Adv. Appl. Prob. **52,** 1164–1196 (2020)
*doi:*10.1017/apr.2020.36

DRAW-DOWN PARISIAN RUIN FOR SPECTRALLY NEGATIVE LÉVY PROCESSES

WENYUAN WANG,* *Xiamen University*
XIAOWEN ZHOU,** *Concordia University*

Abstract

Draw-down time for a stochastic process is the first passage time of a draw-down level that depends on the previous maximum of the process. In this paper we study the draw-down-related Parisian ruin problem for spectrally negative Lévy risk processes. Intuitively, a draw-down Parisian ruin occurs when the surplus process has continuously stayed below the dynamic draw-down level for a fixed amount of time. We introduce the draw-down Parisian ruin time and solve the corresponding two-sided exit problems via excursion theory. We also find an expression for the potential measure for the process killed at the draw-down Parisian time. As applications, we obtain new results for spectrally negative Lévy risk processes with dividend barrier and with Parisian ruin.

Keywords: Spectrally negative Lévy process; reflected process; Parisian ruin; draw-down time; draw-down Parisian ruin; potential measure; excursion theory; risk process

2010 Mathematics Subject Classification: Primary 60G51
Secondary 60E10; 60J35

1. Introduction

The concept of Parisian stopping time was first proposed in [5] for option pricing in mathematical finance. The papers [13] and [12] later introduced Parisian ruin time for linear Brownian motion and the Cramér–Lundberg risk processes to model the ruin problem with implementation delay, where expressions for the Parisian ruin probability were provided. Intuitively, for a risk process, the Parisian ruin time is the first time when the surplus process has stayed below level 0 continuously for a time period of a predetermined duration r.

The Parisian ruin problem has since been studied extensively under the framework of spectrally negative Lévy processes. By considering spectrally negative Lévy processes of bounded and unbounded variation, [9] found the respective expressions for the Parisian ruin probability. The authors of [21] revisited the Parisian ruin probability and provided an expression which is considerably simpler than that of [9], and unifies the results for spectrally negative Lévy processes of bounded and of unbounded variation. In [20], the result of [21] was further extended to refracted Lévy processes. The Parisian-ruin-related dividend optimization problem was investigated in [10], where the barrier dividend strategy turned out to be the optimal strategy. Work on a variant of the above model in which the duration r is random can be found

Received 25 May 2019; revision received 18 April 2020.
* Postal address: School of Mathematical Sciences, Xiamen University, Fujian 361005, People's Republic of China. Email: wwywang@xmu.edu.cn
** Postal address: Department of Mathematics and Statistics, Concordia University, Montreal, Canada. Email: xiaowen.zhou@concordia.ca

in [17], [2], and [14]. Recent work concerning the Parisian ruin with an ultimate bankruptcy level can be found in [8], [11], and [6].

Results on Parisian ruin are often expressed using the scale functions and the marginal density for the spectrally negative Lévy process. The approaches in the previous literature on Parisian ruin for Lévy risk processes typically involve arguments concerning fluctuation identities if the underlying Lévy process has sample paths of bounded variation. Approximation and limiting arguments are further needed to handle the case of unbounded variation.

More recently, in [22] a novel approach is adopted by connecting the desired Parisian ruin fluctuation quantity with the solution to the Kolmogorov forward equation for a spectrally negative Lévy process to find the joint Laplace transform of the Parisian ruin time and the Parisian ruin position, as well as an expression for the q-potential measure of the process killed at the Parisian ruin time.

Since Parisian ruin is defined using excursions of the underlying process, one would expect excursion theory to play a role in its investigation. But we are not aware of any previous studies of Parisian ruin problems via excursion theory.

A general draw-down time for a stochastic process is a downward first passage time of a dynamic level that depends on the previous supremum of the process. It generalizes the classical ruin time of first passage from a fixed level and helps to understand the path-dependent relative downward fluctuations from the previous supremum for the underlying process.

The draw-down time was first studied for diffusions in [18]. Some early work on draw-down time for spectrally negative Lévy processes can be found in [24]. In [1], draw-down exit problems were studied for taxed spectrally negative Lévy processes using both excursion theory and an approximation approach. More recent fluctuation results concerning the draw-down times for spectrally negative Lévy processes such as the associated joint distribution, the potential measure, and creeping behaviors were obtained in [19] via excursion theory. Many ruin-time-related results for spectrally negative Lévy risk processes can be generalized to the associated draw-down time setting, and at the same time, the obtained expressions are in terms of scale functions that remain semi-explicit. We refer to [27] for recent work on draw-down reflected spectrally negative Lévy processes.

In the study of ruin problems with implementation delay in actuarial risk theory, it is interesting to take into consideration the historical performance of the surplus process and adjust the delay accordingly. Given the previous results on both the Parisian ruin probability and the draw-down time, it comes naturally to introduce the general draw-down feature to the Parisian ruin problem for spectrally negative Lévy risk processes. In this way the Parisian ruins can be associated to the previous historical high of the process, which makes it possible to pose more elaborate Parisian ruin problems and leads to better understanding of fluctuation behaviors for Parisian ruin. In this paper we are going to implement this idea and generalize the known results on Parisian ruin time to those concerning the general draw-down Parisian ruin time.

We recently noticed that the two-sided exit problem involving the draw-down Parisian ruin time was studied in [26] for the classical draw-down process using fluctuation theory; the author investigated only the classical draw-down rather than the general draw-down. We remark that the solution to the two-sided exit problem in [26] can be recovered from one of our results; see Remark 4.2 concerning the general draw-down.

More precisely, for spectrally negative Lévy risk processes we find solutions to the two-sided exit problems associated to the draw-down Parisian ruin times. We also find an expression for the potential measure associated to the draw-down Parisian ruin time. In addition, we obtain recursive expressions for moments of accumulated time-discounted increments of the running

https://doi.org/10.1017/apr.2020.36 Published online by Cambridge University Press

supremum up to the draw-down Parisian ruin time. As applications, we recover a previous result and obtain new results on Parisian ruin for a spectrally negative Lévy risk process with a constant dividend barrier.

To prove the main results, we adopt the excursion theory approach, which we find very handy for draw-down fluctuation arguments for spectrally negative Lévy processes. To this end, we first identify the associated exit quantity under the excursion measure for the excursion process of a reflected spectrally negative Lévy process from its running supremum. Since the draw-down-related quantities can be expressed using the excursion process, the desired results then follow from compensation formulas. A similar approach can be found in [19]. To the best of our knowledge, this paper represents the first attempt at applying excursion theory to the study of Parisian ruin problems.

The rest of the paper is arranged as follows. After the introduction in Section 1, in Section 2 we briefly review the spectrally negative Lévy process, the associated scale functions, the draw-down time, and several draw-down fluctuation results. Section 3 introduces the excursion process of the spectrally negative Lévy process reflected from its previous supremum, together with results on the excursion measure related to the Parisian ruin time. The main results and proofs are contained in Section 4. In Section 5, we apply the main results to spectrally negative Lévy processes with Parisian ruin and dividend barrier, thereby recovering previously known results and obtaining new results.

2. Preliminaries on spectrally negative Lévy processes and Parisian ruin problems

We first briefly introduce spectrally negative Lévy processes, the associated scale functions, and some fluctuation identities. Write $X \equiv \{X(t); t \geq 0\}$, defined on a probability space with probability laws $\{\mathbb{P}_x; x \in (-\infty, \infty)\}$ and natural filtration $\{\mathcal{F}_t; t \geq 0\}$, for a spectrally negative Lévy process that is not a purely increasing linear drift or the negative of a subordinator. Denote its running supremum process by

$$\bar{X}(t) := \sup_{0\leq s\leq t} X(s), \quad t \geq 0.$$

The Laplace exponent of X is given by

$$\psi(\theta) := \ln \mathbb{E}_x\Big(\mathrm{e}^{\theta(X(1)-x)}\Big) = \gamma\theta + \frac{1}{2}\sigma^2\theta^2 - \int_{(0,\infty)} \big(1 - \mathrm{e}^{-\theta x} - \theta x \mathbf{1}_{(0,1)}(x)\big)\, \nu(\mathrm{d}x),$$

where the Lévy measure ν satisfies $\int_{(0,\infty)} \big(1 \wedge x^2\big)\, \nu(\mathrm{d}x) < \infty$. It is known that $\psi(\theta)$ is finite for $\theta \in [0, \infty)$, and it is strictly convex and infinitely differentiable. As in [3], the q-scale functions $\{W^{(q)}; q \geq 0\}$ of X are defined as follows. For each $q \geq 0$, $W^{(q)} : [0, \infty) \to [0, \infty)$ is the unique strictly increasing and continuous function with Laplace transform

$$\int_0^\infty \mathrm{e}^{-\theta x} W^{(q)}(x)\mathrm{d}x = \frac{1}{\psi(\theta) - q}, \qquad \text{for } \theta > \Phi_q,$$

where Φ_q is the largest solution of the equation $\psi(\theta) = q$. Further define $W^{(q)}(x) = 0$ for $x < 0$, and write W for the 0-scale function $W^{(0)}$. Note that $W^{(q)}(0+) = 0$ if and only if the process X has sample paths of unbounded variation.

For $p, p + q \geq 0$, $y > 0$, and $x \in (-\infty, \infty)$, define two more scale functions as

$$W_y^{(p,q)}(x) := W^{(p)}(x) + q \int_y^x W^{(p+q)}(x - w)W^{(p)}(w)\mathrm{d}w$$

https://doi.org/10.1017/apr.2020.36 Published online by Cambridge University Press

and

$$Z^{(p)}(x) := 1 + p\int_0^x W^{(p)}(w)\mathrm{d}w.$$

For any $x \in \mathbb{R}$ and $\vartheta \geq 0$, define an exponential change of measure for the spectrally negative Lévy process by

$$\left.\frac{\mathrm{d}\mathbb{P}_x^{\vartheta}}{\mathrm{d}\mathbb{P}_x}\right|_{\mathcal{F}_t} = \mathrm{e}^{\vartheta(X(t)-x)-\psi(\vartheta)t}.$$

Furthermore, note that under the probability measures $\mathbb{P}_x^{\vartheta}$, the process X remains a spectrally negative Lévy process. From now on we denote by $W_{\vartheta}^{(q)}$ and W_{ϑ} the q-scale function and 0-scale function, respectively, under the measure $\mathbb{P}_x^{\vartheta}$.

For the process X, define its first up-crossing time and down-crossing time of level $a \in (-\infty, \infty)$ by

$$\tau_a^+ := \inf\{t \geq 0 : X(t) > a\} \quad \text{and} \quad \tau_a^- := \inf\{t \geq 0 : X(t) < a\},$$

respectively. It can be found in [16] that

$$\mathbb{E}_x\left(\mathrm{e}^{-q\tau_a^+}\mathbf{1}_{\{\tau_a^+<\tau_0^-\}}\right) = \frac{W^{(q)}(x)}{W^{(q)}(a)}, \quad x \in (-\infty, a]. \tag{1}$$

In addition, it follows from [28] that

$$\lim_{x\to\infty}\frac{W^{(q)\prime}(x)}{W^{(q)}(x)} = \Phi_q \quad \text{and} \quad \lim_{y\to\infty}\frac{W^{(q)}(x+y)}{W^{(q)}(y)} = \mathrm{e}^{\Phi_q x}. \tag{2}$$

A function $\xi : (-\infty, \infty) \to (-\infty, \infty)$ is called a draw-down function if $\xi(x) < x$ for all the values of x that are of concern. Define the ξ-draw-down time τ_ξ of X as

$$\tau_\xi := \inf\{t \geq 0 : X(t) < \xi(\bar{X}(t))\},$$

with the convention that $\inf \emptyset := \infty$. We call $\xi(\bar{X}(\tau_\xi))$ the associated draw-down level. By [19], for $\bar{\xi}(z) := z - \xi(z)$ we have

$$\mathbb{E}_x\left(\mathrm{e}^{-q\tau_a^+}\mathbf{1}_{\{\tau_a^+<\tau_\xi\}}\right) = \exp\left\{-\int_x^a \frac{W^{(q)\prime}(\bar{\xi}(z))}{W^{(q)}(\bar{\xi}(z))}\mathrm{d}z\right\}, \quad x \in (-\infty, a]. \tag{3}$$

For $r > 0$ the Parisian ruin time is defined by

$$\kappa_r := \inf\{t > r : t - g_t > r\} \text{ with } \inf\emptyset := \infty,$$

where

$$g_t := \sup\{0 \leq s \leq t : X(s) \geq 0\} \text{ with } \sup\emptyset := 0.$$

Given the draw-down function ξ, we define the ξ-draw-down Parisian ruin time of X as

$$\kappa_r^\xi := \inf\{t > r : t - g_t^\xi > r\} \text{ with } \inf\emptyset := \infty,$$

where

$$g_t^\xi := \sup\{0 \leq s \leq t : X(s) \geq \xi(\bar{X}(s))\} \text{ with } \sup\emptyset := 0.$$

https://doi.org/10.1017/apr.2020.36 Published online by Cambridge University Press

From [21] and [10], we have

$$\mathbb{P}_x(\kappa_r < \infty) = 1 - \mathbb{E}(X(1)) \frac{\int_0^\infty W(x+z) z \mathbb{P}(X(r) \in \mathrm{d}z)}{\int_0^\infty z \mathbb{P}(X(r) \in \mathrm{d}z)}, \quad x \in (-\infty, \infty),$$

and

$$\mathbb{E}_x\left(\mathrm{e}^{-q\tau_a^+} \mathbf{1}_{\{\tau_a^+ < \kappa_r\}}\right) = \frac{\ell_r^{(q)}(x)}{\ell_r^{(q)}(a)}, \quad x \in (-\infty, a], \tag{4}$$

where

$$\begin{aligned}\ell_r^{(q)}(x) &:= \int_0^\infty \mathrm{e}^{-\Phi_q z + qr} W^{(q)}(x+z) \frac{z}{r} \mathbb{P}^{\Phi_q}(X(r) \in \mathrm{d}z) \\ &= \int_0^\infty W^{(q)}(x+z) \frac{z}{r} \mathbb{P}(X(r) \in \mathrm{d}z).\end{aligned}$$

Note that $\ell_r^{(q)}$ can be treated as a scale function associated to the Parisian ruin.

Write $\ell_r := \ell_r^{(0)}$ for simplicity. Then

$$\mathbb{P}_x(\kappa_r < \infty) = 1 - \frac{\mathbb{E}(X(1))}{\int_0^\infty \frac{z}{r} \mathbb{P}(X(r) \in \mathrm{d}z)} \ell_r(x).$$

For $b \in (0, \infty)$, let

$$D(t) := \left(\bar{X}(t) - b\right) \vee 0, \quad t \geq 0, \tag{5}$$

denote the accumulated amount of dividends paid until time t of the barrier strategy with barrier at level b.

In this paper, we are interested in the following fluctuation quantities related to the draw-down Parisian ruin time:

(i) The draw-down Parisian-ruin-time-related two-sided exit problem

$$\mathbb{E}_x\left(\mathrm{e}^{-q\tau_a^+} \mathbf{1}_{\{\tau_a^+ < \kappa_r^\xi \wedge \tau_\eta\}}\right), \quad x \in (-\infty, \infty), \; a \in [x, \infty),$$

where η is another draw-down function such that $\eta(z) < \xi(z) < z$ for all $z \leq a$.

(ii) The joint Laplace transform involving the draw-down Parisian ruin time, the position of X at the draw-down Parisian ruin time, and its running supremum until the draw-down Parisian ruin time:

$$\mathbb{E}_x\left(\mathrm{e}^{-q\left(\kappa_r^\xi - r\right)} \mathrm{e}^{\lambda X(\kappa_r^\xi) - \psi(\lambda) r} \varphi(\bar{X}(\kappa_r^\xi)) \mathbf{1}_{\{\kappa_r^\xi < \tau_a^+\}}\right), \quad x \in (-\infty, a], \; a \in (-\infty, \infty),$$

where $\varphi : (-\infty, \infty) \to (-\infty, \infty)$ is an arbitrary bounded measurable function.

(iii) The potential measure of X involving the draw-down Parisian ruin time,

$$\int_0^\infty \mathrm{e}^{-q(t-r)} \mathbb{E}_x\left(f(X(t), \bar{X}(t)); t < \kappa_r^\xi \wedge \tau_a^+\right) \mathrm{d}t, \quad x \in (-\infty, a], \; a \in (-\infty, \infty),$$

where f is an arbitrary bounded bivariate function which is differentiable with respect to the first argument.

https://doi.org/10.1017/apr.2020.36 Published online by Cambridge University Press

(iv) The kth moment of the following integral involving D, which can be interpreted as the accumulated amount of time-discounted dividends up to the draw-down Parisian ruin time:

$$V_k^{\xi}(x;b) := \mathbb{E}_x\Big([D_b]^k\Big), \quad x \in (-\infty, \infty),\ b \in (0, \infty),$$

with

$$D_b := D_{\xi,b} := \int_{0-}^{\kappa_r^{\xi}} \mathrm{e}^{-qt}\,\mathrm{d}D(t).$$

Further let $V_k(x) := V_k^{\xi}(x;x),\ \ x \in (-\infty, \infty)$. For $\xi(x) = (x-b) \vee 0$ with $b > 0$, D_b can be interpreted as the accumulated discounted dividends paid according to the barrier dividend strategy with barrier at level b until the draw-down Parisian ruin time.

We assume the differentiability of $\ell_r^{(q)}$ whenever needed. In fact, by (2) and the definition of $\ell_r^{(q)}$, $\ell_r^{(q)}$ inherits the same differentiability as $W^{(q)}$. It is known that, when X has sample paths of unbounded variation, or when X has sample paths of bounded variation and the Lévy measure has no atoms, the scale function $W^{(q)}$ (and hence $\ell_r^{(q)}$) is continuously differentiable over $(0, \infty)$ (resp. $(-\infty, \infty)$). Moreover, if X has a nontrivial Gaussian component, then $W^{(q)}$ (and hence $\ell_r^{(q)}$) is twice continuously differentiable over $(0, \infty)$ (resp. $(-\infty, \infty)$). The interested reader is referred to [4] and [15] for more detailed discussions on the smoothness of scale functions.

3. Excursion process and Parisian-ruin-related quantities under excursion measure

In this section, we briefly recall basic concepts in excursion theory for the reflected process $\{\bar{X}(t) - X(t);\ t \geq 0\}$, and we refer to [3] for more details. We also obtain new Parisian-ruin-related results on the excursion measure.

