J. Fluid Mech. (2015), *vol.* 771, *pp.* 16–35. © Cambridge University Press 2015
doi:10.1017/jfm.2015.162

The lift on an aerofoil in grid-generated turbulence

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(Received 16 October 2014; revised 20 January 2015; accepted 9 March 2015;
first published online 14 April 2015)

The three-dimensional effects of turbulence cannot be neglected when the spanwise wavelength of the incident turbulence is not effectively infinite with respect to the chord, which may invalidate the strip assumption. Based on three-dimensional theory, a general approach, expressed in terms of a two-dimensional Fourier transform of the correlation of the buffeting force, is proposed to identify the two-wavenumber spectrum and aerodynamic admittance of the lift force on an aerofoil. It is essential that the approach presented can be validated in wind tunnel experiments. The coherence of the lift force on an aerofoil in grid-generated turbulence is obtained by simultaneous measurements of unsteady surface pressures around several chordwise strips on a stiff sectional model, which controls the accuracy of results. For the purpose of the Fourier transform, three empirical coherence models of the lift force are presented to fit the experimental results. Compared with the linearized theory, the two-wavenumber aerodynamic admittance can describe well the pressure distribution and the pattern of energy transition in an isotropic turbulence field. Thus, the failure mechanism of the traditional strip assumption can be demonstrated explicitly. In addition, the results obtained also validate the theory proposed by Graham (*Aeronaut. Q.*, vol. 21, 1970, pp. 182–198; *Aeronaut. Q.*, vol. 22, 1971, pp. 83–100). The present approach can be extended to study the three-dimensionality of the buffeting force on line-like structures with arbitrary cross-configurations, such as long-span bridges and high-rise buildings.

Key words: aerodynamics, flow–structure interactions, isotropic turbulence

1. Introduction

The lift force on an aerofoil for non-uniform motion is one of the classic aerodynamics issues. Sears (1941) proposed a linearized aerodynamic theory to calculate the unsteady aerodynamic lift acting on a thin aerofoil in a one-dimensional sinusoidal gust. Sears' analysis yielded the famous one-dimensional transfer function, the Sears' function, which relates the lift per unit span to the incident downwash amplitude. Liepmann (1952) introduced statistical concepts to formulate and study the buffeting of an aerofoil of infinite span due to turbulent fluctuations. By treating

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the atmospheric turbulence as a statistically stationary process and neglecting the effects of the three-dimensionality in the turbulence, the lift acting on the aerofoil is considered as the response of the linear system to a single random input (wind fluctuations). Thus, the spectrum of the lift coefficient per unit span is given as

$$S_{C_L}(k) = \frac{(2\pi)^3}{U^2} \left| \chi(2\pi kb) \right|^2 S_w(2\pi k), \tag{1.1}$$

where $k = n/U$, n is frequency (unit: Hz), U is mean wind velocity, b is half-cord, $|\chi(2\pi kb)|^2$ is Sears' function, usually called the aerodynamic admittance, and $S_w(2\pi k)$ is the spectrum of the vertical turbulence velocity.

Taking the spanwise variations into account, this statistical approach was extended by Liepmann (1955) and Diederich (1956) to study the three-dimensional problem of a finite aerofoil in homogeneous turbulence by introducing the indicial admittance $h(t, y)$ and two-wavenumber spectrum of turbulence $S_w(k_1, k_2)$. Liepmann believed that the admittance $\chi(k_1, k_2)$, the two-dimensional Fourier integral of indicial admittance $h(t, y)$, had to be derived from thin-wing theory and represented essentially a generalization of Sears' results to wings of finite span. In practical gust-loading problems, a simplified assumption was often made that the spatial description of the gusts can be taken as representative of the spatial distribution of the lift force when a thin aerofoil passes through a transversely fully coherent gust (the spanwise wavelength of the incident turbulence is infinite), which is essentially the strip assumption (Fung 1969). In this case, the two-wavenumber aerodynamic admittance $\chi(k_1, k_2)$ can always be replaced by Sears' function. However, the strip assumption is certainly not correct, because it eliminates all effects due to the semi-infinite extensions (Hakkinen & Richardson 1957). The lift force on a thin aerofoil with a larger chord compared to the vertical scale of gusts was also observed to be more correlated than the incident fluctuations by Nettleton (Etkin 1971), which indicated the invalidation of the strip assumption. In such a situation, the spanwise variations of the lift forces cannot be ignored when the chordwise dimension is similar to or larger than the scale of the turbulence. This phenomenon was further confirmed by some wind tunnel tests concerning the correlation of the lift force on a bluff body (Hjorth-Hansen, Jakobsen & Strømmen 1992; Sankaran & Jancauskas 1993; Jakobsen 1997; Kimura *et al.* 1997; Larose 1997; Larose & Mann 1998; Ma 2007). Bearman (1971, 1972) considered that the larger spanwise coherence of the unsteady aerodynamic force could be generated due to the distortion of the turbulence approaching the flat-plate, including the stretching and rotation of the vortex line filaments. In order to describe the spatial distribution of the unsteady force on an aerofoil, it is necessary to apply three-dimensional theory to take the spanwise variations of turbulence into consideration.

A more general formalism of the problem for application to three-dimensionally varying turbulence was suggested by Ribner (1956). The mean-square lift caused by a homogeneous turbulent field can be represented as a superposition of plane sinusoidal waves of all orientations and wavelengths:

$$\left\langle L^2 \right\rangle = \iiint_{-\infty}^{\infty} |\chi(\boldsymbol{k})|^2 S_w(\boldsymbol{k}) \mathrm{d}k_1 \mathrm{d}k_2 \mathrm{d}k_3, \tag{1.2}$$

where $\boldsymbol{k}$ is the wavenumber vector having components k_1, k_2, k_3 in the x, y and z directions respectively. For a surface of infinitesimal thickness, $|\chi(\boldsymbol{k})|^2$ can be written as

$$|\chi(\boldsymbol{k})|^2 = |\chi(k_1, k_2)|^2 . \tag{1.3}$$

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Inspired by Ribner's (see Ribner 1956) suggestion, Graham (1970, 1971) numerically calculated the exact two-wavenumber aerodynamic admittance based on lift-surface theory for an aerofoil of infinite span length due to gusts with arbitrary horizontal wavevectors (k_1, k_2), which can describe the three-dimensionality of lift force. Independently, Filotas (1969*a*) derived an approximate close-formed expression for the two-wavenumber aerodynamic transfer function of a aerofoil passing through an inclined sinusoidal gust, which was accurate in the high-frequency limit or very low-frequency limit. In addition, Filotas (1969*b*) compared his theoretical admittance function with the experimental results of Hakkinen & Richardson (1957) and Lamson (1957). However, these results show a great deal of scatter. Experimental verification of Graham's theoretical results for a fixed single aerofoil with a NACA 0015 section was provided by Jackson, Graham & Maull (1973). In their experiment, the spectrum of the lift induced by grid-generated turbulence was measured and the one-wavenumber aerodynamic admittance instead of $|\chi(k_1, k_2)|^2$ was obtained to validate Graham's theory indirectly. Unfortunately, a direct approach to identify the two-wavenumber aerodynamic admittance of lift has not been developed yet. Based on previous studies, the three-dimensional theory, taking the effects of three-dimensionality of the gusts into account, is adopted here to develop a general approach to identify the two-wavenumber spectrum and aerodynamic admittance of the lift force on an aerofoil in a uniform grid-generated turbulent field with a chord larger than the vertical length scale of the oncoming turbulence. One of the main aims of the present work is to validate Graham's theory directly and reveal the three-dimensional characteristics of the lift force on an aerofoil.

The outline of our paper is as follows. In § 2.1, we begin by presenting the mathematical formulation to describe the three-dimensional homogeneous turbulence field. The proposed theory of the two-wavenumber aerodynamic admittance and the lift force on an aerofoil is given in § 2.2. In § 3, an experiment on a NACA 0015 aerofoil in grid-generated turbulence is carried out to validate the present approach. In this section, we also discuss the sensitivity of the results to the choice of different empirical models proposed in § 2.2. In § 4, the application of the proposed approach is discussed and the extension to a bluff body is also demonstrated. Finally in § 5, we conclude with a short discussion.

