THE JOURNAL OF SYMBOLIC LOGIC
Volume 83, Number 1, March 2018

THE EIGHTFOLD WAY

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Abstract. Three central combinatorial properties in set theory are the tree property, the approachability property and stationary reflection. We prove the mutual independence of these properties by showing that any of their eight Boolean combinations can be forced to hold at κ^{++}, assuming that $\kappa = \kappa^{<\kappa}$ and there is a weakly compact cardinal above κ.

If in addition κ is supercompact then we can force κ to be $\aleph_\omega$ in the extension. The proofs combine the techniques of adding and then destroying a nonreflecting stationary set or a κ^{++}-Souslin tree, variants of Mitchell's forcing to obtain the tree property, together with the Prikry-collapse poset for turning a large cardinal into $\aleph_\omega$.

§1. Introduction. The combinatorial principle $\square_\mu$ was introduced by Jensen [18] and plays a central role in combinatorial set theory. It exerts influence over the combinatorics of μ^+ in several different ways. Notably it implies that:

- The tree property fails at μ^+, i.e., there is a μ^+-Aronszajn tree.
- The approachability property holds at μ^+, i.e., $\mu^+ \in I[\mu^+]$.
- Stationary reflection fails at μ^+.

The main result of this article is that for many values of μ these three consequences of $\square_\mu$ are "orthogonal", in the sense that any of their eight possible Boolean combinations is consistent.

1.1. Square and weak square. The principle $\square_\mu$ states that there is a sequence $\langle C_\alpha : \alpha < \mu^+ \rangle$ such that C_α is club in α with $\mathrm{ot}(C_\alpha) \leq \mu$, and $\beta \in \lim(C_\alpha) \implies C_\beta = C_\alpha \cap \beta$, for all α and β. Jensen showed that if $V = L$ then $\square_\mu$ holds for all uncountable cardinals μ, and this theorem has been extended to many larger L-like inner models.

The principle $\square_\mu$ is very often used in inductive constructions of "noncompact" or "nonreflecting" objects of size μ^+. Typically the idea is that we use C_α to guide the construction at stage α, and the coherence of the clubs gives a club set of stages below α where we were guided by an initial segment of C_α, guaranteeing success at stage α.

The following list of results by Jensen illustrates this theme:

Received December 23, 2016.
2010 *Mathematics Subject Classification.* Primary 03E35, Secondary 03E55.
Key words and phrases. square, Aronszajn tree, tree property, stationary reflection, approachability, Mitchell forcing, Prikry forcing.

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0022-4812/18/8301-0020
DOI:10.1017/jsl.2017.69

https://doi.org/10.1017/jsl.2017.69 Published online by Cambridge University Press

Fact 1.1. *Let $\square_\mu$ hold. Then*:

(1) *There is a special μ^+-Aronszajn tree.*
(2) *If $\diamondsuit(\mu^+)$ holds, then there is a μ^+-Souslin tree.*
(3) *Every stationary subset of μ^+ contains a nonreflecting stationary set.*

In this article we will mostly concentrate on the case when the cardinal μ is regular. The principle $\square_\mu$ has a different flavour for μ singular; in particular it follows from core model arguments that the failure of $\square_\mu$ for μ singular has a very high consistency strength. This contrasts with the case for μ regular, where Solovay showed that if $\lambda > \mu$ is Mahlo then forcing with the Lévy collapse $\mathrm{Coll}(\mu, < \lambda)$ produces a model where $\square_\mu$ fails.

Jensen also introduced a weaker principle $\square^*_\mu$. This states that there is a sequence of nonempty sets $\langle \mathcal{C}_\alpha : \alpha < \mu^+ \rangle$ such that $|\mathcal{C}_\alpha| \leq \mu$, and every $C \in \mathcal{C}_\alpha$ is club in α with $\mathrm{ot}(C) \leq \mu$ and $C \cap \beta \in \mathcal{C}_\beta$ for all $\beta \in \lim(C)$. It is easy to see that if $\mu = \mu^{<\mu}$ then $\square^*_\mu$ holds, and Jensen showed that $\square^*_\mu$ is equivalent to the existence of a special Aronszajn tree.

1.2. Stationary reflection and the approachability ideal $I[\lambda]$. To build models where a regular cardinal λ exhibits some amount of stationary reflection, it is important to understand the extent to which forcing preserves stationary subsets of λ, often in a context where λ is not a cardinal in the forcing extension. It is well-known that both λ-closed and λ-cc forcing posets preserve all stationary subsets of λ, and arguments from the theory of proper forcing imply that countably closed forcing posets preserve all stationary subsets of $\lambda \cap \mathrm{cof}(\omega)$.

It is natural to ask when ν^+-closed forcing posets preserve stationary subsets of $\lambda \cap \mathrm{cof}(\nu)$, where ν is an uncountable regular cardinal with $\nu^+ < \lambda$. In connection with this question, Shelah [24], [25] introduced a natural ideal $I[\lambda]$ (the *approachability ideal*), defined as follows. Whenever $\vec{x} = \langle x_\eta : \eta < \lambda \rangle$ is a sequence of bounded subsets of λ and $\alpha < \lambda$, then say that α is *approachable with respect to* $\vec{x}$ if there is $A \subseteq \alpha$ unbounded with $\mathrm{ot}(A) = \mathrm{cf}(\alpha)$, and $A \cap \beta \in \{x_\eta : \eta < \alpha\}$ for all $\beta < \alpha$. Let $S(\vec{x})$ denote the ordinals approachable relative to $\vec{x}$. A subset S of λ is in $I[\lambda]$ if and only if there exists $\vec{x}$ such that almost every (i.e., club many) $\alpha \in S$ is approachable with respect to $\vec{x}$.

