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Modal structuralism simplified

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ABSTRACT
Since Benacerraf's 'What Numbers Could Not Be,' there has been a growing interest in mathematical structuralism. An influential form of mathematical structuralism, modal structuralism, uses logical possibility and second order logic to provide paraphrases of mathematical statements which don't quantify over mathematical objects. These modal structuralist paraphrases are a useful tool for nominalists and realists alike. But their use of second order logic and quantification into the logical possibility operator raises concerns. In this paper, I show that the work of both these elements can be done by a single natural generalization of the logical possibility operator.

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1. Introduction

Since Benacerraf's 'What Numbers Could Not Be,' Benacerraf (1965) there has been a growing interest in mathematical structuralism. One of the most influential forms of structuralism is the modal structuralism developed in Geoffrey Hellman's *Mathematics Without Numbers* (Hellman 1994). Modal Structuralism is a nominalist philosophy of mathematics which maintains that mathematicians can systematically express truths even if there are no mathematical objects, by interpreting statements about mathematical objects as modal claims about what is logically possible. Specifically, Hellman uses claims about logical possibility and second order logic to provide intuitively correct truth conditions for mathematical utterances without quantifying over mathematical objects like numbers and sets.

I don't ultimately find nominalism persuasive, and won't defend it against standard objections. However, I think that Hellman's modal structuralist paraphrases reveal a close relationship between logical possibility and pure mathematics which is of interest to realists and nominalists alike.[1] For they show us

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how to systematically pair ordinary (platonistic) mathematical sentences with modal sentences which have exactly the truth value *a platonist would want to ascribe to* the original, but make claims about logical possibility rather than quantifying over mathematical objects. So, for example, Hellman's paraphrase of 'there are infinitely many primes' is a modal sentence which is (intuitively) true at all possible worlds and does not quantify over mathematical objects.

This is useful to, for example, deflationary realists who want to (somehow) ground mathematical existence facts in logical possibility[2] as well as to nominalists who want to deny the existence of mathematical objects. Also, one part of Hellman's story (his treatment of set theory) provides a natural way of developing an independently popular view about set theory called potentialism. Philosophers like Charles Parsons, who have no truck with blanket nominalism about mathematical objects, have been motivated by specific (i.e. specific-to-set-theory) apparent paradoxes concerning the height of the hierarchy of sets to understand higher set theory as an investigation of extendability (Parsons 2007). Thus, one might want to accept something like Hellman's approach to set theory while being a straightforward realist about other mathematical objects and structures.

In this paper, I will show how to streamline Hellman's modal structuralist paraphrases for mathematics by appealing to a single, intuitively motivated, notion of logical possibility given certain facts – thus avoiding the need for second order quantification.[3] In addition to its intrinsic interest, this simplification provides expository and philosophical benefits over Hellman's approach.

First, existing potentialist and modal structuralist paraphrases for sentences of set theory (including Hellman's) involve quantifying in to the $\diamond$ of logical possibility. That is, they use sentences like $\exists x \diamond R(x)$, where the logical possibility operator is applied to a formula with free variables. There are significant controversies about the truth conditions, and indeed meaningfulness, of such statements. For example, there is disagreement about whether any two things that are actually distinct are necessarily distinct. There is also disagreement about what to say about statements which quantify into a world where an object doesn't exist. For example, Kripke's approach (which Hellman invokes) allows sentences like $(\exists x)\diamond[(\forall y)Fox(y) \wedge \neg Fox(x)]$ to be true, a consequence which Williamson and others have argued is extremely counterintuitive.[4] These controversies can raise doubts about whether our intuitions about quantifying in are reliable while, to my knowledge, no analogous paradoxes arise in the system I lay out.[5]

There is also a Quinean strand of argument which claims that quantifying into modal contexts is meaningless.[6] Thus, it seems, at least, rhetorically desirable to demonstrate that Hellman's program (as well as potentialist set theory) doesn't require quantifying in or similarly controversial notions.

Second, Hellman himself (Hellman 1996) has raised worries about whether his use of second-order logic is nominalistically acceptable, and my modifications show that his program[7] can be accomplished without second order logic, using only concepts he relies on elsewhere in his program. This is not to say that my modifications definitely render Hellman's approach nominalistically acceptable. Indeed, one might even take my demonstration that logical possibility can fill in for second order logic as an argument against the nominalistic acceptability of logical possibility itself. Rather, I show that Hellman can avoid any extra burden imposed specifically by his use of second order logic. Either my modifications render Hellman's account nominalistically acceptable or the very notion of logical possibility employed by Hellman is inherently nominalistically unacceptable and his program fails regardless of the role of second order logic.

In later work Hellman considers[8] a modification to his core story which avoids second order quantification.[9] However, this story relies on an additional assumption (that 'arbitrary [mereological] sums of any individuals independently recognized' exist) which my story and Hellman's original story avoid.[10] This later proposal also does not avoid the issues about quantifying in noted above.

2. Modal structuralism: the core picture

The key idea behind modal structuralism is to reformulate mathematical claims about abstract (non-set-theoretic) objects, like the natural numbers, as claims about how it is possible for objects to be related to one another. For example, something like the twin prime conjecture may be paraphrased as the claim that it would be possible for there to be objects with the structure of the natural numbers and that, necessarily, in any such structure there are infinitely many twin primes. Note that the notion of possibility here isn't that of metaphysical possibility. For, as Charles Parsons points out, our willingness to talk in terms of large mathematical structures (e.g. the reals or the Hilbert space of square integrable functions) does not seem to be hostage to our conviction that it would be *metaphysically* possible for there to be that many non-mathematical objects (Parsons 2007). Thus, it seems like the notion of possibility which the modal structuralist is reaching for is something more like mathematical or logical possibility.

In articulating his modal structuralism, Hellman invokes a primitive notion of logical possibility which he does relatively little to describe. He does say that, '[when evaluating logical possibility] we are not automatically constrained to hold material or natural laws fixed.' So it may be logically possible that $(\exists x)(\text{pig}(x) \wedge \text{flies}(x))$, but physically impossible. And he adds that, "we are free to entertain the possibility of additional objects – even material objects – of a given type", which allows us to say that it's *logically* possible for there to be infinitely many objects even if there are only finitely many objects. Beyond this, however, he just suggests that his applications of logical possibility will make the notion he has in mind clear.

I will abbreviate the claim that it is logically possible that ϕ as $\Diamond\phi$, and the claim that it is logically necessary that ϕ (i.e. $\neg\Diamond\neg\phi$ as $\Box\phi$). With this notion of logical possibility in place, the modal structuralist proposes to understand a mathematician's claim that ψ holds in some mathematical structure (such as the natural numbers), as really asserting a conjunction of two claims. First, it is logically possible for there to be some objects with the relevant structure (e.g. there could be an ω sequence of objects). And second, it is logically necessary that if there were such objects they would satisfy the description ψ (e.g. if there were an ω sequence of objects, a version of ψ would be true in it).