2. Description of the approach

2.1. *Three-dimensional theory of homogeneous turbulence*

When the length scale of turbulence is not large compared with the dimensions of the aerofoil, the one-dimensional turbulence model needs to be replaced by appropriate formulae for two-dimensional lateral turbulence. Thus, the corresponding distance r in the chordwise direction x will only be assumed to have components (r_1, r_2) in the chordwise and spanwise (x, y) directions, respectively. In this case, the three-dimensional correlation function for the vertical turbulent component can be defined as

$$R_w(r_1, r_2) = \langle w(x, y) w(x + r_1, y + r_2)\rangle, \tag{2.1}$$

where R_w is the autocorrelation function of the vertical turbulence velocity, $w(x, y)$ is the vertical turbulence velocity at position (x, y), and $\langle\ \rangle$ signifies an ensemble average. In most cases, R_w should be independent of the orientation of the two points relative to each other and depends only on the distance between them. Therefore, by assuming the vertical turbulence to be asymmetric, (2.1) can be expressed in terms of the one-dimensional correlation function as follows:

$$R_w(r_1, r_2) = R_w\left(\sqrt{r_1^2 + r_2^2}\right). \tag{2.2}$$

Likewise the wavenumber k will be replaced by its components (k_1, k_2). In order to obtain the two-wavenumber spectrum of vertical turbulence, the following Fourier transform relation proposed by Taylor (1965) will be applied:

$$S_w(k_1, k_2) = 2\sigma_w^2 \iint_{-\infty}^{\infty} \cos(2\pi k_1 r_1 + 2\pi k_2 r_2) R_w\left(\sqrt{r_1^2 + r_2^2}\right) \mathrm{d}r_1 \mathrm{d}r_2, \tag{2.3}$$

where σ_w^2 is the mean square of the turbulence velocity in the vertical direction, and k_1 and k_2 are chordwise and spanwise wavenumbers, respectively; $k_1 = n/U$ (n is frequency in units of Hz and U is mean wind velocity). It should be noted that the turbulent velocity could be assumed to be the same at any position over the span of the aerofoil if the scale of the turbulence is large enough compared with the aerofoil (approaching $k_2 = 0$). Thus, (2.3) would become the one-wavenumber case as follows:

$$S_w(k) = 4\sigma_w^2 \int_0^{\infty} \cos(2\pi k r) R_w(r) \mathrm{d}r. \tag{2.4}$$

By switching to polar coordinates, the double integral (2.3) can be reduced to (Taylor 1965)

$$S_w(k_1, k_2) = 4\pi\sigma_w^2 \int_0^{\infty} \mathrm{J}_0(2\pi k r) r R_w(r) \mathrm{d}r \tag{2.5}$$

where J_0 is a Bessel function of the first kind, $r = \sqrt{r_1^2 + r_2^2}$ and $k = \sqrt{k_1^2 + k_2^2}$. Obviously, $S_w(k_1, k_2)$ is the Hankel transformation of the corresponding autocorrelation function $R_w(r)$. Thus, it is convenient to obtain the two-wavenumber spectrum of turbulence once $R_w(r)$ has been obtained. Bullen (1961) has proposed the following lateral autocorrelation function:

$$R_w(r) = \left[(r/a)^n / 2^n / \Gamma(n)\right] \left[2\mathrm{K}_n(r/a) - (r/a)\mathrm{K}_{n-1}(r/a)\right], \tag{2.6}$$

where a and n are parameters governing the shape and scale of the expression, K_n and K_{n-1} are modified Bessel functions of the second kind and $\Gamma(n)$ is the Gamma function. If the turbulent flow is isotropic, the scale of the turbulence L and parameter a have the following relation:

$$L = \left[\sqrt{\pi}\,\Gamma\left(n + \tfrac{1}{2}\right) / \Gamma(n)\right] a. \tag{2.7}$$

In order to obtain the one- and two-wavenumber spectra of turbulence, the Fourier and Hankel transformations including the modified Bessel function of the second kind are required. Gradshteyn & Ryzhik (2007) proposed a series of general formulae of infinite integral involving a complicated fractional function and Bessel functions as follows:

$$\mathscr{F}(b) = \int_0^{\infty} x^{\mu} \mathrm{K}_{\mu}(cx) \cos(bx) \mathrm{d}x = \tfrac{1}{2}\sqrt{\pi}\,(2c)^{\mu}\, \Gamma\left(\mu + \tfrac{1}{2}\right) \left(b^2 + c^2\right)^{-\mu-(1/2)} \tag{2.8}$$

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with the basic condition as $[b>0, c>0, \mathrm{Re}\,\mu>-(1/2)]$, and

$$\mathscr{F}(b)=\int_0^\infty x^{\mu+\nu+1}\mathrm{J}_\nu(cx)\,\mathrm{K}_\mu(bx)\mathrm{d}x=2^{\mu+\nu}c^\nu b^\mu\frac{\Gamma(\mu+\nu+1)}{\left(b^2+c^2\right)^{\mu+\nu+1}} \tag{2.9}$$

with the basic condition as $[\mathrm{Re}\,\nu>|\mathrm{Re}\,\mu|-1,\ \mathrm{Re}\,b>|\mathrm{Im}\,c|]$. In the above equations, K_μ are modified Bessel functions of the second kind, J_ν are Bessel functions of the first kind, Re represents the real part and Im the imaginary part. Obviously, (2.8) and (2.9) are Fourier and Hankel transformations of K_μ, respectively.

Substituting (2.6) into (2.5), the one- and two-wavenumber spectra of the vertical turbulence velocity can be obtained by means of the above relations (2.8) and (2.9), respectively:

$$S_w(k_1)=\frac{2\sigma_w^2L\left[1+8\pi^2a^2k_1^2(n+1)\right]}{[1+(2\pi ak_1)]^{n+(3/2)}}, \tag{2.10}$$

$$S_w(k_1,k_2)=\frac{32\sigma_w^2\pi^3a^4n(n+1)\left(k_1^2+k_2^2\right)}{\left[1+4\pi^2a^2\left(k_1^2+k_2^2\right)\right]^{n+2}}. \tag{2.11}$$

As mentioned above, n is introduced to determine the spectral shape of the turbulence. If $n=1/2$, the turbulence models described by (2.10) become identical to the empirical formula proposed by Dryden *et al.* (1937); if $n=1/3$, (2.10) will be consistent with the spectrum model proposed by von Kármán (1948).

The two-wavenumber spectrum (2.11) demonstrates the spatial three-dimensionality of the turbulence field, so the spatial coherence of the vertical turbulent velocity can be defined by the following non-dimensional two-wavenumber spectrum:

$$\Phi_w(k_1,k_2)=\frac{S_w(k_1,k_2)}{S_w(k_1)}. \tag{2.12}$$

2.2. *Two-wavenumber spectra analysis of the lift force on a thin aerofoil*

The idea of three-dimensional theory can be extended to identify the spatial distribution of unsteady gust loading. Using the coordinate systems defined in figure 1 and assuming the turbulent field to be a stationary random function of position in the X–Y plane, Taylor's hypothesis is validated for the aerofoil through the turbulent field $w(x,y)$. Therefore, the time series could be converted into a 'space series', $x=Ut$. For a thin aerofoil with a lift slope C_L', the unsteady lift at an arbitrary spanwise position y can be defined as in Diederich (1956) and Hakkinen & Richardson (1957):

$$F_L(x,y)=\rho UbC_L'\iint_{-\infty}^{\infty}\phi(\xi,|\eta-y|)w(x-\xi,\eta)\mathrm{d}\xi\,\mathrm{d}\eta, \tag{2.13}$$

where ρ is air density, b is the half-chord of the aerofoil, and $\phi(\xi,y-\eta)$ is defined as the influence function for lift per unit span at y produced by a unit impulse of the turbulence located at position (ξ,η) with respect to the aerofoil. The correlation of the lift for two identical strips of the aerofoil at different spanwise locations can be defined as

$$R_L(\Delta x,\Delta y)=\langle F_L(x,y)F_L(x+\Delta x,y+\Delta y)\rangle, \tag{2.14}$$

where, $\Delta x=U\Delta t$, and Δy is the spanwise separation between two unit strips on the aerofoil (see figure 1). Substituting (2.13) into (2.14) yields the correlation function of the lift:

$$\begin{aligned}R_L=&\ (\rho UbC_L')^2\iiiint_{-\infty}^{\infty}\phi(\xi_1,y-\eta_1)\,\phi(\xi_2,y+\Delta y-\eta_2)\\&\times\langle w(x-\xi_1,\eta_1)\,w(x+\Delta x-\xi_2,\eta_2)\rangle\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\eta_1\mathrm{d}\eta_2.\end{aligned} \tag{2.15}$$

