid d_0 \leqslant_T x \;\&\; x \leqslant_h d_0\}$, *for some* $d_0 \in \mathcal{D}$.

Or, from Borel Turing determinacy, [4.3]:

- *If* $(A_k)_{k<\omega}$ *is a sequence of cofinal analytic subsets of* $\mathcal{D}$, *then* $\bigcap_k A_k$ *is cofinal in* $\mathcal{D}$.

I wish to thank Tony Martin for inspiring exchanges on the present results. He provided the argument for Remark 2.3, below, and observed that my first proof of Theorem 4.6 was needlessly complex. Parts of §4 go back to the author's dissertation [12], it is a pleasure to acknowledge Robert Solovay's direction. Lastly, thanks are due to the referee for thoughtful suggestions.

§1. Preliminaries and notation. The effective descriptive set theory we shall need, as well as basic hyperarithmetic theory, is from Moschovakis' [11], whose terminology and notation we follow. For the theory of admissible sets, we refer to Barwise's [1]. Standard facts about the $\mathbb{L}$-hierarchy are used without explicit mention: see Devlin's [2], or Van Wesep's [13].

$\mathcal{N} = \omega^\omega = \mathbb{N}^\mathbb{N}$ denotes Baire's space (the set of *reals*), and $\mathcal{D}$ the set of Turing degrees. Subsets of $\mathcal{D}$ shall be identified with the corresponding (Turing closed) sets of reals. $\leqslant_T$, $\leqslant_h$, and $\equiv_T$, $\equiv_h$ denote, respectively, Turing and hyperarithmetic (i.e., Δ^1_1) reducibility, and equivalence.

1.1. The ambient theories. Our base theory is ZF^-, Zermelo–Fraenkel set theory stripped of the Power Set axiom.[3] $\mathcal{N}$ or $\mathcal{D}$ may be proper classes in this context, yet speaking of their 'subsets' (Δ^1_1, Σ^1_1, Borel or analytic) can be handled as usual, as these sets are codable by integers, or reals. Amenities such as $\aleph_1$ or $\mathbb{L}_{\omega_1}$ aren't available but, since our results here are global (i.e., Δ^1_1) rather than local, the reader may use instead the more comfortable $\mathsf{ZF}^- +$ "$\mathcal{P}^2(\omega)$ exists".

KP_∞ is the theory Kripke–Platek + Infinity. Much of the argumentation below involves ω-models of KP_∞—familiarity with their properties is assumed.

[3] All implicit uses of Choice herein are ZF^--provable.

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1.2. Turing determinacy. A set of reals $A \subseteq \mathcal{N}$ is said to be *Turing-cofinal* if, for every $x \in \mathcal{N}$, there is $y \in A$, such that $x \leqslant_T y$. A *Turing cone* is a set $\mathrm{Cone}(c) = \{x \in \mathcal{N} \mid c \leqslant_T x\}$, where $c \in \mathcal{N}$. For a class of sets of reals Γ, $\mathrm{Det}(\Gamma)$ is the statement that infinite games $\mathrm{G}_\omega(A)$ where $A \in \Gamma$ are determined, whereas Turing-Det(Γ) stands for the determinacy of games $\mathrm{G}_\omega(A)$ restricted to Turing closed sets $A \in \Gamma$. Recall the following easy, yet central:

Martin's Lemma [6]. For a Turing closed set $A \subseteq \mathcal{N}$, the infinite game $\mathrm{G}_\omega(A)$ is determined *iff* A or its complement contains a Turing cone. ⊣

1.3. Constructibility and condensation. For an ordinal $\lambda > 0$, and $X \subseteq \mathbb{L}_\lambda$, $\mathsf{H}^{\mathbb{L}_\lambda}(X)$ denotes the set of elements of $\mathbb{L}_\lambda$ definable from parameters in X, and $\overline{\mathsf{H}}^{\mathbb{L}_\lambda}(X)$ its transitive collapse. For $X = \varnothing$, one simply writes $\mathsf{H}^{\mathbb{L}_\lambda}$ and $\overline{\mathsf{H}}^{\mathbb{L}_\lambda}$. Gödel's Condensation Lemma is the relevant tool here. Note that, since $\mathbb{L}_\lambda = \overline{\mathsf{H}}^{\mathbb{L}_\lambda}(\lambda) = \mathsf{H}^{\mathbb{L}_\lambda}(\lambda)$, all elements of $\mathbb{L}_\lambda$ are definable in $\mathbb{L}_\lambda$ from ordinal parameters.

1.4. Reflection. The following reflection principle will be used a few times, to make for shorter proofs.[4] A property $\Phi(X)$ of subsets $X \subseteq \mathcal{N}$ is said to be "Π^1_1 *on* Σ^1_1" if, for any Σ^1_1 relation $U \subseteq \mathcal{N} \times \mathcal{N}$, the set $\{x \in \mathcal{N} \mid \Phi(U_x)\}$ is Π^1_1.

A simple example of such a property: let $A \subseteq \mathcal{N}$ be Σ^1_1, and set: $\Theta(X) \Leftrightarrow X \cap A = \varnothing$. $\Theta(X)$ is a Π^1_1 on Σ^1_1 property.

Theorem. *Let $\Phi(X)$ be a Π^1_1 on Σ^1_1 property. For any Σ^1_1 set $S \subseteq \mathcal{N}$ such that $\Phi(S)$ there is a Δ^1_1 set $D \supseteq S$ such that $\Phi(D)$.*

Proof. See Kechris' [5, §35] for a boldface version, easily transcribed to lightface. ⊣

§2. Weak-Turing-Determinacy. Examining what's needed to derive the existence of transitive models from Turing determinacy hypotheses, it is possible to isolate a seemingly weaker statement. For $1 \leqslant \rho < \omega_1^{\mathsf{CK}}$, let $x \equiv_\rho y$ denote Δ^0_ρ-equivalence on $\mathcal{N}$, that is: $x \in \Delta^0_\rho(y)$ & $y \in \Delta^0_\rho(x)$. $\equiv_1$ is just Turing equivalence.