For $x \in (-\infty, \infty)$, the process $\{L(t) := \bar{X}(t) - x,\ t \geq 0\}$ is a local time at 0 for the Markov process $\{\bar{X}(t) - X(t); t \geq 0\}$ under $\mathbb{P}_x$. The corresponding inverse local time is defined as

$$L^{-1}(t) := \inf\{s \geq 0 : L(s) > t\} = \sup\{s \geq 0 : L(s) \leq t\}.$$

Further, let $L^{-1}(t-) := \lim_{s\uparrow t} L^{-1}(s)$. Define a Poisson point process $\{(t, \varepsilon_t);\ t \geq 0\}$ by

$$\varepsilon_t(s) := X(L^{-1}(t)) - X(L^{-1}(t-) + s), \qquad s \in (0, L^{-1}(t) - L^{-1}(t-)] \tag{6}$$

whenever the lifetime of ε_t is strictly positive, i.e. $L^{-1}(t) - L^{-1}(t-) > 0$. If $L^{-1}(t) - L^{-1}(t-) = 0$, define $\varepsilon_t := \Upsilon$ with Υ being an additional isolated point. It is known that ε is a Poisson point process taking values in the space of excursion paths with characteristic measure n if $\{\bar{X}(t) - X(t);\ t \geq 0\}$ is recurrent; otherwise, $\{\varepsilon_t;\ t \leq L(\infty)\}$ is a Poisson point process stopped at the first excursion of infinite lifetime. Here, n is a σ-finite measure on the space $\mathcal{E}$ of excursions, i.e. the space $\mathcal{E}$ of càdlàg functions f satisfying

$$f : (0, \zeta) \to (0, \infty) \ \text{ for some } \ \zeta \in (0, \infty] \ \text{ and } f(\zeta) \in (0, \infty) \text{ if } \zeta < \infty,$$

where $\zeta \equiv \zeta(f)$ denotes the excursion length or lifetime; see Definition 6.13 of [16] for the definition of $\mathcal{E}$. Denote by $\varepsilon(\cdot)$, or ε for short, a generic excursion belonging to the space $\mathcal{E}$ of canonical excursions. The excursion height of a canonical excursion ε is denoted by $\bar{\varepsilon} = \sup_{t\in[0,\zeta]} \varepsilon(t)$.

https://doi.org/10.1017/apr.2020.36 Published online by Cambridge University Press

With a little abuse of notation, for $a \in (0, \infty)$ and $t \in [0, \zeta]$, let

$$g_t^a(\varepsilon) := \begin{cases} \inf\{s \in [0, \zeta] : s \leq t, \varepsilon(w) \geq a \text{ for all } w \in [s, t]\} & \text{if } \varepsilon(t) \geq a, \\ t & \text{otherwise;} \end{cases}$$

and

$$d_t^a(\varepsilon) := \begin{cases} \sup\{s \in [0, \zeta] : s \geq t, \varepsilon(w) \geq a \text{ for all } w \in [t, s)\} & \text{if } \varepsilon(t) \geq a, \\ t & \text{otherwise.} \end{cases}$$

Write $\zeta_t^a(\varepsilon) := d_t^a(\varepsilon) - g_t^a(\varepsilon)$ for the length of the maximum time interval (containing t) when the canonical excursion ε stays above the level a. Further define

$$\alpha_a^+(\varepsilon) := \inf\{g_t^a(\varepsilon) : t \in [0, \zeta], \zeta_t^a(\varepsilon) > r\},$$

with the convention that $\inf \emptyset := \zeta$. Intuitively, $\alpha_a^+(\varepsilon)$ is the starting time of the first time interval of length more than r when the excursion path stays continuously above level a.

The following result gives the excursion measure of the event that there exists a time interval with length at least r during which either the excursion process continuously stays above level $z > 0$, or there is an excursion with height strictly greater than $z + y$ for some $y > 0$.

Proposition 3.1. *For any $z, y \in (0, \infty)$, we have*

$$n\big(\alpha_z^+(\varepsilon) < \zeta \text{ or } \overline{\varepsilon} > z + y\big) = \frac{W'(z)\phi(y, r) + \chi'(z, y, r)}{W(z)\phi(y, r) + \chi(z, y, r)}, \tag{7}$$

where the derivative of χ is with respect to the first argument, and the Laplace transforms of $\phi(y, r)$ and $\chi(x, y, r)$ (in r) are given, respectively, by

$$\int_0^\infty e^{-\theta r} \chi(x, y, r)\, dr = \frac{1}{\theta} \left(\frac{W_y^{(\theta, -\theta)}(x + y)}{W^{(\theta)}(y)} - \frac{W(x) Z^{(\theta)}(y)}{W^{(\theta)}(y)} \right)$$

and

$$\int_0^\infty e^{-\theta r} \phi(y, r)\, dr = \frac{Z^{(\theta)}(y)}{\theta W^{(\theta)}(y)}.$$

Proof. It follows from Theorem 1 of [11] that

$$1 - \mathbb{P}_x\left(\kappa_r \wedge \tau_{-y}^- < \infty\right) = \mathbb{E}\left(X(1)\right) \left(W(x) + \frac{\chi(x, y, r)}{\phi(y, r)} \right) \tag{8}$$

for any fixed positive y. By the strong Markov property, for $a > x$ we have

$$\begin{aligned} \mathbb{P}_x\big(\kappa_r \wedge \tau_{-y}^- < \infty\big) &= \mathbb{P}_x\big(\tau_a^+ < \kappa_r \wedge \tau_{-y}^- < \infty\big) + \mathbb{P}_x\big(\kappa_r \wedge \tau_{-y}^- < \tau_a^+\big) \\ &= \mathbb{P}_x\big(\tau_a^+ < \kappa_r \wedge \tau_{-y}^-\big)\, \mathbb{P}_a\big(\kappa_r \wedge \tau_{-y}^- < \infty\big) + 1 - \mathbb{P}_x\big(\tau_a^+ < \kappa_r \wedge \tau_{-y}^-\big), \end{aligned}$$

which together with (8) implies

$$\begin{aligned} \mathbb{P}_x\big(\tau_a^+ < \kappa_r \wedge \tau_{-y}^-\big) &= \frac{1 - \mathbb{P}_x\big(\kappa_r \wedge \tau_{-y}^- < \infty\big)}{1 - \mathbb{P}_a\big(\kappa_r \wedge \tau_{-y}^- < \infty\big)} \\ &= \frac{W(x)\phi(y, r) + \chi(x, y, r)}{W(a)\phi(y, r) + \chi(a, y, r)}. \end{aligned} \tag{9}$$

Because $\{(t, \varepsilon_t); t \geq 0\}$ defined via (6) is a Poisson point process with intensity measure $\mathrm{d}t \times \mathrm{d}n$, we have

$$\begin{aligned}\mathbb{P}_x\big(\tau_a^+ < \kappa_r \wedge \tau_{-y}^-\big) &= \mathbb{E}_x\Bigg(\prod_{t\leq a-x} \mathbf{1}_{\{\alpha_{x+t}^+(\varepsilon_t)=\zeta(\varepsilon_t),\, \overline{\varepsilon}_t \leq x+t+y\}}\Bigg) \\ &= \exp\left(-\int_0^{a-x} n\big(\alpha_{x+t}^+(\varepsilon) < \zeta \text{ or } \overline{\varepsilon} > x+t+y\big)\ \mathrm{d}t\right) \\ &= \exp\left(-\int_x^{a} n\big(\alpha_{w}^+(\varepsilon) < \zeta \text{ or } \overline{\varepsilon} > w+y\big)\ \mathrm{d}w\right), \end{aligned} \tag{10}$$

where $\overline{\varepsilon}_t$ denotes the excursion height of ε_t. Combining (9) and (10) yields (7). □

We next prove a version of Proposition 3.1 for $y = \infty$.

Corollary 3.1. *For any $x \in (0, \infty)$, we have*

$$n\big(\alpha_x^+(\varepsilon) < \zeta\big) = \frac{\ell_r'(x)}{\ell_r(x)}.$$

Proof. By definition we have

$$\int_0^\infty \mathrm{e}^{-\theta r}\ (W(x)\phi(y, r) + \chi(x, y, r))\ \mathrm{d}r = \frac{W_y^{(\theta,-\theta)}(x+y)}{\theta W^{(\theta)}(y)}. \tag{11}$$

The definition of $W_y^{(\theta,-\theta)}(x+y)$ together with (2) yields

$$\begin{aligned}\lim_{y\uparrow\infty} \frac{W_y^{(\theta,-\theta)}(x+y)}{\theta W^{(\theta)}(y)} &= \mathrm{e}^{\Phi_\theta x}\left(\frac{1}{\theta} - \int_0^x W(w)\,\mathrm{e}^{-\Phi_\theta w}\,\mathrm{d}w\right) \\ &= \int_0^\infty W(x+w)\,\mathrm{e}^{-\Phi_\theta w}\,\mathrm{d}w, \end{aligned} \tag{12}$$

which coincides with the Laplace transform (in r) of $\ell_r(x)$ as follows:

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-\theta r}\ell_r(x)\,\mathrm{d}r &= \int_0^\infty \mathrm{e}^{-\theta r}\int_0^\infty W(x+z)\frac{z}{r}\mathbb{P}\,(X(r)\in \mathrm{d}z)\ \mathrm{d}r \\ &= \int_0^\infty \mathrm{e}^{-\theta r}\int_0^\infty W(x+z)\mathbb{P}(\tau_z^+ \in \mathrm{d}r)\mathrm{d}z \\ &= \int_0^\infty W(x+z)\mathbb{E}(\mathrm{e}^{-\theta \tau_z^+})\mathrm{d}z \\ &= \int_0^\infty W(x+w)\,\mathrm{e}^{-\Phi_\theta w}\,\mathrm{d}w, \end{aligned} \tag{13}$$

where we have used Kendall's identity,

$$\frac{z}{r}\mathbb{P}\,(X(r)\in \mathrm{d}z)\ \mathrm{d}r = \mathbb{P}(\tau_z^+ \in \mathrm{d}r)\mathrm{d}z, \qquad z, r \geq 0.$$

By (11), (12), (13), and the continuity of Laplace transforms, we have

$$\lim_{y\uparrow\infty}\ (W(x)\phi(y, r) + \chi(x, y, r)) = \ell_r(x).$$

https://doi.org/10.1017/apr.2020.36 Published online by Cambridge University Press

In fact, the same arguments as above lead to

$$\lim_{y\uparrow\infty}\big(W'(x)\phi(y,r)+\chi'(x,y,r)\big)=\ell_r'(x).$$

Therefore,

$$\begin{aligned}n\big(\alpha_x^+(\varepsilon)<\zeta\big)&=\lim_{y\uparrow\infty}n\big(\alpha_x^+(\varepsilon)<\zeta \ or\ \overline{\varepsilon}>x+y\big)\\&=\lim_{y\uparrow\infty}\frac{W'(x)\phi(y,r)+\chi'(x,y,r)}{W(x)\phi(y,r)+\chi(x,y,r)}=\frac{\ell_r'(x)}{\ell_r(x)},\end{aligned}$$

which is the desired result. □

Remark 3.1. Applying Corollary 3.1 and a property of Poisson random measure, we have for $x<a$ that

$$\begin{aligned}\mathbb{P}_x\left(\tau_a^+<\kappa_r\right)&=\mathbb{P}_x\left(\alpha_{x+t}^+(\varepsilon_t)=\zeta(\varepsilon_t),\ \ t\le a-x\right)\\&=\mathrm{e}^{-\int_0^{a-x}n(\alpha_{x+t}^+(\varepsilon)<\zeta)\mathrm{d}t}=\mathrm{e}^{-\int_0^{a-x}\frac{\ell_r'(x+t)}{\ell_r(x+t)}\mathrm{d}t}=\frac{\ell_r(x)}{\ell_r(a)}.\end{aligned}$$

Then (4) can be recovered via a change of measure.

Denote by ϵ_g the excursion (away from 0) for the reflected process $\{\bar{X}(s)-X(s);s\ge 0\}$ starting at time g in the time scale for X, i.e. $g=L^{-1}(t-)<L^{-1}(t)$ for some $t\ge 0$ and $\epsilon_g:=\varepsilon_t$. In addition, denote by $\zeta_g:=\zeta(\varepsilon_t)$ and $\bar{\epsilon}_g:=\bar{\varepsilon}_t$ its lifetime and its excursion height, respectively, and write $\alpha_a^+(\epsilon_g):=\alpha_a^+(\varepsilon_t)$; see Section IV.4 of [3]. The following result gives the joint Laplace transform involving α_a^+ under the excursion measure.

Proposition 3.2. *For any $q,\lambda\in[0,\infty)$ and $a,r\in(0,\infty)$, we have*

$$\begin{aligned}&n\Big(\mathrm{e}^{-q\alpha_a^+(\varepsilon)}\mathrm{e}^{\lambda(a-\varepsilon(\alpha_a^+(\varepsilon)+r))-\psi(\lambda)r}\mathbf{1}_{\{\alpha_a^+(\varepsilon)<\zeta\}}\Big)\\&=\frac{\ell_r^{(q)\prime}(a)}{\ell_r^{(q)}(a)}\left(\mathrm{e}^{\lambda a}-(\psi(\lambda)-q)\left(\mathrm{e}^{\lambda a}\int_0^a W^{(q)}(z)\mathrm{e}^{-\lambda z}\mathrm{d}z+\int_0^r\mathrm{e}^{-\psi(\lambda)s}\,\ell_s^{(q)}(a)\mathrm{d}s\right)\right)\\&\quad-\lambda\mathrm{e}^{\lambda a}+(\psi(\lambda)-q)\left(\lambda\mathrm{e}^{\lambda a}\int_0^a W^{(q)}(z)\mathrm{e}^{-\lambda z}\mathrm{d}z+W^{(q)}(a)\right.\\&\quad\left.+\int_0^r\mathrm{e}^{-\psi(\lambda)s}\,\ell_s^{(q)\prime}(a)\mathrm{d}s\right).\end{aligned}\tag{14}$$

Proof. Given $q,\lambda\ge 0$, $r,a>0$, and $a\ge x$, by Theorem 3.1 in [22] we have

$$\begin{aligned}&\mathbb{E}_x\Big(\mathrm{e}^{-q(\kappa_r-r)}\mathrm{e}^{\lambda X(\kappa_r)-\psi(\lambda)r}\mathbf{1}_{\{\kappa_r<\tau_a^+\}}\Big)\\&=\mathrm{e}^{\lambda x}-(\psi(\lambda)-q)\left(\mathrm{e}^{\lambda x}\int_0^x W^{(q)}(z)\mathrm{e}^{-\lambda z}\mathrm{d}z+\int_0^r\mathrm{e}^{-\psi(\lambda)s}\,\ell_s^{(q)}(x)\mathrm{d}s\right)\\&\quad-\frac{\ell_r^{(q)}(x)}{\ell_r^{(q)}(a)}\left(\mathrm{e}^{\lambda a}-(\psi(\lambda)-q)\left(\mathrm{e}^{\lambda a}\int_0^a W^{(q)}(z)\mathrm{e}^{-\lambda z}\mathrm{d}z\right.\right.\\&\quad\left.\left.+\int_0^r\mathrm{e}^{-\psi(\lambda)s}\,\ell_s^{(q)}(a)\mathrm{d}s\right)\right).\end{aligned}\tag{15}$$

https://doi.org/10.1017/apr.2020.36 Published online by Cambridge University Press

By (4) and the compensation formula (see for example Corollary 4.11 of [3] or Theorem 4.4 of [16]), one gets

$$
\begin{aligned}
&\mathbb{E}_x\Big(\mathrm{e}^{-q(\kappa_r-r)}\mathrm{e}^{\lambda X(\kappa_r)-\psi(\lambda)r}\mathbf{1}_{\{\kappa_r<\tau_a^+\}}\Big)\\
&=\mathbb{E}_x\Bigg(\sum_g \mathrm{e}^{-qg}\prod_{h<g}\mathbf{1}_{\{\alpha^+_{x+L(h)}(\epsilon_h)=\,\zeta_h,\,x+L(g)\le a\}}\mathrm{e}^{-q\alpha^+_{x+L(g)}(\epsilon_g)}\\
&\quad\times \mathrm{e}^{\lambda\left(x+L(g)-\epsilon_g\left(\alpha^+_{x+L(g)}(\epsilon_g)+r\right)\right)-\psi(\lambda)r}\mathbf{1}_{\{\alpha^+_{x+L(g)}(\epsilon_g)<\zeta_g\}}\Bigg)\\
&=\mathbb{E}_x\Bigg(\int_0^\infty \mathrm{e}^{-qw}\prod_{h<w}\mathbf{1}_{\{\alpha^+_{x+L(h)}(\epsilon_h)=\,\zeta_h,\,x+L(w)\le a\}}\int_{\mathcal{E}}\mathrm{e}^{-q\alpha^+_{x+L(w)}(\varepsilon)}\\
&\quad\times \mathrm{e}^{\lambda\left(x+L(w)-\varepsilon\left(\alpha^+_{x+L(w)}(\varepsilon)+r\right)\right)-\psi(\lambda)r}\mathbf{1}_{\{\alpha^+_{x+L(w)}(\varepsilon)<\zeta\}}\,n(\,\mathrm{d}\varepsilon)\,\mathrm{d}L(w)\Bigg)\\
&=\mathbb{E}_x\Bigg(\int_0^{a-x}\mathrm{e}^{-qL^{-1}(w-)}\prod_{h<L^{-1}(w-)}\mathbf{1}_{\{\alpha^+_{x+L(h)}(\epsilon_h)=\,\zeta_h\}}\\
&\quad\times\int_{\mathcal{E}}\mathrm{e}^{-q\alpha^+_{x+w}(\varepsilon)}\mathrm{e}^{\lambda\left(x+w-\varepsilon\left(\alpha^+_{x+w}(\varepsilon)+r\right)\right)-\psi(\lambda)r}\mathbf{1}_{\{\alpha^+_{x+w}(\varepsilon)<\zeta\}}\,n(\,\mathrm{d}\varepsilon)\,\mathrm{d}w\Bigg)\\
&=\int_x^a\mathbb{E}_x\Big(\mathrm{e}^{-q\tau_w^+}\mathbf{1}_{\{\tau_w^+<\kappa_r\}}\Big)\;n\Big(\mathrm{e}^{-q\alpha_w^+(\varepsilon)}\mathrm{e}^{\lambda\left(w-\varepsilon\left(\alpha_w^+(\varepsilon)+r\right)\right)-\psi(\lambda)r}\mathbf{1}_{\{\alpha_w^+(\varepsilon)<\zeta\}}\Big)\;\mathrm{d}w\\
&=\int_x^a\frac{\ell_r^{(q)}(x)}{\ell_r^{(q)}(w)}\,n\Big(\mathrm{e}^{-q\alpha_w^+(\varepsilon)}\mathrm{e}^{\lambda\left(w-\varepsilon\left(\alpha_w^+(\varepsilon)+r\right)\right)-\psi(\lambda)r}\mathbf{1}_{\{\alpha_w^+(\varepsilon)<\zeta\}}\Big)\;\mathrm{d}w,
\end{aligned}
$$

where $\epsilon_h(h\le g)$ denotes the excursion (away from 0) with left endpoint h for the reflected process $\{\bar{X}(t)-X(t);t\ge 0\}$, and ζ_h and $\bar{\epsilon}_h$ denote its lifetime and excursion height, respectively. Note that by (15),

$$
\begin{aligned}
&\frac{\ell_r^{(q)}(x)}{\ell_r^{(q)}(a)}\,n\Big(\mathrm{e}^{-q\alpha_a^+(\varepsilon)}\mathrm{e}^{\lambda\left(a-\varepsilon\left(\alpha_a^+(\varepsilon)+r\right)\right)-\psi(\lambda)r}\mathbf{1}_{\{\alpha_a^+(\varepsilon)<\zeta\}}\Big)\\
&=\frac{\ell_r^{(q)}(x)\,\ell_r^{(q)\prime}(a)}{\left(\ell_r^{(q)}(a)\right)^2}\bigg(\mathrm{e}^{\lambda a}-(\psi(\lambda)-q)\bigg(\mathrm{e}^{\lambda a}\int_0^a W^{(q)}(z)\mathrm{e}^{-\lambda z}\mathrm{d}z+\int_0^r\mathrm{e}^{-\psi(\lambda)s}\,\ell_s^{(q)}(a)\mathrm{d}s\bigg)\bigg)\\
&\quad-\frac{\ell_r^{(q)}(x)}{\ell_r^{(q)}(a)}\bigg(\lambda\mathrm{e}^{\lambda a}-(\psi(\lambda)-q)\bigg(\lambda\mathrm{e}^{\lambda a}\int_0^a W^{(q)}(z)\mathrm{e}^{-\lambda z}\mathrm{d}z+W^{(q)}(a)\\
&\quad+\int_0^r\mathrm{e}^{-\psi(\lambda)s}\,\ell_s^{(q)\prime}(a)\mathrm{d}s\bigg)\bigg).
\end{aligned}
$$

We thus obtain (14). □

https://doi.org/10.1017/apr.2020.36 Published online by Cambridge University Press

The following result gives an expression for the potential measure of the excursion process until time $\alpha_a^+ + r$ under the excursion measure.