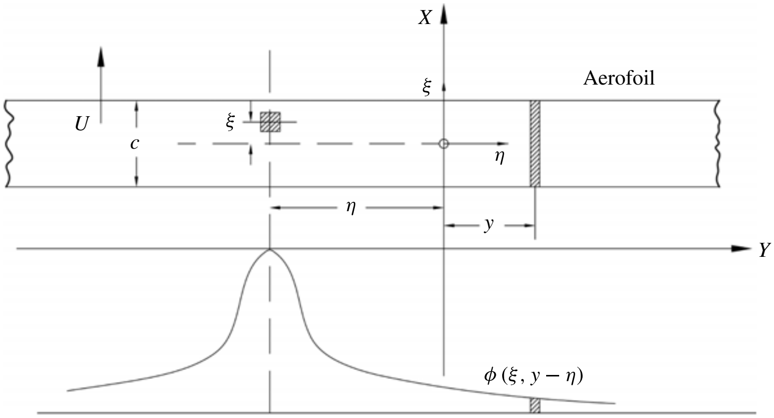

FIGURE 1. Coordinate systems and sketch of the lift influence function. *XY* is a global coordinate system and $\xi\eta$ is a local coordinate system on the aerofoil.

If the turbulence field is homogeneous in the X–Y plane, the ensemble average over the velocity products $R_w[\Delta x+(\xi_1-\xi_1), (\eta_1-\eta_2)]$ is independent of x and can be expressed in terms of a Fourier transform of the two-wavenumber power spectrum $S_w(k_1, k_2)$ as

$$R_w\left[\Delta x+(\xi_1-\xi_1), (\eta_1-\eta_2)\right] \\ = \iint_{-\infty}^{\infty} S_w(k_1, k_2)\exp\{\mathrm{i}\,[k_1\,(\Delta x+\xi_1-\xi)+k_2(\eta_1-\eta_2)]\}\mathrm{d}k_1\mathrm{d}k_2. \quad (2.16)$$

Substituting (2.16) into (2.15), and interchanging the order of integration gives

$$\begin{aligned} R_L(\Delta x, \Delta y) = {} & \left(\rho UbC_L'\right)^2 \iint_{-\infty}^{\infty} \mathrm{d}k_1\mathrm{d}k_2 S_w(k_1, k_2)\exp\{\mathrm{i}\,(k_1\Delta x+k_2\Delta y)\} \\ & \times \iint_{-\infty}^{\infty} \phi\,(\xi_1, y-\eta_1)\exp\left\{\mathrm{i}\left[k_1\xi_1+k_2\,(y-\eta_1)\right]\right\}\mathrm{d}\xi_1\mathrm{d}(y-\eta_1) \\ & \times \iint_{-\infty}^{\infty} \phi\,(\xi_2, y+\Delta y-\eta_2) \\ & \times \exp\left\{-\mathrm{i}\left[k_1\xi_2+k_2\,(y+\Delta y-\eta_2)\right]\right\}\mathrm{d}\xi_2\mathrm{d}\,(y+\Delta y-\eta_2)\,. \end{aligned} \quad (2.17)$$

When the two-wavenumber aerodynamic transfer function is defined as

$$\chi\,(k_1, k_2) = \iint_{-\infty}^{\infty} \phi\,(\xi_1, y-\eta_1)\exp\left\{\mathrm{i}\left[k_1\xi_1+k_2\,(y-\eta_1)\right]\right\}\mathrm{d}\xi_1\mathrm{d}\,(y-\eta_1) \quad (2.18)$$

the lift correlation function (2.17) can be written

$$R_L(\Delta x, \Delta y) = \left(\rho UbC_L'\right)^2 \iint_{-\infty}^{\infty} |\chi\,(k_1, k_2)|^2\, S_w(k_1, k_2)\exp\{\mathrm{i}\,(k_1\Delta x+k_2\Delta y)\}\mathrm{d}k_1\mathrm{d}k_2, \quad (2.19)$$

where $|\chi\,(k_1, k_2)|^2$ is the magnitude of the aerodynamic transfer function, which is also defined as the two-wavenumber aerodynamic admittance function (3D AAF). It is apparent that $R_L(\Delta x, \Delta y)$ is a function of the separations Δx and Δy and not of the location or orientation of the strips. For the lift, the two-wavenumber spectrum can be expressed in the form of a Fourier transform, i.e.

$$S_L(k_1, k_2) = 4 \iint_0^\infty R_L(\Delta x, \Delta y) \cos(2\pi k_1 \Delta x + 2\pi k_2 \Delta y) \mathrm{d}\Delta x \mathrm{d}\Delta y. \tag{2.20}$$

Comparing (2.19) with (2.20), we have the following lift power spectrum from the uniqueness of the Fourier transform:

$$S_L(k_1, k_2) = \left(\rho U b C_L'\right)^2 |\chi(k_1, k_2)|^2 S_w(k_1, k_2), \tag{2.21}$$

where $S_w(k_1, k_2) = \Phi_w(k_1, k_2) S_w(k_1)$.

For $\Delta y = 0$, it is apparent that the traditional one-wavenumber of lift force can be written as

$$S_L(k_1) = 2 \int_0^\infty R_L(\Delta x, 0) \cos(2\pi k_1 \Delta x) \mathrm{d}\Delta x. \tag{2.22}$$

The one-wavenumber spectrum of the lift force at an arbitrary strip along the span of the aerofoil can be obtained from the two-wavenumber spectrum by integrating out the variable, thus

$$S_L(k_1) = \int_{-\infty}^\infty S_L(k_1, k_2) \mathrm{d}k_2. \tag{2.23}$$

This relation is also valid with respect to the vertical turbulence velocity, and then we have

$$S_w(k_1) = \int_{-\infty}^\infty S_w(k_1, k_2) \mathrm{d}k_2. \tag{2.24}$$

Substituting (2.21) into (2.23), the one-wavenumber spectrum of the lift force can now be written

$$S_L(k_1) = \left(\rho U b C_L'\right)^2 |\chi(k_1)|^2 S_w(k_1), \tag{2.25}$$

where

$$|\chi(k_1)|^2 = \int_{-\infty}^\infty |\chi(k_1, k_2)|^2 \Phi_w(k_1, k_2) \mathrm{d}k_2; \tag{2.26}$$

$|\chi(k_1)|^2$ is defined as the one-wavenumber lift aerodynamic admittance (1D AAF) of an arbitrary strip on the aerofoil, which is usually obtained directly by the pressure measurement method in a wind tunnel test. The relationship between the one- and two-wavenumber aerodynamic admittance can be determined by (2.26), which has been applied by Jackson *et al.* (1973) to validate Graham's two-wavenumber aerodynamic transfer function (3D AAF) indirectly. By linearized aerodynamic theory and the Kutta–Joukowski condition, Sears (1941) found an expression for the aerodynamic admittance of the lift on a thin aerofoil immersed in a sinusoidal gust field as follows:

$$|\chi(k_1)|^2 = \left| \frac{\mathrm{J}_0\left(\tilde{k}_1\right) \mathrm{K}_1\left(\mathrm{i}\tilde{k}_1\right) + \mathrm{i}\mathrm{J}_1\left(\tilde{k}_1\right) \mathrm{K}_0\left(\mathrm{i}\tilde{k}_1\right)}{\mathrm{K}_1\left(\mathrm{i}\tilde{k}_1\right) + \mathrm{K}_0\left(\mathrm{i}\tilde{k}_1\right)} \right|^2, \tag{2.27}$$

where $\tilde{k}_1 = 2\pi k_1 B/2$, B being the chord of the aerofoil; J_0, J_1 are Bessel functions of the first kind and K_0, K_1 are modified Bessel functions of the second kind. For $\tilde{k}_1 > 2$, this function is roughly circular but with a slowly decreasing radius as $\tilde{k}_1$ increases.