Definition 2.1. For a class Γ, and $2 \leqslant \rho < \omega_1^{\mathsf{CK}}$, define Weak-Turing-$\mathrm{Det}_\rho(\Gamma)$: *Every Turing-cofinal set of reals $A \in \Gamma$ has two Turing distinct elements $x, y \in A$ such that $x \equiv_\rho y$.*

For any recursive $\rho \geqslant 2$, Weak-Turing-$\mathrm{Det}_\rho(\Delta^1_1)$ will suffice to derive the existence of transitive models of ZF^- with uncountable cardinals. The property lifts from Δ^1_1 to Σ^1_1—note that it is, *a priori*, asymmetric.

Theorem 2.2. *Let $2 \leqslant \rho < \omega_1^{\mathsf{CK}}$,*

$$\text{Weak-Turing-Det}_\rho(\Delta^1_1) \implies \text{Weak-Turing-Det}_\rho(\Sigma^1_1).$$

Proof. Assume Weak-Turing-$\mathrm{Det}_\rho(\Delta^1_1)$. Let $S \in \Sigma^1_1$ and suppose there are no Turing distinct $x, y \in S$ such that $x \equiv_\rho y$, that is

$$\forall x, y\,(x, y \in S \;\&\; x \equiv_\rho y \implies x \equiv_T y).$$

[4]Longer ones can always be produced using Δ^1_1 selection + Σ^1_1 separation.

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This is a statement $\Phi(S)$, where $\Phi(X)$ is easily checked to be a Π^1_1 on Σ^1_1 property. Reflection yields a Δ^1_1 set $D \supseteq S$ such that $\Phi(D)$. By Weak-Turing-Det$_\rho(\Delta^1_1)$, D is not Turing-cofinal; *a fortiori*, S isn't either. $\dashv$

REMARK 2.3. One may be tempted to substitute for Weak-Turing-Det$_\rho(\Delta^1_1)$ a simpler hypothesis:
Every Turing-cofinal Δ^1_1 set of reals has Turing distinct elements x, y, such that $x \equiv_h y$. It turns out to be too weak and, indeed, provable in Analysis. (Tony Martin, private communication: building on his paper [8], he shows that every uncountable Δ^1_1 set of reals contains two Turing distinct reals, in every hyperdegree $\geqslant$ Kleene's $\mathcal{O}$.)

The simpler, weaker, condition does suffice however when asserted about the class Σ^1_1, see Theorem 3.13, below.

§3. Transitive models from Weak-Turing-Determinacy.

We state the main result, and a simple special case. The proof is postponed towards the end of the present section.

THEOREM 3.1. *Let $2 \leqslant \rho < \omega_1^{\mathsf{CK}}$, and assume* Weak-Turing-Det$_\rho(\Delta^1_1)$. *For every $\nu < \omega_1^{\mathsf{CK}}$, there is a transitive model*: $M \models \mathsf{ZF}^- +$ "$\aleph_\nu$ exists".

COROLLARY 3.2. *If every cofinal Δ^1_1 set of Turing degrees contains both a degree and its jump, then for every $\nu < \omega_1^{\mathsf{CK}}$, there is a transitive model*: $M \models \mathsf{ZF}^- +$ "$\aleph_\nu$ exists".

• TERM MODELS.

Given a complete[5] theory $U \supseteq \mathsf{KP}_\infty + (\mathbb{V} = \mathbb{L})$, one constructs its term model. To be specific: owing to the presence of the axiom $\mathbb{V} = \mathbb{L}$, to every formula $\psi(v)$ is associated $\overline{\psi}(v)$ such that $U \vdash \exists v\, \psi(v) \Leftrightarrow \exists!\, v\, \overline{\psi}(v)$, just take for $\overline{\psi}(v)$ the formula $\psi(v) \wedge (\forall w <_{\mathbb{L}} v) \neg\psi(w)$.

Let now $(\varphi_n(v))_{n<\omega}$ be a recursive in U enumeration of the formulas $\varphi(v)$, in the single free variable v, having $U \vdash \exists!\, v\, \varphi(v)$. Using, as metalinguistic device, $(\imath v)\varphi(v)$ for "*the unique v such that $\varphi(v)$*" set:

$$M_U = \{n \in \omega \mid \forall \ell < n,\ U \vdash (\imath v)\varphi_n \neq (\imath v)\varphi_\ell\},$$

and define on M_U the relation $\in_U$:

$$m \in_U n \ \Leftrightarrow\ U \vdash (\imath v)\varphi_m \in (\imath v)\varphi_n.$$

$(M_U, \in_U)$ is a prime model of U and, U being complete, $(M_U, \in_U) \leqslant_T U$. Using the canonical 1-1 enumeration $\omega \to M_U$, substitute ω for M_U and remap $\in_U$ accordingly. The resulting model $\mathcal{M}_U = (\omega, \in^{\mathcal{M}_U})$ shall be called the *term model* of U. The function $U \mapsto \mathcal{M}_U$ is recursive, and $\mathcal{M}_U \equiv_h U$, uniformly.

Whenever $\mathcal{M}_U$ is an ω-model, we say that $a \subseteq \omega$ is realized in $\mathcal{M}_U$ if there is $\mathring{a} \in \omega$ such that $a = \{k \in \omega \mid k^{\mathcal{M}_U} \in^{\mathcal{M}_U} \mathring{a}\}$. We state, for later reference, a couple of standard facts.

PROPOSITION 3.3. *Let U be as above. If $\mathcal{M}_U$ is an ω-model, and $a \subseteq \omega$ is realized in $\mathcal{M}_U$, then*:

[5] Complete extensions are always meant to be consistent, and deductively closed.