Proposition 3.3. *For any $q, \lambda \in [0, \infty)$, $a, r \in (0, \infty)$ and any bounded differentiable function f, we have*

$$\begin{aligned}
&W^{(q)}(0+)\,\mathrm{e}^{qr}f(a) + n\bigg(\int_0^{\zeta} \mathrm{e}^{-q(t-r)}f(a-\varepsilon(t)\,)\mathbf{1}_{\{\alpha_a^+(\varepsilon)>t-r\}}\mathrm{d}t\bigg)\\
&= \frac{\ell_r^{(q)\prime}(a)}{\ell_r^{(q)}(a)}\bigg(\int_0^r \mathrm{e}^{q(r-s)}\mathbb{E}_a\,(f(X(s)))\mathrm{d}s - \int_0^a W^{(q)}(a-z)\mathbb{E}_z\,(f(X(r)))\mathrm{d}z\\
&\quad - \int_0^r \mathbb{E}\,(f(X(r-s)))\,\ell_s^{(q)}(a)\mathrm{d}s\bigg)\\
&\quad - \int_0^r \mathrm{e}^{q(r-s)}\mathbb{E}\,\big(f'(a+X(s))\big)\mathrm{d}s + \int_0^a W^{(q)\prime}(a-z)\mathbb{E}_z\,(f(X(r)))\mathrm{d}z\\
&\quad + W^{(q)}(0+)\,\mathbb{E}_a\,(f(X(r))) + \int_0^r \mathbb{E}\,(f(X(r-s)))\,\ell_s^{(q)\prime}(a)\mathrm{d}s.
\end{aligned} \tag{16}$$

Proof. Let e_q be an exponentially distributed random variable with mean $1/q$ independent of X. For $q, \lambda \geq 0$ and $r, b > 0$ with $x \leq a$, we have

$$\begin{aligned}
&\int_0^{\infty} \mathrm{e}^{-q(t-r)}\mathbb{E}_x\big(f(X(t));\, t<\kappa_r \wedge \tau_a^+\big)\,\mathrm{d}t\\
&= \mathbb{E}_x\bigg(\int_0^{\kappa_r\wedge\tau_a^+} \mathrm{e}^{-q(t-r)}f(X(t))\,\mathrm{d}\bigg(\int_0^t \mathbf{1}_{\{X(s)=\bar{X}(s)\}}\mathrm{d}s\bigg)\bigg)\\
&\quad + \frac{1}{q}\mathbb{E}_x\Big(\mathrm{e}^{qr}f(X(e_q))\,\mathbf{1}_{\{e_q<\kappa_r\wedge\tau_a^+\}}\mathbf{1}_{\{X(e_q)<\bar{X}(e_q)\}}\Big)\\
&:= h_1(x) + h_2(x).
\end{aligned} \tag{17}$$

Note that

$$\int_0^t \mathbf{1}_{\{X(s)=\bar{X}(s)\}}\mathrm{d}s = W^{(q)}(0+)\,\bar{X}(t)$$

and that $X(t)=\bar{X}(t)$ implies $t = L^{-1}(L(t))$ almost surely. Thus, the function $h_1(x)$ can be further expressed as follows:

$$\begin{aligned}
&W^{(q)}(0+)\,\mathbb{E}_x\bigg(\int_0^{\infty} \mathrm{e}^{-q(L^{-1}(L(t))-r)}f(x+L(t))\mathbf{1}_{\{x+L(t)\leq a,\, L^{-1}(L(t))<\kappa_r\}}\mathrm{d}L(t)\bigg)\\
&= W^{(q)}(0+)\,\mathbb{E}_x\bigg(\int_0^{a-x} \mathrm{e}^{-q(L^{-1}(w)-r)}f(x+w)\mathbf{1}_{\{L^{-1}(w)<\kappa_r\}}\mathrm{d}w\bigg)\\
&= W^{(q)(0+)}\,\mathrm{e}^{qr}\int_0^{a-x} \mathbb{E}_x\Big(\mathrm{e}^{-qL^{-1}(w)}\mathbf{1}_{\{L^{-1}(w)<\kappa_r\}}\Big)f(x+w)\mathrm{d}w\\
&= W^{(q)}(0+)\,\mathrm{e}^{qr}\int_0^{a-x} \mathbb{E}_x\Big(\mathrm{e}^{-q\tau_{x+w}^+}\mathbf{1}_{\{\tau_{x+w}^+<\kappa_r\}}\Big)f(x+w)\mathrm{d}w
\end{aligned}$$

https://doi.org/10.1017/apr.2020.36 Published online by Cambridge University Press

$$= W^{(q)}(0+)\,\mathrm{e}^{qr} \int_x^a \mathbb{E}_x\Big(\mathrm{e}^{-q\tau_w^+}\mathbf{1}_{\{\tau_w^+<\kappa_r\}}\Big) f(w)\mathrm{d}w$$

$$= W^{(q)}(0+)\,\mathrm{e}^{qr} \int_x^a \frac{\ell_r^{(q)}(x)}{\ell_r^{(q)}(w)} f(w)\mathrm{d}w, \tag{18}$$

where (4) is used in the final equality.

To further develop $h_2(x)$, note that $X(e_q) < \bar{X}(e_q)$ if and only if there is an excursion with left endpoint g such that $e_q \in (g, g+\zeta_g)$. Hence, by the compensation formula and the memoryless property of the exponential random variable, $h_2(x)$ can be rewritten as

$$\frac{1}{q}\mathbb{E}_x\Bigg(\sum_g \mathrm{e}^{qr}\prod_{h<g}\mathbf{1}_{\{\alpha^+_{x+L(h)}(\epsilon_h)=\,\zeta_h,\, x+L(g)\le a\}}$$
$$\times f(x+L(g)-\epsilon_g\left(e_q-g\right))\mathbf{1}_{\{\alpha^+_{x+L(g)}(\epsilon_g)>e_q-g-r,\, 0<e_q-g<\zeta_g\}}\Bigg)$$

$$= \frac{1}{q}\mathbb{E}_x\Bigg(\sum_g \mathrm{e}^{-q(g-r)}\prod_{h<g}\mathbf{1}_{\{\alpha^+_{x+L(h)}(\epsilon_h)=\,\zeta_h,\, x+L(g)\le a\}}$$
$$\times f(x+L(g)-\epsilon_g(e_q))\mathbf{1}_{\{\alpha^+_{x+L(g)}(\epsilon_g)>e_q-r,\, e_q<\zeta_g\}}\Bigg)$$

$$= \frac{1}{q}\mathbb{E}_x\Bigg(\int_0^\infty \mathrm{e}^{-q(w-r)}\prod_{h<w}\mathbf{1}_{\{\alpha^+_{x+L(h)}(\epsilon_h)=\,\zeta_h,\, x+L(w)\le a\}}$$
$$\times \int_{\mathcal{E}} f(x+L(w)-\varepsilon\left(e_q\right))\mathbf{1}_{\{\alpha^+_{x+L(w)}(\varepsilon)>e_q-r,\, e_q<\zeta\}} n(\mathrm{d}\varepsilon)\,\mathrm{d}L(w)\Bigg)$$

$$= \frac{1}{q}\mathbb{E}_x\Bigg(\int_x^a \mathrm{e}^{-q(\tau_w^+-r)}\mathbf{1}_{\{\tau_w^+<\kappa_r\}}\int_{\mathcal{E}} f(w-\varepsilon\left(e_q\right))\mathbf{1}_{\{\alpha^+_w(\varepsilon)>e_q-r,\, e_q<\zeta\}} n(\mathrm{d}\varepsilon)\,\mathrm{d}w\Bigg)$$

$$= \int_x^a \mathbb{E}_x\Big(\mathrm{e}^{-q\tau_w^+}\mathbf{1}_{\{\tau_w^+<\kappa_r\}}\Big)\; n\bigg(\int_0^\zeta \mathrm{e}^{-q(t-r)} f(w-\varepsilon(t))\mathbf{1}_{\{\alpha^+_w(\varepsilon)>t-r\}}\mathrm{d}t\bigg)\;\mathrm{d}w$$

$$= \int_x^a \frac{\ell_r^{(q)}(x)}{\ell_r^{(q)}(w)} n\bigg(\int_0^\zeta \mathrm{e}^{-q(t-r)} f(w-\varepsilon(t))\mathbf{1}_{\{\alpha^+_w(\varepsilon)>t-r\}}\mathrm{d}t\bigg)\,\mathrm{d}w. \tag{19}$$

It follows from (17), (18), and (19) that

$$\int_0^\infty \mathrm{e}^{-q(t-r)}\mathbb{E}_x\big(f(X(t));\, t<\kappa_r\wedge\tau_a^+\big)\,\mathrm{d}t$$
$$= W^{(q)}(0+)\,\mathrm{e}^{qr}\int_x^a \frac{\ell_r^{(q)}(x)}{\ell_r^{(q)}(w)} f(w)\mathrm{d}w$$
$$+ \int_x^a \frac{\ell_r^{(q)}(x)}{\ell_r^{(q)}(w)}\, n\bigg(\int_0^\zeta \mathrm{e}^{-q(t-r)} f(w-\varepsilon(t))\mathbf{1}_{\{\alpha^+_w(\varepsilon)>t-r\}}\mathrm{d}t\bigg)\,\mathrm{d}w. \tag{20}$$

https://doi.org/10.1017/apr.2020.36 Published online by Cambridge University Press

Meanwhile, by Theorem 4.4 of [22] we know that

$$\begin{aligned}&\int_0^\infty \mathrm{e}^{-q(t-r)}\mathbb{E}_x\big(f(X(t));t<\kappa_r\wedge\tau_a^+\big)\,\mathrm{d}t\\ =&\int_0^r \mathrm{e}^{q(r-s)}\mathbb{E}_x(f(X(s)))\mathrm{d}s-\int_0^x W^{(q)}(x-z)\mathbb{E}_z\left(f(X(r))\right)\mathrm{d}z\\ &-\int_0^r \mathbb{E}\left(f(X(r-s))\right)\ell_s^{(q)}(x)\mathrm{d}s\\ &-\frac{\ell_r^{(q)}(x)}{\ell_r^{(q)}(a)}\left(\int_0^r \mathrm{e}^{q(r-s)}\mathbb{E}_a\left(f(X(s))\right)\mathrm{d}s-\int_0^a W^{(q)}(a-z)\mathbb{E}_z\left(f(X(r))\right)\mathrm{d}z\right.\\ &\left.-\int_0^r \mathbb{E}\left(f(X(r-s))\right)\ell_s^{(q)}(a)\mathrm{d}s\right).\end{aligned}\tag{21}$$

Combining (20) and (21), we obtain (16). □

Remark 3.2. For any $b\in(-\infty,\infty)$, if we replace $f(x)$ with $f(x,b)$ in Proposition 3.3, by similar arguments we have

$$\begin{aligned}&W^{(q)}(0+)\,\mathrm{e}^{qr}f(a,b)+n\left(\int_0^\zeta \mathrm{e}^{-q(t-r)}f(a-\varepsilon(t),b)\mathbf{1}_{\{\alpha_a^+(\varepsilon)>t-r\}}\mathrm{d}t\right)\\ =&\frac{\ell_r^{(q)\prime}(a)}{\ell_r^{(q)}(a)}\left(\int_0^r \mathrm{e}^{q(r-s)}\mathbb{E}_a\left(f(X(s),b)\right)\mathrm{d}s-\int_0^a W^{(q)}(a-z)\mathbb{E}_z\left(f(X(r),b)\right)\mathrm{d}z\right.\\ &\left.-\int_0^r \mathbb{E}\left(f(X(r-s),b)\right)\ell_s^{(q)}(a)\mathrm{d}s\right)\\ &-\int_0^r \mathrm{e}^{q(r-s)}\mathbb{E}\left(\frac{\partial}{\partial x}f(a+X(s),b)\right)\mathrm{d}s+\int_0^a W^{(q)\prime}(a-z)\mathbb{E}_z\left(f(X(r),b)\right)\mathrm{d}z\\ &+W^{(q)}(0+)\,\mathbb{E}_a\left(f(X(r),b)\right)+\int_0^r \mathbb{E}\left(f(X(r-s),b)\right)\ell_s^{(q)\prime}(a)\mathrm{d}s\Bigg).\end{aligned}\tag{22}$$

Remark 3.3. Letting $f(x):=\mathrm{e}^{\lambda x-\psi(\lambda)r}$ in Proposition 3.3, we have

$$\begin{aligned}&n\left(\int_0^\zeta \mathrm{e}^{-q(t-r)}\mathrm{e}^{\lambda(a-\varepsilon(t))-\psi(\lambda)r}\mathbf{1}_{\{\alpha_a^+(\varepsilon)>t-r\}}\,\mathrm{d}t\right)+W^{(q)}(0+)\,\mathrm{e}^{qr}\mathrm{e}^{\lambda a-\psi(\lambda)r}\\ =&\frac{\ell_r^{(q)\prime}(a)}{\ell_r^{(q)}(a)}\left(\frac{\mathrm{e}^{\lambda a}\left(1-\mathrm{e}^{-(\psi(\lambda)-q)r}\right)}{\psi(\lambda)-q}-\mathrm{e}^{\lambda a}\int_0^a W^{(q)}(z)\mathrm{e}^{-\lambda z}\mathrm{d}z-\int_0^r \mathrm{e}^{-\psi(\lambda)s}\,\ell_s^{(q)}(a)\mathrm{d}s\right)\\ &-\frac{\lambda\mathrm{e}^{\lambda a}\left(1-\mathrm{e}^{-(\psi(\lambda)-q)r}\right)}{\psi(\lambda)-q}+\lambda\mathrm{e}^{\lambda a}\int_0^a W^{(q)}(z)\mathrm{e}^{-\lambda z}\mathrm{d}z+W^{(q)}(a)\\ &+\int_0^r \mathrm{e}^{-\psi(\lambda)s}\,\ell_s^{(q)\prime}(a)\mathrm{d}s.\end{aligned}$$

https://doi.org/10.1017/apr.2020.36 Published online by Cambridge University Press

4. Main results

In this section we present several results concerning the draw-down Parisian ruin. The first result below solves a draw-down Parisian-ruin-based two-sided exit problem. It generalizes Theorem 1 of [11].

Theorem 4.1. *Given any a, let η be another draw-down function such that $\eta(z)<\xi(z)<z$ for $z\leq a$. For any $x\in(-\infty, a)$, we have*

$$\mathbb{E}_x\Big(\mathrm{e}^{-q\tau_a^+}\mathbf{1}_{\{\tau_a^+<\kappa_r^\xi\wedge\tau_\eta\}}\Big)$$
$$=\exp\left(-\int_x^a \frac{W^{(q)\prime}(\overline{\xi}(w))\phi_{\Phi_q}(\xi(w)-\eta(w), r)+\chi^{(q)\prime}(\overline{\xi}(w), \xi(w)-\eta(w), r)}{W^{(q)}(\overline{\xi}(w))\phi_{\Phi_q}(\xi(w)-\eta(w), r)+\chi^{(q)}(\overline{\xi}(w), \xi(w)-\eta(w), r)}\,\mathrm{d}w\right),$$

where the Laplace transforms of $\phi_{\Phi_q}(y, r)$ and $\chi^{(q)}(x, y, r):=\mathrm{e}^{\Phi_q x}\chi_{\Phi_q}(x, y, r)$ with respect to r are given, respectively, by

$$\int_0^\infty \mathrm{e}^{-\theta r}\chi^{(q)}(x, y, r)\,\mathrm{d}r$$
$$=\frac{1}{\theta}\left(\frac{W_y^{(\theta+q,-\theta)}(x+y)}{W^{(\theta+q)}(y)}-\frac{W^{(q)}(x)\mathrm{e}^{\Phi_q y}\left(1+\theta\int_0^y \mathrm{e}^{-\Phi_q w}W^{(\theta+q)}(w)\mathrm{d}w\right)}{W^{(\theta+q)}(y)}\right) \tag{23}$$

and

$$\int_0^\infty \mathrm{e}^{-\theta r}\phi_{\Phi_q}(y, r)\,\mathrm{d}r=\frac{\mathrm{e}^{\Phi_q y}\left(1+\theta\int_0^y \mathrm{e}^{-\Phi_q w}W^{(\theta+q)}(w)\mathrm{d}w\right)}{\theta W^{(\theta+q)}(y)}; \tag{24}$$

here, $y\in(0,\infty)$, the derivative of $\chi^{(q)}$ is taken on the first argument, and ϕ_{Φ_q} and χ_{Φ_q} play the roles of ϕ and χ for the process $(X, \mathbb{P}_x^{\Phi_q})$.