Hellman uses second-order quantification to give categorical descriptions[11] of such structures, e.g. the ω sequence mentioned above. Employing these descriptions allows Hellman's paraphrase strategy to ensure (assuming second order logic works in the usual way) that all well-formed claims about these structures are either true or false. For example, let PA_2 be the standard second order categorical axiomatization of the natural numbers in terms of a successor relation S[12] (conjoined into a single sentence) and let ϕ be a sentence about the natural numbers. Using $\phi(\mathbb{N}/X)(S/f)$ to denote the result of replacing every instance of $\mathbb{N}$ in ϕ with the second order variable X and every instance of the successor relation[13] S with the second order relation variable f Hellman's paraphrase of the mathematical claim ϕ becomes[14]:

$$\Diamond\left[(\exists X)(\exists f)PA_2(\mathbb{N}/X)(S/f)\right]\wedge \\ \Box\left[(\forall X)(\forall f)\left(PA_2(\mathbb{N}/X)(S/f)\rightarrow\phi(\mathbb{N}/X)(S/f)\right)\right]$$

The first half of this sentence says that it is logically possible for some objects to form an ω-sequence (with some relation f acting as the successor function). The second half says that it is logically necessary that if some objects (those in X) form an ω-sequence (under f) then ϕ (modified to use X and f instead of $\mathbb{N}$ and S) is true of them.

This paraphrase strategy (assuming logical possibility and second order quantification operate as Hellman expects[15]) captures the intended truth conditions for most statements in pure mathematics. However, Hellman also wishes to provide paraphrases for statements of applied mathematics. Consider the claim that there are a prime number of rats. One cannot give correct truth conditions for this claim by only talking about what is logically possible simpliciter – for the truth of 'there are a prime number of rats' is not determined only by facts about what is logically possible. It also reflects contingent facts about the world.

Hellman addresses this problem by replacing appeals to logical possibility with appeals to logical possibility given the 'material'[16] facts. So, for example, where the platonist takes 'there are a prime number of rats' to mean something like 'there is a function which bijectively maps the rats to the natural numbers below some prime p', Hellman will translate this claim approximately as follows. It is logically possible, given the material facts, that there are objects which behave like numbers (in the sense of satisfying PA_2). And it is logically necessary, given the material facts, that if there are objects which behave like numbers then there is a function which bijectively maps the rats to the natural numbers below p.

Hellman considers two approaches to understanding this crucial notion of logical possibility given the material facts. The first is to leave it as a primitive, "reject[ing] the demand" for further explanation of what it means to hold material facts fixed. The second is to cash out the notion of 'holding the material facts fixed' by using an actuality operator @, read as 'it is actually the case that.' In either case, we see that Hellman is already committed to something like a notion of logical possibility holding some facts fixed. The reader should bear this in mind when considering the particular notion of logical possibility I offer below.

3. Logical possibility sharpened and generalized

I will now introduce my preferred notion of *logical possibility given certain facts*. Let me begin by calling to mind some features of the standard notion of logical possibility which I take Hellman to be developing.

3.1. The conventional notion of logical possibility

It seems that we have an intuitive notion of logical possibility which applies to claims like $(\exists x)(red(x) \wedge round(x))$ and makes sentences like the following come out true.

- It is logically possible that $(\exists x)(red(x) \wedge round(x))$
- It is not logically possible that $(\exists x)(red(x) \wedge \neg red(x))$
- It is logically necessary that $(\forall x)(red(x)) \rightarrow \neg(\exists x)(\neg red(x))$

Philosophers representing a range of different philosophies of mathematics have made use of this notion[17] and are comfortable applying it to non-first order sentences as well. If you are skeptical that there is such a notion, note that it is definable in terms of the even more common notion of validity (something is logically possible iff its negation is not logically necessary iff the inference from the empty premise to its negation is not valid).

To evaluate whether a claim ϕ requires something logically possible, we hold fixed the operation of logical vocabulary (like $\exists$, $\wedge$, $\vee$, $\neg$), but abstract away from any further constraints imposed by metaphysical necessity on the behavior of particular relations. Thus, we consider **all possible ways for relations to apply *whether or not these ways are describable in our language***. For example, it

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is logically possible that $(\exists x)(\text{Raven}(x) \wedge \text{Vegetable}(x))$, even if it would be metaphysically impossible for anything to be both a raven and a vegetable. During this evaluation we also abstract away from constraints on the size of the universe,[18] so that $\Diamond(\exists x)(\exists y)(\neg x = y)$ would be true even if the actual universe contained only a single object.

This notion of logical possibility is generally regarded as a fundamental notion[19] conceptually distinct from syntactic consistency, i.e. the impossibility of proving a contradiction. Instead, it corresponds to our intuitive sense that certain mathematical theories (like second-order Peano Arithmetic) require something coherent, while others (like Frege's inconsistent theory of extensions) do not – a sense which is not restricted merely to first-order descriptions.

A core idea I will develop is that the above notion of logical possibility can be naturally generalized. A (pure) logical possibility operator doesn't allow information to 'leak out', so merely adding such an operator to first order logic does little to increase its 'power.' This can make it appear somewhat surprising that, as we shall see, the tame-looking further step of considering logical possibility holding certain facts fixed (a concept Hellman already appeals to) is enough to let us relinquish our use of second order quantification. However, we observed above that the concept of logical possibility goes far beyond what is capturable in first order logic, so it's not totally shocking that we can unlock that power by letting some information pass through (but not free variables).

3.2. Logical possibility generalized

Let us now develop the notion of logical possibility discussed in the previous section. Consider a sentence like, 'Given what cats and baskets there are, it is logically impossible that each cat slept in a distinct basket.' There's an intuitive reading on which this sentence will be true if and only if there are more cats than baskets.[20] This reading employs a notion of logical possibility *holding certain facts fixed* (in this case, facts about what cats and baskets there are). Remember, Hellman's use of logical possibility given the material facts commits him to the coherence of something very much like this notion.

Accordingly, I think we can intuitively understand a conditional logical possibility operator $\Diamond$ which takes a sentence ϕ and a finite (potentially empty) list of relation symbols $R_1 \ldots R_n$ and produces a sentence $\Diamond(R_1 \ldots R_n)\phi$ which says that it is logically possible for ϕ to be true, given how the relations $R_1 \ldots R_n$ apply. For ease of reading, I will sink the specification of relevant relations into the subscript as follows: $\Diamond_{R_1 \ldots R_n}\phi$

Thus, for example, the claim, 'given what cats and baskets there are, it is logically impossible that each cat slept in a distinct basket' becomes:

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(CATS)

$$\Box_{\text{cat,basket}}\neg\Big((\forall x)\Big[\text{cat}(x) \rightarrow (\exists y)\,\big(\text{basket}(y) \wedge \text{sleptIn}(x,y)\big)\Big] \wedge (\forall z)(\forall w)(\forall w')\Big[\text{basket}(z) \wedge \text{cat}(w) \wedge \text{cat}(w') \wedge \text{sleptIn}(w,z) \wedge \text{sleptIn}(w',z) \rightarrow w = w'\Big]\Big)$$

Finally, note that by using this notion we can also make *nested* logical possibility claims, i.e. claims about the logical possibility of scenarios which are themselves described in terms of logical possibility. I have in mind sentences like the following:

(◇CATS)

$$\Diamond\Box_{\text{cat,basket}}\neg\Big((\forall x)\Big[\text{cat}(x) \rightarrow (\exists y)\,\big(\text{basket}(y) \wedge \text{sleptIn}(x,y)\big)\Big] \wedge (\forall z)(\forall w)(\forall w')\Big[\text{basket}(z) \wedge \text{cat}(w) \wedge \text{cat}(w') \; \text{sleptIn}(w,z) \wedge \text{sleptIn}(w',z) \rightarrow w = w'\Big]\Big)$$

This sentence says that it would be logically possible for there to be cats and baskets such that it would be logically necessary, given (the structural facts about) what cats and baskets there are *in that scenario*, that some cat lacked its own basket to sleep in. Note that in a nested claim with this form ($\Diamond\Box_R\psi$), the subscript freezes the facts about how the relation R applies in the scenario being considered, which may *not* be the state of affairs in the actual world. For example, ◇CATS expresses a metaphysically necessary truth. For, whatever the actual world is like, it will always be logically possible for there to be, say, 3 cats and 2 baskets. This scenario is one in which it is logically necessary (holding fixed the structural facts about what cats and baskets there are) that: if each cat slept in a basket then multiple cats slept in the same basket. So it is metaphysically necessary that ◇CATS even if the actual world contains more baskets than cats.