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(1) *For all* $x \leqslant_h a$, *x is realized in* $\mathcal{M}_U$.
(2) $a \leqslant_T U$. *Hence U is not realized in* $\mathcal{M}_U$, *lest the Turing jump* U' *be realized in* $\mathcal{M}_U$, *causing* $U' \leqslant_T U$. ⊣

Note that if $U = \mathrm{Th}(\mathbb{L}_\alpha)$, where α is admissible, then $\mathcal{M}_U$ is a copy of $\mathsf{H}^{\mathbb{L}_\alpha}$. Hence $\mathcal{M}_U \cong \mathbb{L}_\beta$, for some $\beta \leqslant \alpha$. The following easy proposition is quite familiar.

Proposition 3.4. *Assume* $\mathbb{V} = \mathbb{L}$. *For cofinally many countable admissible* α*'s,* $\mathbb{L}_\alpha = \mathsf{H}^{\mathbb{L}_\alpha}$, *equivalently*: $\mathcal{M}_{\mathrm{Th}(\mathbb{L}_\alpha)} \cong \mathbb{L}_\alpha$.

Proof. Suppose not. Let λ be the sup of the admissible α's having $\mathbb{L}_\alpha = \mathsf{H}^{\mathbb{L}_\alpha}$, and let $\kappa > \lambda$ be the first admissible such that λ is countable in $\mathbb{L}_\kappa$. Since λ is definable and countable in $\mathbb{L}_\kappa$, $\lambda \cup \{\lambda\} \subseteq \mathsf{H}^{\mathbb{L}_\kappa}$. It follows readily that $\mathbb{L}_\kappa = \overline{\mathsf{H}}^{\mathbb{L}_\kappa} = \mathsf{H}^{\mathbb{L}_\kappa}$, a contradiction. ⊣

• Cardinality in the constructible levels.
Set theory within the confines of $\mathbb{L}_\lambda$, λ an arbitrary limit ordinal, imposes some contortions. For technical convenience, the notion of cardinal needs to be slightly twisted—for a time only.

Definition 3.5.
(1) For an ordinal α, $\mathrm{Card}(\alpha) = \min_{\xi \leqslant \alpha}$(*there is a surjection* $\xi \to \alpha$).
(2) α is a cardinal if $\alpha = \mathrm{Card}(\alpha)$.
(3) $\mathrm{Card}_\lambda \subseteq \mathbb{L}_\lambda$ is the class of infinite cardinals as computed in $\mathbb{L}_\lambda$.

3.6. Note that, for λ limit, from a surjection $g\colon \gamma \to \alpha$ in $\mathbb{L}_\lambda$, one can extract $a \subseteq \gamma$ and $\lhd \subseteq a \times a$ such that $g{\restriction}a\colon (a, \lhd) \rightarrowtail (\alpha, \in)$,[6] and both $(a, \lhd)$, $g{\restriction}a$ are in $\mathbb{L}_\lambda$. Further, if λ is admissible, in $\mathbb{L}_\lambda$ the altered notion of cardinality and the standard one coincide.

Convention 3.7. For simplicity's sake, the assertion "$\aleph_\nu$ exists in $\mathbb{L}_\lambda$" should be understood as:
There is an isomorphism $\nu + 1 \rightarrowtail J$, *where J is an initial segment of* Card_λ.
Note that its negation is equivalent in KP_∞ to: *There is* $\kappa \leqslant \nu$ such that $\mathrm{Card}_\lambda \cong \kappa$. The notation $\aleph_\nu^{\mathbb{L}_\lambda}$ carries the obvious meaning.

We need the following result, readily proved using the Jensen techniques of [4]. A direct proof is provided in the Appendix.

Proposition 3.8. *For* λ *a limit ordinal, if* $\mathbb{L}_\lambda \models$ "$\boldsymbol{\mu} > \omega$ is a successor cardinal" *then* $\mathbb{L}_\mu \models \mathsf{ZF}^-$.

• The theories T_ν.
Let $\mathcal{M}$ be an ω-model of KP_∞. The wellfounded part of $\mathbb{O}\mathrm{n}^{\mathcal{M}}$ 'includes' ω_1^{CK}. For $\nu < \omega_1^{\mathsf{CK}}$, pick e_ν a recursive index for a wellordering $<_{e_\nu}$ of a subset of ω, of length ν. Using e_ν, statements about ν can tentatively be expressed in KP_∞. In $\mathcal{M}$, the truth of such statements is independent of the choice of e_ν. Indeed, $<_{e_\nu}$ is realized in $\mathcal{M}$, and its realization is isomorphic in $\mathcal{M}$ to the $\mathcal{M}$-ordinal of order-type ν, to be denoted $\nu^{\mathcal{M}}$. For a formula $\varphi(x, \ldots)$, we write $\mathcal{M} \models \varphi(\underline{\nu}, \ldots)$, instead of a 'translated' $\mathcal{M} \models \varphi^*(e_\nu, \ldots)$.

[6]Herein, '$\rightarrowtail$' denotes isomorphism map.

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Definition 3.9. For $\nu < \omega_1^{\mathsf{CK}}$, T_ν is the theory

$$\mathsf{KP}_\infty + (\mathbb{V} = \mathbb{L}) + \text{“for all limit } \lambda, \aleph_{\nu+1} \text{ doesn't exist in } \mathbb{L}_\lambda\text{”}.$$

This definition is clearly lacking: a recursive index e_ν coding the ordinal ν is not made explicit. This is immaterial, as we shall be interested only in ω-models of T_ν. They possess the following rigidity property.