Proof. By (7) and an argument similar to that of (10) we have

$$\mathbb{P}_x\big(\tau_a^+<\kappa_r^\xi\wedge\tau_\eta\big)$$
$$=\mathbb{E}_x\left(\prod_{t\leq a-x}\mathbf{1}_{\{\alpha^+_{\overline{\xi}(x+t)}(\varepsilon_t)=\zeta(\varepsilon_t),\ \overline{\varepsilon}_t\leq\overline{\eta}(x+t)\}}\right)$$
$$=\exp\left(-\int_x^a n\Big(\alpha^+_{\overline{\xi}(w)}(\varepsilon)<\zeta \text{ or } \overline{\varepsilon}>\overline{\xi}(w)+\xi(w)-\eta(w)\Big)\,\mathrm{d}w\right)$$
$$=\exp\left(-\int_x^a \frac{W'(\overline{\xi}(w))\phi(\xi(w)-\eta(w), r)+\chi'(\overline{\xi}(w), \xi(w)-\eta(w), r)}{W(\overline{\xi}(w))\phi(\xi(w)-\eta(w), r)+\chi(\overline{\xi}(w), \xi(w)-\eta(w), r)}\,\mathrm{d}w\right), \tag{25}$$

where $\overline{\eta}(w):=w-\eta(w)$. By (25) together with a change of measure, one has

$$\mathbb{E}_x\Big(\mathrm{e}^{-q\tau_a^+}\mathbf{1}_{\{\tau_a^+<\kappa_r^\xi\wedge\tau_\eta\}}\Big)$$
$$=\mathrm{e}^{-\Phi_q(a-x)}\mathbb{P}_x^{\Phi_q}\left(\tau_a^+<\kappa_r^\xi\wedge\tau_\eta\right)$$
$$=\mathrm{e}^{-\Phi_q(a-x)}\exp\left(-\int_x^a n_{\Phi_q}\left(\alpha^+_{\overline{\xi}(w)}(\varepsilon)<\zeta \text{ or } \overline{\varepsilon}>\overline{\eta}(w)\right)\,\mathrm{d}w\right)$$

$$= \mathrm{e}^{-\Phi_q(a-x) - \int_x^a \frac{W'_{\Phi_q}(\overline{\xi}(w))\phi_{\Phi_q}(\xi(w)-\eta(w),r) + \chi'_{\Phi_q}(\overline{\xi}(w),\xi(w)-\eta(w),r)}{W_{\Phi_q}(\overline{\xi}(w))\phi_{\Phi_q}(\xi(w)-\eta(w),r) + \chi_{\Phi_q}(\overline{\xi}(w),\xi(w)-\eta(w),r)}\,\mathrm{d}w}$$

$$= \mathrm{e}^{- \int_x^a \frac{W^{(q)\prime}(\overline{\xi}(w))\phi_{\Phi_q}(\xi(w)-\eta(w),r) + \chi^{(q)\prime}(\overline{\xi}(w),\xi(w)-\eta(w),r)}{W^{(q)}(\overline{\xi}(w))\phi_{\Phi_q}(\xi(w)-\eta(w),r) + \chi^{(q)}(\overline{\xi}(w),\xi(w)-\eta(w),r)}\,\mathrm{d}w},$$

where

$$W_{\Phi_q}(x) = \mathrm{e}^{-\Phi_q x} W^{(q)}(x), \qquad \chi_{\Phi_q}(x, y, r) = \mathrm{e}^{-\Phi_q x} \chi^{(q)}(x, y, r),$$

and n_{Φ_q} represents the excursion measure under the new probability measure $\mathbb{P}_x^{\Phi_q}$. □

We next provide a version of Theorem 4.1 for $\eta \equiv -\infty$.

Corollary 4.1. *For $x \in (-\infty, a)$, we have*

$$\mathbb{E}_x\Big(\mathrm{e}^{-q\tau_a^+}\mathbf{1}_{\{\tau_a^+ < \kappa_r^\xi\}}\Big) = \exp\bigg(-\int_x^a \frac{\ell_r^{(q)\prime}(\overline{\xi}(w))}{\ell_r^{(q)}(\overline{\xi}(w))}\,\mathrm{d}w\bigg). \tag{26}$$

Proof. By definition we have

$$W_y^{(\theta+q,-\theta)}(x+y) = W^{(\theta+q)}(x+y) - \theta\int_0^x W^{(q)}(w)W^{(\theta+q)}(x+y-w)\,\mathrm{d}w,$$

which together with (2) yields

$$\begin{aligned}
&\lim_{y\uparrow\infty} \frac{W_y^{(\theta+q,-\theta)}(x+y)}{\theta W^{(\theta+q)}(y)} \\
&= \mathrm{e}^{\Phi_{\theta+q}x}\bigg(\frac{1}{\theta} - \int_0^x W^{(q)}(w)\,\mathrm{e}^{-\Phi_{\theta+q}w}\,\mathrm{d}w\bigg) \\
&= \mathrm{e}^{\Phi_{\theta+q}x}\bigg(\frac{1}{\theta} - \int_0^\infty W^{(q)}(w)\,\mathrm{e}^{-\Phi_{\theta+q}w}\,\mathrm{d}w + \int_x^\infty W^{(q)}(w)\,\mathrm{e}^{-\Phi_{\theta+q}w}\,\mathrm{d}w\bigg) \\
&= \int_0^\infty W^{(q)}(x+w)\,\mathrm{e}^{-\Phi_{\theta+q}w}\,\mathrm{d}w.
\end{aligned} \tag{27}$$

This coincides with the Laplace transform (in r) of $\mathrm{e}^{-qr}\ell_r^{(q)}(x)$ as follows:

$$\begin{aligned}
\int_0^\infty \mathrm{e}^{-\theta r}\mathrm{e}^{-qr}\ell_r^{(q)}(x)\,\mathrm{d}r &= \int_0^\infty \mathrm{e}^{-(\theta+q)r}\int_0^\infty W^{(q)}(x+z)\frac{z}{r}\mathbb{P}\left(X(r)\in \mathrm{d}z\right)\,\mathrm{d}r \\
&= \int_0^\infty \mathrm{e}^{-(\theta+q)r}\int_0^\infty W^{(q)}(x+z)\mathbb{P}(\tau_z^+ \in \mathrm{d}r)\mathrm{d}z \\
&= \int_0^\infty W^{(q)}(x+z)\mathbb{E}(\mathrm{e}^{-(\theta+q)\tau_z^+})\mathrm{d}z \\
&= \int_0^\infty W^{(q)}(x+w)\,\mathrm{e}^{-\Phi_{\theta+q}w}\,\mathrm{d}w,
\end{aligned} \tag{28}$$

where we have used Kendall's identity,

$$\frac{z}{r}\mathbb{P}\left(X(r)\in \mathrm{d}z\right)\,\mathrm{d}r = \mathbb{P}(\tau_z^+ \in \mathrm{d}r)\mathrm{d}z, \qquad z, r \ge 0.$$

https://doi.org/10.1017/apr.2020.36 Published online by Cambridge University Press

From (23) and (24) one knows that

$$\int_0^\infty \mathrm{e}^{-\theta r}\left(W^{(q)}(x)\phi_{\Phi_q}(y,r)+\chi^{(q)}(x,y,r)\right)\mathrm{d}r=\frac{W_y^{(\theta+q,-\theta)}(x+y)}{\theta W^{(\theta+q)}(y)}. \tag{29}$$

Combining (27), (28), and (29), one can conclude that

$$\lim_{y\uparrow\infty}\left(W^{(q)}(x)\phi_{\Phi_q}(y,r)+\chi^{(q)}(x,y,r)\right)=\mathrm{e}^{-qr}\ell_r^{(q)}(x).$$

By the same arguments, we have

$$\lim_{y\uparrow\infty}\left(W^{(q)\prime}(x)\phi_{\Phi_q}(y,r)+\chi^{(q)\prime}(x,y,r)\right)=\mathrm{e}^{-qr}\ell_r^{(q)\prime}(x).$$

Hence, we have

$$\lim_{c\uparrow\infty}\frac{W^{(q)\prime}(\overline{\xi}(w))\phi_{\Phi_q}(c+\xi(w),r)+\chi^{(q)\prime}(\overline{\xi}(w),c+\xi(w),r)}{W^{(q)}(\overline{\xi}(w))\phi_{\Phi_q}(c+\xi(w),r)+\chi^{(q)}(\overline{\xi}(w),c+\xi(w),r)}=\frac{\ell_r^{(q)\prime}(\overline{\xi}(w))}{\ell_r^{(q)}(\overline{\xi}(w))}.$$

It then follows easily from Theorem 4.1 that

$$\begin{aligned}\mathbb{E}_x\left(\mathrm{e}^{-q\tau_a^+}\mathbf{1}_{\{\tau_a^+<\kappa_r^\xi\}}\right)&=\lim_{c\uparrow\infty}\mathbb{E}_x\left(\mathrm{e}^{-q\tau_a^+}\mathbf{1}_{\{\tau_a^+<\kappa_r^\xi\wedge\tau_{-c}^-\}}\right)\\&=\exp\left(-\int_x^a\frac{\ell_r^{(q)\prime}(\overline{\xi}(w))}{\ell_r^{(q)}(\overline{\xi}(w))}\,\mathrm{d}w\right),\end{aligned}$$

which is the desired result. □

Remark 4.1. Letting $\xi\equiv 0$ in (26), one recovers (18) of [10]:

$$\mathbb{E}_x\left(\mathrm{e}^{-q\tau_a^+}\mathbf{1}_{\{\tau_a^+<\kappa_r\}}\right)=\exp\left(-\int_x^a\frac{\ell_r^{(q)\prime}(w)}{\ell_r^{(q)}(w)}\,\mathrm{d}w\right)=\frac{\ell_r^{(q)}(x)}{\ell_r^{(q)}(a)}.$$

Letting $\xi(x)=kx-d$ with $k\in(-\infty,1)$ and $d\in(0,\infty)$ in (26), we have

$$\mathbb{E}_x\left(\mathrm{e}^{-q\tau_a^+}\mathbf{1}_{\{\tau_a^+<\kappa_r^\xi\}}\right)=\left(\frac{\ell_r^{(q)}((1-k)x+d)}{\ell_r^{(q)}((1-k)a+d)}\right)^{\frac{1}{1-k}}.$$

By (26), one can also obtain the draw-down Parisian ruin probability

$$\mathbb{P}_x\left(\kappa_r^\xi<\infty\right)=1-\exp\left(-\int_x^\infty\frac{\ell_r'(\overline{\xi}(z))}{\ell_r(\overline{\xi}(z))}\,\mathrm{d}z\right).$$

https://doi.org/10.1017/apr.2020.36 Published online by Cambridge University Press

The following result presents the joint Laplace transform involving the draw-down Parisian ruin time, the position of X at the draw-down Parisian ruin time, and its running supremum until the draw-down Parisian ruin time. It generalizes Theorem 3.1 in [22].

Theorem 4.2. *For any $q, \lambda \in [0, \infty)$, $x \in (-\infty, \infty)$, $a \geq x$ and any bounded measurable function $\varphi\colon (-\infty, \infty) \mapsto (-\infty, \infty)$, we have*

$$\begin{aligned}
&\mathbb{E}_x\left(\mathrm{e}^{-q\left(\kappa_r^{\xi}-r\right)}\mathrm{e}^{\lambda X(\kappa_r^{\xi})-\psi(\lambda)r}\varphi(\bar{X}(\kappa_r^{\xi}))\mathbf{1}_{\{\kappa_r^{\xi}<\tau_a^+\}}\right)\\
&=\int_x^a \mathrm{e}^{\lambda\xi(w)}\varphi(w)\exp\left(-\int_x^w \frac{\ell_r^{(q)\prime}(\overline{\xi}(z))}{\ell_r^{(q)}(\overline{\xi}(z))}\mathrm{d}z\right)\left[\frac{\ell_r^{(q)\prime}(\overline{\xi}(w))}{\ell_r^{(q)}(\overline{\xi}(w))}\left(\mathrm{e}^{\lambda\overline{\xi}(w)}-(\psi(\lambda)-q)\right.\right.\\
&\quad\times\left.\left(\mathrm{e}^{\lambda\overline{\xi}(w)}\int_0^{\overline{\xi}(w)} W^{(q)}(z)\mathrm{e}^{-\lambda z}\mathrm{d}z+\int_0^r \mathrm{e}^{-\psi(\lambda)s}\,\ell_s^{(q)}(\overline{\xi}(w))\mathrm{d}s\right)\right)\\
&\quad-\lambda\mathrm{e}^{\lambda\overline{\xi}(w)}+(\psi(\lambda)-q)\left(\lambda\mathrm{e}^{\lambda\overline{\xi}(w)}\int_0^{\overline{\xi}(w)} W^{(q)}(z)\mathrm{e}^{-\lambda z}\mathrm{d}z\right.\\
&\quad\left.\left.+W^{(q)}(\overline{\xi}(w))+\int_0^r \mathrm{e}^{-\psi(\lambda)s}\,\ell_s^{(q)\prime}(\overline{\xi}(w))\mathrm{d}s\right)\right]\mathrm{d}w. \qquad (30)
\end{aligned}$$

Proof. By (26) and the compensation formula, we have

$$\begin{aligned}
&\mathbb{E}_x\left(\mathrm{e}^{-q\left(\kappa_r^{\xi}-r\right)}\mathrm{e}^{\lambda X(\kappa_r^{\xi})-\psi(\lambda)r}\varphi(\bar{X}(\kappa_r^{\xi}))\mathbf{1}_{\{\kappa_r^{\xi}<\tau_a^+\}}\right)\\
&=\mathbb{E}_x\left(\sum_g \mathrm{e}^{-qg}\varphi(x+L(g))\prod_{h<g}\mathbf{1}_{\{\alpha^+_{\overline{\xi}(x+L(h))}(\epsilon_h)=\zeta_h,\,x+L(g)\leq a\}}\mathrm{e}^{-q\alpha^+_{\overline{\xi}(x+L(g))}(\epsilon_g)}\right.\\
&\qquad\left.\times\,\mathrm{e}^{\lambda\left(x+L(g)-\epsilon_g\left(\alpha^+_{\overline{\xi}(x+L(g))}(\epsilon_g)+r\right)\right)-\psi(\lambda)r}\mathbf{1}_{\{\alpha^+_{\overline{\xi}(x+L(g))}(\epsilon_g)<\zeta_g\}}\right)\\
&=\mathbb{E}_x\left(\int_0^\infty \mathrm{e}^{-qw}\varphi(x+L(w))\prod_{h<w}\mathbf{1}_{\{\alpha^+_{\overline{\xi}(x+L(h))}(\epsilon_h)=\zeta_h,\,x+L(w)\leq a\}}\int_{\mathcal{E}}\mathrm{e}^{-q\alpha^+_{\overline{\xi}(x+L(w))}(\varepsilon)}\right.\\
&\qquad\left.\times\,\mathrm{e}^{\lambda\left(x+L(w)-\varepsilon\left(\alpha^+_{\overline{\xi}(x+L(w))}(\varepsilon)+r\right)\right)-\psi(\lambda)r}\mathbf{1}_{\{\alpha^+_{\overline{\xi}(x+L(w))}(\varepsilon)<\zeta\}}\,n(\mathrm{d}\varepsilon)\,\mathrm{d}L(w)\right)\\
&=\mathbb{E}_x\left(\int_0^{a-x}\mathrm{e}^{-qL^{-1}(w-)}\varphi(x+w)\prod_{h<L^{-1}(w-)}\mathbf{1}_{\{\alpha^+_{\overline{\xi}(x+L(h))}(\epsilon_h)=\zeta_h\}}\right.\\
&\qquad\left.\times\int_{\mathcal{E}}\mathrm{e}^{-q\alpha^+_{\overline{\xi}(x+w)}(\varepsilon)}\mathrm{e}^{\lambda\left(x+w-\varepsilon\left(\alpha^+_{\overline{\xi}(x+w)}(\varepsilon)+r\right)\right)-\psi(\lambda)r}\mathbf{1}_{\{\alpha^+_{\overline{\xi}(x+w)}(\varepsilon)<\zeta\}}\,n(\mathrm{d}\varepsilon)\,\mathrm{d}w\right)\\
&=\int_x^a \varphi(w)\mathbb{E}_x\left(\mathrm{e}^{-q\tau_w^+}\mathbf{1}_{\{\tau_w^+<\kappa_r^{\xi}\}}\right)\\
&\qquad\times n\left(\mathrm{e}^{-q\alpha^+_{\overline{\xi}(w)}(\varepsilon)}\mathrm{e}^{\lambda\left(w-\varepsilon\left(\alpha^+_{\overline{\xi}(w)}(\varepsilon)+r\right)\right)-\psi(\lambda)r}\mathbf{1}_{\{\alpha^+_{\overline{\xi}(w)}(\varepsilon)<\zeta\}}\right)\mathrm{d}w
\end{aligned}$$

$$= \int_x^a \mathrm{e}^{\lambda \xi(w)} \varphi(w) \exp\left(-\int_x^w \frac{\ell_r^{(q)\prime}(\overline{\xi}(z))}{\ell_r^{(q)}(\overline{\xi}(z))}\,\mathrm{d}z\right)$$
$$\times\, n\left(\mathrm{e}^{-q\alpha^+_{\overline{\xi}(w)}(\varepsilon)} \mathrm{e}^{\lambda\left(\overline{\xi}(w)-\varepsilon\left(\alpha^+_{\overline{\xi}(w)}(\varepsilon)+r\right)\right)-\psi(\lambda)r} \mathbf{1}_{\{\alpha^+_{\overline{\xi}(w)}(\varepsilon)<\zeta\}}\right)\,\mathrm{d}w,$$

which together with (14) yields (30). □

Remark 4.2. By Theorem 4.2, for any $q \in [0, \infty)$, $x \in (-\infty, \infty)$, and $a \in [x, \infty)$ we have

$$\mathbb{E}_x\left(\mathrm{e}^{-q\left(\kappa_r^{\xi}-r\right)}\mathbf{1}_{\{\kappa_r^{\xi}<\tau_a^+\}}\right)$$
$$= \int_x^a \exp\left(-\int_x^w \frac{\ell_r^{(q)\prime}(\overline{\xi}(z))}{\ell_r^{(q)}(\overline{\xi}(z))}\,\mathrm{d}z\right)\left[\frac{\ell_r^{(q)\prime}(\overline{\xi}(w))}{\ell_r^{(q)}(\overline{\xi}(w))}\left(Z^{(q)}(\overline{\xi}(w)) + q\int_0^r \ell_s^{(q)}(\overline{\xi}(w))\mathrm{d}s\right)\right.$$
$$\left. - q\left(W^{(q)}(\overline{\xi}(w)) + \int_0^r \ell_s^{(q)\prime}(\overline{\xi}(w))\mathrm{d}s\right)\right]\mathrm{d}w.$$

Letting $a \uparrow \infty$ and then choosing $\xi(w) = w \vee y - a$ with $0 \le y - x < a < \infty$ gives

$$\mathbb{E}_x\left(\mathrm{e}^{-q\left(\kappa_r^{\xi}-r\right)}\mathbf{1}_{\{\kappa_r^{\xi}<\infty\}}\right)$$
$$= \int_x^y \frac{\partial}{\partial w}\left[\frac{-\ell_r^{(q)}(x-y+a)}{\ell_r^{(q)}(w-y+a)}\left(Z^{(q)}(w-y+a) + q\int_0^r \ell_s^{(q)}(w-y+a)\mathrm{d}s\right)\right]\mathrm{d}w$$
$$+ \frac{\ell_r^{(q)}(x-y+a)}{\ell_r^{(q)}(a)}\int_y^\infty \exp\left(-\int_y^w \frac{\ell_r^{(q)\prime}(a)}{\ell_r^{(q)}(a)}\,\mathrm{d}z\right)\left[\frac{\ell_r^{(q)\prime}(a)}{\ell_r^{(q)}(a)}\left(Z^{(q)}(a) + q\int_0^r \ell_s^{(q)}(a)\mathrm{d}s\right)\right.$$
$$\left. - q\left(W^{(q)}(a) + \int_0^r \ell_s^{(q)\prime}(a)\mathrm{d}s\right)\right]\mathrm{d}w$$
$$= \left[\frac{-\ell_r^{(q)}(x-y+a)}{\ell_r^{(q)}(w-y+a)}\left(Z^{(q)}(w-y+a) + q\int_0^r \ell_s^{(q)}(w-y+a)\mathrm{d}s\right)\right]\Bigg|_{w=x}^{w=y}$$
$$+ \frac{\ell_r^{(q)}(x-y+a)}{\ell_r^{(q)\prime}(a)}\left[\frac{\ell_r^{(q)\prime}(a)}{\ell_r^{(q)}(a)}\left(Z^{(q)}(a) + q\int_0^r \ell_s^{(q)}(a)\mathrm{d}s\right) - q\left(W^{(q)}(a) + \int_0^r \ell_s^{(q)\prime}(a)\mathrm{d}s\right)\right]$$
$$= Z^{(q)}(x-y+a) + q\int_0^r \ell_s^{(q)}(x-y+a)\mathrm{d}s - q\frac{\ell_r^{(q)}(x-y+a)}{\ell_r^{(q)\prime}(a)}\left(W^{(q)}(a) + \int_0^r \ell_s^{(q)\prime}(a)\mathrm{d}s\right),$$

which recovers Theorem 2.1 of [26]. It can then be checked via a change of measure that Proposition 2.3 of [26] can also be recovered from our Theorem 4.2.