In what follows, I will often use mathematical-looking symbols or schematic-looking symbols (e.g. $\mathbb{N}, S$) for relations appearing in logical possibility statements rather than actual relations like 'happy()' and 'loves()'. However, these symbols should be regarded merely as an abbreviation, so when I write $\Box_P(\forall x)(P(x) \rightarrow Q(x))$ it is shorthand for something like $\Box_{Happy}(\forall x)(Happy(x) \rightarrow Elephant(x))$.

Note that the specific choice of relations does not mater, as when a relation occurs inside a □ or ◇ which does not subscript that relation, it contributes to the truth conditions for this sentence in exactly the same way that any other relation with the same arity would. For example, the sentence $\Diamond_{\text{Dog}}(\exists x)(\exists y)(\text{Dog}(x) \wedge \text{Cat}(y) \wedge \neg x = y)$ will hold if and only if $\Diamond_{\text{Dog}}(\exists x)(\exists y)(\text{Dog}(x) \wedge \text{Lemur}(y) \wedge \neg x = y)$ does.

This reflects the fact that questions about logical possibility abstract away from all specific facts about the relations in question (other than their arity). Logical possibility involves considering all possibilities for the relations mentioned in the statement under consideration, whether we can describe them or not (this is the analog of requiring second order quantifiers to range over all possible collections). I emphasize this fact, because I will translate claims about mathematical objects using claims about how it would be logically possible for some arbitrarily chosen relations to apply (as Putnam does in Putnam 1967, 10–11) instead of using variables bound by second order quantifiers as Hellman does.

Some readers may still have questions about how holding relations fixed works. One could think about $\Diamond_{R_1 \ldots R_n}\phi$ claims as holding fixed the *particular objects* in the extension of the relations $R_1 \ldots R_n$ – and then asking whether one could supplement them with other objects (and choose extensions for all other relations) so as to make ϕ true.[21] However, I take the intuitive notion of preserving the *structural facts* about how some relations apply (that is, the facts about what might be called the mathematical structure of the objects with respect to some relations as opposed to facts about any particular objects) to make sense without appeal to any notion of de re properties or object identity across logically possible scenarios.

In terms of the CATS example, preserving the structural facts about how cat and basket apply requires considering scenarios which agree with the actual world on the number of objects satisfying cat(), the number of objects satisfying basket() and the number of things in the extension of both cat() and basket(). This does not require preserving facts about identity. For example, if one cat died and an additional kitten was born, the structural facts about how cat and basket apply would remain unaltered.

Speaking in set theoretic terms, we might say that the 'structural facts about $R_1 \ldots R_n$' are those facts which determine the isomorphism class of the objects falling under[22] some R_j. However, I take conditional logical possibility to be a primitive notion which we can learn directly.

Note that this notion of relativized logical possibility is stronger than Hellman's notion of unrelativized logical possibility supplemented by an actuality operator in one important way. In Appendix 4, I show that we can capture the same content Hellman expresses using his actuality operator by relativizing all our possibility operators to the relations whose extension in the actual world we wish to discuss.[23] In contrast, merely using Hellman's actuality operator does not allow us to express claims about what is logically possible relative to scenarios which are themselves merely logically possible but not actual. This feature turns out to be very useful, as we will see.

4. Reformulating Hellman's simple paraphrases

Now we turn to demonstrating that Hellman's paraphrases of mathematical claims can be captured using only conditional logical possibility claims and first order vocabulary.

4.1. Strategy

My translations will have approximately the same logical form as Hellman's. Given a description D of a mathematical structure and a statement ϕ about this structure, my translation for ϕ will still assert that it would be logically possible for the structure described by D to be realized, and that it is logically necessary that if some objects have this structure than (a suitably modified version of) ϕ will be true of them. However, we will need to replace all of Hellman's use of second order logic in his translations of mathematical statements with logical possibility claims.

To illustrate this strategy, consider the case of mathematical statements about the natural numbers (I describe how to generalize this approach in Appendix 2). Recall that one can uniquely describe the intended structure of the natural numbers by combining the first four Peano Axioms (Weisstein 2013) (which can be expressed using only first order logical vocabulary) with a second order Axiom of Induction, which can be expressed as follows[24]:

$$(\forall X)\left(X(0) \wedge (\forall n)(X(n) \rightarrow X(S(n))) \rightarrow (\forall n)(\mathbb{N}(n) \rightarrow X(n))\right)$$

Informally, this axiom says that if some property X applies to 0 and is closed under successor,[25] then it applies to all the numbers. We can express the same idea using $\diamond$ (and a predicate we abbreviate as P[26]) as follows.

$$\Box_{\mathbb{N},S}[P(0) \wedge (\forall x)(\forall y)(P(x) \wedge S(x,y) \rightarrow P(y))] \rightarrow (\forall x)(\mathbb{N}(x) \rightarrow P(x))$$

This formula says that, given the facts about what is a number and a successor, (i.e. how $\mathbb{N}$ and S apply), it would be logically impossible for P to apply to 0[27] and be closed under the successor operation but not apply to all the numbers. Call the result of conjoining this sentence with the four first order axioms of Peano arithmetic $PA_{\diamond}$.

Now we can slot this into Hellman's paraphrase strategy, and so replace his translation of any first order sentence of number theory ϕ.[28] Thus,

$$\diamond\left[(\exists X)(\exists f)\mathrm{PA}_2(\mathbb{N}/X)(S/f)\right] \wedge \Box\left[(\forall X)(\forall f)\left(\mathrm{PA}_2(\mathbb{N}/X)(S/f) \rightarrow \phi(\mathbb{N}/X)(S/f)\right)\right]$$

becomes:

$$\Diamond\left[\mathsf{PA}_{\Diamond}(\mathbb{N}/P)(S/R)\right] \wedge \Box\left[\left(\mathsf{PA}_{\Diamond}(\mathbb{N}/P)(S/R) \to \phi(\mathbb{N}/P)(S/R)\right)\right]$$

where P is an arbitrary one place relation and R is an arbitrary two place relation. As noted above, logical possibility claims reflect facts about all possible ways that a predicate P could apply - whether describable or not. Thus, my translation of a sentence ϕ about the natural numbers intuitively has the same truth value as Hellman's translation of that sentence (assuming second order quantification and logical possibility work as Hellman expects).[29] In the remainder of this paper, I will simply speak of the truth-values of Hellman's translations or Hellman's intended truth-values, but in both cases I mean the truth-values his translations would have if the above assumption were true. A similar story can be told for mathematical structures other than the natural numbers, as I show in Appendix 2.

Hellman argues for the bivalence of his translations by appealing to the categoricity of the second order descriptions of the mathematical structures under consideration. In other words, given any sentence ψ in the appropriate language, either it or its negation will be necessitated by Hellman's description D of the relevant structure. If you accept that my translations of mathematical sentences have the same truth-values as Hellman's translations of these sentences, then my translations of sentences about these mathematical structures will also be bivalent. However, we need not go through Hellman to see that my translations yield bivalence in cases where it is intuitively desired (i.e. when we seem to have a suitably definite conception of the relevant mathematical structure).