Of particular interest in the following discussion is the square of the amplitude, which has been approximated by Liepmann (1952) as

$$|\chi(k_1)|^2 \approx \frac{1}{1+2\pi\tilde{k}_1}, \tag{2.28}$$

which bears a close resemblance to the exact function throughout $\tilde{k}_1 > 0$. For cases in which neither the aerofoil span nor the spanwise wavelength of the incident turbulence is effectively infinite with respect to the chord (Blake 1986), a variety of attempts have been made to find a proper aerodynamic transfer function to describe the pressure distribution on the aerofoil. Reissner & Stevens (1947*a*,*b*) introduced a correction factor to the basic function of the two-dimensional theory to study the effect of three-dimensionality of the flow and gave the corresponding methods for the numerical evaluation of the results. Based on lift-surface theory, Graham (1970, 1971) numerically computed the exact loading functions for a thin aerofoil of infinite span and varying $\tilde{k}_2$ due to arbitrary yawed sinusoidal gusts in incompressible flow. Furthermore, Mugridge (1970, 1971) determined an approximate closed-form expression for the lift aerodynamic admittance in terms of a correlation factor to the traditional Sears' function, which is of high accuracy for the lower wavenumber range ($k_1 < 1/\pi$) compared with Graham's exact result, that is

$$|\chi(k_1,k_2)|^2 \approx \frac{1}{1+2\pi\tilde{k}_1}\left|F\left(\tilde{k}_1,\tilde{k}_2\right)\right|^2, \tag{2.29}$$

where the correlation function is

$$|F(k_1,k_2)|^2 = \left[\frac{\tilde{k}_1^2+2/\pi^2}{\tilde{k}_1^2+\tilde{k}_2^2+2/\pi^2}\right]. \tag{2.30}$$

An approximate closed-form expression for the 3D AAF of the lift force derived by Filotas (1969*a*) is given as

$$\chi(k_1,k_2) = \frac{\exp\left\{-\mathrm{i}\tilde{k}\left[\sin\beta - \dfrac{\pi\beta\,(1+0.5\cos\beta)}{1+2\pi\tilde{k}\,(1+0.5\cos\beta)}\right]\right\}}{\sqrt{1+\pi\tilde{k}\left(1+\sin^2\beta+\pi\tilde{k}\cos\beta\right)}}, \tag{2.31}$$

where $\tilde{k} = \sqrt{\tilde{k}_1^2+\tilde{k}_2^2}$ and $\sin\beta = \tilde{k}_1/\tilde{k}$. This formula was derived for an aerofoil of infinite span and its magnitude was approximated by Filotas (1969*b*) as

$$|\chi(k_1,k_2)|^2 = \frac{\sqrt{\tilde{k}_1^2+\tilde{k}_2^2}}{\sqrt{\tilde{k}_1^2+\tilde{k}_2^2+\pi\left(\pi\tilde{k}_2^3+\tilde{k}_2^2+\pi\tilde{k}_1\tilde{k}_2+2\tilde{k}_1^2\right)}}. \tag{2.32}$$

Filotas' approximate expression, based on linearized incompressible lifting surface theory, is asymptotically exact in the limiting cases where the reduced frequency is either very small or very large. For use in approximations, Blake (1986) proposed a closed-form expression by fitting Graham's exact calculation as follows:

$$|\chi(k_1,k_2)|^2 = \frac{1}{1+2\pi\tilde{k}_1}\left[\frac{1+3.2\left(2\tilde{k}_1\right)^{1/2}}{1+2.4\left(2\tilde{k}_1\right)^2+3.2\left(2\tilde{k}_1\right)^{1/2}}\right]. \tag{2.33}$$

Blake's approximation agreed with Graham's exact values to within 20 % when $\tilde{k}_1 > \tilde{k}_2/2$. Previous studies of the two-wavenumber unsteady aerodynamic force on an aerofoil (Reissner & Stevens 1947*a*,*b*; Filotas 1969*a*,*b*; Graham 1970, 1971; Mugridge 1970, 1971), especially the two-wavenumber transfer function (aerodynamic admittance), were mainly derived theoretically based on potential flow and lift-surface theory. In this paper, a direct approach is proposed to study the two-wavenumber spectrum and aerodynamic admittance of the lift on a stationary aerofoil immersed in grid-generated turbulent flow by wind tunnel test. For a given type of turbulence field, the cross-spectrum of the lift force can be measured either by pressure scanners or by a force-measuring balance. Therefore, the two-wavenumber lift spectrum on an aerofoil could be expressed in terms of the Fourier transform of the experimentally determined spanwise cross-spectrum of the lift force $S_L(k_1, \Delta y)$ as

$$S_L(k_1, k_2) = 2\int_0^{\infty} S_L(k_1, \Delta y)\cos(2\pi k_2 \Delta y)\mathrm{d}\Delta y, \tag{2.34}$$

where $S_L(k_1, \Delta y)$ is the cross-spectrum of the lift force between two locations along the span separated by Δy. By introducing the coherence function of the lift force, $S_L(k_1, \Delta y)$ can be defined as

$$S_L(k_1, \Delta y) = \mathrm{Coh}_L(k_1, \Delta y) S_L(k_1), \tag{2.35}$$

where $\mathrm{Coh}_L(k_1, \Delta y)$ is the spanwise coherence of the lift force. Inserting (2.35) into (2.34) yields

$$S_L(k_1, k_2) = \Phi_L(k_1, k_2) S_L(k_1), \tag{2.36}$$

where $\Phi_L(k_1, k_2)$ is the two-wavenumber coherence of the lift force. Substituting (2.25) into (2.36), $S_L(k_1, k_2)$ can be expressed in terms of $\Phi_L(k_1, k_2)$ and the one-wavenumber aerodynamic admittance $|\chi(k_1)|^2$, such that

$$S_L(k_1, k_2) = \left(\rho U b C_L'\right)^2 |\chi(k_1)|^2 \Phi_L(k_1, k_2) S_w(k_1). \tag{2.37}$$

Comparing (2.21) with (2.37), the physical property of the two-wavenumber aerodynamic admittance can be indicated by the following relation:

$$\frac{|\chi(k_1, k_2)|^2}{|\chi(k_1)|^2} = \frac{\Phi_L(k_1, k_2)}{\Phi_w(k_1, k_2)}. \tag{2.38}$$

In contrast with the one-wavenumber aerodynamic admittance, the two-wavenumber aerodynamic admittance can describe the effects of secondary spanwise flow and consequently the redistribution of the surface pressure. In other words, the ratio between $|\chi(k_1, k_2)|^2$ and $|\chi(k_1)|^2$ reflects the degree to which the lift force on an aerofoil departs from the strip assumption, which varies with frequency k_1 and k_2. Thus, the influence factor describing the three-dimensionality of the turbulence can be defined as

$$\vartheta(k_1, k_2) = \frac{|\chi(k_1, k_2)|^2}{|\chi(k_1)|^2}. \tag{2.39}$$

Assuming the validity of the strip assumption, which means the spanwise length is indefinite ($k_2 = 0$), the 3D AAF of lift force is independent of k_2 and can be represented by Sears' function. Consequently, the 1D AAF $|\chi(k_1)|^2$ also matches Sears' function according to (2.26). Therefore, the influence factor is unity, which explains the traditional conclusion that the correlation of the buffeting force can be represented by that of the turbulence. In fact, this assumption always fails as the characteristic length of the structures approaches the scales of the vertical gusts. Several wind tunnel tests on aerofoils and bluff bodies have confirmed that the unsteady lift was more correlated than the incident wind (Hjorth-Hansen *et al.* 1992; Sankaran & Jancauskas 1993; Jakobsen 1997; Kimura *et al.* 1997; Larose 1997; Larose & Mann 1998; Ma 2007), which indicated the invalidity of the strip assumption. In this case, $\mathrm{Coh}_L(k_1, \Delta y)$ cannot be represented by the coherence of turbulence. Moreover, it cannot be deduced theoretically, similar to isotropic turbulence. So it is better to determine $\mathrm{Coh}_L(k_1, \Delta y)$ by wind tunnel tests. According to (2.21) and (2.36), the 3D AAF $|\chi(k_1, k_2)|^2$ can be expressed in terms of $\Phi_L(k_1, k_2)$, so it is necessary to develop a relatively accurate mathematical coherence model of the buffeting force. Fortunately, a series of empirical coherence models of the buffeting force (Hjorth-Hansen *et al.* 1992; Jakobsen 1997; Kimura *et al.* 1997; Larose & Mann 1998) can be applied. To be convenient for Fourier transform, Jakobsen's, Kimura *et al.*'s and Dryden *et al.* (1937)'s empirical models are adopted to describe the coherence of the buffeting lift force. Jakobsen's model is defined in exponential form, leaving some floating parameters to fit the results obtained from the wind tunnel test. The formulation is modified slightly to be convenient for Fourier transformation as follows:

$$\mathrm{Coh}^{1/2}_{L_{Jakobsen}}(k_1, \Delta y) = \exp\{-A\Delta y\}, \tag{2.40}$$

where

$$A = \left(\sqrt{c_2^2 + (c_3 k_1)^2}\right)^{c_1} \tag{2.41}$$

and c_1, c_2, c_3 are floating parameters that need to be fitted.