The following result gives the potential measure of X involving the draw-down Parisian ruin time. It generalizes Theorem 4.4 in [22].

https://doi.org/10.1017/apr.2020.36 Published online by Cambridge University Press

Theorem 4.3. *For any $q, \lambda \geq 0$, $r > 0$, $a \geq x$ and any bounded bivariate function f(x,y) that is differentiable with respect to x, we have*

$$\int_0^\infty \mathrm{e}^{-q(t-r)}\mathbb{E}_x\big(f(X(t), \bar{X}(t)); t < \kappa_r^\xi \wedge \tau_a^+\big)\,\mathrm{d}t$$

$$= \int_x^a \exp\left(-\int_x^w \frac{\ell_r^{(q)\prime}(\overline{\xi}(z))}{\ell_r^{(q)}(\overline{\xi}(z))}\,\mathrm{d}z\right)\left[\frac{\ell_r^{(q)\prime}(\overline{\xi}(w))}{\ell_r^{(q)}(\overline{\xi}(w))}\left(\int_0^r \mathrm{e}^{q(r-s)}\mathbb{E}\,(f(w+X(s), w))\mathrm{d}s\right.\right.$$

$$- \int_0^{\overline{\xi}(w)} W^{(q)}(\overline{\xi}(w)-z)\mathbb{E}\,(f(z+\xi(w)+X(r), w))\mathrm{d}z$$

$$\left. - \int_0^r \mathbb{E}\,(f(\xi(w)+X(r-s), w))\,\ell_s^{(q)}(\overline{\xi}(w))\mathrm{d}s\right)$$

$$- \int_0^r \mathrm{e}^{q(r-s)}\mathbb{E}\left(\frac{\partial}{\partial x}f(w+X(s), w)\right)\mathrm{d}s$$

$$+ \int_0^{\overline{\xi}(w)} W^{(q)\prime}(\overline{\xi}(w)-z)\mathbb{E}\,(f(z+\xi(w)+X(r), w))\mathrm{d}z$$

$$\left. + W^{(q)}(0+)\,\mathbb{E}\,(f(w+X(r), w)) + \int_0^r \mathbb{E}(f(\xi(w)+X(r-s), w))\ell_s^{(q)\prime}(\overline{\xi}(w))\mathrm{d}s\right]\mathrm{d}w.$$

Proof. For $q, \lambda \geq 0$ and $r > 0$ with $a \geq x$, we have

$$\begin{aligned}
&\int_0^\infty \mathrm{e}^{-q(t-r)}\mathbb{E}_x\big(f(X(t), \bar{X}(t)); t < \kappa_r^\xi \wedge \tau_a^+\big)\,\mathrm{d}t \\
&= \mathbb{E}_x\left(\int_0^{\kappa_r^\xi \wedge \tau_a^+} \mathrm{e}^{-q(t-r)}f(X(t), \bar{X}(t))\,\mathrm{d}\left(\int_0^t \mathbf{1}_{\{X(s)=\bar{X}(s)\}}\mathrm{d}s\right)\right) \\
&\quad + \frac{1}{q}\,\mathbb{E}_x\Big(\mathrm{e}^{qr}f(X(e_q), \bar{X}(e_q))\,\mathbf{1}_{\{X(e_q)<\bar{X}(e_q),\ e_q<\kappa_r^\xi\wedge\tau_a^+\}}\Big) \\
&:= I_1(x) + I_2(x).
\end{aligned} \tag{31}$$

Note that

$$\int_0^t \mathbf{1}_{\{X(s)=\bar{X}(s)\}}\mathrm{d}s = W^{(q)}(0+)\,\bar{X}(t)$$

and that $X(t) = \bar{X}(t)$ implies $t = L^{-1}(L(t))$ almost surely. The function $I_1(x)$ can be rewritten as

$$\begin{aligned}
&W^{(q)}(0+)\,\mathbb{E}_x\bigg(\int_0^\infty \mathrm{e}^{-q(L^{-1}(L(t))-r)}f(x+L(t), x+L(t)) \\
&\quad \times \mathbf{1}_{\{x+L(t)\leq a,\ L^{-1}(L(t))<\kappa_r^\xi\}}\mathrm{d}L(t)\bigg) \\
&= W^{(q)}(0+)\,\mathbb{E}_x\left(\int_0^{a-x} \mathrm{e}^{-q(L^{-1}(w)-r)}f(x+w, x+w)\mathbf{1}_{\{L^{-1}(w)<\kappa_r^\xi\}}\mathrm{d}w\right)
\end{aligned}$$

https://doi.org/10.1017/apr.2020.36 Published online by Cambridge University Press

$$= W^{(q)}(0+)\,\mathrm{e}^{qr}\int_0^{a-x}\mathbb{E}_x\Big(\mathrm{e}^{-qL^{-1}(w)}\mathbf{1}_{\{L^{-1}(w)<\kappa_r^{\xi}\}}\Big)f(x+w,\,x+w)\mathrm{d}w$$

$$= W^{(q)}(0+)\,\mathrm{e}^{qr}\int_x^{a}\mathrm{e}^{-\int_x^w\frac{\ell_r^{(q)\prime}(\overline{\xi}(z))}{\ell_r^{(q)}(\overline{\xi}(z))}\,\mathrm{d}z}f(\overline{\xi}(w)+\xi(w),\,\overline{\xi}(w)+\xi(w))\mathrm{d}w,\tag{32}$$

where (4) is used in the last equality. Using the compensation formula, the function $I_2(x)$ can be rewritten as

$$\frac{1}{q}\mathbb{E}_x\Bigg(\sum_g \mathrm{e}^{qr}\prod_{h<g}\mathbf{1}_{\{\alpha^+_{\overline{\xi}(x+L(h))}(\epsilon_h)=\zeta_h,\,x+L(g)\le a\}}$$

$$\times f(x+L(g)-\epsilon_g\left(e_q-g\right),\,x+L(g))\mathbf{1}_{\{\alpha^+_{\overline{\xi}(x+L(g))}(\epsilon_g)>e_q-g-r,\,0<e_q-g<\zeta_g\}}\Bigg)$$

$$=\frac{1}{q}\mathbb{E}_x\Bigg(\sum_g \mathrm{e}^{-q(g-r)}\prod_{h<g}\mathbf{1}_{\{\alpha^+_{\overline{\xi}(x+L(h))}(\epsilon_h)=\zeta_h,\,x+L(g)\le a\}}$$

$$\times f(x+L(g)-\epsilon_g(e_q),\,x+L(g))\mathbf{1}_{\{\alpha^+_{\overline{\xi}(x+L(g))}(\epsilon_g)>e_q-r,\,e_q<\zeta_g\}}\Bigg)$$

$$=\frac{1}{q}\mathbb{E}_x\Bigg(\int_0^\infty \mathrm{e}^{-q(w-r)}\prod_{h<w}\mathbf{1}_{\{\alpha^+_{\overline{\xi}(x+L(h))}(\epsilon_h)=\zeta_h,\,x+L(w)\le a\}}$$

$$\times\int_{\mathcal{E}}f(x+L(w)-\varepsilon\left(e_q\right),\,x+L(w))\mathbf{1}_{\{\alpha^+_{\overline{\xi}(x+L(w))}(\varepsilon)>e_q-r,\,e_q<\zeta\}}n(\,\mathrm{d}\varepsilon)\,\mathrm{d}L(w)\Bigg)$$

$$=\frac{1}{q}\mathbb{E}_x\Bigg(\int_x^a \mathrm{e}^{-q(\tau_w^+-r)}\mathbf{1}_{\{\tau_w^+<\kappa_r^\xi\}}\int_{\mathcal{E}}f(w-\varepsilon\left(e_q\right),\,w)\mathbf{1}_{\{\alpha^+_{\overline{\xi}(w)}(\varepsilon)>e_q-r,\,e_q<\zeta\}}n(\,\mathrm{d}\varepsilon)\,\mathrm{d}w\Bigg)$$

$$=\int_x^a\exp\left(-\int_x^w\frac{\ell_r^{(q)\prime}(\overline{\xi}(z))}{\ell_r^{(q)}(\overline{\xi}(z))}\,\mathrm{d}z\right)$$

$$\times\; n\bigg(\int_0^\zeta \mathrm{e}^{-q(t-r)}f(\overline{\xi}(w)-\varepsilon(t)+\xi(w),\,\overline{\xi}(w)+\xi(w))\mathbf{1}_{\{\alpha^+_{\overline{\xi}(w)}(\varepsilon)>t-r\}}\mathrm{d}t\bigg)\,\mathrm{d}w.\tag{33}$$

Combining (31), (32), (33), and (22) leads to the desired result. □

We have the following version of Theorem 4.3 when f is independent of y.

Corollary 4.2. *For any $q,\lambda\ge 0$, $r>0$, $a\ge x$ and any bounded differentiable function f, we have*

$$\int_0^\infty \mathrm{e}^{-q(t-r)}\mathbb{E}_x\big(f(X(t));\,t<\kappa_r^\xi\wedge\tau_a^+\big)\,\mathrm{d}t$$

$$=\int_x^a\exp\left(-\int_x^w\frac{\ell_r^{(q)\prime}(\overline{\xi}(z))}{\ell_r^{(q)}(\overline{\xi}(z))}\,\mathrm{d}z\right)\left[\frac{\ell_r^{(q)\prime}(\overline{\xi}(w))}{\ell_r^{(q)}(\overline{\xi}(w))}\left(\int_0^r \mathrm{e}^{q(r-s)}\mathbb{E}\left(f(w+X(s))\right)\mathrm{d}s\right.\right.$$

$$\left.-\int_0^{\overline{\xi}(w)}W^{(q)}(\overline{\xi}(w)-z)\mathbb{E}\left(f(z+\xi(w)+X(r))\right)\mathrm{d}z\right.$$

https://doi.org/10.1017/apr.2020.36 Published online by Cambridge University Press

$$- \int_0^r \mathbb{E}\left(f(\xi(w)+X(r-s))\right) \ell_s^{(q)}(\overline{\xi}(w))\mathrm{d}s\Big)$$

$$- \int_0^r \mathrm{e}^{q(r-s)}\mathbb{E}\left(f'(w+X(s))\right)\mathrm{d}s + \int_0^{\overline{\xi}(w)} W^{(q)\prime}(\overline{\xi}(w)-z)\mathbb{E}\left(f(z+\xi(w)+X(r))\right)\mathrm{d}z$$

$$+ W^{(q)}(0+)\,\mathbb{E}\left(f(w+X(r))\right) + \int_0^r \mathbb{E}\left(f(\xi(w)+X(r-s))\right) \ell_s^{(q)\prime}(\overline{\xi}(w))\mathrm{d}s\Big)\Big]\,\mathrm{d}w.$$

The following result gives the Laplace transform of the potential measure of X killed upon up-crossing $a\,(\geq x)$ or draw-down Parisian ruin.

Corollary 4.3. *For any $q, \lambda \geq 0$, $r > 0$, and $a \geq x$, we have*

$$\mathbb{E}_x\left(\int_0^{\kappa_r^{\xi}\wedge\tau_a^+} \mathrm{e}^{-q(t-r)}\mathrm{e}^{\lambda X(t)-\psi(\lambda)r}\mathrm{d}t\right)$$

$$= \int_x^a \mathrm{e}^{\lambda\xi(w)} \exp\left(-\int_x^w \frac{\ell_r^{(q)\prime}(\overline{\xi}(z))}{\ell_r^{(q)}(\overline{\xi}(z))}\,\mathrm{d}z\right)\left[\frac{\ell_r^{(q)\prime}(\overline{\xi}(w))}{\ell_r^{(q)}(\overline{\xi}(w))}\left(\frac{\mathrm{e}^{\lambda\overline{\xi}(w)}\left(1-\mathrm{e}^{-(\psi(\lambda)-q)r}\right)}{\psi(\lambda)-q}\right.\right.$$

$$\left. - \mathrm{e}^{\lambda\overline{\xi}(w)} \int_0^{\overline{\xi}(w)} W^{(q)}(z)\mathrm{e}^{-\lambda z}\mathrm{d}z - \int_0^r \mathrm{e}^{-\psi(\lambda)s}\,\ell_s^{(q)}(\overline{\xi}(w))\mathrm{d}s\right)$$

$$- \frac{\lambda\mathrm{e}^{\lambda\overline{\xi}(w)}\left(1-\mathrm{e}^{-(\psi(\lambda)-q)r}\right)}{\psi(\lambda)-q} + \lambda\mathrm{e}^{\lambda\overline{\xi}(w)} \int_0^{\overline{\xi}(w)} W^{(q)}(z)\mathrm{e}^{-\lambda z}\mathrm{d}z$$

$$\left. + W^{(q)}(\overline{\xi}(w)) + \int_0^r \mathrm{e}^{-\psi(\lambda)s}\,\ell_s^{(q)\prime}(\overline{\xi}(w))\mathrm{d}s\right]\mathrm{d}w.$$

Proof. Letting $f(x) := \mathrm{e}^{\lambda x-\psi(\lambda)r}$ in Theorem 4.3, or using the compensation formula together with Remark 3.3, one can get the desired result. □

Recall the definition of $V_k^{\xi}(x; b)$ at the end of Section 2. The following result generalizes (20) in [10] and Propositions 1 and 2 in [25].

Theorem 4.4. *For any $q \geq 0$ and $k \geq 1$, we have*

$$V_k^{\xi}(x; b) = \int_b^{\infty} kV_{k-1}(z) \exp\left(-\int_x^z \frac{\ell_r^{(kq)\prime}(\overline{\xi}(w))}{\ell_r^{(kq)}(\overline{\xi}(w))}\mathrm{d}w\right)\mathrm{d}z, \quad x \in (-\infty, b],$$

where

$$V_k(x) = \int_x^{\infty} kV_{k-1}(z) \exp\left(-\int_x^z \frac{\ell_r^{(kq)\prime}(\overline{\xi}(w))}{\ell_r^{(kq)}(\overline{\xi}(w))}\mathrm{d}w\right)\mathrm{d}z, \qquad x \in (-\infty, \infty),$$

with $V_0(x) \equiv 1$.