To see how this plays out in the case of the natural numbers, note that Hellman's translations are intuitively bivalent because he uses second order logic to express the idea that the numbers are as few as can be (and thereby rule out nonstandard models which add 'points at infinity'), by saying that any second order X applying to 0 and closed under successor applies to all the natural numbers. My translations do that same work by asserting that it would be logically impossible for a predicate to apply to 0 and the successor of every number it applies to without applying to all numbers. Intuitively this has the same effect that Hellman intends his second order description to have, while not presuming anything about the behavior of second order quantifiers.

5. Hellman's potentialist set theory

Now let us turn to Hellman's translations for statements of (pure) set theory, which have a significantly different structure from his translations of claims about ordinary mathematical structures.

5.1. Motivations for potentialism

If we had a categorical description of the intended structure of the hierarchy of sets (in the language of second order logic), we could nominalistically paraphrase sentences in set theory using the strategy from the last section.

However, there are well-known reasons for doubting that we have any coherent and adequate conception of absolute infinity (the supposed height of the hierarchy of sets). The concern here is not simply that it might be impossible to cash the notion of absolute infinity out in other terms. After all, every theory will have to take some notions as primitive. Rather, the worry is that our intuitive notion isn't even coherent – in the way that our naive conception of set is incoherent (as demonstrated by Russell's paradox).

One might like to say that the hierarchy of sets goes all the way up – so no restrictive ideas of where it stops are needed to understand its behavior. However, if the sets really do go 'all the way up' in this sense, then it would seem that the ordinals should satisfy the following closure principle.

> For any way some things could be well-ordered, there is an ordinal corresponding[30] to it.

But the ordinals themselves are well ordered, and there can be no ordinal corresponding to this well-ordering. If the sets are a definite totality, i.e. a logically possible collection of objects, this is a contradiction. Thus, this naive closure principle can't be correct.

In response, we might try to find some other characterization of the sets as a definite structure (in particular, some other characterization of the intended height of the hierarchy of sets[31]). However, it's not clear that any intuitive conception of the intended height of the sets remains once the paradoxical well-ordering principle above is retracted. As Wright and Shapiro put it Shapiro and Wright (2006), all our reasons for thinking that sets exist in the first place appear to suggest that, for any given height which an actual mathematical structure could have, the sets should continue up past this height. Thus, taking set theory at face value can seem to force us to posit an unprincipled fact about where the sets stop.[32] This problem isn't limited to realists, but applies to all philosophers (including modal structuralists) who take set theory to be the study of a single definite structure.

5.2. The potentialist approach to set theory

Potentialists, including Hellman, respond to this problem by taking a potentialist approach to set theory (along lines suggested by Putnam Putnam (1967)). On this approach, mathematicians' claims which appear to quantify over sets should really be[33] understood as claims about how it is (in some sense) *possible* to extend initial segments of the hierarchy of sets, i.e. collections of objects which satisfy our intuitive conception of the width of the hierarchy of sets but not the paradox-generating height requirement. Hellman, unsurprisingly, understands the relevant notion of possibility in terms of logical possibility (and I will follow him in so doing).[34]

The potentialist takes set theorists' singly-quantified existence claims, like $(\exists x)(x = x)$, to really be saying that it would be possible for a collection of objects V_0 to satisfy (a version of) ZFC_2 while containing a suitable object x (in this case, an x such that $x = x$). The potentialist takes set theorists' universal statements with a single quantifier like $(\forall x)(x = x)$, to really say that it is necessary that any object x in a collection of objects satisfying ZFC_2 would have the relevant property.

The potentialist handles nested quantification using claims about how collections of objects satisfying a version of ZFC_2 could be extended. For example, Hellman would offer the following translation of $(\forall x)(\exists y)(x \in y)$: necessarily if V_1 satisfies ZFC_2 and includes a set x, it is logically possible for there to be an extension, V_2,[35] of V_1, also satisfying ZFC_2 and containing a set y such that $x \in y$ (in the sense of $\in$ relevant to V_2).[36] Writing this out formally using Hellman's notion of logical possibility gives us the following sentence (implicitly restricting V_1 and V_2 to range over collections of objects satisfying a version of ZFC_2 and using $\geq$ to denote extension):

$$\Box(\forall V_1)(\forall x)[x \in V_1 \rightarrow \Diamond(\exists V_2)(\exists y)(y \in V_2 \wedge V_2 \geq V_1, \wedge x \in y)]$$

Note that by adopting this potentialist understanding of set theory, we avoid commitment to arbitrary limits on the intended height of the hierarchy of sets. We also avoid the assumption that there is (or could be) any single structure which contains ordinals witnessing all possible well-orderings, though every possible well-ordering is realized in some possible initial segment of the sets.

6. Formulating potentialist set theory

Now let us turn to the problem of articulating a suitable replacement for Hellman's potentialist paraphrases which avoids second order quantification. I will explain my version of these potentialist paraphrases informally, but the interested reader should see Appendix 4 for more details. The appendix also reviews why bivalence holds for my translations of sentences in set theory.[37]

To articulate potentialist paraphrases of set theory in terms of conditional logical possibility, we must first express the claim that some objects behave like a standard width initial segment of the hierarchy of sets. Hellman expresses this idea by using ZFC_2, a second order version of the ZFC axioms of set theory. One can show that ZFC_2 suffices to pin down the intended width of the hierarchy of sets (though not their height). It's not too hard to write a version of ZFC_2 in terms of my notion of conditional logical possibility, by using a version of the trick for replacing second order quantification with claims about logically

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possible extendability demonstrated in Section 4 and generalized in Appendix 2. This approach lets us write out a sentence (as it were $ZFC_{\diamond}[set_i, \in_i]$) using the logical possibility operator which says that the objects satisfying set_i under the relation $\in_i$ capture the behavior of an initial segment of the sets.

We now must duplicate the complex statements about extendability used to handle nested quantification in Hellman's paraphrases. It is straightforward to define the claim that $\text{set}_{i+1}, \in_{i+1}$ extends $\text{set}_i, \in_i$ using only the logical possibility operator and first order vocabulary. This allows us to talk about possible extensions of initial segments of the sets. However, to fully represent potentialist paraphrases, we also need to mirror Hellman's claims which fix an object x from among those which some relations $\text{set}_i, \in_i$ apply to, and talk about how an element y in a potential extension $\text{set}_{i+1}, \in_{i+1}$ relates to x. As stated, this claim involves quantifying in, but we must find another method.

The key idea behind my strategy is to require that each initial segment of objects satisfying $\text{set}_i, \in_i$ be considered together with a relation R_i which assigns each 'variable' from some countable collection[38] to an object satisfying set_i. Thus, R_i behaves like an assignment function which associates each variable with some object within the initial segment V_i. We can then preserve the behavior of this assignment function in relevant modal contexts by adding R_i to the subscripts on relevant $\Box$s and $\diamond$s and demanding that R_{i+1} agree with R_i everywhere except on the particular variable we want to select from $\text{set}_{i+1}, \in_{i+1}$. This allows us to preserve our choice of some sets x, y and z within $\text{set}_i, \in_i$ while considering ways that one could choose an additional object w from within some logically possible $\text{set}_{i+1}, \in_{i+1}$ extending $\text{set}_i, \in_i$. The overall effect will be to duplicate what Hellman achieves via quantifying in, through the use of the relations R_i.