The following are standard facts about $I[\lambda]$ (see for example [4] for proofs):

- $I[\lambda]$ is a λ-complete normal ideal on λ.
- If $\lambda = \lambda^{<\lambda}$ and $\vec{x}$ enumerates $[\lambda]^{<\lambda}$, then the set of α approachable with respect to $\vec{x}$ is stationary, and is the largest element (modulo clubs) of $I[\lambda]$. More generally if $\lambda = \lambda^{<\mu}$ for some regular $\mu < \lambda$ and $\vec{x}$ enumerates $[\lambda]^{<\mu}$, then the set of $\alpha \in \lambda \cap \mathrm{cof}(\mu)$ approachable with respect to $\vec{x}$ is stationary, and is the largest subset (modulo clubs) of $\lambda \cap \mathrm{cof}(\mu)$ in $I[\lambda]$.
- If $\lambda = \mu^+$ and $\square^*_\mu$ holds (in particular if $\mu = \mu^{<\mu}$) then $\mu^+ \in I[\mu^+]$.
- If $\lambda = \mu^+$ and μ is regular, then $\mu^+ \cap \mathrm{cof}(< \mu) \in I[\mu^+]$.
- If ν is regular with $\nu^+ < \lambda$, then $S \in I[\lambda]$ for some stationary $S \subseteq \lambda \cap \mathrm{cof}(\nu)$.
- If ν is regular with $\nu < \lambda$ and $S \in I[\lambda]$ is stationary with $S \subseteq \lambda \cap \mathrm{cof}(\nu)$, then the stationarity of S is preserved by ν^+-closed forcing posets.

The ideal $I[\lambda]$ has proved to be intimately connected with many topics in combinatorial set theory, for example PCF theory [27], saturated ideals [10], and the extent of diamond [22].

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1.3. Forcing facts. We shall need some fairly standard forcing facts.

FACT 1.2 (Easton's Lemma). *Let κ be regular uncountable. Let $\mathbb{P}$ be κ-cc. Let $\mathbb{Q}$ be κ-closed. Let G be $\mathbb{P}$-generic over V. Let H be $\mathbb{Q}$-generic over V. Then*:

(1) *$\mathbb{P}$ remains κ-cc in $V[H]$.*
(2) *$\mathbb{Q}$ is $(<\kappa)$-distributive in $V[G]$.*
(3) *$G \times H$ is $\mathbb{P} \times \mathbb{Q}$-generic over V.*

FACT 1.3. *Let λ be a regular uncountable cardinal. Then there exist a forcing poset $\mathbb{P}_{\mathrm{NRSS}}$ and a $\mathbb{P}_{\mathrm{NRSS}}$-name for a forcing poset $\dot{\mathbb{Q}}_{\mathrm{NRSS}}$ such that*:

- *Forcing with $\mathbb{P}_{\mathrm{NRSS}}$ adds a nonreflecting stationary subset S of λ.*
- *Forcing with $\mathbb{Q}_{\mathrm{NRSS}}$ over the $\mathbb{P}_{\mathrm{NRSS}}$-generic extension destroys the stationarity of S.*
- *The poset $\mathbb{P}_{\mathrm{NRSS}} * \dot{\mathbb{Q}}_{\mathrm{NRSS}}$ has a λ-closed dense subset.*

Conditions in $\mathbb{P}_{\mathrm{NRSS}}$ are functions p from a proper initial segment of λ to 2, such that for every $\alpha \leq \mathrm{dom}(p)$ of uncountable cofinality there is C club in α with $p \restriction C$ identically zero. Conditions in $\mathbb{Q}_{\mathrm{NRSS}}$ are closed bounded subsets of λ disjoint from the stationary set added by $\mathbb{P}_{\mathrm{NRSS}}$. The λ-closed dense subset consists of pairs $(p, \hat{c})$ where c is a closed bounded subset of λ, $\mathrm{dom}(p) = \max(c)$, and $p \restriction c$ is identically zero.

FACT 1.4 (Kunen, [19]). *Let λ be a regular uncountable cardinal. Then there exist a forcing poset $\mathbb{P}_{\mathrm{Souslin}}$ and a $\mathbb{P}_{\mathrm{Souslin}}$-name for a forcing poset $\dot{\mathbb{Q}}_{\mathrm{Souslin}}$ such that*:

- *Forcing with $\mathbb{P}_{\mathrm{Souslin}}$ adds a λ-Souslin tree T.*
- *$\mathbb{P}_{\mathrm{Souslin}}$ forces that $\mathbb{Q}_{\mathrm{Souslin}}$ is λ-cc.*
- *Forcing with $\mathbb{Q}_{\mathrm{Souslin}}$ over the $\mathbb{P}_{\mathrm{Souslin}}$-generic extension adds a branch through T.*
- *The poset $\mathbb{P}_{\mathrm{Souslin}} * \dot{\mathbb{Q}}_{\mathrm{Souslin}}$ has a λ-closed dense subset.*

Conditions in $\mathbb{P}_{\mathrm{Souslin}}$ are normal trees of successor height less than λ, with a strong homogeneity property (cf. [19, pp. 69]). $\mathbb{Q}_{\mathrm{Souslin}}$ is just forcing with the generic Souslin tree added by $\mathbb{P}_{\mathrm{Souslin}}$. The homogeneity property ensures that $\mathbb{P}_{\mathrm{Souslin}} * \mathbb{Q}_{\mathrm{Souslin}}$ has a dense subset which is λ-closed.