Proof. For $\epsilon > 0$ and any integer $n \geq 1$, we have

$$\mathbb{E}_b\left(\left(\int_0^{\tau_{b+\epsilon}^+} \mathrm{e}^{-qs}D(s)\mathrm{d}s\right)^n \mathbf{1}_{\{\tau_{b+\epsilon}^+<\kappa_r^{\xi}\}}\right) = o(\epsilon) \tag{34}$$

https://doi.org/10.1017/apr.2020.36 Published online by Cambridge University Press

and

$$\mathbb{E}_b\left(\left(\left(\int_0^{\kappa_r^\xi} \mathrm{e}^{-qs}\mathrm{d}D\,(s)\right)^n \mathbf{1}_{\{\kappa_r^\xi<\tau_{b+\epsilon}^+\}}\right)=o(\epsilon). \tag{35}$$

Actually, $X\left(\tau_{b+\epsilon}^+\right)=b+\epsilon$ implies $D(s)\le\epsilon$ for all $s\in[0,\,\tau_{b+\epsilon}^+]$. Hence, the left-hand side of (34) is less than

$$\begin{aligned}&\epsilon^n\mathbb{E}_b\left[\left(\int_0^{\tau_{b+\epsilon}^+}\mathrm{e}^{-qs}\mathrm{d}s\right)^n\mathbf{1}_{\left\{\tau_{b+\epsilon}^+<\kappa_r^\xi\right\}}\right]\\ &\le\frac{\epsilon^n}{q^n}\left(\mathbb{E}_b\left[\mathbf{1}_{\left\{\tau_{b+\epsilon}^+<\kappa_r^\xi\right\}}\right]-\mathbb{E}_b\left[\mathrm{e}^{-q\tau_{b+\epsilon}^+}\mathbf{1}_{\left\{\tau_{b+\epsilon}^+<\kappa_r^\xi\right\}}\right]\right)\\ &=\frac{\epsilon^n}{q^n}\left(\exp\left(-\int_b^{b+\epsilon}\frac{\ell_r'\left(\bar\xi\,(w)\right)}{\ell_r\left(\bar\xi\,(w)\right)}\mathrm{d}w\right)-\exp\left(-\int_b^{b+\epsilon}\frac{\ell_r^{(q)\prime}\left(\bar\xi\,(w)\right)}{\ell_r^{(q)}\left(\bar\xi\,(w)\right)}\mathrm{d}w\right)\right)=o(\epsilon).\end{aligned}$$

which gives (34). By integration by parts, the left-hand side of (35) can be rewritten as

$$\begin{aligned}&\mathbb{E}_b\left[\left(\mathrm{e}^{-q\kappa_r^\xi}D(\kappa_r^\xi)+q\int_0^{\kappa_r^\xi}\mathrm{e}^{-qs}D\,(s)\mathrm{d}s\right)^n\mathbf{1}_{\{\kappa_r^\xi<\tau_{b+\epsilon}^+\}}\right]\\ &\le\mathbb{E}_b\left[\left(\epsilon\mathrm{e}^{-q\kappa_r^\xi}+\epsilon\int_0^{\kappa_r^\xi}q\mathrm{e}^{-qs}\mathrm{d}s\right)^n\mathbf{1}_{\{\kappa_r^\xi<\tau_{b+\epsilon}^+\}}\right]\\ &=\epsilon^n\left(1-\exp\left(-\int_b^{b+\epsilon}\frac{\ell_r'\left(\bar\xi\,(w)\right)}{\ell_r\left(\bar\xi\,(w)\right)}\mathrm{d}w\right)\right)=o(\epsilon),\end{aligned}$$

which gives (35). By (35), one has

$$V_k(b)=\mathbb{E}_b\left([D_b]^k\,\mathbf{1}_{\{\tau_{b+\epsilon}^+<\kappa_r^\xi\}}\right)+o(\epsilon).$$

Using the strong Markov property and the binomial theorem, one can rewrite the term $\mathbb{E}_b\left([D_b]^k\,\mathbf{1}_{\{\tau_{b+\epsilon}^+<\kappa_r^\xi\}}\right)$ as

$$\begin{aligned}&\mathbb{E}_b\left[\sum_{i=0}^k C_k^i\left(\int_0^{\tau_{b+\epsilon}^+}\mathrm{e}^{-qs}\mathrm{d}D\,(s)\right)^i\left(\int_{\tau_{b+\epsilon}^+}^{\kappa_r^\xi}\mathrm{e}^{-qs}\mathrm{d}D\,(s)\right)^{k-i}\mathbf{1}_{\{\tau_{b+\epsilon}^+<\kappa_r^\xi\}}\right]\\ &=\mathbb{E}_b\left[\sum_{i=0}^k C_k^i\left(\int_0^{\tau_{b+\epsilon}^+}\mathrm{e}^{-qs}\mathrm{d}D\,(s)\right)^i\mathbf{1}_{\left\{\tau_{b+\epsilon}^+<\kappa_r^\xi\right\}}\left(\mathrm{e}^{-q\tau_{b+\epsilon}^+}\right)^{k-i}V_{k-i}(b+\epsilon)\right]\\ &=\sum_{i=0}^k C_k^iV_{k-i}(b+\epsilon)\mathbb{E}_b\left[\left(\int_0^{\tau_{b+\epsilon}^+}\mathrm{e}^{-qs}\mathrm{d}D\,(s)\right)^i\mathrm{e}^{-(k-i)q\tau_{b+\epsilon}^+}\mathbf{1}_{\{\tau_{b+\epsilon}^+<\kappa_r^\xi\}}\right]\end{aligned}$$

https://doi.org/10.1017/apr.2020.36 Published online by Cambridge University Press

$$
\begin{aligned}
&= \sum_{i=0}^{k} C_k^i V_{k-i}(b+\epsilon)\mathbb{E}_b\Bigg[\mathrm{e}^{-(k-i)q\tau_{b+\epsilon}^+} \sum_{j=0}^{i} C_i^j \epsilon^j \mathrm{e}^{-jq\tau_{b+\epsilon}^+} \\
&\quad \times \left(q \int_0^{\tau_{b+\epsilon}^+} \mathrm{e}^{-qs} D(s)\mathrm{d}s \right)^{i-j} \mathbf{1}_{\{\tau_{b+\epsilon}^+ < \kappa_r^\xi\}} \Bigg] \\
&= \sum_{i=0}^{k} C_k^i V_{k-i}(b+\epsilon) \sum_{j=0}^{i} C_i^j \epsilon^j \mathbb{E}_b\Bigg[\mathrm{e}^{-q(k-i+j)\tau_{b+\epsilon}^+} q^{i-j} \\
&\quad \times \left(\int_0^{\tau_{b+\epsilon}^+} \mathrm{e}^{-qs} D(s)\mathrm{d}s \right)^{i-j} \mathbf{1}_{\{\tau_{b+\epsilon}^+ < \kappa_r^\xi\}} \Bigg],
\end{aligned}
\tag{36}
$$

where the identity

$$
\begin{aligned}
&\mathbb{E}_b\left(\left(\int_{\tau_{b+\epsilon}^+}^{\kappa_r^\xi} \mathrm{e}^{-qs} \mathrm{d}D(s) \right)^i \middle| \mathcal{F}_{\tau_{b+\epsilon}^+} \right) \\
&\quad = \mathbb{E}_b\left(\left(\int_{\tau_{b+\epsilon}^+}^{\kappa_r^\xi} \mathrm{e}^{-qs} \mathrm{d}\left((\bar{X}(s) - (b+\epsilon)) \vee 0 \right) \right)^i \middle| \mathcal{F}_{\tau_{b+\epsilon}^+} \right) \\
&\quad = \mathrm{e}^{-iq\tau_{b+\epsilon}^+} V_i(b+\epsilon), \qquad i \geq 0,
\end{aligned}
$$

is used for the first equality in (36). By (34) and (35), one can keep only those summands with $j = i = 1$ or $j = i = 0$ in (36), and then

$$
\begin{aligned}
V_k(b) &= V_k(b+\epsilon)\, \mathbb{E}_b\left[\mathrm{e}^{-kq\tau_{b+\epsilon}^+} \mathbf{1}_{\left\{\tau_{b+\epsilon}^+ < \kappa_r^\xi\right\}} \right] \\
&\quad + kV_{k-1}(b+\epsilon)\, \mathbb{E}_b\left[\epsilon \mathrm{e}^{-kq\tau_{b+\epsilon}^+} \mathbf{1}_{\left\{\tau_{b+\epsilon}^+ < \kappa_r^\xi\right\}} \right] + o(\epsilon) \\
&= (V_k(b+\epsilon) + k\epsilon V_{k-1}(b+\epsilon)) \exp\left(-\int_b^{b+\epsilon} \frac{\ell_r^{(kq)\prime}(\bar{\xi}(w))}{\ell_r^{(kq)}(\bar{\xi}(w))} \mathrm{d}w \right) + o(\epsilon),
\end{aligned}
$$

which can be rearranged as

$$
\begin{aligned}
0 &= \left(\frac{V_k(b+\epsilon) - V_k(b)}{\epsilon} + kV_{k-1}(b+\epsilon) \right) \exp\left(-\int_b^{b+\epsilon} \frac{\ell_r^{(kq)\prime}(\bar{\xi}(w))}{\ell_r^{(kq)}(\bar{\xi}(w))} \mathrm{d}w \right) \\
&\quad + V_k(b) \frac{-1 + \exp\left(-\int_b^{b+\epsilon} \frac{\ell_r^{(kq)\prime}(\bar{\xi}(w))}{\ell_r^{(kq)}(\bar{\xi}(w))} \mathrm{d}w \right)}{\epsilon} + o(1).
\end{aligned}
\tag{37}
$$

Letting $\epsilon \to 0$ in (37) we get

$$
0 = V_k'(b) + kV_{k-1}(b) - V_k(b) \frac{\ell_r^{(kq)\prime}(\bar{\xi}(b))}{\ell_r^{(kq)}(\bar{\xi}(b))}.
\tag{38}
$$

https://doi.org/10.1017/apr.2020.36 Published online by Cambridge University Press

By the standard method of variation of constants, one can obtain the solution of (38) with boundary conditions $V_k(\infty) = 0$ and $V_0(x) = 1$ as

$$V_k(b) = \int_b^\infty kV_{k-1}(z) \exp\left(-\int_b^z \frac{\ell_r^{(kq)\prime}\left(\bar{\xi}(w)\right)}{\ell_r^{(kq)}\left(\bar{\xi}(w)\right)} \mathrm{d}w\right) \mathrm{d}z.$$

For $x \in (-\infty, b]$ and $k \geq 1$, we have

$$V_k^{\xi}(x; b) = \mathbb{E}_x\left[\left(\int_{\tau_b^+}^{\kappa_r^{\xi}} \mathrm{e}^{-qs}\, \mathrm{d}\left(\bar{X}(s) - b\right)\right)^k \mathbf{1}_{\{\tau_b^+ < \kappa_r^{\xi}\}}\right]$$

$$= \mathbb{E}_x\left(\mathrm{e}^{-kq\tau_b^+} \mathbf{1}_{\{\tau_b^+ < \kappa_r^{\xi}\}}\right) V_k(b)$$

$$= \exp\left(-\int_x^b \frac{\ell_r^{(kq)\prime}\left(\bar{\xi}(w)\right)}{\ell_r^{(kq)}\left(\bar{\xi}(w)\right)} \mathrm{d}w\right) \int_b^\infty kV_{k-1}(z) \exp\left(-\int_b^z \frac{\ell_r^{(kq)\prime}\left(\bar{\xi}(w)\right)}{\ell_r^{(kq)}\left(\bar{\xi}(w)\right)} \mathrm{d}w\right) \mathrm{d}z.$$

The proof is complete. □

Remark 4.3. In the above proof, we borrow a binomial argument from [25]. But our arguments related to the Parisian draw-down time are more involved because we need to keep track of the running supremum process of X. In addition, we employ a differential equation argument.

Remark 4.4. For $k = 1$ we present an alternative, more transparent argument. Given $a > b$, by (26) we have

$$\begin{aligned}
V_1(b) &= \mathbb{E}_b\left(\int_0^\infty \mathbf{1}_{\{L(t) < L(\tau_a^+ \wedge \kappa_r^{\xi})\}} \mathrm{e}^{-qt}\, \mathrm{d}L(t)\right) \\
&= \int_0^\infty \mathbb{E}_b\left(\mathrm{e}^{-qL^{-1}(y)} \mathbf{1}_{\{y < L(\tau_a^+) = a-b,\, y < L(\kappa_r^{\xi})\}}\right) \mathrm{d}y \\
&= \int_0^{a-b} \mathbb{E}_b\left(\mathrm{e}^{-qL^{-1}(y)} \mathbf{1}_{\{L^{-1}(y) < \kappa_r^{\xi}\}}\right) \mathrm{d}y \\
&= \int_b^a \mathbb{E}_b\left(\mathrm{e}^{-q\tau_z^+} \mathbf{1}_{\{\tau_z^+ < \kappa_r^{\xi}\}}\right) \mathrm{d}z \\
&= \int_b^a \exp\left(-\int_b^z \frac{\ell_r^{(q)\prime}(\bar{\xi}(w))}{\ell_r^{(q)}(\bar{\xi}(w))} \mathrm{d}w\right) \mathrm{d}z.
\end{aligned} \tag{39}$$

5. Application of draw-down Parisian ruin results to a spectrally negative Lévy process reflected at its past supremum

Recall the dividend process D defined in (5). Let the corresponding risk process with dividends deducted according to the barrier strategy with barrier level b be defined as

$$Y(t) := X(t) - D(t), \qquad t \geq 0.$$

For fixed $b \in (0, \infty)$, if we choose the general draw-down function ξ such that

$$\xi(z) := \xi_b(z) = (z - b) \vee 0, \qquad z \in (-\infty, \infty),$$

https://doi.org/10.1017/apr.2020.36 Published online by Cambridge University Press

then we have

$$\kappa_r^{\xi} := \inf\{t > r : t - g_t^{\xi} > r\}, \quad \text{where} \quad g_t^{\xi} := \sup\{0 \le s \le t : Y(s) \ge 0\};$$

i.e., $\kappa_r^{\xi} = \kappa_r^{\xi_b}$ degenerates to the classical Parisian ruin time for the risk process Y. In addition, by the definition of τ_{ξ} we have

$$\tau_{\xi_b} := \inf\{t \ge 0 : Y(t) \le 0\},$$

which is the ruin time for the risk process Y.

The following result gives the potential measure of Y upon the up-crossing time of level a or the Parisian ruin time of Y.

Corollary 5.1. *For* $b \in (0, \infty)$, $\xi = \xi_b$, $q, \lambda \ge 0$, $r > 0$, $a \ge x$, *and bounded differentiable function* f, *we have*

$$\begin{aligned}
&\int_0^\infty \mathrm{e}^{-q(t-r)} \mathbb{E}_x\big(f(Y(t)); t < \kappa_r^{\xi_b} \wedge \tau_a^+\big)\, \mathrm{d}t \\
&= \Bigg[-\frac{\ell_r^{(q)}(x)}{\ell_r^{(q)}(w)} \bigg(\int_0^r \mathrm{e}^{q(r-s)} \mathbb{E}\,(f(w + X(s)))\mathrm{d}s - \int_0^w W^{(q)}(w-z) \mathbb{E}\,(f(z + X(r)))\mathrm{d}z \\
&\quad - \int_0^r \mathbb{E}\,(f(X(r-s)))\, \ell_s^{(q)}(w)\mathrm{d}s \bigg) \Bigg] \Bigg|_{w=x}^{w=a\wedge b} \\
&\quad + \frac{\ell_r^{(q)}(x)}{\ell_r^{(q)\prime}(b)} \left(1 - \mathrm{e}^{-\frac{\ell_r^{(q)\prime}(b)}{\ell_r^{(q)}(b)}(a-b)} \right) \Bigg[\frac{\ell_r^{(q)\prime}(b)}{\ell_r^{(q)}(b)} \bigg(\int_0^r \mathrm{e}^{q(r-s)} \mathbb{E}\,(f(b + X(s)))\mathrm{d}s \\
&\quad - \int_0^b W^{(q)}(b-z) \mathbb{E}\,(f(z + X(r)))\mathrm{d}z - \int_0^r \mathbb{E}\,(f(X(r-s)))\, \ell_s^{(q)}(b)\mathrm{d}s \bigg) \\
&\quad - \int_0^r \mathrm{e}^{q(r-s)} \mathbb{E}\,\big(f'(b + X(s))\big)\mathrm{d}s + \int_0^b W^{(q)\prime}(b-z) \mathbb{E}\,(f(z + X(r)))\mathrm{d}z \\
&\quad + W^{(q)}(0+)\, \mathbb{E}\,(f(b + X(r))) + \int_0^r \mathbb{E}\,(f(X(r-s)))\, \ell_s^{(q)\prime}(b)\mathrm{d}s \Bigg] \mathbf{1}_{(b,\infty)}(a).
\end{aligned}$$

Proof. Replacing ξ and $f(x,y)$ respectively with ξ_b and $f(x - (y-b) \vee 0)$ in Theorem 4.3 yields

$$\begin{aligned}
&\int_0^\infty \mathrm{e}^{-q(t-r)} \mathbb{E}_x\big(f(Y(t)); t < \kappa_r^{\xi_b} \wedge \tau_a^+\big)\, \mathrm{d}t \\
&= \int_x^a \exp\left(-\int_x^w \frac{\ell_r^{(q)\prime}(z\wedge b)}{\ell_r^{(q)}(z\wedge b)}\, \mathrm{d}z \right) \Bigg[\frac{\ell_r^{(q)\prime}(w\wedge b)}{\ell_r^{(q)}(w\wedge b)} \bigg(\int_0^r \mathrm{e}^{q(r-s)} \mathbb{E}\,(f(w\wedge b + X(s)))\mathrm{d}s \\
&\quad - \int_0^{w\wedge b} W^{(q)}(w\wedge b - z) \mathbb{E}\,(f(z + X(r)))\mathrm{d}z \\
&\quad - \int_0^r \mathbb{E}\,(f(X(r-s)))\, \ell_s^{(q)}(w\wedge b)\mathrm{d}s \bigg)
\end{aligned}$$

https://doi.org/10.1017/apr.2020.36 Published online by Cambridge University Press

$$-\int_0^r \mathrm{e}^{q(r-s)}\mathbb{E}\left(f'(w\wedge b+X(s))\right)\mathrm{d}s+\int_0^{w\wedge b} W^{(q)\prime}(w\wedge b-z)\mathbb{E}\left(f(z+X(r))\right)\mathrm{d}z$$
$$\left.+W^{(q)}(0+)\mathbb{E}\left(f(w\wedge b+X(r))\right)+\int_0^r \mathbb{E}\left(f(X(r-s))\right)\ell_s^{(q)\prime}(w\wedge b)\mathrm{d}s\right]\mathrm{d}w$$
$$=\int_x^a \frac{\partial}{\partial w}\left[\frac{-\ell_r^{(q)}(x)}{\ell_r^{(q)}(w)}\left(\int_0^r \mathrm{e}^{q(r-s)}\mathbb{E}\left(f(w+X(s))\right)\mathrm{d}s-\int_0^w W^{(q)}(w-z)\mathbb{E}\left(f(z+X(r))\right)\mathrm{d}z\right.\right.$$
$$\left.\left.-\int_0^r \mathbb{E}\left(f(X(r-s))\right)\ell_s^{(q)}(w)\mathrm{d}s\right)\right]\mathrm{d}w\,\mathbf{1}_{[x,b]}(a)$$
$$+\int_x^b \frac{\partial}{\partial w}\left[\frac{-\ell_r^{(q)}(x)}{\ell_r^{(q)}(w)}\left(\int_0^r \mathrm{e}^{q(r-s)}\mathbb{E}\left(f(w+X(s))\right)\mathrm{d}s-\int_0^w W^{(q)}(w-z)\mathbb{E}\left(f(z+X(r))\right)\mathrm{d}z\right.\right.$$
$$\left.\left.-\int_0^r \mathbb{E}\left(f(X(r-s))\right)\ell_s^{(q)}(w)\mathrm{d}s\right)\right]\mathrm{d}w\,\mathbf{1}_{(b,\infty)}(a)$$
$$+\frac{\ell_r^{(q)}(x)}{\ell_r^{(q)}(b)}\int_b^a \mathrm{e}^{-\frac{\ell_r^{(q)\prime}(b)}{\ell_r^{(q)}(b)}(w-b)}\mathrm{d}w\left[\frac{\ell_r^{(q)\prime}(b)}{\ell_r^{(q)}(b)}\left(\int_0^r \mathrm{e}^{q(r-s)}\mathbb{E}\left(f(b+X(s))\right)\mathrm{d}s\right.\right.$$
$$\left.-\int_0^b W^{(q)}(b-z)\mathbb{E}\left(f(z+X(r))\right)\mathrm{d}z-\int_0^r \mathbb{E}\left(f(X(r-s))\right)\ell_s^{(q)}(b)\mathrm{d}s\right)$$
$$-\int_0^r \mathrm{e}^{q(r-s)}\mathbb{E}\left(f'(b+X(s))\right)\mathrm{d}s+\int_0^b W^{(q)\prime}(b-z)\mathbb{E}\left(f(z+X(r))\right)\mathrm{d}z$$
$$\left.+W^{(q)}(0+)\mathbb{E}\left(f(b+X(r))\right)+\int_0^r \mathbb{E}\left(f(X(r-s))\right)\ell_s^{(q)\prime}(b)\mathrm{d}s\right]\mathbf{1}_{(b,\infty)}(a),$$

which gives the desired result. □

The following result gives the joint Laplace transform involving the Parisian ruin time of Y.