Kimura *et al.* (1997) referred to the von Kármán root coherence function and modified the length scale of turbulence and the frequency based on wind tunnel tests. Thus, the following empirical coherence model of the lift force is proposed in a form similar to Kimura *et al.*'s approach, taking the Fourier transform into consideration:

$$\mathrm{Coh}^{1/2}_{L_{Kimura}}(k_1, \Delta y) = \frac{2^{1/6}}{\Gamma(5/6)}\left(\eta^{5/6}\mathrm{K}_{5/6}(\eta) - \frac{\eta^{11/6}\mathrm{K}_{1/6}(\eta)}{B_1}\right), \tag{2.42}$$

where

$$\eta = A_1 \Delta y, \tag{2.43}$$

$$A_1 = \frac{C}{\alpha_1 L_w^x}\sqrt{1 + (2\pi/C)^2\left[(k_1 U)^{\beta_1}\left(\frac{\alpha_1 L_w^x}{U}\right)\right]^2}, \tag{2.44}$$

$$B_1 = 1 + \frac{8}{3}(2\pi/C)^2\left[(k_1 U)^{\beta_1}\left(\frac{\alpha_1 L_w^x}{U}\right)\right]^2 \tag{2.45}$$

$$C = \sqrt{\pi}\frac{\Gamma(5/6)}{\Gamma(1/3)}, \tag{2.46}$$

and U is mean wind velocity, $\mathrm{K}_{5/6}$, $\mathrm{K}_{1/6}$ are modified Bessel functions of the second kind, α_1, β_1 are floating parameters determined by wind tunnel tests, and L_w^x is the integral length scale of turbulence. It should be noted that the coherence of vertical gusts can be represented by (2.42) where the parameters $\alpha_1 = 2$ and $\beta_1 = 1$.

https://doi.org/10.1017/jfm.2015.162 Published online by Cambridge University Press

Therefore, the Fourier transform of (2.42) is absolutely consistent with (2.12) with $n = 1/3$, which provides another approach to obtain the two-wavenumber spectrum of the turbulence. This approach is of significance for natural atmospheric turbulence. Referring to the theoretical study proposed by Vickery (1965), Dryden *et al.*'s coherence model is defined similarly to Kimura *et al.*'s model as

$$\mathrm{Coh}^{1/2}_{L_{Dryden}}(k_1, \Delta y) = \theta \mathrm{K}_1(\theta) - \frac{\theta^2}{B_2}\mathrm{K}_0(\theta), \tag{2.47}$$

where

$$\theta = A_2 \Delta y, \tag{2.48}$$

$$A_2 = \frac{1}{\alpha_2 L_w^x}\sqrt{1 + (2\pi)^2\left[(k_1 U)^{\beta_2}\left(\frac{\alpha_2 L_w^x}{U}\right)\right]^2}, \tag{2.49}$$

$$B_2 = 1 + 3\left[2\pi\,(k_1 U)^{\beta_2}\left(\frac{\alpha_2 L_w^x}{U},\right)\right]^2 \tag{2.50}$$

and U and L_w^x are defined as in (2.42), K_0, K_1 are modified Bessel functions of the second kind, and α_2, β_2 are floating parameters determined by wind tunnel tests. In addition to (2.8), the following Fourier transform relation is necessary to obtain the two-wavenumber spectrum of the lift force (Jakobsen's empirical model):

$$\mathscr{F}(b) = \int_0^\infty \mathrm{e}^{-ax}\cos(bx)\mathrm{d}x = a\left(a^2 + b^2\right)^{-1}. \tag{2.51}$$

Substituting (2.40)–(2.50) into (2.34)–(2.36), and applying the Fourier transform relations (2.51) and (2.8), respectively, the two-wavenumber coherence of the lift on an aerofoil can be obtained:

$$\Phi_{L_{Jakobsen}}(k_1, k_2) = \frac{2A}{\left[A^2 + (2\pi k_2)^2\right]}, \tag{2.52}$$

$$\Phi_{L_{Kimura}}(k_1, k_2) = \frac{2\Gamma(4/3)}{\Gamma(5/6)}\sqrt{\pi}A_1^{5/3}\left\{\frac{\left[1 + 5/(3B_1)\right](2\pi k_2)^2 + (1 - 1/B_1)A_1^2}{\left[(2\pi k_2)^2 + A_1^2\right]^{7/3}}\right\}, \tag{2.53}$$

$$\Phi_{L_{Dryden}}(k_1, k_2) = 2\Gamma(3/2)\sqrt{\pi}A_2^2\left\{\frac{(1 + 2/B_2)(2\pi k_2)^2 + (1 - 1/B_2)A_2^2}{\left[(2\pi k_2)^2 + A_2^2\right]^{5/2}}\right\}. \tag{2.54}$$

In terms of the two-wavenumber coherence of the lift force and the corresponding one-wavenumber spectrum, $S_L(k_1, k_2)$ can be obtained conveniently. Then, applying (2.11) and (2.21), the two-wavenumber aerodynamic admittance $|\chi(k_1, k_2)|^2$ can be derived directly as follows:

$$|\chi(k_1, k_2)|^2 = \frac{S_L(k_1, k_2)}{(\rho U b C_L')^2 S_w(k_1, k_2)}. \tag{2.55}$$

3. Experimental validation

Wind tunnel tests were carried out in a closed-circuit-type wind tunnel with a 2.4 m (width) × 2 m (height) working section (XNJD-1). The turbulence is generated by uniform grid with a mesh size of 0.33 m × 0.33 m and a bar size of 0.07 m, installed 4.2 m upstream of the model. The flow field characteristics are measured by a Cobra Probe and the wind fluctuations in three directions could be measured simultaneously.

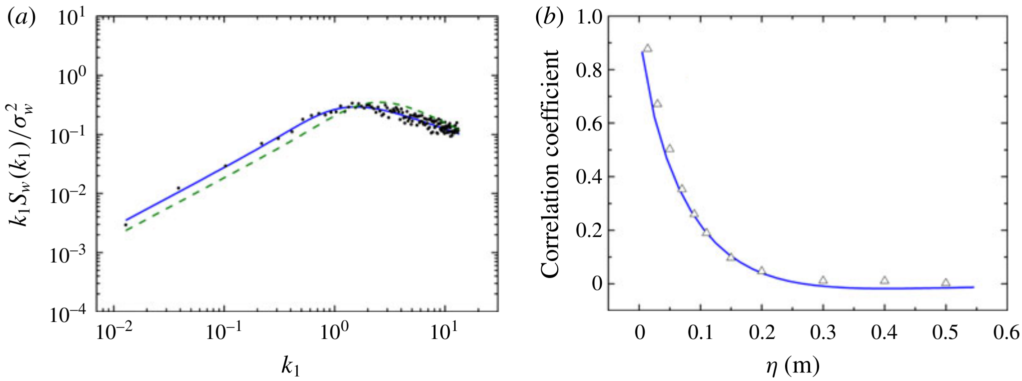

FIGURE 2. (Colour online) (*a*) Spectrum of the vertical wind fluctuations for grid turbulence (●). Solid line: fit of von Kármán's spectrum using (2.10) with $n = 1/3$; dashed line: fit of Dryden *et al.*'s spectrum using (2.10) with $n = 1/2$. (*b*) Comparison of spanwise correlation coefficient of vertical turbulence, w, between Bullen's theoretical results (solid line) and experimental results (△), where η is spanwise distance.

