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UNIRATIONALITY OF CUBIC HYPERSURFACES

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Abstract Segre proved that a smooth cubic surface over Q is unirational if and only if it has a rational point. We prove that the result also holds for cubic hypersurfaces over any field, including finite fields.

Keywords: cubic hypersurface; unirational; rational points

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1. Introduction

A remarkable result of [**6**] says that a smooth cubic surface over $\mathbb{Q}$ is unirational if and only if it has a rational point. [**4**, II.2] observed that similar arguments work for higher dimensional cubic hypersurfaces satisfying a certain genericity assumption over any infinite field. [**2**, 2.3.1] extended the result of Segre to any normal cubic hypersurface (other than cones) over a field of characteristic zero. It is also clear that the result should hold for all sufficiently large finite fields, though the details were not worked out in general. [**4**, IV.8] settles the cubic surface case for finite fields with at least 34 elements. The aim of this note is to observe that a variant of the Segre–Manin method works for all fields and for all cubics.

Theorem 1.1. *Let k be a field and $X \subset \mathbb{P}^{n+1}$ a smooth cubic hypersurface of dimension $n \geqslant 2$ over k. Then the following are equivalent:*

(1) *X is unirational (over k); and*

(2) *X has a k-point.*

Similar results hold for singular cubic hypersurfaces, with a few exceptions.

Theorem 1.2. *Let k be a perfect field and $X \subset \mathbb{P}^{n+1}$ an irreducible cubic hypersurface of dimension $n \geqslant 2$ over k which is not a cone over an $(n-1)$-dimensional cubic. Then the following are equivalent:*

(1) *X is unirational (over k);*

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(2) *X has a k-point; and*

(3) *X has a smooth k-point.*

Non-perfect fields. Over non-perfect fields of characteristic 3, there are *non-singular* cubic hypersurfaces of arbitrary dimension which are not unirational but do have a k point (Proposition 4.1). (Note that over a non-perfect field, smooth and non-singular are not equivalent notions.) I have not been able to find similar examples in characteristic 2.

An inspection of the proof shows that if k is not perfect, the three parts of Theorem 1.2 are equivalent if one of the following conditions holds:

(1) $\operatorname{char} k \geqslant 5$;

(2) $\operatorname{char} k = 3$ and X has no triple points over $\bar{k}$; and

(3) $\operatorname{char} k = 2$ and there is a smooth k point $p \in X$ such that projection from p is separable.

Question 1.3. Unirationality of varieties is very poorly understood in general and there are very basic open questions. We do not even have a list of unirational surfaces and very few examples are known in higher dimensions. For instance, let X be a smooth projective variety over k such that X is unirational over $\bar{k}$. Assume for simplicity that k is infinite and consider the following properties:

(1) X is unirational (over k);

(2) $X(k)$ is dense in X; and

(3) X has a k-point.

It is clear that each property implies the next. They are equivalent for cubic hypersurfaces by Theorem 1.1. It is extremely unlikely that they are always equivalent, but no counter examples are known.

Proof of (1) $\Rightarrow$ (2) $\Rightarrow$ (3) for Theorem 1.2. Theorem 1.2 (1) $\Rightarrow$ Theorem 1.2 (2) is clear for infinite fields. For finite fields it follows from [**5**].

Assume that Theorem 1.2 (2) holds and let $x \in X$ be a k-point. We are done if X is smooth at x. Otherwise x is a double point and we can choose affine coordinates such that $x = (0, \dots, 0)$ and X is given by an equation $q(x_1, \dots, x_n) + c(x_1, \dots, x_n) = 0$ where q is quadratic and c is cubic. Assume that there is a point $(p_1, \dots, p_n) \in k^n$ such that $q(p_1, \dots, p_n) \neq 0$. Then the line connecting the origin and $(p_1, \dots, p_n)$ intersects X in a single point outside the origin and this is a smooth k-point of X. Thus we are done unless q vanishes everywhere on k^n.