Corollary 5.2. *For $q,\lambda\in[0,\infty)$, $a\in(-\infty,\infty)$, and $x\in(-\infty,a)$, we have*

$$\mathbb{E}_x\left(\mathrm{e}^{-q\left(\kappa_r^{\xi_b}-r\right)}\mathrm{e}^{\lambda Y(\kappa_r^{\xi_b})}\mathbf{1}_{\{\kappa_r^{\xi_b}<\tau_a^+\}}\right)=\mathrm{e}^{\psi(\lambda)r}\left[-\frac{\ell_r^{(q)}(x)}{\ell_r^{(q)}(w)}\left(\mathrm{e}^{\lambda w}-(\psi(\lambda)-q)\right.\right.$$
$$\left.\left.\left.\times\left(\mathrm{e}^{\lambda w}\int_0^w W^{(q)}(z)\mathrm{e}^{-\lambda z}\mathrm{d}z+\int_0^r \mathrm{e}^{-\psi(\lambda)s}\,\ell_s^{(q)}(w)\mathrm{d}s\right)\right)\right]\right|_{w=x}^{w=a\wedge b}$$
$$+\mathrm{e}^{\psi(\lambda)r}\frac{\ell_r^{(q)}(x)}{\ell_r^{(q)\prime}(b)}\left(1-\mathrm{e}^{-\frac{\ell_r^{(q)\prime}(b)}{\ell_r^{(q)}(b)}(a-b)}\right)\mathbf{1}_{(b,\infty)}(a)\left[\frac{\ell_r^{(q)\prime}(b)}{\ell_r^{(q)}(b)}\left(\mathrm{e}^{\lambda b}-(\psi(\lambda)-q)\right.\right.$$
$$\left.\times\left(\mathrm{e}^{\lambda b}\int_0^b W^{(q)}(z)\mathrm{e}^{-\lambda z}\mathrm{d}z+\int_0^r \mathrm{e}^{-\psi(\lambda)s}\,\ell_s^{(q)}(b)\mathrm{d}s\right)\right)$$
$$\left.-\lambda\mathrm{e}^{\lambda b}+(\psi(\lambda)-q)\left(\lambda\mathrm{e}^{\lambda b}\int_0^b W^{(q)}(z)\mathrm{e}^{-\lambda z}\mathrm{d}z+W^{(q)}(b)+\int_0^r \mathrm{e}^{-\psi(\lambda)s}\,\ell_s^{(q)\prime}(b)\mathrm{d}s\right)\right].$$

https://doi.org/10.1017/apr.2020.36 Published online by Cambridge University Press

Proof. Letting $\xi(w)=\xi_b(w)$ and $\varphi(w)=\mathrm{e}^{-\lambda\xi_b(w)+\psi(\lambda)r}$ in Theorem 4.2 and using a similar argument as the proof of Corollary 5.1, one arrives at the desired result. □

The following result, on the kth moment of the discounted total dividends paid according to the barrier strategy with barrier b until the Parisian ruin time for Y, is a direct consequence of Theorem 4.4. In particular, the corresponding result with $k=1$ recovers the identity (20) in [10].

Corollary 5.3. *For $b\in(0,\infty)$, $\xi=\xi_b$, $q\geq 0$, and $k\geq 1$, we have*

$$V_k^{\xi_b}(x;b)=k!\,\frac{\ell_r^{(kq)}(x)}{\ell_r^{(kq)}(b)}\prod_{i=1}^{k}\frac{\ell_r^{(iq)}(b)}{\ell_r^{(iq)\prime}(b)},\qquad x\in(-\infty,b],$$

and

$$V_k(x)=\ell_r^{(kq)}(x)\left(\int_x^b\frac{kV_{k-1}(z)}{\ell_r^{(kq)}(z)}\mathrm{d}z+\frac{k!}{\ell_r^{(kq)\prime}(b)}\prod_{i=1}^{k-1}\frac{\ell_r^{(iq)}(b)}{\ell_r^{(iq)\prime}(b)}\right),\qquad x\in(-\infty,b],$$

with $V_0(x)\equiv 1$. In particular, for $k=1$ we have

$$V_1^{\xi_b}(x;b)=\frac{\ell_r^{(q)}(x)}{\ell_r^{(q)\prime}(b)},\qquad x\in(-\infty,b].$$

Proof. For $x\in(-\infty,b]$, we have

$$\begin{aligned}V_k(x)&=\int_x^b kV_{k-1}(z)\exp\left(-\int_x^z\frac{\ell_r^{(kq)\prime}(w)}{\ell_r^{(kq)}(w)}\mathrm{d}w\right)\mathrm{d}z\\&\quad+\exp\left(-\int_x^b\frac{\ell_r^{(kq)\prime}(w)}{\ell_r^{(kq)}(w)}\mathrm{d}w\right)\int_b^\infty kV_{k-1}(z)\exp\left(-\int_b^z\frac{\ell_r^{(kq)\prime}(b)}{\ell_r^{(kq)}(b)}\mathrm{d}w\right)\mathrm{d}z\\&=\ell_r^{(kq)}(x)\left(\int_x^b\frac{kV_{k-1}(z)}{\ell_r^{(kq)}(z)}\mathrm{d}z+\frac{k!}{\ell_r^{(kq)\prime}(b)}\prod_{i=1}^{k-1}\frac{\ell_r^{(iq)}(b)}{\ell_r^{(iq)\prime}(b)}\right).\end{aligned}$$

The proof is thus complete. □

Let

$$D_{\tau_{\xi_b}}:=\int_0^{\tau_{\xi_b}}\mathrm{e}^{-qt}\mathrm{d}D(t),$$

represent the present value of the accumulated dividends paid until the time of ruin for Y. In addition, for each $k\geq 1$, we introduce the kth moment of $D_{\tau_{\xi_b}}$ as

$$U_k(x;b):=\mathbb{E}_x\left(\left[D_{\tau_{\xi_b}}\right]^k\right).$$

The following result recovers Propositions 1 and 2 in [25].

Corollary 5.4. *For $b\in(0,\infty)$, $q\geq 0$, and $k\geq 1$, we have*

$$U_k(x;b)=k!\,\frac{W^{(kq)}(x)}{W^{(kq)}(b)}\prod_{i=1}^{k}\frac{W^{(iq)}(b)}{W^{(iq)\prime}(b)},\qquad x\in(-\infty,b].$$

https://doi.org/10.1017/apr.2020.36 Published online by Cambridge University Press

In particular, when $k = 1$ we have

$$U_1(x; b)=\frac{W^{(q)}(x)}{W^{(q)\prime}(b)}, \qquad x \in (-\infty, b].$$

Proof. Note that $\tau_{\xi_b} = \lim_{r\to 0} \kappa_r^{\xi_b}$ with probability 1. Note from (26) that

$$\mathbb{E}_x\Big(\mathrm{e}^{-q\tau_b^+}\mathbf{1}_{\{\tau_b^+<\kappa_r^{\xi_b}\}}\Big) = \exp\bigg(-\int_x^b \frac{\ell_r^{(q)\prime}(w)}{\ell_r^{(q)}(w)}\,\mathrm{d}w\bigg) = \frac{\ell_r^{(q)}(x)}{\ell_r^{(q)}(b)}, \qquad x \in (-\infty, b], \tag{40}$$

and

$$\mathbb{E}_x\Big(\mathrm{e}^{-q\tau_a^+}\mathbf{1}_{\{\tau_a^+<\kappa_r^{\xi_b}\}}\Big) = \exp\bigg(-\int_x^b \frac{\ell_r^{(q)\prime}(w)}{\ell_r^{(q)}(w)}\,\mathrm{d}w\bigg)\exp\bigg(-\int_b^a \frac{\ell_r^{(q)\prime}(b)}{\ell_r^{(q)}(b)}\,\mathrm{d}w\bigg)$$
$$= \frac{\ell_r^{(q)}(x)}{\ell_r^{(q)}(b)}\,\mathrm{e}^{-\frac{\ell_r^{(q)\prime}(b)}{\ell_r^{(q)}(b)}(a-b)}, \qquad x \in (-\infty, b],\ a \in (b, \infty). \tag{41}$$

By (40) together with (1), we have

$$\lim_{r\to 0}\frac{\ell_r^{(q)}(x)}{\ell_r^{(q)}(b)} = \lim_{r\to 0}\mathbb{E}_x\Big(\mathrm{e}^{-q\tau_b^+}\mathbf{1}_{\{\tau_b^+<\kappa_r^{\xi_b}\}}\Big)$$
$$=\mathbb{E}_x\Big(\mathrm{e}^{-q\tau_b^+}\mathbf{1}_{\{\tau_b^+<\tau_{\xi_b}\}}\Big) = \mathbb{E}_x\Big(\mathrm{e}^{-q\tau_b^+}\mathbf{1}_{\{\tau_b^+<\tau_0^-\}}\Big) = \frac{W^{(q)}(x)}{W^{(q)}(b)}, \quad x \in (-\infty, b]. \tag{42}$$

By (3) we have

$$\mathbb{E}_x\Big(\mathrm{e}^{-q\tau_a^+}\mathbf{1}_{\{\tau_a^+<\tau_{\xi_b}\}}\Big) = \exp\bigg(-\int_x^a \frac{W^{(q)\prime}(\bar{\xi}_b(w))}{W^{(q)}(\bar{\xi}_b(w))}\,\mathrm{d}w\bigg)$$
$$= \frac{W^{(q)}(x)}{W^{(q)}(b)}\,\mathrm{e}^{-\frac{W^{(q)\prime}(b)}{W^{(q)}(b)}(a-b)}, \quad x \in (-\infty, b],\ a \in (b, \infty). \tag{43}$$

Combining (41), (42), and (43), we arrive at

$$\lim_{r\to 0}\frac{\ell_r^{(q)\prime}(b)}{\ell_r^{(q)}(b)} = \frac{W^{(q)\prime}(b)}{W^{(q)}(b)}. \tag{44}$$

The desired results follow from a combination of (42), (44), and Corollary 5.3. □

If we choose ξ such that $\xi \equiv 0$, then the draw-down Parisian ruin time κ_r^{ξ} of X degenerates to the Parisian ruin time κ_r of X. The following result gives a generalized version of the potential measure for the process X killed upon up-crossing level a ($\geq x$) or the Parisian ruin time of X.

Corollary 5.5. *For $\xi \equiv 0$, $q, \lambda \geq 0$, $r > 0$, $a \geq x$, and bounded bivariate function f(x,y) which is differentiable with respect to x, we have*

$$\int_0^\infty \mathrm{e}^{-q(t-r)}\mathbb{E}_x\big(f(X(t), \bar{X}(t)); t < \kappa_r \wedge \tau_a^+\big)\,\mathrm{d}t$$
$$= \int_x^a \frac{\ell_r^{(q)}(x)}{\ell_r^{(q)}(w)}\Bigg[\frac{\ell_r^{(q)\prime}(w)}{\ell_r^{(q)}(w)}\bigg(\int_0^r \mathrm{e}^{q(r-s)}\mathbb{E}\,(f(w+X(s), w))\mathrm{d}s$$
$$-\int_0^w W^{(q)}(w-z)\mathbb{E}\,(f(z+X(r), w))\mathrm{d}z$$

https://doi.org/10.1017/apr.2020.36 Published online by Cambridge University Press

$$- \int_0^r \mathbb{E}\left(f(X(r-s), w)\right) \ell_s^{(q)}(w)\mathrm{d}s\Bigg)$$

$$- \int_0^r \mathrm{e}^{q(r-s)}\mathbb{E}\left(\frac{\partial}{\partial x}f(w+X(s), w)\right)\mathrm{d}s + \int_0^w W^{(q)\prime}(w-z)\mathbb{E}\left(f(z+X(r), w)\right)\mathrm{d}z$$

$$+ W^{(q)}(0+)\,\mathbb{E}\left(f(w+X(r), w)\right) + \int_0^r \mathbb{E}\left(f(X(r-s), w)\right) \ell_s^{(q)\prime}(w)\mathrm{d}s\Bigg)\Bigg]\,\mathrm{d}w.$$

Proof. This is a direct consequence of Theorem 4.3. □

The following result gives the solution for the kth moment of the accumulated discounted dividend payout until the Parisian ruin time κ_r for X.

Corollary 5.6. *For $\xi \equiv 0$, $q \geq 0$, and $k \geq 1$, we have*

$$V_k^{\xi}(x; b) = \ell_r^{(kq)}(x) \int_b^{\infty} \frac{kV_{k-1}(z)}{\ell_r^{(kq)}(z)}\mathrm{d}z, \qquad x \in (-\infty, b],$$

where

$$V_k(x) = \ell_r^{(kq)}(x) \int_x^{\infty} \frac{kV_{k-1}(z)}{\ell_r^{(kq)}(z)}\mathrm{d}z, \qquad x \in (-\infty, \infty),$$

with $V_0(x) \equiv 1$.

Proof. The desired result is a direct application of Theorem 4.4, letting $\xi \equiv 0$. □

6. Examples

To illustrate how to apply the main results in Section 4 to obtain expressions or numerical values for quantities related to the draw-down Parisian ruin, we compute in this section the functions $\ell_r^{(q)}$, $\chi^{(q)}$, and ϕ_{Φ_q} for two examples of spectrally negative Lévy processes: the drifted Brownian motion and the Cramér–Lundberg risk model with exponential claims.

6.1. Small claims: Brownian motion

If $X(t) = \mu t + \sigma B(t)$ is a Brownian motion with drift $\mu \in (-\infty, \infty)$ and volatility $\sigma \in (0, \infty)$, then

$$W^{(q)}(x) = \frac{1}{\sigma^2\delta_q}\left[\mathrm{e}^{(-w+\delta_q)x} - \mathrm{e}^{-(w+\delta_q)x}\right],$$

with $\delta_q := \sigma^{-2}\sqrt{\mu^2 + 2q\sigma^2}$ and $w := \mu/\sigma^2$. Hence,

$$\mu + (-w+\delta_q)\sigma^2 = -\mu + (w+\delta_q)\sigma^2 = \sqrt{\mu^2 + 2q\sigma^2}$$

and

$$\ell_r^{(q)}(x) = \int_0^{\infty} W^{(q)}(x+z)\frac{z}{r}\mathbb{P}\left(X(r) \in \mathrm{d}z\right)$$

$$= \frac{1}{\sigma^2\delta_q r}\frac{1}{\sqrt{2\pi\sigma^2 r}}\int_0^{\infty}\left[\mathrm{e}^{(-w+\delta_q)(x+z)} - \mathrm{e}^{-(w+\delta_q)(x+z)}\right] z\mathrm{e}^{-\frac{(z-\mu r)^2}{2\sigma^2 r}}\mathrm{d}z$$

$$= \frac{1}{\sigma^2\delta_q r}\frac{1}{\sqrt{2\pi\sigma^2 r}}\left(\mathrm{e}^{(-w+\delta_q)x}\int_0^{\infty} z\mathrm{e}^{-\frac{(z-\mu r-(-w+\delta_q)\sigma^2 r)^2 + \mu^2 r^2 - (\mu + (-w+\delta_q)\sigma^2)^2 r^2}{2\sigma^2 r}}\mathrm{d}z\right.$$

https://doi.org/10.1017/apr.2020.36 Published online by Cambridge University Press

$$- \mathrm{e}^{-(w+\delta_q)x} \int_0^\infty z \mathrm{e}^{-\frac{(z-\mu r+(w+\delta_q)\sigma^2 r)^2+\mu^2 r^2-(\mu-(w+\delta_q)\sigma^2)^2 r^2}{2\sigma^2 r}} \mathrm{d}z \Big)$$

$$= \frac{1}{\sigma^2 \delta_q r} \frac{1}{\sqrt{2\pi\sigma^2 r}} \Big(\mathrm{e}^{(-w+\delta_q)x} \int_0^\infty z \mathrm{e}^{-\frac{(z-\sqrt{\mu^2+2q\sigma^2}r)^2-2q\sigma^2 r^2}{2\sigma^2 r}} \mathrm{d}z$$

$$- \mathrm{e}^{-(w+\delta_q)x} \int_0^\infty z \mathrm{e}^{-\frac{(z+\sqrt{\mu^2+2q\sigma^2}r)^2-2q\sigma^2 r^2}{2\sigma^2 r}} \mathrm{d}z \Big)$$

$$= \frac{\mathrm{e}^{qr}}{\sigma^2 \delta_q r} \bigg[\mathrm{e}^{(-w+\delta_q)x} \bigg(\frac{\sigma\sqrt{r}}{\sqrt{2\pi}} \mathrm{e}^{-\frac{\mu^2+2q\sigma^2}{2\sigma^2} r} + \sqrt{\mu^2+2q\sigma^2} r \mathcal{N}\bigg(\frac{\sqrt{\mu^2+2q\sigma^2}\sqrt{r}}{\sigma} \bigg) \bigg)$$

$$- \mathrm{e}^{-(w+\delta_q)x} \bigg(\frac{\sigma\sqrt{r}}{\sqrt{2\pi}} \mathrm{e}^{-\frac{\mu^2+2q\sigma^2}{2\sigma^2} r} - \sqrt{\mu^2+2q\sigma^2} r \mathcal{N}\bigg(\frac{-\sqrt{\mu^2+2q\sigma^2}\sqrt{r}}{\sigma} \bigg) \bigg) \bigg],$$

where $\mathcal{N}$ is the cumulative distribution function of a standard normal random variable.