7. Conclusion

In this paper I have shown how to streamline Hellman's modal structuralist approach to mathematics, by invoking a notion of logical possibility given certain facts. We saw that Hellman already accepts a notion of logical possibility holding the material facts fixed. Given this, it is only natural that he should also accept my notion of conditional logical possibility. However, once one does this there is no need to invoke second order quantification as an additional primitive.

The streamlining I propose also helps us evaluate the two apparent problems for modal structuralism mentioned in the introduction. We have seen that it is possible to completely eliminate the controversial practice of quantifying in from Hellman's paraphrases.

I think the technical work in this paper demonstrates that there is no unavoidable *special* problem for modal structuralism caused by its reliance on second order logic. This is not to say that modal structuralism is ontologically innocent. Although logical possibility intuitively appears ontologically innocent, whether my simplification defends modal structuralism's ontological innocence or reveals that (despite our intuitions) logical possibility is itself unsuitable for nominalist use depends on the right answer to certain controversial background questions. Specifically, it depends on whether we ought to take any other notion which does the same work as second order logic to be equally ontologically committal.

If similarity of mathematical behavior doesn't require (or make a strong case for) similarity of ontological role, then my simplification allows modal structuralism to shake off the aspersions that have been cast on its nominalistic credentials. If it does, then we can respond by either giving up on the nominalistic acceptability of modal structuralism and admitting that the seemingly innocent notion of logical possibility (and possibly many other notions we don't suspect) is actually ontologically committal or by reevaluating our reasons for thinking that second order logic is ontologically committal (since the results of this paper show that although second order logic is similar to set theory which looks ontologically committal, it is also similar to logical possibility which looks non-committal).[39]

In conclusion, we've seen that by adopting a small generalization of Hellman's notion of logical possibility (the meaningfulness of which he has already endorsed), we can significantly streamline modal structuralism – and perhaps solve some other problems as well.

Notes

1. The fact that you can capture logical possibility for first order sentences using set theory is well known. Modal structuralist paraphrases attempt to show that you can go the other way around and capture truth conditions for set theory as a whole in terms of logical possibility.
2. By deflationary (ontological) realists, I mean philosophers who accept the existence of mathematical objects but don't take these objects/existence facts to be metaphysically fundamental (in terms of grounding). Such philosophers could re-interpret Hellman's paraphrases as bi-conditionals which explain how existence facts about mathematical objects are systematically grounded in facts about logical possibility (just as one might say that existence facts about cities are systematically grounded in facts about what people are doing, while believing that cities really exist).
3. I will focus on pure mathematics in this paper, but we will see that the same strategy can be used to streamline what Hellman says about applied mathematics as well.
4. While this debate is commonly conducted in terms of metaphysical possibility, it naturally raises similar concerns for logical possibility.
5. Specifically, my account of mathematics is compatible with taking Williamson to show that any notion of possibility that allows quantifying in (such as metaphysical possibility) must have a fixed domain – provided one thinks it doesn't make sense to quantify into logical possibility. Of course, it's not compatible with taking Williamson to show that every modal notion must have a fixed domain.

6. I take Quine's problem with quantifying in, in 'Reference and Modality', to be that he dislikes the 'Aristotelian essentialism' of taking some properties to belong to an object like the number 7 essentially (e.g. being less than 9) while others apply only contingently (e.g. being the number of planets). As we will see, my system eschews cross-world object identification of any kind (e.g. cross-world equality or counterpart relations) as well as quantifying in. Thus criticisms like Quine's can't even get off the ground.
7. It is striking that eliminating the second order quantifiers seems to result in no significant loss of expressive power, i.e. if a structure is definable using Hellman's system then it is definable in my system as well.
8. See Hellman (1994) and Hellman's later paper Hellman (1996).
9. These modified paraphrases work by assuming the logical possibility of an (infinite) collection of atoms and then considering mereological fusions of atoms and plural quantification over these fusions to mimic three layers of sets (and functions) over this original infinite collection.
10. This assumption is controversial, as it not only commits us to the existence of the mereological fusion of Lewis' nose and the Eiffel tower (and the Chrysler corporation and the Obamas' marriage, if one believes in such non-concrete objects) but requires we believe the same holds true in all logically possible scenarios. Furthermore (even if Hellman is right about mereology), it can seem unattractive to say that the true content of, say, real analysis commits one to a generous Lewisian position on the problem of special composition (Lewis 1986) – or that mereological principles hold with logical necessity since, e.g. this conflicts with the intuition that it would be logically possible for there to be exactly 4 objects. For example, if arbitrary fusions exist, there could be 2 atoms and hence 3 total objects, or 3 atoms and hence 7 objects, but couldn't be exactly 4 objects.
11. Note that, by Lowenheim–Skolem considerations, no categorical description of common mathematical structures such as the natural numbers can be given using first order logic alone.
12. That is, PA_2 is the result of replacing the induction schema in Peano Arithmetic with a single induction axiom formulated in second order logic as described in Weisstein (2013).
13. Although PA and PA_2 are often formulated using a successor function, it is easy enough to transform them into claims about a successor relation, by adding an axiom asserting that every member of $\mathbb{N}$ has a unique successor in $\mathbb{N}$.
14. Note, $\mathbb{N}$ is understood to express the property of being a number and S the successor relation, so we cannot use them as variables by writing something like $(\exists \mathbb{N})(\exists S)PA_2$.
15. Obviously, if it isn't really possible for there to be something satisfying PA_2 (for example because second order logic is ontologically committal and the necessary second order objects can't exist) then the paraphrase Hellman provides for statements in arithmetic would fail.
16. Hellman's notion of material facts seems to include (at least) the fundamental physical facts, and definitely does not include facts about mathematical objects.
17. For example, see the discussion of the corresponding notion of consequence in Field (1989) and Rayo (2013) alongside that of Hellman (1996).
18. See Etchemendy (1990) on the tension between standard Tarskian reinterpretation-based accounts of logical possibility and the intuitive notion of logical possibility regarding this point.

19. At first glance, one might be tempted to simply identify claims about logical possibility with claims about the existence of a set theoretic model. However, philosophers such as Hartry Field have convincingly argued that, 'We should think of the intuitive notion of validity not as literally defined by the model theoretic account, or in any other manner; rather, we should think of it as a primitive notion.' Field (2008) Very crudely, the issue is this: a key aspect of our notion of logical possibility/validity is that what's actual must be logically possible. But, if we identify logical possibility with the existence of a set theoretic model, then it looks puzzling why the inference from actual to possible is permissible. After all the total universe can't be represented as a set theoretic model (as it contains all the sets, and hence is proper class sized) even though it is actual.
20. Admittedly, there's another reading of this sentence on which it expresses a necessary falsehood. However, this is not the reading I have in mind.
21. This will give the right verdict if we assume that actually distinct objects are distinct in all logically possible scenarios.
22. I say that an object x 'falls under' R_j iff it appears in some tuple in the extension of R_j, i.e. $\exists y_1, \ldots, y_m R_j(y_1 \ldots y_m) \wedge (x = y_1 \vee x = y_2 \vee \ldots \vee x = y_m)$.
23. Note, however, that the translations of a statement in applied mathematics may need to use relation symbols which do not occur in the original statement (as per the Putnamian strategy of replacing mathematical vocabulary with arbitrary otherwise unused relation symbols with the right arity discussed below).
24. Let $P(0)$ be shorthand for $(\exists z)(\forall w)\left(\mathbb{N}(z) \wedge \neg S(w, z) \wedge P(z)\right)$.
25. That is, X applies to a number that is not the successor of any number, and it applies to the successor of every number it applies to.
26. P can be any one place predicate different from the predicate abbreviated by $\mathbb{N}$.
27. By this I mean the unique number that isn't a successor.
28. The strategy in Appendix 2 allows us to translate second order sentences of number theory as well.
29. Note that I will not attempt to formally prove that my translations have the same truth values Hellman intends his translations to have. Just as a formal proof is of little value in verifying you've correctly formalized an English sentence into predicate calculus, so too it is of little value in verifying my translations have the same truth-values as Hellman's translations (given Hellman's assumptions about second order logic etc.). Any formal proof would have to make assumptions about what statements are equivalent on the intended interpretation of the two languages – the very aspect most open to doubt.
30. By this I mean an ordinal with the same order-type as the well ordering in question.
31. Note that the axioms of ZFC or even ZFC_2 don't suffice to categorically determine the height of the set theoretic hierarchy. For example, if (as most mathematicians assume) the hierarchy of sets extends beyond the first inaccessible then the initial segment of the hierarchy below that inaccessible will satisfy ZFC/ZFC_2.
32. That is, it seems that facts about where the realist hierarchy of sets stops would not be determined by anything in our conception of the sets (and maybe not even by anything we can have knowledge of at all).
33. Strictly speaking, I take it, Putnam would say these claims *can* be so understood.
34. However a number of other approaches are possible. See, for example, the closely related accounts given by Linnebo Linnebo (2010) and Parsons Parsons (1977).