However, if a homogeneous polynomial f of degree d vanishes on k^n and $|k| \geqslant d$, then f is identically zero. □

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2. Unirationality constructions

The interesting part is to show unirationality starting with a smooth k-point. The construction is presented in three stages, successive version working in greater and greater generality. At least in retrospect, all of this is only a slight modification of the works of Segre.

First unirationality construction. Let $X \subset \mathbb{P}^{n+1}$ be a cubic and $p \in X$ a point. Let C_p denote the intersection of X with the tangent plane at p. We expect that usually C_p is an irreducible cubic with a double point at p. If this is indeed the case, then the inverse of the projection from p gives a birational map $\pi_p : \mathbb{P}^{n-1} \dashrightarrow C_p$. If $p \in X(k)$, then C_p is birational to $\mathbb{P}^{n-1}$ over k.

Assume next that we have two points $p, q \in X$ and C_p, C_q are both irreducible with a double point at p (resp. q). Define the 'third intersection point' map

$$\phi : C_p \times C_q \dashrightarrow X$$

as follows. Take $u \in C_p, v \in C_q$. If the line connecting u, v is not contained in X, it has a unique third intersection point with X; call it $\phi(u, v)$. Under very mild genericity assumptions (Lemma 3.2) this is a well defined dominant map. Thus we get that X is unirational via

$$\Phi : \mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \overset{\pi_p \times \pi_q}{\dashrightarrow} C_p \times C_q \overset{\phi}{\dashrightarrow} X.$$

Definition 2.1 (restriction of scalars). Let L/K be a finite degree field extension. Restriction of scalars (or Weil restriction) is a way to associate to an L-variety U a K-variety $\mathfrak{R}_{L/K}U$ such that there is a natural identification of the L-points of U with the K-points of $\mathfrak{R}_{L/K}U$. This dictates that $\dim \mathfrak{R}_{L/K}U = \deg(L/K) \cdot \dim U$ (see [**1**, 7.6] for details).

This can be done very explicitly in the affine case as follows. Let $U \subset \mathbb{A}^n$ be an affine variety defined over L. Choose equations $f_i(x_1, \dots, x_n)$ for U and let $e_1, \dots, e_d \in L$ be a K-basis. Choose new coordinates $y_{ij} : i = 1, \dots, n,\ j = 1, \dots, d$, and set $x_i = \sum_j e_j y_{ij}$. We can then write

$$f_k(x_1, \dots, x_n) = \sum_{\ell} e_\ell f_{k\ell}(y_{ij}), \quad \text{where } f_{k\ell} \in K[y_{ij}].$$

Let $\mathfrak{R}_{L/K}U$ be the subvariety of $\mathbb{A}^{nd}$ defined by the equations $f_{k\ell} = 0$.

In particular we see that $\mathfrak{R}_{L/K}(\mathbb{A}^n) \cong \mathbb{A}^{dn}$. In the projective case, $\mathfrak{R}_{L/K}(\mathbb{P}^n)$ is inconvenient to describe by explicit equations but we at least get that $\mathfrak{R}_{L/K}(\mathbb{P}^n)$ is birational to $\mathbb{P}^{dn}$ over K. (They are not isomorphic for $d > 1$.)

Second unirationality construction. Let $X \subset \mathbb{P}^{n+1}$ be a cubic defined over k and $k' \supset k$ a quadratic extension. Let $p \in X(k')$ be a point and $\bar{p} \in X(k')$ its conjugate. (Let us ignore that $k' \supset k$ may be inseparable in characteristic 2.) We have conjugate birational maps $\pi_p : \mathbb{P}^{n-1} \dashrightarrow C_p$ and $\pi_{\bar{p}} : \mathbb{P}^{n-1} \dashrightarrow C_{\bar{p}}$. If $u \in \mathbb{P}^{n-1}(k')$, then $\pi_p(u)$ and $\pi_{\bar{p}}(\bar{u})$ are conjugate points of X, thus the line connecting them is defined over k. Hence

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$\phi(u, \bar{u}) \in X(k)$. Putting this invariantly, we obtain a rational map (defined over k)

$$\Phi : \mathfrak{R}_{k'/k}\mathbb{P}^{n-1} \dashrightarrow X$$

which is dominant under mild genericity assumptions.