Lemma 3.10. *Let $\nu < \omega_1^{\mathsf{CK}}$, and $\mathcal{M}_1$, $\mathcal{M}_2$ be ω-models of T_ν. Let $u \in \mathbb{O}\mathrm{n}^{\mathcal{M}_1}$, and $w, w_* \in \mathbb{O}\mathrm{n}^{\mathcal{M}_2}$, for any two isomorphisms $f : \mathbb{L}_u^{\mathcal{M}_1} \simeq \mathbb{L}_w^{\mathcal{M}_2}$ and $f_* : \mathbb{L}_u^{\mathcal{M}_1} \simeq \mathbb{L}_{w_*}^{\mathcal{M}_2}$, $f = f_*$.*

Proof. By an easy reduction, it suffices to prove this for u, a limit $\mathcal{M}_1$-ordinal.

Let $<_1$ denote the ordering of $\mathbb{O}\mathrm{n}^{\mathcal{M}_1}$ in $\mathcal{M}_1$, and set $\mathcal{C}_u = \{c <_1 u \mid \mathcal{M}_1 \models \boldsymbol{c} \in \mathrm{Card}_{\boldsymbol{u}}\}$.

The relevant claim here is that $(\mathcal{C}_u, <_1)$ is wellordered. Indeed, since $\mathcal{M}_1 \models \mathsf{T}_{\underline{\nu}}$,

$$\mathcal{M}_1 \models \text{“}\aleph_{\underline{\nu}+1} \text{ doesn't exist in } \mathbb{L}_{\boldsymbol{u}}\text{”}.$$

Hence, as observed in 3.7, there is $k \in \mathbb{O}\mathrm{n}^{\mathcal{M}_1}$ with

$$\mathcal{M}_1 \models \boldsymbol{k} \leqslant \underline{\nu} + 1 \ \&\ \mathrm{Card}_{\boldsymbol{u}} \cong \boldsymbol{k}.$$

The isomorphism in $\mathcal{M}_1$ induces an actual isomorphism $(\mathcal{C}_u, <_1) \cong (\{x \mid x <_1 k\}, <_1)$. Since $\mathcal{M}_1$ is an ω-model, $\nu^{\mathcal{M}_1}$ (and hence, k) is in its wellfounded part, thus the claim.

First, we check that f and f_* agree on the $\mathcal{M}_1$-ordinals $o <_1 u$, using induction on $\mathcal{C}_u$. Clearly, for $o \leqslant_1 \omega^{\mathcal{M}_1}$, $f(o) = f_*(o)$. Set $\kappa_u(o) = \mathrm{Card}(o)$, *as evaluated in* $\mathbb{L}_u^{\mathcal{M}_1}$, and show by induction on $c \in \mathcal{C}_u$:

$$\text{for all } o <_1 u,\ \ \kappa_u(o) \leqslant_1 c \implies f(o) = f_*(o).$$

The inductive hypothesis, for $c' <_1 c$, yields, for all $o <_1 c$, $f(o) = f_*(o)$, hence $f(c) = f_*(c)$. Let now $o <_1 u$ have $\kappa_u(o) = c$. Inside $\mathbb{L}_u^{\mathcal{M}_1}$, $(o, \in)$ is isomorphic to an ordering $s = (a, \lhd)$, where $a \subseteq c$ and $\lhd \subseteq c \times c$, (see 3.6). Since f and f_* agree on the $\mathcal{M}_1$-ordinals up to c, one readily gets $f(s) = f_*(s)$. In $\mathcal{M}_2$ now, the common value $f(s)$ is isomorphic to both the ordinals $f(o)$ and $f_*(o)$, hence $f(o) = f_*(o)$.

This entails $w = w_*$ and $\mathbb{L}_w^{\mathcal{M}_2} = \mathbb{L}_{w_*}^{\mathcal{M}_2}$. Now, any $x \in \mathbb{L}_u^{\mathcal{M}_1}$ is definable in $\mathbb{L}_u^{\mathcal{M}_1}$ from $\mathcal{M}_1$-ordinals (see 1.3), thus $f(x)$ and $f_*(x)$ satisfy in $\mathbb{L}_w^{\mathcal{M}_2}$ the same definition from equal parameters, hence $f(x) = f_*(x)$. ⊣

• Pseudo-wellfounded models.

A relation $\lhd \subseteq \omega \times \omega$ is said to be *pseudo-wellfounded* if every nonempty $\Delta_1^1(\lhd)$ subset of ω has a $\lhd$-minimal element. By the standard computation, this is a Σ_1^1 property.[7] Indeed, we may define it, for $\boldsymbol{E} \subseteq \omega \times \omega$, as:

$$\text{pseudo-WF}(\boldsymbol{E}) \overset{\text{def}}{\iff} (\forall X \leqslant_h \boldsymbol{E})\big(X \neq \varnothing \implies (\exists k \in X)(\forall m \in X)\neg(m\, \boldsymbol{E}\, k)\big).$$

[7] We shall use, in complexity computations, the classic result of Kleene: *Given a* Σ_1^1 predicate $\mathcal{S}(x, y, -)$, the predicate $(\forall y \leqslant_h x)\mathcal{S}(x, y, -)$ is Σ_1^1—and dually for Π_1^1. See [11, §4D.3] for a more general result.

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DEFINITION 3.11. For $\nu < \omega_1^{\mathsf{CK}}$, $\mathcal{S}_\nu$ is the set of theories:

$$\mathcal{S}_\nu = \{\, U \mid U \text{ is a complete extension of } \mathsf{T}_\nu, \text{ and } \mathcal{M}_U \text{ is pseudo-wellfounded}\}.$$

Easily, $\mathcal{S}_\nu$ is Σ^1_1. Indeed, the first clause in its definition is arithmetical, while the second reads "pseudo-WF$(\in^{\mathcal{M}_U})$," where the function $U \mapsto \in^{\mathcal{M}_U}$ is recursive.