The model, with a NACA 0015 cross-section and chord length of 0.5 m, is made of glass fibre with several transverse ribs to increase its stiffness. The 0.7 m long centre portion (action model) of the model is instrumented to measure the unsteady surface pressures on six strips along the span. Two 0.4 m long pseudo-models and end plates are installed at each end of the action model to ensure two-dimensionality of the flow and to render end effects unimportant. The unsteady surface pressures on the aerofoil section are measured synchronously by DMS 3400 pressure scanners, which are mounted inside the model to keep the tubing length (0.2 m) to a minimum, ensuring a good frequency response up to 256 Hz. For the convenience of calculating aerodynamic admittance, the sampling frequency of the Cobra Probe and fluctuating pressures are all set to 256 Hz. A total of 11 spanwise separations ranging from 0.014 to 0.5 m are investigated to define the spanwise distribution of the turbulence. The tests in the wind tunnel were typically conducted at a mean wind speed of 11.5 m s^{-1}.

As shown in (2.55), we know that an adequate description of the two-wavenumber spectrum of the turbulence is of great significance to identify the 3D AAF $|\chi(k_1, k_2)|^2$. The two-wavenumber spectrum of the vertical fluctuations is related to $S_w(k_1)$ as shown in (2.24) and $R_w(r)$, which is its Hankel transform. Thus, the one-wavenumber spectrum $S_w(k_1)$ is compared in figure 2(*a*) with the theoretical result (2.10). Compared with Dryden *et al.*'s spectrum model, it is found that the measured results agree particularly well with (2.10) with $n = 1/3$ (von Kármán's spectrum). Figure 2(*b*) demonstrates that the measured correlation coefficients of the w component are consistent with the theoretical result (2.6) with $n = 1/3$ proposed by Bullen (1961), which indicates that the grid-generated field satisfies the homogeneous isotropic turbulence assumption. The measured results of the turbulence field confirm that the indirectly measured two-wavenumber spectrum of w is of high accuracy.

The spectrum of the lift measured on an arbitrary strip is plotted in figure 3(*a*). Figure 3(*b*) shows that the lift force is better correlated than the vertical fluctuations, which is consistent with Nettleton's experimental results (Etkin 1971) and also confirms the invalidity of the strip assumption. In order to further assess the correlation of the lift force and the turbulence, the width of the correlation is evaluated.

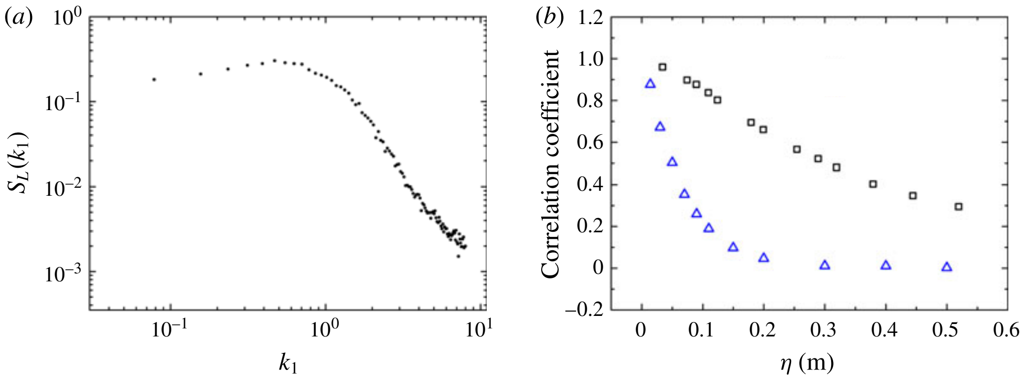

FIGURE 3. (Colour online) (*a*) Spectrum of the buffeting lift force on an arbitrary strip. (*b*) Comparisons between the spanwise correlation coefficients of the lift force (□) and the w component of turbulence (△), where η is spanwise distance.

Empirical model	Jakobsen's model			Kimura *et al.*'s model		Dryden *et al.*'s model	
Parameter	c_1	c_2	c_3	α_1	β_1	α_2	β_2
	0.9265	1.8302	1.2404	9.5655	0.6523	10.1800	0.6790

TABLE 1. Parameters of the empirical models fitted by the measured results.

The spanwise width of the correlation defined by Larose (1997) is represented by the integral length scales as follows:

$$L^y = \int_0^\infty R_{12}(\Delta y)\mathrm{d}\Delta y, \tag{3.1}$$

where R_{12} is the correlation coefficient. The correlation width of the lift force and vertical turbulence can be obtained from figure 3(*b*). It is found that L_L^y is 3.9 times L_w^y ($L_w^y = 0.054$ m), which approaches Nettleton's result $L_L^y = 3.6L_w^y$ (Larose 1997).

In the present approach, the coherence of the lift force is of great significance and needs to be fitted by the proposed empirical coherence model. In other words, the accuracy of the identified 3D AAF mainly depends on the description of the spanwise correlation of the lift force. Therefore, the three proposed empirical models are fitted by the measured results in figure 4 to evaluate the sensitivity of the results to the choice of model. The parameters in the empirical models, obtained by a nonlinear least-squares fit of the measured coherence of the lift force, are given in table 1.

Kimura *et al.*'s and Dryden *et al.*'s models are close to the experimentally measured values when the spanwise distance $\Delta y/L_w^y$ ranges from 1.3 to 8. Dryden *et al.*'s model has exceptionally good agreement with the measured results for lower frequency when $\Delta y/L_w^y < 2$. Kimura *et al.*'s model agrees relatively well with the experimental results either at low frequency or in high frequency when $\Delta y/L_w^y < 8$. In contrast, Jakoben's model poorly describes the spanwise correlation of the lift force for $\Delta y/L_w^y < 2$, and its accuracy improves gradually with an increase of $\Delta y/L_w^y$ from 3 to 6. Compared with Kimura *et al.*'s and Dryden *et al.*'s models, Jakobsen's model has similar accuracy when $\Delta y/L_w^y$ ranges from 6 to 8, and excellent agreement is observed as $\Delta y/L_w^y$ approaches 13. In conclusion, Jakobsen's model is more suitable for large spanwise distance, and Kimura *et al.*'s and Dryden *et al.*'s models have

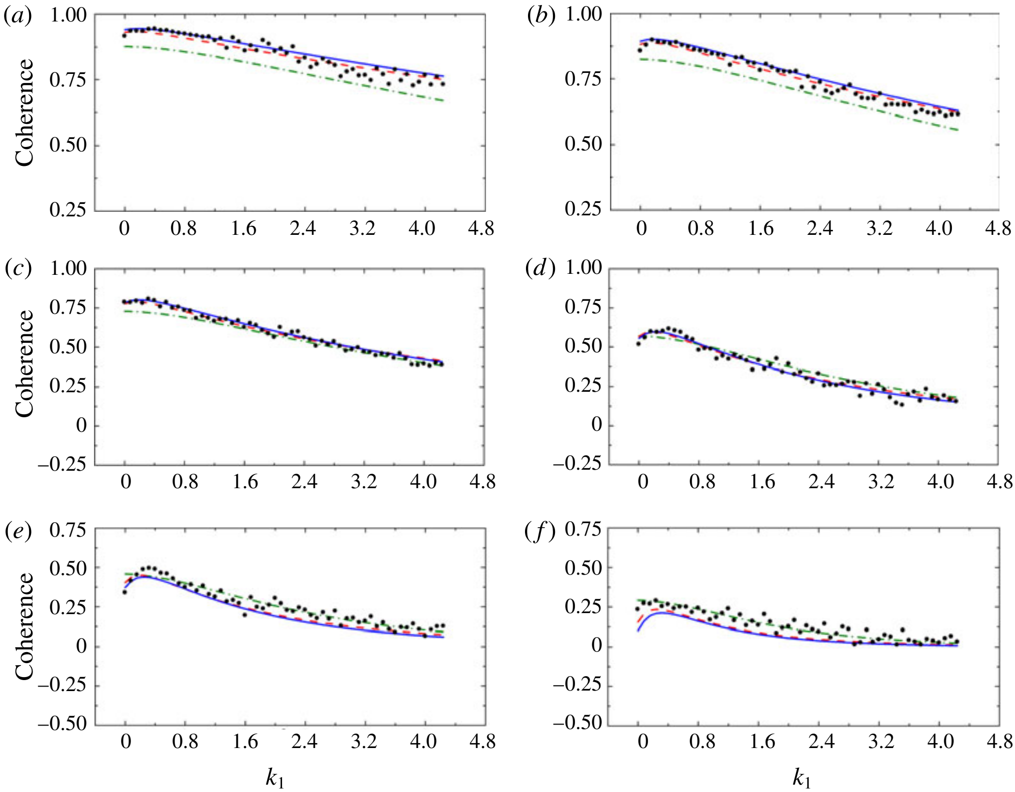

FIGURE 4. (Colour online) Comparisons between direct measurements of the spanwise coherence of the lift force (●) and the empirical models of Jakobsen (dash-dotted), Kimura *et al.* (dashed) and Dryden *et al.* (solid). (*a*) $\Delta y/L_w^y = 1.3$; (*b*) $\Delta y/L_w^y = 2$; (*c*) $\Delta y/L_w^y = 3.3$; (*d*) $\Delta y/L_w^y = 6$; (*e*) $\Delta y/L_w^y = 8$; (*f*) $\Delta y/L_w^y = 13$.