Even in the surface case, there are some examples when neither the first nor the second unirationality construction applies.

Example 2.2. Let S be the cubic surface $(x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0)$. By [**3**] over the fields $\mathbb{F}_2, \mathbb{F}_4, \mathbb{F}_{16}$ all the points are on the 27 lines. Hence the second unirationality construction does not work over $\mathbb{F}_2$ and $\mathbb{F}_4$. (It does work over $\mathbb{F}_{16}$.)

Final unirationality construction. Assume now that X is a cubic defined over k and $x \in X$ is a smooth k-point. Let L be a line through x. If L is not contained in X, then it intersects X in a point pair $\{p, q\}$. These points are usually not in k, but they are conjugate over k and lie in a quadratic extension $k' = k'(L)$ of k. Hence, under some genericity assumptions, we obtain a dominant map

$$\Phi : \mathfrak{R}_{k'/k}\mathbb{P}^{n-1} \dashrightarrow X$$

which shows that X is unirational. There are very few problems if k is infinite, since then a general choice of L should work. (Extra work is needed in characteristic 2.) The situation is less clear over finite fields since there may not be enough room to choose L general (see, for example, Example 2.2).

To avoid this difficulty, we do not choose any line, rather we work with all lines simultaneously. We should obtain a map

$$\Psi : \bigcup_{x \in L \subset \mathbb{P}^{n+1}} \mathfrak{R}_{k'(L)/k}\mathbb{P}^{n-1} \dashrightarrow X.$$

We are in good shape if we can identify the left-hand side with a product $\mathbb{P}^n \times \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$, at least birationally. Once this problem is settled, it is enough to check dominance over the algebraic closure where the previous arguments work. It seems best to give an explicit algebraic description.

Algebraic description of Ψ. We work in affine coordinates, assuming that the origin is a smooth point of X. Thus the equation of X can be written as

$$F = L(x_1, \ldots, x_{n+1}) + Q(x_1, \ldots, x_{n+1}) + C(x_1, \ldots, x_{n+1}),$$

where L is linear, Q is quadratic and C is cubic. We may assume that $\partial F/\partial x_{n+1}$ is not identically zero (for instance we can even assume that $L = x_{n+1}$).

We write down a rational map

$$\Psi : \mathbb{A}^{3n-2}(u_1, \ldots, u_n, v_1, \ldots, v_{n-1}, w_1, \ldots, w_{n-1}) \dashrightarrow X.$$

Later we check that it is dominant with a few exceptions.

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Consider the universal line through the origin $(\tau u_1, \dots, \tau u_n, \tau)$. It intersects X in two further points which correspond to the roots of the quadratic equation

$$L(u_1, \dots, u_n, 1) + \tau Q(u_1, \dots, u_n, 1) + \tau^2 C(u_1, \dots, u_n, 1) = 0.$$

The equation is irreducible if X is irreducible. Let its roots be $t_1, t_2 \in \overline{k(u_1, \dots, u_n)}$.

The equation of the tangent space of X at $\boldsymbol{p} = (p_1, \dots, p_{n+1}) \in X$ is

$$\frac{\partial F}{\partial x_1}(\boldsymbol{p})(x_1 - p_1) + \dots + \frac{\partial F}{\partial x_{n+1}}(\boldsymbol{p})(x_{n+1} - p_{n+1}) = 0.$$