In the context of Facts 1.3 and 1.4, if $\lambda = \lambda^{<\lambda}$ then we may find a dense subset of $\mathbb{P} * \dot{\mathbb{Q}}$ with cardinality λ, and argue that $\mathbb{P} * \dot{\mathbb{Q}}$ is equivalent to $\mathrm{Add}(\lambda, 1)$. We also note that:

- The forcing poset $\mathbb{Q}_{\mathrm{Souslin}}$ has λ-cc because it is a λ-Souslin tree.
- The definitions of the forcing posets $\mathbb{P}_{\mathrm{Souslin}}$ and $\mathbb{P}_{\mathrm{NRSS}}$ depend only on the bounded subsets of λ, so these posets will be computed in the same way by a generic extension which adds no bounded subsets of λ.
- If we force with $\mathbb{P}_{\mathrm{NRSS}} \times \mathbb{P}_{\mathrm{Souslin}}$, then we may view the result as a two-step iterated forcing with $\mathbb{P}_{\mathrm{NRSS}}$ then $\mathbb{P}_{\mathrm{Souslin}}$ as defined in the extension by $\mathbb{P}_{\mathrm{NRSS}}$, and vice versa. It follows easily that S is a Souslin tree and T a nonreflecting stationary set in the extension by $\mathbb{P}_{\mathrm{NRSS}} \times \mathbb{P}_{\mathrm{Souslin}}$.
- Forcing with $\mathbb{P}_{\mathrm{NRSS}} \times \mathbb{P}_{\mathrm{Souslin}}$ followed by $\mathbb{Q}_{\mathrm{NRSS}} \times \mathbb{Q}_{\mathrm{Souslin}}$ is equivalent to $\mathrm{Add}(\lambda, 1)$.

FACT 1.5 (Baumgartner, [2]). *Let $\mu < \lambda$ with μ regular and λ weakly compact. Then after forcing with the Lévy collapse* $\mathrm{Coll}(\mu, <\lambda)$, *we have that $\lambda = \mu^+$ and every stationary subset of $\mu^+ \cap \mathrm{cof}(<\mu)$ reflects at a point of cofinality μ.*

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The following fact is crucial to our analysis of the tree and approachability properties.

FACT 1.6 (Branch Lemmas). *Suppose that T is a tree of height δ where δ has cofinality κ^+. Then*:

- (Silver, [28]). *If the levels of T have size less than 2^κ and $\mathbb{P}$ is κ^+-closed then $\mathbb{P}$ adds no new cofinal branches through T.*
- (Unger, [30]). *If $\mathbb{P}$ is a forcing whose square $\mathbb{P}^2$ has the κ^+-cc then $\mathbb{P}$ adds no new cofinal branches through T.*

REMARK 1.7. Using Easton's Lemma, Fact 1.6 implies that if T is a tree of height κ^+ whose levels have size less than 2^κ, $\mathbb{P}$ is a poset which is κ^+-closed, and $\mathbb{Q}$ is a poset whose square $\mathbb{Q}^2$ is κ^+-cc, then $\mathbb{P} \times \mathbb{Q}$ adds no new cofinal branches through T.

§2. The main results.

Let μ be a successor cardinal. We will consider three combinatorial assertions about μ^+:

TP: The cardinal μ^+ has the tree property.
RP: Every stationary subset of $\mu^+ \cap \mathrm{cof}(< \mu)$ reflects at a point of cofinality μ.
AP: The cardinal μ^+ has the approachability property, that is $\mu^+ \in I[\mu^+]$.

THEOREM 2.1. *Let κ be a regular cardinal with $\kappa^{<\kappa} = \kappa$ and let $\mu = \kappa^+$. Then (assuming the existence of suitable large cardinals above κ) for each Boolean combination Φ of* TP, AP *and* RP *there exists a generic extension in which cardinals up to and including μ are preserved and Φ holds.*

THEOREM 2.2. *Let κ be a measurable cardinal, $\lambda > \kappa$ weakly compact and let $\mu = \kappa^+$. Then assuming that κ remains measurable after forcing with* $\mathrm{Add}(\kappa, \lambda)$, *for each Boolean combination Φ of* TP, AP *and* RP *there exists a generic extension in which cardinals up to and including μ are preserved, κ is singular of cofinality ω, and Φ holds.*

REMARK 2.3. Theorem 2.2 slightly understates our results. Certain "easy" Boolean combinations can be achieved starting with an arbitrary singular cardinal κ. We discuss this further below.

2.1. Easy cases.

not-TP + AP ± RP. If $2^\kappa = \kappa^+$ then $\mu = \mu^{<\mu}$, so that $\square^*_\mu$ holds and we have not-TP and AP. The following two arguments work uniformly for all κ, without the need to distinguish the cases "κ is regular" and "κ is singular".

► Construction for not-TP + AP + not-RP: Start by forcing with $\mathrm{Add}(\kappa^+, 1)$, so that in the extension $2^\kappa = \kappa^+$. Then force to add a nonreflecting stationary subset of $\kappa^{++} \cap \mathrm{cof}(\omega)$.

► Construction for not-TP + AP + RP: Start with $\lambda > \kappa$ and λ weakly compact, and force with $\mathrm{Coll}(\kappa^+, < \lambda)$. In the extension $2^\kappa = \kappa^+$, $\lambda = \kappa^{++}$ and every stationary subset of $\kappa^{++} \cap \mathrm{cof}(\leq \kappa)$ reflects to a point of cofinality κ^+.