similar accuracy when $\Delta y/L_w^y < 8$, and are important for lower frequency and higher frequency, respectively.

In order to study the three-dimensionality of the turbulence, the influence factor $\vartheta(k_1, k_2)$ is presented in figure 5(*a*). When the reduced frequencies k_1B and k_2B are close to zero, $\vartheta(k_1, k_2)$ reaches its maximum, which means that the aerofoil tends to extract energy from larger-scale vortices in the turbulence. The two-wavenumber coherence of w is obtained from (2.12) as shown in figure 5(*b*). The contours show that the correlation of the vertical turbulence mainly depends on narrow-band reduced frequencies in the chordwise and spanwise directions ($0.2 < k_1B < 1.1$, $0.2 < k_2B < 1$). Taking Kimura *et al.*'s empirical model for instance, the two-wavenumber coherence of lift can be derived from (2.42) as given in figure 5(*c*), and has similar characteristics to the incident turbulence w. Comparing the coherence of w with lift force, it is found that the correlation of the lift force depends on a larger range in the chordwise direction ($0.03 < k_1B < 1.2$) and lower reduced frequency in the spanwise direction ($0.03 < k_2B < 0.3$). This phenomenon can precisely be interpreted by the influence factor $\vartheta(k_1, k_2)$.

Based on (2.11) and (2.35), the two-wavenumber spectrum of vertical turbulence and lift force on the aerofoil model can be obtained as shown in figure 6(*a*,*b*). The variation trends of $S_w(k_1, k_2)$ and $S_L(k_1, k_2)$ (taking Kimura *et al.*'s empirical model

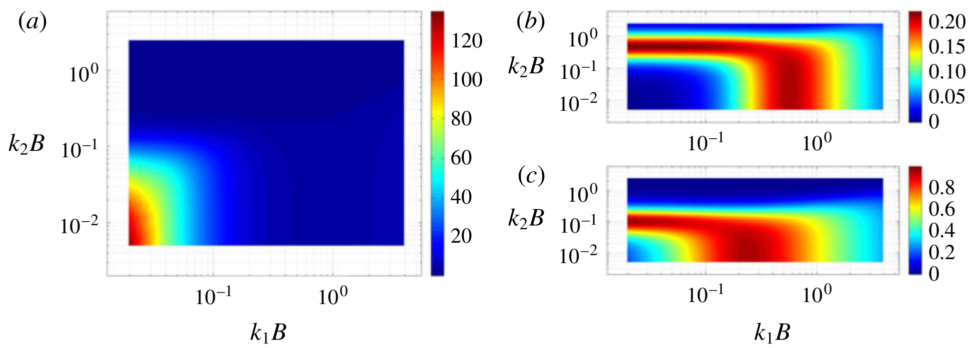

FIGURE 5. Contours of the influence factor, spectrum of vertical turbulence and lift force. (*a*) Two-wavenumber influence factor $\vartheta(k_1, k_2)$ associated with reduced wavenumbers. (*b*) Two-wavenumber coherence of vertical turbulence $\Phi_w(k_1, k_2)$. (*c*) Two-wavenumber coherence of lift force $\Phi_L(k_1, k_2)$.

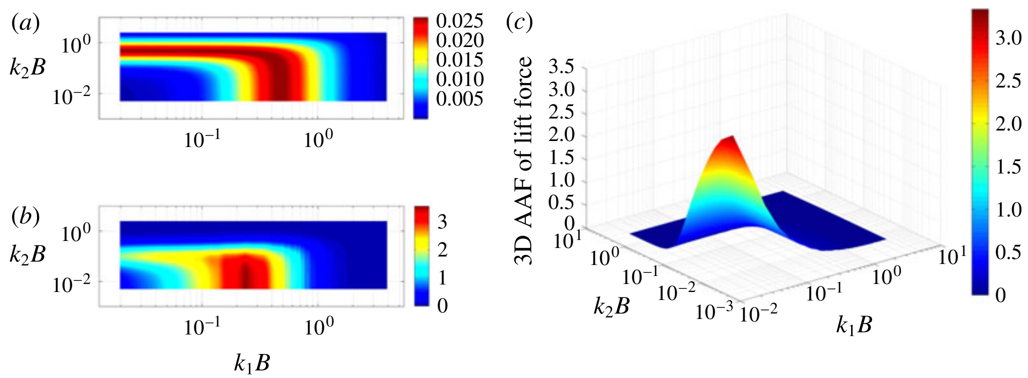

FIGURE 6. (*a*) Contours of the two-wavenumber spectrum of vertical turbulence $S_w(k_1, k_2)$. (*b*) Contours of the two-wavenumber spectrum of lift force $S_L(k_1, k_2)$. (*c*) Measured 3D AAF of lift force $|\chi(k_1, k_2)|^2$.

for instance, though in fact the three empirical models have similar results) are in accordance with $\mathrm{Coh}_w(k_1, k_2)$ and $\mathrm{Coh}_L(k_1, k_2)$ as presented in figures 5(*b*) and 5(*c*), respectively. Furthermore, the 3D AAF $|\chi(k_1, k_2)|^2$ can be obtained directly from (2.55) as presented in figure 6(*c*), which indicates that the aerofoil may be regarded as a low-pass filter, transferring the energy from higher to lower frequency. According to Bearman's theory, this pattern of energy transition may be explained as the spanwise vortex line filaments being stretched and rotated when the turbulence passes though the aerofoil. Consequently, the energy is transferred to the larger-scale vortices (lower frequency), while the smaller-scale vortices diminish in this process.

In § 2.2, we presented a series of theoretical 3D AAF models of thin aerofoils (Filotas 1969*a*,*b*; Graham 1970, 1971; Mugridge 1970, 1971; Blake 1986). It is not convenient to compare the measured 3D AAF with the theoretical results directly, so we validate our proposed theory by comparing the 1D AAF obtained by the measured 3D AAF, defined as in (2.26), with those obtained by empirical 3D AAF models. Since Graham did not provide an explicit expression for 3D AAF, Filotas', Mugridge's and Blake's 3D AAF models are applied. Figure 7 compares the directly

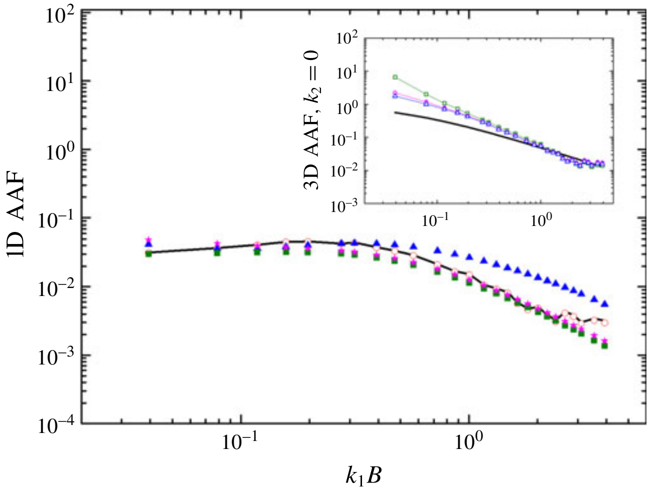

FIGURE 7. (Colour online) Comparisons between experimentally measured 1D AAF (solid line) from (2.25) and theoretical calculations from (2.26): ○, the 1D AAF calculated from the measured 3D AAF; ▲, Mugridge's 3D AAF (2.29); ■, Filotas's 3D AAF (2.31); ★, Blake's 3D AAF (2.33). The inset shows the comparison between the theoretical 3D AAF (Graham's result, solid line) with $k_2 = 0$, which is consistent with Sears' function, and the experimentally obtained results based on the empirical models of Jakobsen (□), Kimura *et al.* (○) and Dryden *et al.* (△).