Thus the universal tangent line at $(t_1 u_1, \dots, t_1 u_n, t_1)$ can be described parametrically as

$$\begin{aligned} x_1 &= t_1 u_1 + \sigma(v_1 + t_1 w_1), \dots, x_{n-1} = t_1 u_{n-1} + \sigma(v_{n-1} + t_1 w_{n-1}), \\ x_n &= t_1 u_n + \sigma, \\ x_{n+1} &= t_1 - \sigma \left(\frac{\partial F}{\partial x_{n+1}}(t_1 \boldsymbol{u}, t_1) \right)^{-1} \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(t_1 \boldsymbol{u}, t_1)(v_i + t_1 w_i), \end{aligned}$$

where we set $v_n = 1, w_n = 0$. Substituting the above parametric representation into F, we obtain a cubic equation in σ

$$\sum_{j=0}^{3} \sigma^j H_j \quad \text{where } H_j \in k(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}, t_1).$$

$H_0 = H_1 = 0$ since we have a tangent line, thus the third intersection point corresponds to the value $\sigma = -H_2/H_3$. Thus we obtain a point

$$Q_1 \in k(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}, t_1)^{n+1}.$$

Replacing t_1 by its conjugate t_2 we obtain another point Q_2. The line connecting Q_1 and Q_2 can be given parametrically as

$$L(\lambda) = \frac{\lambda - t_2}{t_1 - t_2} Q_1 + \frac{\lambda - t_1}{t_2 - t_1} Q_2,$$

and this is a parametrization over $k(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$. Evaluating F on the line we have that t_1, t_2 are roots, so

$$F(L(\lambda)) = (A\lambda + B)(C\lambda^2 + Q\lambda + L).$$

Thus if we expand

$$F(L(\lambda)) = \sum_{j=0}^{3} \lambda^j G_j, \quad \text{then } G_j \in k(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}),$$

and the third root is

$$-\frac{B}{A} = -\frac{G_2}{G_3} + \frac{Q}{C}.$$

Substituting this into the parametrization of the line gives

$$\Psi(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) \in X(k(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})).$$

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Depending on our definition of unirationality, we also need to check the following lemma.

Lemma 2.3. *For a k-variety X the following are equivalent:*

(1) *there is a dominant map $\phi_m : \mathbb{A}^m \dashrightarrow X$ for some m; and*

(2) *there is a dominant map $\phi_m : \mathbb{A}^m \dashrightarrow X$ for $m = \dim X$.*

Proof. Assume that $m > \dim X$. There is a dense open set $U \subset \mathbb{A}^m$ such that $\phi_m|_U$ is open with $m - \dim X$ dimensional fibres. Let $\boldsymbol{u} \in U$ be a point. If $\boldsymbol{u} \in Z \subset \mathbb{A}^m$ is a hypersurface which does not contain the irreducible component of the fibre of $\phi_m|_U$ through $\boldsymbol{u}$, then $\phi_m|_Z : Z \dashrightarrow X$ is dominant.

If k is infinite and $m > \dim X$, then we can choose Z to be a general hyperplane.

Assume next that k is finite. Fix a prime $\ell \neq \operatorname{char} k$ and let k' be the composite of all algebraic extensions of degree ℓ^s of k. k' is infinite, hence we can choose a point $\boldsymbol{u} = (u_1, \dots, u_m) \in U(k')$. By permuting the coordinates we may assume that $\deg k(u_m)/k \leqslant \deg k(u_1)/k$, or, equivalently, $k(u_m) \subset k(u_1)$. This implies that u_m can be written as a polynomial of u_1, hence the ideal $I(\boldsymbol{u}) \subset k[x_1, \dots, x_m]$ contains a polynomial of the form $x_m - p(x_1)$. This implies that $I(\boldsymbol{u})$ is generated by polynomials of the form $x_m - P(x_1, \dots, x_{m-1})$. Thus we can choose $Z = (x_m = P(x_1, \dots, x_{m-1}))$ for suitable P. □

3. Proof of Theorem 1.2 (3) ⇒ Theorem 1.2 (1)

Let X be an irreducible cubic hypersurface. The set of all triple points of $X(\bar{k})$ is a linear space and it is defined over k if k is perfect. Thus X is a cone over a cubic hypersurface without triple points. Therefore, X has no triple points over $\bar{k}$.