Note further: for $U \in \mathcal{S}_\nu$, $\mathcal{M}_U$ is an ω-model. The sets $\mathcal{S}_\nu$ play a central role in the proof. They are sparse, in the following sense.

PROPOSITION 3.12. *For $\nu < \omega_1^{\mathsf{CK}}$, no two distinct members of $\mathcal{S}_\nu$ have the same hyperdegree.*

PROOF. Let $U_1, U_2 \in \mathcal{S}_\nu$ have $U_1 \equiv_h U_2$, and let $\mathcal{M}_1, \mathcal{M}_2$ stand for $\mathcal{M}_{U_1}, \mathcal{M}_{U_2}$. We'll obtain $U_1 = U_2$ by showing $\mathcal{M}_1 \cong \mathcal{M}_2$. Define a relation between 'ordinals' $u \in \mathcal{M}_1$ and $w \in \mathcal{M}_2$:

$$u \simeq w \;\Leftrightarrow\; \exists f\,(f : \mathbb{L}_u^{\mathcal{M}_1} \stackrel{\sim}{\rightarrow} \mathbb{L}_w^{\mathcal{M}_2}).$$

Set $I_1 = \mathrm{Dom}(\simeq)$, and $I_2 = \mathrm{Im}(\simeq)$. I_1 and I_2 are initial segments of $\mathbb{O}\mathrm{n}^{\mathcal{M}_1}$ and $\mathbb{O}\mathrm{n}^{\mathcal{M}_2}$, respectively. Using Lemma 3.10, the relation "$u \simeq w$" defines a bijection $I_1 \to I_2$ which is, indeed, the restriction of an isomorphism

$$\boldsymbol{F} : \bigcup_{u\in I_1} \mathbb{L}_u^{\mathcal{M}_1} \stackrel{\sim}{\rightarrow} \bigcup_{w\in I_2} \mathbb{L}_w^{\mathcal{M}_2}.$$

Note that, by the same lemma,

$$u \simeq w \;\Leftrightarrow\; \exists!\, f\,(f : \mathbb{L}_u^{\mathcal{M}_1} \stackrel{\sim}{\rightarrow} \mathbb{L}_w^{\mathcal{M}_2}).$$

The RHS here reads: $\exists!\, f\,\mathcal{I}(f, U_1, u, U_2, w)$, where $\mathcal{I}$ is a Δ^1_1 predicate, hence:

$$u \simeq w \;\Leftrightarrow\; \exists f \leqslant_h U_1 \oplus U_2 (\, f : \mathbb{L}_u^{\mathcal{M}_1} \stackrel{\sim}{\rightarrow} \mathbb{L}_w^{\mathcal{M}_2}).$$

By the standard computation, the relation "$u \simeq w$" is $\Delta^1_1(U_1 \oplus U_2)$ $[= \Delta^1_1(U_1) = \Delta^1_1(U_2)]$. Consequently, I_1 and I_2 are also $\Delta^1_1(U_1)$ $[= \Delta^1_1(U_2)]$. $\mathcal{M}_1, \mathcal{M}_2$ being pseudo-wellfounded, $\mathbb{O}\mathrm{n}^{\mathcal{M}_1} - I_1$ and $\mathbb{O}\mathrm{n}^{\mathcal{M}_2} - I_2$ each, if nonempty, has a minimum. Denote m_1, m_2 the respective potential minima, and consider the cases:

– $\mathbb{O}\mathrm{n}^{\mathcal{M}_1} - I_1$ and $\mathbb{O}\mathrm{n}^{\mathcal{M}_2} - I_2$ are both nonempty. This isn't possible, as $\boldsymbol{F}$ would be the isomorphism $\boldsymbol{F} : \mathbb{L}_{m_1}^{\mathcal{M}_1} \stackrel{\sim}{\rightarrow} \mathbb{L}_{m_2}^{\mathcal{M}_2}$, entailing $m_1 \in I_1$ and $m_2 \in I_2$.

– $I_1 = \mathbb{O}\mathrm{n}^{\mathcal{M}_1}$ and $\mathbb{O}\mathrm{n}^{\mathcal{M}_2} - I_2 \neq \varnothing$. Here $\mathcal{M}_1 = \bigcup_{u\in I_1} \mathbb{L}_u^{\mathcal{M}_1}$, and $\boldsymbol{F} : \mathcal{M}_1 \stackrel{\sim}{\rightarrow} \mathbb{L}_{m_2}^{\mathcal{M}_2}$. U_1 is now the theory of $\mathbb{L}_{m_2}^{\mathcal{M}_2}$, hence is realized in $\mathcal{M}_2$. Since $U_2 \equiv_h U_1$, by Prop. 3.3(1), U_2 is also realized in $\mathcal{M}_2$ (that's $\mathcal{M}_{U_2}$). This contradicts (2) of the same proposition.

– The third case, symmetric of the previous one, is equally impossible.

– The remaining case: $I_1 = \mathbb{O}\mathrm{n}^{\mathcal{M}_1}$ and $I_2 = \mathbb{O}\mathrm{n}^{\mathcal{M}_2}$. Here $\mathcal{M}_1 = \bigcup_{u\in I_1} \mathbb{L}_u^{\mathcal{M}_1}$ and $\mathcal{M}_2 = \bigcup_{w\in I_2} \mathbb{L}_w^{\mathcal{M}_2}$, thus $\boldsymbol{F} : \mathcal{M}_1 \stackrel{\sim}{\rightarrow} \mathcal{M}_2$ is the desired isomorphism. ⊣

PROOF OF THEOREM 3.1. Our hypothesis is Weak-Turing-Det$_\rho(\Delta^1_1)$, and we may work entirely in $\mathbb{L}$.

Fix any $\nu < \omega_1^{\mathsf{CK}}$, towards a transitive model of $\mathsf{ZF}^- +$ "$\aleph_\nu$ exists".