2.2. Harder cases.

(TP or not-AP) + RP; not-TP + not-AP + not-RP. To arrange the tree property or failure of the approachability property we will use variants of Mitchell forcing [21]. For the moment we suppress most of the details, which we defer until Section 3. All of our Mitchell forcing variants will have the following properties in common:

(1) The conditions are defined from cardinal parameters κ and λ, where $\kappa < \lambda$ with κ regular and λ a large cardinal.
(2) The conditions are (at least morally) bounded subsets of λ, and the definition of the forcing conditions and the ordering will not change in a generic extension with the same bounded subsets of λ.
(3) They preserve cardinals up to and including κ^+, and force that $2^\kappa = \lambda = \kappa^{++}$. See Section 3.4.
(4) Assuming that λ is at least weakly compact, they force that every stationary subset of $\lambda \cap \mathrm{cof}(\leq \kappa)$ reflects at some point of cofinality κ^+. See Section 3.7.
(5) Assuming that λ is at least weakly compact, they preserve the tree property at λ. See Section 3.6.

To prove Theorem 2.1 we will assume that $\kappa^{<\kappa} = \kappa$, and in this case our Mitchell forcing variants will be κ-closed. To prove Theorem 2.2 we will assume that κ is measurable, and in this case our Mitchell forcing variants will singularise κ.

For the purposes of this section we will use forcing posets which we call $\mathbb{M}_0$ and $\mathbb{M}_1$, whose exact definitions will be given later in Section 3. They will have in common the properties 1–5 listed above, but an important difference (see Section 3.5) is that:

(1) The poset $\mathbb{M}_0$ forces that $\lambda \in I[\lambda]$.
(2) Assuming that λ is at least Mahlo, the poset $\mathbb{M}_1$ forces that $\lambda \notin I[\lambda]$.

► Construction for TP + AP + RP: Start with λ weakly compact and force with $\mathbb{M}_0$.

► Construction for TP + not-AP + RP: Start with λ weakly compact and force with $\mathbb{M}_1$.

► Construction for not-TP + not-AP + RP: Start with λ weakly compact, do an Easton support iteration adding a Cohen subset to each inaccessible less than λ, and denote the resulting model by V. It is routine to check λ is Mahlo in V and weakly compact in $V^{\mathrm{Add}(\lambda,1)}$.

We use the poset $\mathbb{P}_{\mathrm{Souslin}}$ for adding a λ-Souslin tree from Fact 1.4. We will force with $\mathbb{P}_{\mathrm{Souslin}} \times \mathbb{M}_1$, and note that since $\mathbb{P}_{\mathrm{Souslin}}$ adds no bounded subsets of λ we may view this as an iteration where we force with $\mathbb{P}_{\mathrm{Souslin}}$ and then with $\mathbb{M}_1$ as computed in $V^{\mathbb{P}_{\mathrm{Souslin}}}$.

- Since $\mathbb{P}_{\mathrm{Souslin}}$ embeds into $\mathrm{Add}(\lambda, 1)$, λ is still Mahlo after forcing with $\mathbb{P}_{\mathrm{Souslin}}$ and hence we have not-AP in the final model.
- Since $(\mathbb{M}_1)^2$ is λ-cc in $V^{\mathbb{P}_{\mathrm{Souslin}}}$, it follows from Fact 1.6 that the Souslin tree added by $\mathbb{P}_{\mathrm{Souslin}}$ is still an Aronszajn tree after forcing with $\mathbb{M}_1$ and hence we have not-TP in the final model.
- Recall that there is $\mathbb{Q}_{\mathrm{Souslin}} \in V^{\mathbb{P}_{\mathrm{Souslin}}}$ such that $\mathbb{P}_{\mathrm{Souslin}} * \dot{\mathbb{Q}}_{\mathrm{Souslin}}$ is equivalent to forcing with $\mathrm{Add}(\lambda, 1)$. Forcing with $\mathbb{Q}_{\mathrm{Souslin}}$ we obtain the model $V^{\mathrm{Add}(\lambda,1)\times\mathbb{M}_1}$, which we may again view as the extension by an iteration where we force with $\mathrm{Add}(\lambda, 1)$ and then with $\mathbb{M}_1$ as computed in $V^{\mathrm{Add}(\lambda,1)}$.
- Since λ is weakly compact in $V^{\mathrm{Add}(\lambda,1)}$, it follows from Clause (2.2) in the list of properties of our Mitchell variant, that RP holds in the extension by $\mathrm{Add}(\lambda, 1) \times \mathbb{M}_1$. Now, let S be a stationary subset of $\lambda \cap \mathrm{cof}(\leq \kappa)$ in the

https://doi.org/10.1017/jsl.2017.69 Published online by Cambridge University Press

extension by $\mathbb{P}_{\text{Souslin}} \times \mathbb{M}_1$. Then since $\mathbb{Q}_{\text{Souslin}}$ is λ-cc, it preserves the stationarity of S. Since $\mathbb{P}_{\text{Souslin}} * \mathbb{Q}_{\text{Souslin}}$ is equivalent to $\text{Add}(\lambda, 1)$, and since RP holds in the extension of V by $\text{Add}(\lambda, 1) \times \mathbb{M}_1$, S has a stationary initial segment in this extension, and this initial segment is stationary in the intermediate extension by $\mathbb{P}_{\text{Souslin}} \times \mathbb{M}_1$.