Assume next that X is not normal. The non-normal locus has dimension $(n-1)$ and the linear space spanned by it is in X. Thus the non-normal locus is a linear space $L^{n-1} \subset \mathbb{P}^{n+1}$ which is defined over k if k is perfect. Projecting form L realizes X as a $\mathbb{P}^{n-1}$-bundle over $\mathbb{P}^1$, hence rational.

For the rest of the proof assume that X is normal. We need to check three conditions.

First we prove that C_x is irreducible with a double point at x for general $x \in X(\bar{k})$. This is done in Proposition 3.1.

Second, we need to check that the third intersection point map $\phi : C_p \times C_q \dashrightarrow X$ is dominant. It is, however, not enough to check this for a general pair p, q. In our construction p, q are the two intersection points of a line through x, hence dependent. Assume that $\pi_x : X \dashrightarrow \mathbb{P}^n$, the projection from x, is separable. Then for a generic line $x \in L$ we get 2 distinct intersection points and both intersections are transverse. In particular, the tangent space of X at one point does not contain the other point. In Lemma 3.2 we see that this is sufficient to guarantee that $\phi : C_p \times C_q \dashrightarrow X$ is dominant.

Third, we need to consider the case when the projection $\pi_x : X \dashrightarrow \mathbb{P}^n$ is inseparable. This can happen only in characteristic 2. Over a perfect field a purely inseparable map induces a purely inseparable map in the reverse direction, hence in this case X is (purely inseparably) unirational. Nonetheless, we check in Corollary 5.3 that we can always choose a smooth k-point such that projection from it is separable. □

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Proposition 3.1. *Let k be an algebraically closed field and $X \subset \mathbb{P}^{n+1}$ a normal cubic hypersurface over k without triple points. Then C_x is irreducible with a double point at x for general $x \in X$.*

Proof. Let $x \in X$ be arbitrary. If C_x is irreducible with a triple point at x, then C_x is a cone, hence there is an $(n-2)$-dimensional family of lines through x. If C_x is reducible, then either C_x contains an $(n-1)$-dimensional linear space through x or a quadric cone with vertex at x. In either case, there is an $(n-2)$-dimensional family of lines through x. Thus it is enough to prove that for a general $x \in X$ the family of lines in X through x has dimension at most $n-3$. This is equivalent to proving that a general surface section of X contains only finitely many lines.

X has no triple points, hence, by Bertini, a general surface section of X is also normal with no triple points.

If S is a normal cubic surface without triple points, then there are only finitely many lines through each double point. (Choose affine coordinates such that the equation becomes $q(x_1, x_2, x_3) + c(x_1, x_2, x_3) = 0$. The lines through $(0,0,0)$ correspond to the solutions of $(q = c = 0) \subset \mathbb{P}^2$. If there are infinitely many solutions, then q and c have a common factor, thus the surface is reducible.) Each line in the smooth locus has self-intersection -1, hence rigid. Thus S has only finitely many lines. □

Lemma 3.2. *Let $X \subset \mathbb{P}^{n+1}$ be an irreducible cubic hypersurface. Let $x, y \in X$ be smooth points and C_x, C_y the corresponding intersections with the tangent hyperplanes. Assume that*

(1) *C_x and C_y are irreducible; and*

(2) *$x \notin C_y$ and $y \notin C_x$.*

Then the third intersection point map $\phi : C_x \times C_y \dashrightarrow X$ is dominant.

Proof. Let us see first that ϕ is indeed defined. Pick a point $u \in C_x$ which is a smooth point of X. Pick $v \in C_y$ which is a smooth point of X such that v does not lie on $T_u X$. If we now choose a general $w \in C_x$, then v does not lie on $T_w X$ and w does not lie on $T_v X$. Thus the line connecting u, w has a unique third intersection point with X. This shows that ϕ is defined at the pair (v, w).