CLAIM. There is a limit ordinal λ, such that: $\aleph_{\nu+1}$ exists in $\mathbb{L}_\lambda$.

Suppose no such λ exists. It follows that for all admissible $\alpha > \omega$, $\mathbb{L}_\alpha \models \mathsf{T}_\nu$. This entails that $\mathcal{S}_\nu$ is Turing-cofinal: indeed, since $\mathbb{V} = \mathbb{L}$, using Proposition 3.4, given

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$x \subseteq \omega$ there is an $\alpha > \omega$, admissible, such that $x \in \mathbb{L}_\alpha$ and $\mathcal{M}_{\mathrm{Th}(\mathbb{L}_\alpha)} \cong \mathbb{L}_\alpha$. Thus $x \leqslant_T \mathrm{Th}(\mathbb{L}_\alpha)$ and, $\mathcal{M}_{\mathrm{Th}(\mathbb{L}_\alpha)}$ being wellfounded, $\mathrm{Th}(\mathbb{L}_\alpha) \in \mathcal{S}_\nu$.

Invoking now Weak-Turing-Det$_\rho(\Delta^1_1)$ and Theorem 2.2, Weak-Turing-Det$_\rho(\Sigma^1_1)$ holds. Hence, there are distinct $U_1, U_2 \in \mathcal{S}_\nu$ such that $U_1 \equiv_\rho U_2$, contradicting the previous proposition. $\dashv_{\text{CLAIM}}$

Let now λ be as claimed, and set $\mu = \aleph^{\mathbb{L}_\lambda}_{\nu+1}$. In $\mathbb{L}_\lambda$, μ is a successor cardinal hence, by Prop. 3.8, $\mathbb{L}_\mu \models \mathsf{ZF}^-$. Further, for $\xi \leqslant \nu$, $\aleph^{\mathbb{L}_\lambda}_\xi < \mu$ and $\aleph^{\mathbb{L}_\lambda}_\xi$ is an $\mathbb{L}_\mu$-cardinal (now in the usual sense), hence $\mathbb{L}_\mu \models \mathsf{ZF}^- +$ "$\aleph_\nu$ exists". $\dashv$

Note the following byproduct of the previous proposition, and the proof just given (substituting $U_1 \equiv_h U_2$ for $U_1 \equiv_\rho U_2$, in the proof)—in contradistinction to Remark 2.3.

THEOREM 3.13. *Assume every Turing-cofinal* Σ^1_1 *set of reals has two Turing distinct elements* x, y, *such that* $x \equiv_h y$. *For every* $\nu < \omega^{\mathsf{CK}}_1$, *there is a transitive model*: $M \models \mathsf{ZF}^- +$ "$\aleph_\nu$ exists". $\dashv$

An easy consequence of the main result: Weak-Turing-Det$_\rho(\Delta^1_1)$ implies full Δ^1_1 determinacy. The proof proceeds via Martin's Borel determinacy theorem: no direct argument is known for this sort of implication—apparently first observed by Friedman for Turing-Det(Δ^1_1).

THEOREM 3.14. *For* $2 \leqslant \rho < \omega^{\mathsf{CK}}_1$, Weak-Turing-Det$_\rho(\Delta^1_1)$ *implies* $\mathrm{Det}(\Delta^1_1)$.

PROOF. Assume Weak-Turing-Det$_\rho(\Delta^1_1)$. Let $A \subseteq \mathcal{N}$ be Δ^1_1, say $A \in \Sigma^0_\nu$ where $\nu < \omega^{\mathsf{CK}}_1$. Applying Theorem 3.1, there is a transitive $M \models \mathsf{ZF}^- +$ "$\aleph_\nu$ exists". Invoking (non-optimally) Martin's main result from [8] inside M, Σ^0_ν games are determined. The statement "*the game* $\mathrm{G}_\omega(A)$ *is determined*" is Σ^1_2. By Mostowki's absoluteness theorem, being true in M, it holds in the universe: $\mathrm{G}_\omega(A)$ is indeed determined. $\dashv$

§4. Δ^1_1 determinacy and properties of Σ^1_1 sets. We proceed now to show that Δ^1_1 determinacy imparts weak determinacy properties to the class Σ^1_1. In view of Theorem 3.14, there is no point, here, in working from weaker hypotheses.

DEFINITION 4.1. The hyp-Turing cone with vertex $d \in \mathcal{D}$ is the set of degrees

$$\mathrm{Cone}_h(d) = \mathrm{Cone}(d) \cap \Delta^1_1(d) = \{x \in \mathcal{D} \mid d \leqslant_T x \ \&\ x \leqslant_h d\}.$$

Hyp-Turing-Det(Γ) is the statement: *Every cofinal set of degrees* $A \in \Gamma$ *contains a hyp-Turing cone.*

THEOREM 4.2. *Assume* Turing-Det(Δ^1_1). *If* $(S_k)_{k<\omega}$ *is a* Σ^1_1 *sequence of Turing-cofinal sets of degrees, then* $\bigcap_k S_k \neq \varnothing$*—and, indeed,* $\bigcap_k S_k$ *contains a hyp-Turing cone.*

PROOF. Let the S_k's be given as the sections of a Σ^1_1 relation $S \subseteq \omega \times \mathcal{N}$, and assume $\bigcap_k S_k$ contains no hyp-Turing cone: $\forall x \in \mathcal{N}(\mathrm{Cone}_h(x) \not\subseteq \bigcap_k S_k)$, i.e.,

$$\forall x \in \mathcal{N}\, \exists y \leqslant_h x\, (x \leqslant_T y \ \&\ y \notin \textstyle\bigcap_k S_k).$$