35. By this I mean that every element of V_1 is an element of V_2 and the second order relation quantifier Hellman uses to give $\in$ its meaning on V_2 agrees with $\in$ on V_1 (and, indeed, any element of V_2 which is $\in_2$ an element of V_1 is also in V_1).
36. Actually, Hellman has a separate story about how to handle restricted quantification in set theory which I elide for present purposes. See Hellman (1994) Chapter 2 Section 2.
37. That is, if ϕ is a sentence in the language of set theory, either my translation of ϕ or my translation of $\neg\phi$ will express a truth.
38. We can use the definition of the natural numbers to provide a countable collection of 'variables' where we can use definite descriptions to uniquely refer to each such variable. Specifically we can think about the variable symbols in the language of set theory as being canonically associated with numbers, and use the sentence uniquely defining the associated number to refer to the variable.
39. Most readers will probably find it immediately more attractive to say that both second order logic and logical possibility are 'guilty' i.e. ontologically committal. However, I think there's a surprisingly attractive prima facie case for taking the opposite approach. For, as we saw in the discussion of potentialism in 5.2, there appears to be a simple and independently motivated way of grounding set theoretic claims in claims about logical possibility (one which is motivated by the Burali-Forti paradox). But, in contrast, we have seen (in discussing Field's remarks in Section 3) that it does not seem possible to systematically ground facts about logical possibility in facts about set theory. One might argue that these facts militate in favor of taking logical possibility to be the more grounding-fundamental notion of the three, and therefore (perhaps) the one whose apparent ontological commitments reflect the true ontological commitments of everything else that is grounded in it.
40. In this language the non-mathematical objects are taken to be ur-elements as per (McGee 1997).
41. Our set theoretic approximation can give the wrong answers if there are 'more' actual objects than there are sets.
42. Note that if you are a potentialist about set theory in the sense advocated above, these conditions do capture correct truth conditions for logical possibility but can't be used to *define* logical possibility on pain of circularity.
43. By this I mean a partial function ρ from the collection of variables in the language of logical possibility to objects in $\mathscr{M}$, such that the domain of ρ is finite and includes (at least) all free variables in ψ.
44. As usual I take $\Box$ and $\forall$ to be abbreviations for $\neg\Diamond\neg$ and $\neg\exists\neg$ respectively.
45. As usual $\phi[x/v]$ substitutes v for x everywhere where v occurs free in ϕ.
46. Remember ϕ can't have any free variables.
47. For the reasons discussed in footnote 30, I haven't tried to give a formal proof of the fact that second order quantifiers can be replaced with conditional logical possibility operators.
48. Officially, Hellman's paraphrases take the form $\Diamond D \wedge \Box(D \rightarrow \phi)$. But when D is categorical this is equivalent to the form above.
49. That is we can assume all quantifiers are of either the form $(\exists x)(M(x) \wedge \phi)$ or $(\forall x)(M(x) \rightarrow \phi)$.
50. Hellman himself doesn't give a very fleshed out story about how to handle physical quantity statements, like 'there is an object weighing more than 5 grams' or say anything about how to handle probability statements (which are especially challenging insofar as they seem to associate numbers with something like sets of possible worlds, rather than any physical objects).

51. We presume that the sentence to be translated does not unrestrictedly quantify over all material objects, e.g. assert the finiteness of the material world, or if it does there is a single predicate that applies to every material object. Admittedly, there would be trouble if you wanted to translate a single sentence that used all atomic vocabulary of the right arity. However, there is plenty of atomic vocabulary that doesn't occur in the kind of scientific applications of mathematics which Hellman tries to capture (e.g. 'angel', 'blesses' etc.), so this is unlikely to be a practical problem.
52. Unlike Hellman, I don't propose to give potentialist translations of second order statements about set theory, because unbounded second order quantification over a potentialist hierarchy isn't obviously meaningful. Hellman himself admits (in Chapter 2 Section 3 of Hellman (1994)) that his translations of second order sentences don't behave the way we'd intuitively expect, e.g. his translation of second order replacement doesn't motivate his translation of first order replacement. Also, those who like Hellman's treatment of second order set theory can use the techniques proposed here and in Appendix 2, in a fairly straightforward way, to reproduce it.

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Appendix 1. A more formal approach to conditional logical possibility

I take the notion of conditional logical possibility to be primitive and intuitive. However, one can provide approximately correct truth conditions for sentences involving nested applications of subscripted $\Box$ and $\Diamond$ operators, in terms of the more familiar language of set theory with ur-elements.[40]

First let us define a formal language $\mathcal{L}$, which I will call the language of logical possibility (though this language may be not able to express all meaningful claims involving logical possibility). Fix some infinite collection of variables and a collection of relation symbols, and define $\mathcal{L}$ to be the smallest language built from these variables using these relation symbols and equality closed under applications of the normal first order connectives, quantifiers, $\Box$ and $\Diamond$ (where the latter two operators can only be applied to sentences, so there is no quantifying in).

Specifically, if we ignore the possibility of sentences which demand something coherent but wouldn't have a model in the sets, (such as sentences which require the existence of proper class many objects) and take all quantifiers appearing outside a logical possibility operator to be implicitly restricted to some set sized domain of non-mathematical objects[41] we could say the following[42]:

Definition 1: A formula ψ is true relative to a model $\mathscr{M}$ and an assignment ρ which takes the free variables in ψ to elements in the domain of $\mathscr{M}$[43] just if the following conditions obtain[44] (note that only the last clause says something out of the ordinary):