Notice that to compute the draw-down Parisian ruin probability (see Remark 4.1), we only need the expression for $\ell_r := \ell_r^{(0)}$. To find the Laplace transform associated to the two-sided exit solution (26), the kth moment of dividends (see Theorem 4.4), or the potential measure of X (see Theorem 4.3) involving the draw-down Parisian ruin time, we also need the expression for $\ell_r^{(q)}$. If we want to compute the probability density of the draw-down Parisian ruin time, we must first use the expression for $\ell_r^{(q)}$ to compute the right-hand side of (30), then numerically invert it using the algorithms in [7] or the method of Fourier series expansion proposed by [23], which has been proven to be an efficient method for the numerical inversion of Laplace transforms. Furthermore, if we want to compute the Laplace transform associated to the generalized two-sided exit solution (see Theorem 4.1), we need to numerically compute $\chi^{(q)}$ and ϕ_{Φ_q} via the algorithms of Laplace inversion in [7] or [23]. To this end, we need to compute the right-hand sides of (23) and (24). In fact, one has $\Phi_q = \delta_q - w$ and

$$\int_y^{x+y} W^{(q)}(x+y-z) W^{(\theta+q)}(z) \mathrm{d}z$$

$$= \frac{1}{\sigma^4 \delta_q \delta_{\theta+q}} \int_y^{x+y} \big[\mathrm{e}^{(-w+\delta_q)(x+y-z)} - \mathrm{e}^{-(w+\delta_q)(x+y-z)}\big]\big[\mathrm{e}^{(-w+\delta_{\theta+q})z} - \mathrm{e}^{-(w+\delta_{\theta+q})z}\big] \mathrm{d}z$$

$$= \frac{\mathrm{e}^{-w(x+y)}}{\sigma^4 \delta_q \delta_{\theta+q}} \bigg(\mathrm{e}^{\delta_q(x+y)} \bigg(\frac{\mathrm{e}^{(\delta_{\theta+q}-\delta_q)(x+y)} - \mathrm{e}^{(\delta_{\theta+q}-\delta_q)y}}{\delta_{\theta+q}-\delta_q} + \frac{\mathrm{e}^{-(\delta_{\theta+q}+\delta_q)(x+y)} - \mathrm{e}^{-(\delta_{\theta+q}+\delta_q)y}}{\delta_{\theta+q}+\delta_q} \bigg)$$

$$+ \mathrm{e}^{-\delta_q(x+y)} \bigg(- \frac{\mathrm{e}^{(\delta_{\theta+q}+\delta_q)(x+y)} - \mathrm{e}^{(\delta_{\theta+q}+\delta_q)y}}{\delta_{\theta+q}+\delta_q} - \frac{\mathrm{e}^{-(\delta_{\theta+q}-\delta_q)(x+y)} - \mathrm{e}^{-(\delta_{\theta+q}-\delta_q)y}}{\delta_{\theta+q}-\delta_q} \bigg) \bigg)$$

$$= \frac{\mathrm{e}^{-w(x+y)}}{\sigma^4 \delta_q \delta_{\theta+q}} \bigg(\frac{\mathrm{e}^{\delta_{\theta+q}(x+y)} - \mathrm{e}^{\delta_{\theta+q}y+\delta_q x}}{\delta_{\theta+q}-\delta_q} + \frac{\mathrm{e}^{-\delta_{\theta+q}(x+y)} - \mathrm{e}^{-\delta_{\theta+q}y+\delta_q x}}{\delta_{\theta+q}+\delta_q}$$

$$- \frac{\mathrm{e}^{\delta_{\theta+q}(x+y)} - \mathrm{e}^{\delta_{\theta+q}y-\delta_q x}}{\delta_{\theta+q}+\delta_q} - \frac{\mathrm{e}^{-\delta_{\theta+q}(x+y)} - \mathrm{e}^{-\delta_{\theta+q}y-\delta_q x}}{\delta_{\theta+q}-\delta_q} \bigg)$$

$$= \frac{\mathrm{e}^{-w(x+y)}}{\sigma^4 \delta_q \delta_{\theta+q}} \bigg(\frac{\sqrt{\mu^2+2q\sigma^2}}{\theta} \Big(\mathrm{e}^{\delta_{\theta+q}(x+y)} - \mathrm{e}^{-\delta_{\theta+q}(x+y)} \Big) - \frac{\mathrm{e}^{\delta_{\theta+q}y+\delta_q x}}{\delta_{\theta+q}-\delta_q} - \frac{\mathrm{e}^{-\delta_{\theta+q}y+\delta_q x}}{\delta_{\theta+q}+\delta_q}$$

$$+ \frac{\mathrm{e}^{\delta_{\theta+q}y-\delta_q x}}{\delta_{\theta+q}+\delta_q} + \frac{\mathrm{e}^{-\delta_{\theta+q}y-\delta_q x}}{\delta_{\theta+q}-\delta_q} \bigg),$$

https://doi.org/10.1017/apr.2020.36 Published online by Cambridge University Press

which implies

$$\begin{aligned}
&W_y^{(\theta+q,-\theta)}(x+y)\\
&= W^{(\theta+q)}(x+y) - \theta \int_y^{x+y} W^{(q)}(x+y-w)W^{(\theta+q)}(w)\mathrm{d}w\\
&= \frac{1}{\sigma^2\delta_{\theta+q}}\Big(\mathrm{e}^{(-w+\delta_{\theta+q})(x+y)} - \mathrm{e}^{-(w+\delta_{\theta+q})(x+y)}\Big) - \frac{\theta\,\mathrm{e}^{-w(x+y)}}{\sigma^4\delta_q\delta_{\theta+q}}\Big(\frac{\mathrm{e}^{\delta_{\theta+q}y-\delta_q x}}{\delta_{\theta+q}+\delta_q} + \frac{\mathrm{e}^{-\delta_{\theta+q}y-\delta_q x}}{\delta_{\theta+q}-\delta_q}\\
&\quad + \frac{\sqrt{\mu^2+2q\sigma^2}}{\theta}\Big(\mathrm{e}^{\delta_{\theta+q}(x+y)} - \mathrm{e}^{-\delta_{\theta+q}(x+y)}\Big) - \frac{\mathrm{e}^{\delta_{\theta+q}y+\delta_q x}}{\delta_{\theta+q}-\delta_q} - \frac{\mathrm{e}^{-\delta_{\theta+q}y+\delta_q x}}{\delta_{\theta+q}+\delta_q}\Big).
\end{aligned}$$

One also finds that

$$\begin{aligned}
1+\theta\int_0^y \mathrm{e}^{-\Phi_q z}W^{(\theta+q)}(z)\mathrm{d}z &= 1 + \frac{\theta}{\sigma^2\delta_{\theta+q}}\int_0^y \mathrm{e}^{(w-\delta_q)z}\big[\mathrm{e}^{(-w+\delta_{\theta+q})z} - \mathrm{e}^{-(w+\delta_{\theta+q})z}\big]\mathrm{d}z\\
&= 1 + \frac{\theta}{\sigma^2\delta_{\theta+q}}\Big(\frac{\mathrm{e}^{(\delta_{\theta+q}-\delta_q)y}-1}{\delta_{\theta+q}-\delta_q} + \frac{\mathrm{e}^{-(\delta_{\theta+q}+\delta_q)y}-1}{\delta_{\theta+q}+\delta_q}\Big)\\
&= \frac{\theta}{\sigma^2\delta_{\theta+q}}\Big(\frac{\mathrm{e}^{(\delta_{\theta+q}-\delta_q)y}}{\delta_{\theta+q}-\delta_q} + \frac{\mathrm{e}^{-(\delta_{\theta+q}+\delta_q)y}}{\delta_{\theta+q}+\delta_q}\Big),
\end{aligned}$$

which implies

$$\begin{aligned}
&W^{(q)}(x)\mathrm{e}^{\Phi_q y}\Big(1+\theta\int_0^y \mathrm{e}^{-\Phi_q w}W^{(\theta+q)}(w)\mathrm{d}w\Big)\\
&= \frac{\theta}{\sigma^4\delta_q\delta_{\theta+q}}\big[\mathrm{e}^{(-w+\delta_q)x} - \mathrm{e}^{-(w+\delta_q)x}\big]\Big(\frac{\mathrm{e}^{(\delta_{\theta+q}-w)y}}{\delta_{\theta+q}-\delta_q} + \frac{\mathrm{e}^{-(\delta_{\theta+q}+w)y}}{\delta_{\theta+q}+\delta_q}\Big)\\
&= \frac{\theta}{\sigma^4\delta_q\delta_{\theta+q}}\mathrm{e}^{-w(x+y)}\big[\mathrm{e}^{\delta_q x} - \mathrm{e}^{-\delta_q x}\big]\Big(\frac{\mathrm{e}^{\delta_{\theta+q}y}}{\delta_{\theta+q}-\delta_q} + \frac{\mathrm{e}^{-\delta_{\theta+q}y}}{\delta_{\theta+q}+\delta_q}\Big)\\
&= \frac{\theta\,\mathrm{e}^{-w(x+y)}}{\sigma^4\delta_q\delta_{\theta+q}}\Big(\frac{\mathrm{e}^{\delta_{\theta+q}y+\delta_q x}}{\delta_{\theta+q}-\delta_q} - \frac{\mathrm{e}^{\delta_{\theta+q}y-\delta_q x}}{\delta_{\theta+q}-\delta_q} + \frac{\mathrm{e}^{-\delta_{\theta+q}y+\delta_q x}}{\delta_{\theta+q}+\delta_q} - \frac{\mathrm{e}^{-\delta_{\theta+q}y-\delta_q x}}{\delta_{\theta+q}+\delta_q}\Big).
\end{aligned}$$

Therefore, the right-hand side of (23) becomes

$$\begin{aligned}
&\frac{W^{(\theta+q)}(x+y) - \dfrac{\mathrm{e}^{-w(x+y)}}{\sigma^2\delta_{\theta+q}}\big(\mathrm{e}^{\delta_{\theta+q}(x+y)} - \mathrm{e}^{-\delta_{\theta+q}(x+y)} - \mathrm{e}^{\delta_{\theta+q}y-\delta_q x} + \mathrm{e}^{-\delta_{\theta+q}y-\delta_q x}\big)}{\theta W^{(\theta+q)}(y)}\\
&= \frac{\dfrac{\mathrm{e}^{-(w+\delta_q)x}}{\sigma^2\delta_{\theta+q}}\big(\mathrm{e}^{\delta_{\theta+q}y-wy} - \mathrm{e}^{-\delta_{\theta+q}y-wy}\big)}{\theta\dfrac{1}{\sigma^2\delta_{\theta+q}}\big(\mathrm{e}^{(-w+\delta_{\theta+q})y} - \mathrm{e}^{-(w+\delta_{\theta+q})y}\big)} = \frac{1}{\theta}\mathrm{e}^{-(w+\delta_q)x},
\end{aligned}$$

which together with (23) yields

$$\chi^{(q)}(x, y, r) = \mathrm{e}^{-(w+\delta_q)x}, \qquad x\in(-\infty,\infty),\quad y\in(0,\infty),\quad r\in(0,\infty).$$

In addition, the right-hand side of (24) can be re-expressed as

$$\frac{\dfrac{1}{\delta_{\theta+q}-\delta_q}\mathrm{e}^{(\delta_{\theta+q}-w)y}+\dfrac{1}{\delta_{\theta+q}+\delta_q}\mathrm{e}^{-(\delta_{\theta+q}+w)y}}{\mathrm{e}^{(-w+\delta_{\theta+q})y}-\mathrm{e}^{-(w+\delta_{\theta+q})y}},$$

which can be numerically inverted (to get ϕ_{Φ_q}) using the algorithms in [7] or the method of Fourier series expansion in [23].

6.2. Big claims: the Cramér–Lundberg risk model

Let $X(t)=ct-\sum_{n=1}^{N(t)}C_n$ be a linear drift with $c>0$ minus a compound Poisson process with jump intensity η and independent, identically exponentially distributed jump sizes $(C_n)_{n\geq 1}$ with mean $1/\alpha$. Then

$$W^{(q)}(x)=c^{-1}\big(A^{+}\mathrm{e}^{q^{+}(q)x}-A^{-}\mathrm{e}^{q^{-}(q)x}\big),$$

with

$$A_{\pm}:=\frac{\alpha+q^{\pm}(q)}{q^{+}(q)-q^{-}(q)}\quad\text{and}\quad q^{\pm}(q):=\frac{q+\eta-\alpha c\pm\sqrt{(q+\eta-\alpha c)^2+4cq\alpha}}{2c}.$$

One can verify that

$$\begin{aligned}\ell_r^{(q)}(x)&=\mathrm{e}^{-\eta r}\Big(A_{+}\mathrm{e}^{q^{+}(q)(x+cr)}-A_{-}\mathrm{e}^{q^{-}(q)(x+cr)}\Big)+\frac{A_{+}\mathrm{e}^{q^{+}(q)(x+cr)-\eta r}}{cr}\sum_{m=0}^{\infty}\frac{\Big(\frac{\alpha\eta r}{q^{+}(q)+\alpha}\Big)^{m+1}}{m!(m+1)!}\\&\quad\times\bigg[cr\Gamma\big(m+1,cr(q^{+}(q)+\alpha)\big)-\frac{\Gamma(m+2,cr(q^{+}(q)+\alpha))}{q^{+}(q)+\alpha}\bigg]-\frac{A_{-}\mathrm{e}^{q^{-}(q)(x+cr)-\eta r}}{cr}\\&\quad\times\sum_{m=0}^{\infty}\frac{\Big(\frac{\alpha\eta r}{q^{-}(q)+\alpha}\Big)^{m+1}}{m!(m+1)!}\bigg[cr\Gamma\big(m+1,cr(q^{-}(q)+\alpha)\big)-\frac{\Gamma(m+2,cr(q^{-}(q)+\alpha))}{q^{-}(q)+\alpha}\bigg],\end{aligned}$$

where $\Gamma(\beta,x)=\int_0^x\mathrm{e}^{-w}w^{\beta-1}\mathrm{d}w$ is the gamma function. The Laplace transforms of $\chi^{(q)}$ and ϕ_{Φ_q} in r, i.e. the right-hand sides of (23) and (24), can also be derived accordingly. The corresponding computations are omitted because they are fairly lengthy.

Acknowledgements

The authors thank the anonymous referees for very helpful comments and suggestions. W. Wang thanks Concordia University; the work on this paper was completed during his visit there. He acknowledges the support of the National Natural Science Foundation of China (No. 11661074). W. Wang and X. Zhou are supported by a grant from the Natural Sciences and Engineering Research Council of Canada (No. RGPIN-2016-06704). X. Zhou is supported by the National Natural Science Foundation of China (No. 11771018).

References

[1] AVRAM, F., VU, N. AND ZHOU, X. (2017). On taxed spectrally negative Lévy processes with draw-down stopping. *Insurance Math. Econom.* **76**, 69–74.

[2] BAURDOUX, E., PARDO, J., PEREZ, J. AND RENAUD, J. (2016). Gerber–Shiu functionals at Parisian ruin for Lévy insurance risk processes. *J. Appl. Prob.* **53**, 572–584.

[3] BERTOIN, J. (1996). *Lévy Processes*. Cambridge University Press.
[4] CHAN, T., KYPRIANOU, A. AND SAVOV, M. (2011). Smoothness of scale functions for spectrally negative Lévy processes. *Prob. Theory Relat. Fields* **150**, 691–708.
[5] CHESNEY, M., JEANBLANC, M. AND YOR, M. (1997). Brownian excursions and Parisian barrier options. *Adv. Appl. Prob.* **29**, 165–184.
[6] CHEUNG, E. AND WONG, J. (2017). On the dual risk model with Parisian implementation delays in dividend payments. *Europ. J. Operat. Res.* **257**, 159–173.
[7] COHEN, A. (2007). *Numerical Methods for Laplace Transform Inversion*. Springer, New York.
[8] CZARNA, I. (2016). Parisian ruin probability with a lower ultimate bankrupt barrier. *Scand. Actuarial J.* **2016**, 319–337.
[9] CZARNA, I. AND PALMOWSKI, Z. (2011). Ruin probability with Parisian delay for a spectrally negative Lévy risk process. *J. Appl. Prob.* **48**, 984–1002.
[10] CZARNA, I. AND PALMOWSKI, Z. (2014). Dividend problem with Parisian delay for a spectrally negative Lévy risk process. *J. Optimization Theory Appl.* **161**, 239–256.
[11] CZARNA, I. AND RENAUD, J. (2016). A note on Parisian ruin with an ultimate bankruptcy level for Lévy insurance risk processes. *Statis. Prob. Lett.* **113**, 54–61.
[12] DASSIOS, A. AND WU, S. (2009). Parisian ruin with exponential claims. Working paper, London School of Economics. Available at http://stats.lse.ac.uk/angelos.
[13] DASSIOS, A. AND WU, S. (2009). Semi-Markov model for excursions and occupation time of Markov processes. Working paper, London School of Economics. Available at http://stats.lse.ac.uk/angelos.
[14] FROSTIG, E. AND KEREN-PINHASIK, A. (2020). Parisian ruin with Erlang delay and a lower bankruptcy barrier. *Methodology Comput. Appl. Prob.* **22**, 101–134.
[15] KUZNETSOV, A., KYPRIANOU, A. AND RIVERO, V. (2012). The theory of scale functions for spectrally negative Lévy processes. In *Lévy Matters II* (Lecture Notes Math. **2061**), Springer, Heidelberg, pp. 97–186.
[16] KYPRIANOU, A. (2014). *Introductory Lectures on Fluctuations of Lévy Processes with Applications*. Springer, Berlin.
[17] LANDRIAULT, D., RENAUD, J. AND ZHOU, X. (2014). Insurance risk models with Parisian implementation delays. *Methodology Comput. Appl. Prob.* **16**, 583–607.
[18] LEHOCZKY, J. P. (1977). Formulas for stopped diffusion processes with stopping times based on the maximum. *Ann. Prob.* **5**, 601–607.
[19] LI, B., VU, N. AND ZHOU, X. (2019). Exit problems for general draw-down times of spectrally negative Lévy processes. *J. Appl. Prob.* **56**, 441–457.
[20] LKABOUS, M., CZARNA, I. AND RENAUD, J. (2017). Parisian ruin for a refracted Lévy process. *Insurance Math. Econom.* **74**, 153–163.
[21] LOEFFEN, R., CZARNA, I. AND PALMOWSKI, Z. (2013). Parisian ruin probability for spectrally negative Lévy processes. *Bernoulli* **19**, 599–609.
[22] LOEFFEN, R., PALMOWSKI, Z. AND SURYA, B. A. (2018). Discounted penalty function at Parisian ruin for Lévy insurance risk process. *Insurance Math. Econom.* **83**, 190–197.
[23] MOORTHY, M. (1995). Numerical inversion of two-dimensional Laplace transforms—Fourier series representation. *Appl. Numer. Math.* **17**, 119–127.
[24] PISTORIUS, M. (2007). An excursion-theoretical approach to some boundary crossing problems and the Skorokhod embedding for reflected Lévy processes. In *Séminaire de Probabilités XL*, Springer, Berlin, Heidelberg, pp. 287–307.
[25] RENAUD, J. AND ZHOU, X. (2007). Distribution of the present value of dividend payments in a Lévy risk model. *J. Appl. Prob.* **44**, 420–427.
[26] SURYA, B. A. (2019). Parisian excursion below a fixed level from the last record maximum of Lévy insurance risk process. In *2017 MATRIX Annals*, Springer, Cham, pp. 311–326.
[27] WANG, W. AND ZHOU, X. (2020). A draw-down reflected spectrally negative Lévy process. To appear in *J. Theoret. Prob.*
[28] ZHOU, X. (2007). Exit problems for spectrally negative Lévy processes reflected at either the supremum or the infimum. *J. Appl. Prob.* **44**, 1012–1030.