This is a statement $\Phi(S)$, where $\Phi(X)$ is a Π^1_1 on Σ^1_1 property of subsets $X \subseteq \omega \times \mathcal{N}$. Reflection yields a Δ^1_1 relation $D \supseteq S$ such that $\Phi(D)$. Shrink D, if need be, to ensure that its sections D_k are Turing closed, preserving $\Phi(D)$ and $D \supseteq S$. Now, $D_k \supseteq S_k$ and $\bigcap_k D_k$ contains no hyp-Turing cone. A contradiction ensues using Turing-Det(Δ^1_1) + Martin's Lemma: each D_k, being cofinal in $\mathcal{D}$, contains a Turing cone hence, easily, so does $\bigcap_k D_k$. ⊣

The converse is immediate. Indeed, if Turing-Det(Δ^1_1) fails, by Martin's Lemma there is a Δ^1_1 set $A \subseteq \mathcal{D}$, such that both A and $\sim A$ are cofinal in $\mathcal{D}$, and the Δ^1_1 sequence $\langle A, \sim A\rangle$ has empty intersection. Relativizing 4.2, one readily gets:

Corollary 4.3. *Assume Borel Turing determinacy. If* $(A_k)_{k<\omega}$ *is a sequence of cofinal analytic sets of Turing degrees, then* $\bigcap_k A_k$ *is cofinal in* $\mathcal{D}$. ⊣

An interesting special case of 4.2, where the 'sequence' $(S_k)_{k<1}$ is a single Σ^1_1 term.

Theorem 4.4. Turing-Det(Δ^1_1) *implies* Hyp-Turing-Det(Σ^1_1). ⊣

In view of Theorem 3.14, the implication is an equivalence. A similar result obtains for full determinacy.

Definition 4.5. For a game $\mathrm{G}_\omega(A)$, a strategy σ for Player I is called a hyp-winning strategy if $\forall \tau \leqslant_h \sigma\,(\sigma{*}\tau \in A)$, i.e., applying σ, Player I wins against any $\Delta^1_1(\sigma)$ sequence of moves by Player II.

Theorem 4.6. *Assume* Det(Δ^1_1). *For* $S \in \Sigma^1_1$, *one of the following holds for* $\mathrm{G}_\omega(S)$,
(1) *Player I has a hyp-winning strategy.*
(2) *Player II has a winning strategy.*

Proof. Let S be Σ^1_1, and assume Player I has no hyp-winning strategy for $\mathrm{G}_\omega(S)$, that is: $\forall\sigma\,\exists\tau \leqslant_h \sigma\,(\sigma{*}\tau \notin S)$. Much as in the proof of 4.2, Reflection yields a Δ^1_1 set $D \supseteq S$ such that Player I has no hyp-winning strategy for $\mathrm{G}_\omega(D)$, hence no winning strategy. Invoking Det(Δ^1_1), Player II has a winning strategy for $\mathrm{G}_\omega(D)$ which is, *a fortiori*, winning for $\mathrm{G}_\omega(S)$. ⊣

§5. Appendix.

The point of the present section is to sketch a proof of Proposition 3.8, without dissecting the $\mathbb{L}$ construction—albeit with a recourse to admissible sets. Finer results most certainly hold.

$\mathcal{F}$ is the set of formulas, $\mathcal{F} \in \mathbb{L}_{\omega+1}$, and $\models_{\mathbb{L}_\alpha}$ is the satisfaction relation for $\mathbb{L}_\alpha$,

$$\models_{\mathbb{L}_\alpha}(\varphi, \vec{s}) \;\Leftrightarrow\; \varphi \in \mathcal{F} \;\&\; \vec{s} \in \mathbb{L}_\alpha^{<\omega} \;\&\; \mathbb{L}_\alpha \models \varphi[\vec{s}].$$

Apart from the classic Condensation Lemma (see 1.3), we shall need the following familiar result: *For any limit* $\lambda > \omega$, *and* $\beta < \lambda$, $\models_{\mathbb{L}_\beta} \in \mathbb{L}_\lambda$. See [13, §7.1].

Notation. Let $X \gg^\lambda Y$ abbreviate $\exists f \in \mathbb{L}_\lambda (\, f : X \twoheadrightarrow Y)$, where '$\twoheadrightarrow$' stands for surjective map.

Reminder. Here, "μ is an $\mathbb{L}_\lambda$-cardinal" means: "for no $\xi < \mu$, does $\xi \gg^\lambda \mu$" (see 3.5).

Lemma 5.1. *Let* $\lambda > \omega$ *be limit. For* $0 < \alpha \leqslant \gamma < \lambda$, *and* $\mathbb{L}_\beta = \overline{\mathsf{H}}^{\mathbb{L}_\gamma}(\alpha)$, $\alpha^{<\omega} \gg^\lambda \beta$.

Proof. Observe that $\mathbb{L}_\beta = \mathsf{H}^{\mathbb{L}_\beta}(\alpha)$, and $\beta < \lambda$. In $\mathbb{L}_\beta$, every $\xi < \beta$ is the unique solution of some formula $\varphi(v, \vec{\eta})$, where $\vec{\eta} \in \alpha^{<\omega}$. Thus, using $\models_{\mathbb{L}_\beta} \in \mathbb{L}_\lambda$, one

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readily derives $\mathcal{F} \times \alpha^{<\omega} \gg^{\lambda} \beta$. Using an injection $\mathcal{F} \times \alpha^{<\omega} \to \alpha^{<\omega}$ in $\mathbb{L}_\lambda$, one gets $\alpha^{<\omega} \gg^{\lambda} \beta$. ⊣

PROPOSITION 5.2. *Let $\lambda > \omega$ be a limit ordinal, and $\omega < \mu < \lambda$, an $\mathbb{L}_\lambda$-cardinal.*

(1) *For $0 < \alpha < \mu \leqslant \gamma < \lambda$, and $\mathbb{L}_\beta = \overline{\mathsf{H}}^{\mathbb{L}_\gamma}(\alpha) : \beta < \mu$.*
(A downward Löwenheim–Skolem property).