► Construction for not-TP + not-AP + not-RP: We start with λ Mahlo and force with the product $\mathbb{P}_{\text{NRSS}} \times \mathbb{P}_{\text{Souslin}} \times \mathbb{M}_1$. Since none of these posets add bounded subsets of λ, this can be construed as an iteration in any order we please, so easily not-TP and not-RP hold. Also λ is still Mahlo in the extension by $\mathbb{P}_{\text{NRSS}} \times \mathbb{P}_{\text{Souslin}}$, so that in the final model we also have not-AP.

2.3. Hardest cases. TP $\pm$ AP + not-RP. To construct models of TP + not-RP, we will force with $\mathbb{M}_0$ or $\mathbb{M}_1$ as in the previous subsection and then add a nonreflecting stationary set by forcing with $\mathbb{P}_{\text{NRSS}}$ from Fact 1.3. In order to see that the tree property holds and that we have the desired control over the approachability property, we will use some more facts (see Section 3.5) about the posets $\mathbb{M}_i$, namely that the models which they produce are "robust under Cohen forcing". More precisely if λ is at least weakly compact then:

- In the extension by $\mathbb{M}_0 * \text{Add}(\lambda, 1)$, we have TP + AP.
- In the extension by $\mathbb{M}_1 * \text{Add}(\lambda, 1)$, we have TP + not-AP.

To get TP to hold in the extension by $\mathbb{M}_i * \mathbb{P}_{\text{NRSS}}$, we need another *branch lemma* saying that $\mathbb{Q}_{\text{NRSS}}$ does not add branches through a λ-Aronszajn tree. The proofs are slightly different in the cases where κ is regular and κ is singular (note that if we are proving Theorem 2.2 then κ is singular after forcing with $\mathbb{M}_i$), and accordingly we state and prove two versions of the branch lemma.

Both versions use the following standard fact [20]. If γ and δ are cardinals with $\gamma < \delta$, U is a δ-Aronszajn tree and $\mathbb{Q}$ is a forcing poset which adds a branch $\dot{b}$ through U, then for every condition $q \in \mathbb{Q}$ there are a level $\text{Lev}_\beta(U)$ of U and extensions $\langle r_i : i < \gamma \rangle$ of q such that

- Each condition r_i determines where $\dot{b}$ meets $\text{Lev}_\beta(U)$, say as a node s_i.
- For $i \neq j$, $s_i \neq s_j$.

LEMMA 2.4. *Let $\kappa^{<\kappa} = \kappa$ with $2^\kappa = \kappa^{++}$. Let T be a nonreflecting stationary subset of κ^{++}, let U be a κ^{++}-tree and let $\mathbb{Q}$ be the standard poset to add a club disjoint from T. Then $\mathbb{Q}$ does not add a cofinal branch in U.*

PROOF. Let $\dot{b}$ name for a branch through U. Let θ be a large enough regular cardinal and let $M \prec H_\theta$ contain everything relevant with $\kappa^+ + 1 \subseteq M$, $\alpha = M \cap \kappa^{++}$ an ordinal of cofinality κ^+, and ${}^{<\kappa}M \subseteq M$.

Since T is nonreflecting we may choose $C \subseteq \alpha$ a club set of order type κ^+ with C disjoint from T. Now we build a tree of conditions $\langle q_s : s \in 2^{<\kappa} \rangle$ together with a tree of nodes $\langle u_s : s \in 2^{<\kappa} \rangle$ such that:

(1) $q_s, u_s \in M$ for all $s \in 2^{<\kappa}$.
(2) q_s forces that $u_s \in \dot{b}$.
(3) $\max(q_s) \in C$.
(4) If t is an end-extension of s then $q_t \leq q_s$ and $u_s \leq_U u_t$.
(5) For all s, $u_{s\frown 0}$ and $u_{s\frown 1}$ are incomparable in U.

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The construction is simple: at successor stages we appeal to the "branch splitting" fact above, and for s of limit length we define q_s by forming the union of q_t for t a proper initial segment of s and then adding $\beta = \sup_t \max(q_t)$ as the top point: this gives a condition because $\beta \in C$ and hence $\beta \notin T$, and it gives an element of M because ${}^{<\kappa}M \subseteq M$.

When the construction is done we proceed as follows, noting that every node u_s is in M so lies below level α of U. For every $x \in 2^\kappa$ we choose q_x such that $q_x \leq q_{x \restriction j}$ for every $j < \kappa$, and then extend q_x to r_x which decides where $\dot{b}$ meets $\mathrm{Lev}_\alpha(U)$. By construction the nodes r_x are distinct, and since $2^\kappa = \kappa^{++}$ and U is a κ^{++}-tree this is a contradiction. ⊣

Lemma 2.5. *Let κ be singular of cofinality ω with $2^\kappa = \kappa^\omega = \kappa^{++}$. Let T be a nonreflecting stationary subset of κ^{++}, let U be a κ^{++}-Aronszajn tree and let $\mathbb{Q}$ be the standard poset to add a club disjoint from T. Then $\mathbb{Q}$ does not add a cofinal branch in U.*

Proof. Let $\dot{b}$ name for a branch through U, and fix $\langle \kappa_n : n < \omega \rangle$ a sequence of regular cardinals which is increasing and cofinal in κ. We choose M, α and C exactly as in the proof of Lemma 2.4, except that we drop the demand that ${}^{<\kappa}M \subseteq M$. Let I be the tree of finite sequences s such that $s(i) \in \kappa_i$ for $i < \ell(s)$.