In order to prove dominance, we need to show that ϕ has at least one fibre of dimension $n-2$. Pick a point $z \in X$ which is not on $C_x \cup C_y$ and let $\pi : \mathbb{P}^{n+1} \dashrightarrow T_y X$ denote the projection from z. Then $\phi^{-1}(z)$ is the set of pairs (v, w) such that $\pi(v) = w$. Thus we are done if

$$\dim((C_y \cap \pi(C_x)) \setminus (C_y \cap C_x)) = n - 2.$$

For this it is sufficient to find one projection $\pi' : \mathbb{P}^{n+1} \dashrightarrow T_y X$ where this holds. Then the same holds for a general projection and a general projection always corresponds to a point of X. Pick any smooth point $v \in C_y$ and let π' be a projection such that $\pi'(x) = v$. Then $\pi'(C_x)$ and C_y intersect at v but they have different multiplicity there. Hence their intersection has dimension $n-2$. □

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4. An example in characteristic 3

The following example, valid in characteristic 3, shows that Theorem 1.2 does not hold for every field.

Proposition 4.1. *Let k be a field of characteristic 3 and t_i algebraically independent over k. Set $K = k(t_1, \dots, t_n)$ and*

$$Y := \left(y^3 - yz^2 = \sum_{i=1}^{n} t_i x_i^3 \right) \subset \mathbb{P}^{n+1}.$$

Then Y has the following properties:

(1) *Y is non-singular;*

(2) *over $\bar{K}$, Y is a cone over a cuspidal cubic curve;*

(3) *$Y(K) = \{(0,1,0,\dots,0), (1,1,0,\dots,0), (1,-1,0,\dots,0)\}$; and*

(4) *Y is not unirational (over K).*

Proof. Y is the generic fibre of the smooth variety

$$\left(y^3 - yz^2 = \sum_i t_i x_i^3 \right) \subset \mathbb{A}^n_{(t_1,\dots,t_n)} \times \mathbb{P}^{n+1}_{(y,z,x_1,\dots,x_n)}$$

over $\mathbb{A}^n$, thus Y is non-singular. Proposition 4.1 (2) holds since over $\bar{K}$ we can write our equation as

$$\left(y - \sum_i \sqrt[3]{t_i} x_i \right)^3 - yz^2 = 0.$$

In order to see (3) we may as well assume that k is algebraically closed. Assume that we have relatively prime polynomials $f, g, h_i \in k[t_1, \dots, t_n]$ such that

$$f^3 - fg^2 = \sum_i t_i h_i^3.$$

We are done if $h_1 = \dots = h_n = 0$. Otherwise, we can make a substitution $t_i = c_i t$ for $i = 1, \dots, n$ and general c_i to get a solution of

$$f(f-g)(f+g) = t \cdot h^3 \quad \text{with } f, g, h \in k[t] \text{ and } h \neq 0.$$

We may assume that f and g are relatively prime. Thus two of the factors f, $f-g$, $f+g$ are cubes and the third is t times a cube. However, $f + (f-g) + (f+g) = 0$, hence if two are cubes, then so is their sum which is minus the third factor. This is a contradiction.

Since Y has only three points in K, it does not contain any rational curves and so it is definitely not unirational. □

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5. Some results in characteristic 2

Lemma 5.1. *Assume that* $\operatorname{char} k = 2$. *Let* $V \subset \mathbb{P}^{n+1}$ *be the linear span of all points* $p \in X(k)$ *such that projection from* p *is a purely inseparable map* $X \dashrightarrow \mathbb{P}^n$. *Let* $(y_i = 0)$ *be equations of* V *and* x_j *coordinates on* V. *Then the equation of* X *can be written as*

$$f := \sum_j \ell_j(\boldsymbol{y}) x_j^2 + g(\boldsymbol{y}),$$

where the ℓ_j *are linear and* g *is cubic. If* $V \neq \emptyset$, *then* X *is not smooth.*

Proof. We can choose coordinates such that the points

$$p_1 = (1 : 0 : \dots : 0), \dots, p_m = (0 : \dots : \overset{m\text{th}}{1} : 0 : \dots : 0)$$

are in $X(k)$ and projection from p_i is a purely inseparable map for $i = 1, \dots, m$. p_i is inseparable if and only if x_i occurs in the equation of X always with even exponent. This gives the above equation.