(2) *μ is admissible.*

PROOF. We check (1) and (2) simultaneously, by induction on μ.

(1) Set $\overline{\mu} = \min_{\eta \leqslant \mu}(\eta^{<\omega} \gg^{\lambda} \mu)$. Note that, for $\eta \leqslant \overline{\mu}$, $(\eta \gg^{\lambda} \overline{\mu} \implies \eta^{<\omega} \gg^{\lambda} \overline{\mu}^{<\omega})$, it follows that $\overline{\mu}$ is an $\mathbb{L}_\lambda$-cardinal, and clearly $\omega < \overline{\mu} \leqslant \mu$.

We claim that $\overline{\mu} = \mu$. If $\mu = \aleph_1^{\mathbb{L}_\lambda}$, then $\overline{\mu} = \mu$. Else, if $\overline{\mu} < \mu$ then, by induction, $\overline{\mu}$ is admissible, yielding an $\mathbb{L}_{\overline{\mu}}$-definable map $\overline{\mu} \twoheadrightarrow \overline{\mu}^{<\omega}$. Whence $\overline{\mu} \gg^{\lambda} \overline{\mu}^{<\omega} \gg^{\lambda} \mu$, and thus $\overline{\mu} \gg^{\lambda} \mu$, contradicting "$\mu$ is an $\mathbb{L}_\lambda$-cardinal."

Now, given $0 < \alpha < \mu \leqslant \gamma < \lambda$, and $\mathbb{L}_\beta = \overline{\mathsf{H}}^{\mathbb{L}_\gamma}(\alpha)$, the previous lemma yields $\alpha^{<\omega} \gg^{\lambda} \beta$. Hence, since $\alpha < \overline{\mu} = \mu$, $\beta < \mu$.

(2) To show that μ is admissible, only Δ_0 COLLECTION needs checking.

Say $\mathbb{L}_\mu \models \forall x \in \boldsymbol{a} \exists y \varphi(x, y, \vec{\boldsymbol{p}})$, where φ is Δ_0, and $a, \vec{p} \in \mathbb{L}_\mu$. Pick $\alpha < \mu$ with $a, \vec{p} \in \mathbb{L}_\alpha$ and set $\mathbb{L}_\beta = \overline{\mathsf{H}}^{\mathbb{L}_\mu}(\alpha)$: $\mathbb{L}_\beta \models \forall x \in \boldsymbol{a} \exists y \varphi(x, y, \vec{\boldsymbol{p}})$. Applying (1), $\beta < \mu$, thus $b =^{\text{def}} \mathbb{L}_\beta \in \mathbb{L}_\mu$. By Δ_0 absoluteness, $\mathbb{L}_\mu \models \forall x \in \boldsymbol{a} \exists y \in \boldsymbol{b} \varphi(x, y, \vec{\boldsymbol{p}})$. ⊣

PROPOSITION 3.8. For λ limit, $\mathbb{L}_\lambda \models$ "$\boldsymbol{\mu} > \omega$ is a successor cardinal" $\implies$ $\mathbb{L}_\mu \models \mathsf{ZF}^-$.

PROOF. Set $\pi =$ the cardinal preceding μ in $\mathbb{L}_\lambda$. We argue that π is the largest cardinal in $\mathbb{L}_\mu$. Indeed, for $\pi \leqslant \eta < \mu$, pick $\gamma < \lambda$ such that $\exists f \in \mathbb{L}_\gamma (f : \pi \twoheadrightarrow \eta)$, and set $\mathbb{L}_\beta = \overline{\mathsf{H}}^{\mathbb{L}_\gamma}(\eta + 1)$. We get $\exists f \in \mathbb{L}_\beta (f : \pi \twoheadrightarrow \eta)$ and, invoking 5.2(1), $\beta < \mu$. Hence $\mathbb{L}_\mu \models \exists f (f : \boldsymbol{\pi} \twoheadrightarrow \boldsymbol{\eta})$.

Next: μ is regular in $\mathbb{L}_\lambda$. The usual ZFC proof for the regularity of infinite successors goes through here: for each nonzero $\eta < \mu$, using $<_{\mathbb{L}_\mu}$, select $f_\eta \in \mathbb{L}_\mu$, $f_\eta : \pi \twoheadrightarrow \eta$, and note that the sequence $(f_\eta)_{0<\eta<\mu}$ is in $\mathbb{L}_{\mu+1} \subseteq \mathbb{L}_\lambda$, etc.

Finally, to show $\mathbb{L}_\mu \models \mathsf{ZF}^-$: since by 5.2(2) μ is admissible, using the standard definable bijection $\mu \to \mathbb{L}_\mu$, it suffices to verify REPLACEMENT for $\mathbb{L}_\mu$ class-functions $\mu \to \mu$.

Let therefore $F : \mu \to \mu$ be $\mathbb{L}_\mu$-definable, from parameters $\vec{p}$. Given a set of ordinals $s \in \mathbb{L}_\mu$, s is bounded in μ. By regularity of μ in $\mathbb{L}_\lambda$, $F[s]$ is bounded as well. Pick $\alpha < \mu$, with $F[s] \subseteq \alpha$ and $s, \vec{p} \in \mathbb{L}_\alpha$: $F[s]$ is definable over $\mathbb{L}_\mu$ from $s, \vec{p} \in \mathbb{L}_\alpha$, and $\mathbb{L}_\alpha \subseteq \mathsf{H}^{\mathbb{L}_\mu}(\alpha) \prec \mathbb{L}_\mu$. Set $\mathbb{L}_\beta = \overline{\mathsf{H}}^{\mathbb{L}_\mu}(\alpha)$, applying 5.2(1), $\beta < \mu$, and thus $F[s] \in \mathbb{L}_{\beta+1} \subseteq \mathbb{L}_\mu$. ⊣

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