We build a tree of conditions $\langle q_s : s \in I \rangle$ together with a tree of nodes $\langle u_s : s \in I \rangle$ such that:

(1) $q_s, u_s \in M$ for all $s \in I$.
(2) q_s forces that $u_s \in \dot{b}$.
(3) $\max(q_s) \in C$.
(4) If t is an end-extension of s then $q_t \leq q_s$ and $u_s \leq_U u_t$.
(5) For all s, the nodes $\langle u_{s ^\frown i} : i < \kappa_{\ell(s)} \rangle$ are pairwise incomparable in U.

The construction is basically as before, but we need no closure assumption on M since there are no limit stages. We finish as before by choosing lower bounds q_x for $x \in \prod_n \kappa_n$ and producing κ^ω many distinct points in $\mathrm{Lev}_\alpha(U)$. ⊣

► Construction for TP + AP + not-RP: Start with λ weakly compact and force with $\mathbb{M}_0 * \mathbb{P}_{\mathrm{NRSS}}$, where in distinction to our previous cases we compute $\mathbb{P}_{\mathrm{NRSS}}$ in the extension by $\mathbb{M}_0$. Clearly not-RP holds. Since AP holds in the extension by $\mathbb{M}_0$ and is upwards absolute to models with the same cardinals, AP also holds. Finally since TP holds in the extension by $\mathbb{M}_0 * \mathrm{Add}(\lambda, 1)$, it follows immediately from Lemma 2.4 or Lemma 2.5 that TP holds.

► Construction for TP + not-AP + not-RP: This is similar to the previous case, but this time we force with $\mathbb{M}_1 * \mathbb{P}_{\mathrm{NRSS}}$. We use that $\mathbb{P}_{\mathrm{NRSS}}$ preserves stationarity to preserve not-AP.

§3. Variants of Mitchell forcing. In this section we will construct the posets $\mathbb{M}_0$ and $\mathbb{M}_1$ used in proving most of the cases of Theorems 2.1 and 2.2. Before we define the relevant forcing posets, a few remarks:

- Mitchell [21] started with a large cardinal λ, which will be at least a Mahlo cardinal, and defined a forcing poset $\mathbb{M}$ such that in the final model ω_1 is preserved and $2^\omega = \lambda = \omega_2$. The key property of the poset $\mathbb{M}$ is that for many inaccessible $\alpha < \lambda$, there is an intermediate model $V[G^{\mathbb{M}}_\alpha]$ of the final generic

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extension $V[G^{\mathbb{M}}]$ such that $2^\omega = \alpha = \omega_2$ in $V[G^{\mathbb{M}}_\alpha]$, and the "tail forcing" $\mathbb{M}/G^{\mathbb{M}}_\alpha$ does not add any fresh subsets of α. Recall that a set $d \subset \alpha$ is *fresh* if for all $\eta < \alpha$, $d \cap \eta$ is in the ground model. The poset $\mathbb{M}$ is constructed using the standard Cohen poset $\mathrm{Add}(\omega, \lambda)$ to blow up the power set of ω; it is straightforward to replace ω by a regular cardinal κ such that $\kappa^{<\kappa} = \kappa$, and obtain a version of $\mathbb{M}$ which preserves cardinals up to κ^+ and forces $2^\kappa = \lambda = \kappa^{++}$.

Assuming that λ is Mahlo, forcing with $\mathbb{M}$ yields a model where $I[\aleph_2]$ is a proper ideal. The key point is that if $\langle x_\eta : \eta < \lambda\rangle$ enumerates $[\lambda]^{\aleph_0}$ in $V[G^{\mathbb{M}}]$, then there is α as above such that $\langle x_\eta : \eta < \alpha\rangle$ enumerates $[\alpha]^{\aleph_0}$ in $V[G^{\mathbb{M}}_\alpha]$. Then $\alpha \notin S(\vec{x})$, since otherwise an unbounded subset of α witnessing that $\alpha \in S(\vec{x})$ would be fresh. A very similar argument shows that this version of Mitchell's model has no special $\aleph_2$-Aronszajn tree.

To obtain a model with no $\aleph_2$-Aronszajn tree we need to strengthen the assumption on λ. Assuming that λ is weakly compact, suppose for contradiction that T is an $\aleph_2$-Aronszajn tree in $V[G^{\mathbb{M}}]$. Using the Π^1_1-indescribability of λ, we find α as above such that $T \upharpoonright \alpha$ is an $\aleph_2$-Aronszajn tree in $V[G^{\mathbb{M}}_\alpha]$. The tail forcing adds a cofinal branch in $T \upharpoonright \alpha$, but such a branch is fresh, an immediate contradiction.

- With a view to producing a model where both $\aleph_2$ and $\aleph_3$ have the tree property, Abraham [1] introduced several new ideas. In particular he introduced a wider class of "Mitchell type" forcing posets, and analysed the properties of these posets by representing them as projections of products of simpler posets. This gave in particular a new proof that if we build Mitchell's model with λ weakly compact then we obtain a model of the tree property; a key ingredient here is that if $2^\omega = \omega_2$ then countably closed forcing cannot add a new branch to an ω_2-tree. Another of Abraham's innovations was to define versions of Mitchell forcing with "lookahead"; the point here is to do constructions in which a forcing of Mitchell type which enforces the tree property is followed by further forcing, and the tree property is preserved.
- Cummings and Foreman [5] gave a model in which the tree property holds at κ^{++} for κ singular. Initially κ and λ are both large cardinals, and the ground model has been prepared so that forcing with $\mathrm{Add}(\kappa, \lambda)$ preserves the measurability of κ. In this type of forcing the two-step iteration $\mathrm{Add}(\kappa, \lambda) * \mathrm{Prikry}(U)$ (for U some normal measure on κ in the Add extension) plays the same role that $\mathrm{Add}(\kappa, \lambda)$ did in Mitchell's original forcing.