$\partial f/\partial x_j$ is zero, and the equations $\partial f/\partial y_i = 0$ have a common solution. Since $f = 3f = \sum_i (\partial f/\partial y_i)$, these give singular points of $X(\bar{k})$. □

Lemma 5.2. *Let* k *be a perfect field of characteristic 2. Let* X *be a cubic of dimension at least 2 given by an equation*

$$f(\boldsymbol{x}, \boldsymbol{y}) := \sum_j \ell_j(\boldsymbol{y}) x_j^2 + g(\boldsymbol{y}).$$

Then X *has a smooth* k*-point with non-zero* y*-coordinate.*

Proof. Assume first that we have at least two x-variables. If $\ell_1 = c\ell_2$, then

$$\ell_1 x_1^2 + \ell_2 x_2^2 = \ell_1 (x_1 + \sqrt{c} x_2)^2$$

thus we can change coordinates to eliminate one x-variable. Otherwise we can pick $\boldsymbol{y}_0$ such that $\ell_1(\boldsymbol{y}_0) \neq 0$ and $\ell_2(\boldsymbol{y}_0) = 0$. Then

$$(\sqrt{g(\boldsymbol{y}_0)/\ell_1(\boldsymbol{y}_0)}, x_2, \boldsymbol{y}_0) \in X(k)$$

for every x_2. Since

$$\frac{\partial f}{\partial y_i} = x_2^2 + \frac{\partial g}{\partial y_i}$$

the above point is smooth for suitable choice of x_2.

Thus assume that there is only one x-variable and write the equation as $y_1 x_1^2 + g(\boldsymbol{y})$.

Take any $(p_1, \dots, p_n) \in k^n$. If $p_1 = 0$ and $g(p_1, \dots, p_n) = 0$, then $(x_1 : p_1 : \dots : p_n) \in X(k)$ for any x_1 and one of them is a smooth by looking at $\partial f/\partial y_1$.

If $p_1 \neq 0$, then $p_0 := \sqrt{-g(p_1, \dots, p_n)/p_1} \in k$ and $(p_0 : p_1 : \dots : p_n)$ is a smooth point unless

$$g(p_1, \dots, p_n) - p_1 \frac{\partial g}{\partial y_1}(p_1, \dots, p_n) = 0.$$

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Thus we are done unless the following hold:

(1) $g - y_1(\partial g/\partial y_1)$ is non-zero for $y_1 = 0$, and

(2) $g - y_1(\partial g/\partial y_1)$ is zero for $y_1 \neq 0$.

Write $g = \sum y_1^i g_{3-i}(y_2, \ldots, y_n)$. Then

$$g - y_1(\partial g/\partial y_1) = y_1^2 g_1(y_2, \ldots, y_n) + g_3(y_2, \ldots, y_n).$$

g_1 is a linear form thus it has a non-trivial zero $(p_2, \ldots, p_n)$. If $g_3(p_2, \ldots, p_n) = 0$, then set $p_1 = 0$ and if $g_3(p_2, \ldots, p_n) \neq 0$, then set $p_1 = 1$. □

Combining the above lemmas we obtain the following corollary.

Corollary 5.3. *Let k be a perfect field of characteristic 2 and $X \subset \mathbb{P}^{n+1}$ a cubic with a smooth k-point. Assume that $n \geqslant 2$. Then there is a smooth point $x \in X(k)$ such that the projection from x is separable.*

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