- $\psi = R_i(x_1 \ldots x_k)$ and $R_i^{\mathscr{M}}(\rho(x_1), \ldots, \rho(x_k))$ (as usual $R_i^{\mathscr{M}}$ is the interpretation of R_i by $\mathscr{M}$).
- $\psi = x = y$ and $\rho(x) = \rho(y)$.
- $\psi = \neg\phi$ and ϕ is not true relative to $\mathscr{M}, \rho$.
- $\psi = \phi \wedge \psi$ and both ϕ and ψ are true relative to $\mathscr{M}, \rho$.
- $\psi = \phi \vee \psi$ and either ϕ or ψ are true relative to $\mathscr{M}, \rho$.
- $\psi = \exists x \phi(x)$ and there is an assignment ρ' which extends ρ by assigning a value to an additional variable v not in ϕ and $\phi[x/v]$ is true relative to $\mathscr{M}, \rho'$[45].
- $\psi = \Diamond_{R_1 \ldots R_n} \phi$ and there is another model $\mathscr{M}'$ and a bijection θ from

$$Ext(R_1^{\mathscr{M}}, \ldots, R_n^{\mathscr{M}}) \stackrel{\text{def}}{=}$$

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$$\{y \mid \bigvee_{\substack{1 \le i \le n \\ 1 \le j \le k_i}} (\exists x_1, \ldots, x_{k_i}) \left[y = x_j \wedge R_i^{\mathscr{M}}(x_1, \ldots, x_{k_i}) \right] \}$$

to $Ext(R_1^{\mathscr{M}'}, \ldots, R_n^{\mathscr{M}'})$ such that

$$R_i^{\mathscr{M}}(x_1, \ldots, x_{k_i}) \iff R_i^{\mathscr{M}'}(\theta(x_1), \ldots, \theta(x_{k_i}))$$

and ϕ is true relative to $\mathscr{M}'$ and the empty assignment[46].

Note that in the last clause the models $\mathscr{M}$ and $\mathscr{M}'$ need not share any elements. Rather the structure $Ext(R_1^{\mathscr{M}}, \ldots, R_n^{\mathscr{M}})$ (those elements appearing in some tuple in the extension of some $R_i^{\mathscr{M}}$) must be isomorphic (under the relations $R_1 \ldots R_n$) to $(Ext(R_1^{\mathscr{M}'}, \ldots, R_n^{\mathscr{M}'}))$.

Set Theoretic Approximation: A sentence in the language of logical possibility is true *simpliciter* iff it is true relative to a set theoretic model whose domain consists of the actual objects (which the quantifiers in our original non-mathematical language range over) and whose extensions for atomic relations reflects the actual extensions of these relations and the empty assignment function ρ. Note that this definition gives statements lacking any necessity operators the same truth values as they have in the actual world.

Appendix 2. Modal structuralist paraphrases for regular mathematics

In this appendix, I will give a general method for simplifying Hellman's paraphrases of non-set theoretic mathematics.[47] I will follow Hellman in focusing on the case where the mathematical structure under consideration has a categorical second order description D, and provide a translation of Hellman's paraphrases which we may assume is in the following form[48] (where all first order quantifiers in D and ψ are restricted to M[49] and no logical possibility operators appear in D or ψ):

$$\Diamond(\exists M)[D \wedge \psi]$$

We may ignore the difference between quantification over classes and quantification over relations, by regarding class variables as unary relation variables. For visual clarity we will use capital letters for second order quantification over relations. We will also assume that no second order function quantifiers occur in D or ψ, though the same mechanism can be easily extended to handle function quantifiers. Note that as all first order quantifiers are restricted to M, we only need concern ourselves with the behavior of relations and relation variables on elements of M.

We may now define my translation of Hellman's paraphrase $\Diamond(\exists M)[D \wedge \psi]$ to be $\Diamond t(D \wedge \psi)$ where t is defined via the following recursive definition (with $t = t_{()}$).

$$\begin{aligned} t_{(R_1 \ldots R_n)}(\exists P\phi) &= \Diamond_{M, R_1 \ldots R_n} t_{(R_1 \ldots R_{n+1})}(\phi[P/R_{n+1}])] \\ t_{(R_1 \ldots R_n)}(\forall P\phi) &= \Box_{M, R_1 \ldots R_n} t_{(R_1 \ldots R_{n+1})}(\phi[P/R_{n+1}])] \\ t_{(R_1 \ldots R_n)}(\neg\phi) &= \neg t_{(R_1 \ldots R_n)}(\phi) \\ t_{(R_1 \ldots R_n)}(\phi \wedge \psi) &= t_{(R_1 \ldots R_n)}(\phi) \wedge t_{(R_1 \ldots R_n)}(\psi) \\ t_{(R_1 \ldots R_n)}(\phi \vee \psi) &= t_{(R_1 \ldots R_n)}(\phi) \vee t_{(R_1 \ldots R_n)}(\psi) \end{aligned}$$

$$t_{(R_1...R_n)}(\exists x\phi) = (\exists x)[t_{(R_1...R_n)}(\phi)]$$
$$t_{(R_1...R_n)}(\forall x\phi) = (\forall x)[t_{(R_1...R_n)}(\phi)]$$
$$t_{(R_1...R_n)}(R_k(x_1, \ldots x_m)) = R_k(x_1, \ldots x_m)$$
$$t_{(R_1...R_n)}(x_1 = x_2) = x_1 = x_2$$

We now argue that this translation preserves (intended) truth values. Except for the first two lines the translation is entirely homophonic, so as long as those equalities preserve (intended) truth values, the entire translation should do so. However, the first and second equalities simply express the fact that, understand as Hellman intends, second order relation variables on a domain M range over all logically possible relations on M and vice versa. Finally, the same consideration (on a given domain, $\exists M$ ranges over exactly the collections it would be logically possible for a predicate to apply to) tells us that moving between $\Diamond(\exists M)t([D \wedge \psi])$ and $\Diamond t(D \wedge \psi)$ shouldn't change the truth value (again assuming second order quantification operates in the usual fashion as Hellman expects).

Appendix 3. Note about applied mathematics

Although the aim of this paper is to simplify Hellman's story about *pure* mathematics, everything Hellman says about applied mathematics[50] is also expressible using my notion of conditional logical possibility. I only mention this fact because it means that Hellman could adopt my simplifications without significant harm to his proposal.

As noted in Section 2, Hellman paraphrases sentences in applied mathematics, like 'There are a prime number of rats' with sentences of the form:

$$\Box(\text{holding fixed all material facts})\phi$$

where ϕ is a sentence asserting that if there are objects behaving like the numbers, (or whatever mathematical objects are mentioned in the statement to be translated) then these objects are related to the material objects in some (second-order describable) fashion. For instance, ϕ might assert that if some things behave like the natural numbers, then there is a function which pairs up the rats in the actual world in a one-to-one fashion with those natural numbers up to some prime, thereby asserting that there are a prime number of rats.

It is possible to do equivalent work using my notion of conditional logical possibility. First we apply the technique outlined in Appendix 2 to replace second order quantification with conditional logical possibility. We then add all the non-mathematical relations mentioned in the sentence to be translated (in the example 'there are a prime number of rats' this would just be the predicate 'rat()') as subscripts to *all* the $\Box$ and $\Diamond$ operators in the sentence. The resulting sentence now simply holds fixed every material fact it actually makes use of, allowing it to be expressed in terms of conditional logical possibility[51] (without appeal to a notion of holding *all* the material facts fixed).

Appendix 4. Paraphrasing potentialist set theory

Potentialism about set theory replaces claims about a definite totality of sets with claims about how initial segments of the sets can extend each other. Hellman considers initial segments of the sets which satisfy ZFC_2 and uses quantifying into formulate claims about how these segments can be extended. We reformulate Hellman's potentialist understanding of first order set theory.[52] in the language of conditional logical possibility in two steps.

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First, we replace the requirement that the initial segments satisfy ZFC_2 with an equivalent characterization $\mathrm{ZFC}_\Diamond$ in terms of conditional logical possibility, using the technique described in Appendix 2.