3.1. The forcing. We set up a general framework for defining "Mitchell type" forcing posets. This class of posets will be flexible enough to prove all the instances of Theorems 2.1 and 2.2 in which the tree property holds or the approachability property fails.

Let κ and λ be regular cardinals with $\kappa = \kappa^{<\kappa} < \lambda$. We assume that λ is Mahlo (and sometimes weakly compact).

One parameter in the definition of our "Mitchell type" forcing poset will be a poset $\mathbb{P}$ for blowing up the power set of κ to λ. Specifically, we shall use forcing posets $\mathbb{P}$ such that:

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(1) $|\mathbb{P}| = \lambda$.
(2) $\mathbb{P}$ adds λ subsets of κ.
(3) $\mathbb{P}$ preserves κ.
(4) $\mathbb{P}$ has the κ^+-cc.
(5) $\mathbb{P}$ is the union of a $\subseteq$-increasing and continuous sequence $\langle \mathbb{P}_\alpha : \alpha < \lambda \rangle$, and there exists a set $A \subseteq \lambda$ such that:
 (a) A contains almost every point in $\lambda \cap \mathrm{cof}(> \kappa)$.
 (b) For every $\alpha \in A$, $\mathbb{P}_\alpha$ is a complete subposet of $\mathbb{P}$ (so that if $G^{\mathbb{P}}$ is $\mathbb{P}$-generic it induces a filter $G^{\mathbb{P}}_\alpha$ which is $\mathbb{P}_\alpha$-generic).
 (c) For every $\alpha \in A$:
  (i) $\mathbb{P}/G^{\mathbb{P}}_\alpha \times \mathbb{P}/G^{\mathbb{P}}_\alpha$ is κ^+-cc in $V[G^{\mathbb{P}}_\alpha]$.
  (ii) $P(\kappa) \cap V[G^{\mathbb{P}}_\alpha] \subsetneq P(\kappa) \cap V[G^{\mathbb{P}}_{\alpha^*}]$, where α^* is the successor of α in A.

Example 3.1. The two examples of most interest to us are $\mathbb{P} = \mathrm{Add}(\kappa, \lambda)$ and $\mathbb{P} = \mathrm{Add}(\kappa, \lambda) * \mathrm{Prikry}(\dot{U})$ where $\dot{U}$ is a name for a normal measure on κ in $V[\dot{G}^{\mathbb{P}}]$. In the first example, set $A = \lambda$ and $\mathbb{P}_\alpha = \mathrm{Add}(\kappa, \alpha)$ for all α. In the second example, we define A using the fact that for almost every α in $\lambda \cap \mathrm{cof}(> \kappa)$ we have $\Vdash_{\mathbb{P}} \dot{U} \cap V[\dot{G}^{\mathbb{P}}_\alpha] \in V[\dot{G}^{\mathbb{P}}_\alpha]$, and then set $\mathbb{P}_\alpha = \mathrm{Add}(\kappa, \alpha) * \mathrm{Prikry}(\dot{U}_\alpha)$ where $\dot{U}_\alpha = \dot{U} \cap V[\dot{G}^{\mathbb{P}}_\alpha]$. See Unger's article [30] for a careful discussion, including proofs for all the needed properties of $\mathbb{P}$.

Let $\mathbb{B}$ be the regular open algebra of $\mathbb{P}$, and let $\mathbb{B}_\alpha$ be the regular open algebra of $\mathbb{P}_\alpha$ for $\alpha \in A$. Let π_α be the natural projection map from $\mathbb{B}$ to $\mathbb{B}_\alpha$, and note that if $\alpha < \beta$ with $\alpha, \beta \in A$ then $\pi_\alpha \upharpoonright \mathbb{B}_\beta$ is the natural projection map from $\mathbb{B}_\beta$ to $\mathbb{B}_\alpha$. The point of forming the Boolean algebras is that in the case when $\mathbb{P} = \mathrm{Add}(\kappa, \lambda) * \mathrm{Prikry}(\dot{U})$ we will not be able to form a projection of posets from $\mathbb{P}$ to $\mathbb{P}_\alpha$.

Remark 3.2. For use later, we note that if $\mathbb{Q}$ is κ^+-closed and H is $\mathbb{Q}$-generic then:

- By Easton's Lemma, $\mathbb{P}$ is κ^+-cc in $V[H]$.
- The antichains of $\mathbb{P}$ in $V[H]$ all lie in V, and so (since elements of $\mathbb{B}$ can be understood as suprema of antichains) $\mathbb{B}$ is the Boolean completion of $\mathbb{P}$ in $V[H]$.
- Let $\alpha \in A$. Since V and $V[H]$ have the same antichains both for $\mathbb{P}$ and $\mathbb{P}_\alpha$, it is still the case in $V[H]$ that $\mathbb{P}_\alpha$ is a complete subposet of $\mathbb{P}$. Also $\mathbb{B}_\alpha$ is still the Boolean completion of $\mathbb{P}_\alpha$ and π_α is still the natural projection map.
- By Easton's Lemma again, if $G^{\mathbb{P}}_\alpha$ is $\mathbb{P}_\alpha$-generic over V then $G^{\mathbb{P}}_\alpha$ is $\mathbb{P}_\alpha$-generic over $V[H]$, and the quotient $\mathbb{B}/G^{\mathbb{P}}_\alpha$ is computed in the same way by $V[G^{\mathbb{P}}_\alpha]$ and $V[H][G^{\mathbb{P}}_\alpha]$.
- Since $\mathbb{P}_\alpha * (\mathbb{P}/\dot{G}^{\mathbb{P}}_\alpha \times \mathbb{P}/\dot{G}^{\mathbb{P}}_\alpha)$ is κ^+-cc, by Easton's Lemma it is κ^+-cc in $V[H]$, so that $\mathbb{P}/G^{\mathbb{P}}_\alpha \times \mathbb{P}/G^{\mathbb{P}}_\alpha$ is κ^+-cc in $V[H][G^{\mathbb{P}}_\alpha]$.