measured 1D AAF (2.25) with the calculated results derived from four different 3D AAF models (2.26). The 1D AAF calculated from the measured 3D AAF agrees exceptionally well with the directly measured results and matches Filotas' result at very low and very high frequency. This result is also consistent with Mugridge's result at low frequency ($k_1 < 1/\pi$). Compared with Graham's exact 3D AAF, Mugridge's 3D AAF has high accuracy at low frequency, and Filotas' model is asymptotically exact when the frequency is either very small or very large (Mugridge 1971). Therefore, the proposed approach in this paper can be proved to be of high accuracy. Then, Graham's theory can be validated by an experimental approach directly. With respect to Blake's 3D AAF, there are few studies to assess its accuracy explicitly. Compared with the 1D AAF derived by the other two theoretical 3D AAF models, Blake's result has higher accuracy although the reduced frequency is very low.

The inset in figure 7 compares the identified 3D AAF based on the three proposed empirical coherence models of lift force with Sears' function when $k_2 = 0$. The trends of these results are consistent with Sears' function (solid), and approach it at higher frequency. The results indicate that Sears' function deduced from fully coherent gusts (one-dimensional sinusoidal gusts) can also possibly be identified in an isotropic turbulent field (grid-generated turbulence). In addition, the results indicate that the 3D AAF in fully coherent gusts is sensitive to the empirical coherence models. Compared with Jakobsen's model, Dryden *et al.*'s and Kimura *et al.*'s empirical models are more accurate at lower reduced frequency and have similar accuracy to Jakobsen's at higher reduced frequency. The deviation of the results obtained at low reduced frequency may be caused by the following three factors: (i) the effect of aerofoil thickness (Sears' function is derived for a thin aerofoil with no chamber); (ii) the effect of the viscosity of the fluid; (iii) the accuracy of the empirical coherence model of the lift force.

4. Application of the identification approach to other configurations

In the above discussion we confirmed that the proposed theory is of high accuracy for aerofoils. In fact, it is not hard to extend this approach to study the two-wavenumber spectrum and aerodynamic admittance for a bluff body with arbitrary cross-section, such as a streamlined bridge deck. In this case, the contribution of the longitudinal turbulence cannot be neglected. Based on the traditional gust loading model suggested by Scanlan (1978), it is not hard to derive the general two-wavenumber spectrum of the buffeting force, taking the lift force for instance:

$$S_L(k_1, k_2) = (\rho Ub)^2|\chi(k_1, k_2)|^2 \left[4C_L^2 S_u(k_1, k_2) + (C_L' + C_D)^2 S_w(k_1, k_2)\right], \tag{4.1}$$

where C_L, C_D are the lift and drag coefficients, respectively and $S_u(k_1, k_2)$ is the two-wavenumber spectrum of the longitudinal turbulence. Therefore, once the two-wavenumber spectrum of the lift force and the incident wind fluctuations are identified by the proposed approach, the two-wavenumber aerodynamic admittance can be obtained directly. It should be noted that these empirical coherence models of lift force were originally applied to bluff bodies: Jakobsen's model is for a streamlined deck and Kimura *et al.*'s model is for a rectangular cylinder. Hence, these empirical coherence models are universal for line-like structures with arbitrary sections. However, the parameters in the empirical coherence models may not be constant, was they are related to the integral length scale of turbulence and the dimension of the structure. Thus, the parameters should be determined as a function of L_w by large numbers of wind tunnel tests with variations of L_w. For the buffeting drag force and overturning moment, similar results could be derived.

The two-wavenumber number spectrum of longitudinal turbulence can also be expressed in terms of the Fourier transform of the spanwise coherence. Referring to the theoretical model based on von Kármán's theory, the coherence model for longitudinal turbulence can be defined as

$$\mathrm{Coh}_u^{1/2}(k_1, \Delta y) = \frac{2^{1/6}}{\Gamma(5/6)}\left(\eta^{5/6}K_{5/6}(\eta) - \frac{\eta^{11/6}K_{1/6}(\eta)}{2}\right), \tag{4.2}$$

where

$$\eta = A_3\Delta y, \tag{4.3}$$

$$A_3 = \frac{C}{\alpha_3 L_w^x}\sqrt{1 + (2\pi/C)^2\left[(k_1U)^{\beta_3}\left(\frac{\alpha_3 L_w^x}{U}\right)\right]^2}, \tag{4.4}$$

and U, L_w, C are defined as in (2.42), $\mathrm{K}_{5/6}$ and $\mathrm{K}_{1/6}$ are modified Bessel functions of the second kind, and α_3, β_3 are floating parameters determined by wind tunnel tests. For isotropic turbulence, (4.2) is absolutely consistent with the theoretical result proposed by Roberts & Surry (1973) when the parameters $\alpha_3 = 2$, $\beta_3 = 1$. If the turbulent flow is not entirely isotropic, the parameters need to be identified by experimental measurements. Thus, the two-wavenumber coherence of the longitudinal turbulence can be expressed in terms of a Fourier transform as

$$\Phi_u(k_1, k_2) = \frac{\Gamma(4/3)}{3\Gamma(5/6)}\sqrt{\pi}A_3^{5/3}\left\{\frac{11(2\pi k_2)^2 + 3A_3^2}{\left[(2\pi k_2)^2 + A_3^2\right]^{7/3}}\right\}. \tag{4.5}$$

In this case, the coherence of vertical turbulence can be identified by (2.42) based on wind tunnel tests. With respect to line-like structures with arbitrary cross-configurations, the 3D AAF can be identified based on the proposed approach in the following four steps: first, the parameters in empirical coherence models for the turbulence and buffeting force should be determined by experimental measurements as proposed in this paper; secondly, the two-wavenumber spectrum of the turbulence and buffeting force can be obtained by the presented models; thirdly, the static coefficients should be measured by section model tests; finally, the 3D AAF can be identified directly based on the proposed two-wavenumber gust loading models, such as (4.1).

5. Conclusion

Based on three-dimensional theory, a general approach is proposed to identify the two-wavenumber spectrum of lift and aerodynamic admittance on an aerofoil directly, which can be applied conveniently with the use of traditional simultaneous measurements of the surface pressure method on a section model. The wind tunnel tests confirm that this approach is of high accuracy and efficiency. Comparisons between the two-wavenumber spectra of lift force and w indicate that a spanwise vortex line filament would be stretched when the turbulence passing over the aerofoil has a similar or smaller dimension compared with that of aerofoil. Thus, energy is transferred from higher frequency to lower frequency. The 3D AAF can explicitly describe the mechanism of energy transition and the three-dimensionality of the lift force, which also validates Graham's theoretical results directly. Also, the characteristics of other closed-form approximations of the 3D AAF are further confirmed when compared with Graham's exact result.

In fact, the accuracy of this approach mainly depends on the empirical coherence model of the unsteady aerodynamic force. In particular, the 3D AAF for fully coherent gusts is sensitive to the empirical coherence model. However, the parameters in the coherence models may not be constant, but be related to the dimensions of the turbulence and the structure, and they need to be further improved based on a large number of experiments. In addition, the approach proposed in this paper can be extended to study the three-dimensionality of the buffeting force on line-like structures with arbitrary configurations, such as long-span bridges and high-rise buildings, in a more precise way.

Acknowledgements

This work is supported by the National Natural Science Foundation under the grant number 51278435, 51478402 and China National Basic Research Plan '973' under the grant number 2013CB036301. We are indebted to Dr N. Yeung (ARUP, Hong Kong) and Professor G. Huang (SWJTU, China) for their valuable comments and suggestions, which greatly enhanced the quality of this article.

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