Secondly, we can reformulate claims about how initial segments can be extended in a way that eschews quantifying in. Recall that potentialism translates sentences of set theory by replacing quantifiers over the sets with statements about how it would be possible to extend initial segments of the sets and choose elements from those initial segments, e.g. if ϕ is quantifier free then $\exists x \phi(x)$ would translate to $\Diamond[\mathrm{ZFC}_\Diamond(\text{set}, \in) \wedge (\exists x)(\text{set}(x) \wedge \phi(x))]$ where this says that it would be logically possible for there to be an initial segment of the hierarchy of sets containing an object that satisfied ϕ.

To express potentialist truth conditions without quantifying in, I will require that each initial segment $\text{set}_i, \in_i$ be paired with an associated assignment relation R_i which (in effect) assigns each of the countably many variables $x_1, x_2 \ldots$ in the first-order language of set theory to objects within set_i. When we ask about the possibility of extending the current initial segment $(\text{set}_i, \in_i)$ we can place R_i in the subscript of all further $\Box$ and $\Diamond$ expressions to pass along the information about variable assignments. We allow this choice of assignments for variables to be modified to allow variables to be assigned to objects in set_{i+1} (an initial segment extending set_i) by defining another assignment R_{i+1} which must agree with R_i everywhere except for on the (number representing) the variable allowed to range over set_{i+1}.

I will use $\mathscr{V}(V_a)$ to abbreviate the claim that $\text{set}_a, \in_a$ satisfy $\mathrm{ZFC}_\Diamond(\text{set}_a, \in_a)$ and R_a behaves like (the relation corresponding to) an assignment function from the objects satisfying $\mathbb{N}$ to those satisfying set_a. More concretely this amounts to the conjunction of the following three claims:

- $\mathrm{ZFC}_\Diamond(\text{set}_a, \in_a)$, i.e. V_a behaves like an initial segment of the hierarchy of sets.
- $\mathbb{N}, S$ satisfy $\mathrm{PA}_\Diamond$.
- R_a behaves like a function from $\mathbb{N}$ to set_a

Remember that, as discussed on page 12, schematic relation symbols (like $\in$, set_a and P) are used as a mnemonic device in place of suitable non-mathematical relations with the same arity.

Note that my only reason for using $\mathrm{PA}_\Diamond$ is that the natural numbers (under successor) contain infinitely many definable objects, which we can use to represent variables, for example 1 represents x_1, 2 represents x_2 etc. In what follows, I will use $\boldsymbol{n}$, to abbreviate the formula where $\boldsymbol{n}$ is replaced by a variable constrained to be the (unique) n-th successor of 0. Thus, for example, a claim of the form $\phi(\mathbf{1})$ abbreviates $(\forall x)[S(0,x) \rightarrow \phi(x)]$. I will abbreviate the conditionalized logical possibility operators $\Diamond_{\text{set}_n, \in_n, \mathbb{N}, S, R_n}$ and $\Box_{\text{set}_n, \in_n, \mathbb{N}, S, R_n}$ by $\Diamond_{V_n}$ and $\Box_{V_n}$ respectively.

I will use $V_a \geq_i V_b$ to abbreviate the claim that the set_a under $\in_a$ extends the set_b under $\in_b$ and the assignment of variables R_b agrees with R_a everywhere *except on i* (where i is the code for x_i). Put more concretely, this is to say that

- $\mathscr{V}(V_a)$
- $\mathscr{V}(V_b)$
- $(\forall x)[\text{set}_b(x) \rightarrow \text{set}_a(x)]$
- $(\forall x)(\forall y)[\text{set}_a(y) \rightarrow (x \in_b y \leftrightarrow x \in_a y)]$
- $(\forall n)[\mathbb{N}(n) \rightarrow n = \mathbf{i} \vee (\forall y)(R_a(n,y) \leftrightarrow R_b(n,y))]$

We can now translate the set theoretic utterance $(\exists x)(\forall y)(x = y \vee \neg y \in x)$ into a claim about how it is logically possible for $\text{set}_1, \in_1, R_1$ to be extended. First we rewrite this set theoretic statement in a regimented language with numbered variables as

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$(\exists x_1)(\forall x_2)[x_1 = x_2 \vee \neg x_2 \in x_1]$. Then we translate this sentence into:

$$\Diamond(\mathscr{V}(V_1) \wedge \Box_{V_1}[V_2 \geq_2 V_1 \rightarrow (\forall z)(\forall y)(R_2(\mathbf{1}, z) \wedge R_2(\mathbf{2}, y) \rightarrow z = y \vee \neg y \in_2 z)])$$

That is, such $\exists x_1 \forall x_2$ sentences can be understood as making a claim with the following form. There could be a model of set theory $\text{set}_1, \in_1$ and a relation R_1 assigning 1 (representing x_1) to an element of set_1 so that it is necessary (holding fixed $\text{set}_1, \in_1, R_1$ and the numbers) than any model of set theory $\text{set}_2, \in_2$ extending $\text{set}_1, \in_1$ and relation R_2 assigning 2 to an element of set_2 (while agreeing with R_1 about the assignment of 1) makes the interior of the above formula true when x_1, x_2 are replaced by the assignments of 1, 2 by R_2 and $\in$ is replaced with $\in_2$.

The same strategy works more generally to produce paraphrases of arbitrary sentences in the language of pure set theory. We can use recursive applications of the following principles to translate every sentence in the first-order language of set theory into a claim about logically possible extendability.

In particular we define t_n as follows:

- $t_n(x_i \in x_j)$ is the claim that R_n assigns i to an object $\in_n$ the object it assigns to j i.e. $(\forall z)(\forall z')[R_n(\mathbf{i}, z) \wedge R_n(\mathbf{j}, z') \rightarrow z \in_n z']$
- $t_n(x_i = x_j)$ is the claim that R_n assigns i to the same object it assigns j to i.e. $(\forall z)(\forall z')[R_n(\mathbf{i}, z) \wedge R_n(\mathbf{j}, z') \rightarrow z = z']$
- $t_n(\neg\phi) = \neg t_n(\phi)$
- $t_n(\phi \vee \psi) = t_n(\phi) \vee t_n(\psi)$
- for $n \geq 0$, $t_{n+1}((\forall x_i)\phi(x)) : \Box_{V_n}[V_{n+1} \geq_i V_n \rightarrow t_{n+2}(\phi)]$
- for $n \geq 0$, $t_{n+1}((\exists x_i)\phi(x)) : \Diamond_{V_n}[V_{n+1} \geq_i V_n \wedge t_{n+2}(\phi)]$
- $t_0((\forall x)\phi(x)) : \Box[\mathscr{V}(V_0) \rightarrow t_1(\phi)]$
- $t_0((\exists x)\phi(x)) : \Diamond[\mathscr{V}(V_0) \wedge t_1(\phi)]$

The translation of a set theoretic sentence ϕ is $t_0(\phi)$. Note that the validity of the above translation relies on the fact that for any two structures satisfying ZFC_2 one is isomorphic to an initial segment of the other. Hellman invokes a version of this claim in Chapter 2 Section 3 of Hellman (1994) and I think an analogous argument can be made within my formal system, but reasons of space prevent me from demonstrating this here. Also note that in the above definition we can replace V_j with $V_{j \bmod 2}$ without affecting the truth value of the translation. This allows us to translate sentences with arbitrarily many quantifier alternations using a fixed finite number of atomic relations.

Note that this translation honors the intuitive bivalence of the language of set theory. Consider an arbitrary set theoretic sentence ϕ. $t(\phi) = t_0(\phi)$ and $t(\neg\phi) = t_0(\neg\phi) = \neg(t_0(\phi))$. Thus either $t(\phi)$ or $t(\neg\phi)$ will be true.