We use $\mathbb{P}$ to define two versions of Mitchell forcing, $\mathbb{M}_0$ and $\mathbb{M}_1$. They will have the following features in common:

(1) $\mathbb{M}_i$ is a λ-cc forcing poset.
(2) $\mathbb{M}_i$ projects to $\mathbb{P}$, and if $G^{\mathbb{M}_i}$ is $\mathbb{M}_i$-generic then $P(\kappa) \cap V[G^{\mathbb{M}_i}] = P(\kappa) \cap V[G^{\mathbb{P}}]$ where $G^{\mathbb{P}}$ is the projected $\mathbb{P}$-generic filter.

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(3) $\mathbb{M}_i$ preserves cardinals up to and including κ^+, and forces that $2^\kappa = \lambda = \kappa^{++}$.
(4) If λ is weakly compact then $\mathbb{M}_i$ preserves the tree property at λ, and forces that every stationary subset of $\lambda \cap \mathrm{cof}(\leq \kappa)$ reflects at a point of cofinality κ^+.
(5) Conditions in $\mathbb{M}_i$ have three coordinates: the first coordinate is drawn from $\mathbb{P}$ and is responsible for blowing up the power set of κ, the second coordinate is responsible for collapsing cardinals between κ^+ and λ, while the third coordinate is responsible for ensuring that certain combinatorial facts hold in the extension by $\mathbb{M}_i * \mathrm{Add}(\lambda, 1)$.

The key difference between the posets $\mathbb{M}_i$ is in the way the approachability ideal $I[\lambda]$ looks in the corresponding generic extensions. In the extension by $\mathbb{M}_0$, $\lambda \in I[\lambda]$, while in the extension by $\mathbb{M}_1$ we will have that A is stationary and $A \notin I[\lambda]$.

3.2. The forcing poset $\mathbb{M}_0$. Formally we will define $\mathbb{M}_0$ by giving an inductive definition of $\mathbb{M}_0 \restriction \beta$ for $\beta \in A \cup \{\lambda\}$, and then setting $\mathbb{M}_0 = \mathbb{M}_0 \restriction \lambda$.

Conditions in $\mathbb{M}_0 \restriction \beta$ have the form (b, q, r) where:

(1) $b \in \mathbb{B}_\beta$, $b \neq 0$.
(2) $\mathrm{dom}(q) \subseteq A \cap \beta$, and $|\mathrm{dom}(q)| \leq \kappa$.
(3) $q(\alpha)$ is a $\mathbb{P}_\alpha$-name for a condition in $\mathrm{Coll}(\kappa^+, \alpha)^{V^{\mathbb{P}_\alpha}}$.
(4) $\mathrm{dom}(r) \subseteq A \cap \beta$, and $\mathrm{dom}(r)$ is an Easton set of regular cardinals.
(5) $r(\alpha)$ is an $\mathbb{M}_0 \restriction \alpha$-name for a condition in $\mathrm{Add}(\alpha, 1)^{V^{\mathbb{M}_0 \restriction \alpha}}$.

Conditions in $\mathbb{M}_0 \restriction \beta$ are ordered as follows: $(b', q', r') \leq (b, q, r)$ if and only if:

(1) $b' \leq b$ in $\mathbb{B}_\beta$.
(2) $\mathrm{dom}(q) \subseteq \mathrm{dom}(q')$, and $\pi_\alpha(b') \Vdash q'(\alpha) \leq q(\alpha)$ for all $\alpha \in \mathrm{dom}(q)$.
(3) $\mathrm{dom}(r) \subseteq \mathrm{dom}(r')$, and $(\pi_\alpha(b'), q' \restriction \alpha, r' \restriction \alpha) \Vdash r'(\alpha) \leq r(\alpha)$ for every $\alpha \in \mathrm{dom}(r)$.

3.3. The forcing poset $\mathbb{M}_1$. Let A^* be the set of successor points in A. We define $\mathbb{M}_1$ exactly the same as $\mathbb{M}_0$ except that in (2), A is replaced by A^*. More precisely:

Conditions in $\mathbb{M}_1 \restriction \beta$ have the form (b, q, r) where:

(1) $b \in \mathbb{B}_\beta$, $b \neq 0$.
(2) $\mathrm{dom}(q) \subseteq A^* \cap \beta$, and $|\mathrm{dom}(q)| \leq \kappa$.
(3) $q(\alpha)$ is a $\mathbb{P}_\alpha$-name for a condition in $\mathrm{Coll}(\kappa^+, \alpha)^{V^{\mathbb{P}_\alpha}}$.
(4) $\mathrm{dom}(r) \subseteq A \cap \beta$, and $\mathrm{dom}(r)$ is an Easton set of regular cardinals.
(5) $r(\alpha)$ is an $\mathbb{M}_1 \restriction \alpha$-name for a condition in $\mathrm{Add}(\alpha, 1)^{V^{\mathbb{M}_1 \restriction \alpha}}$.

The ordering is exactly as for $\mathbb{M}_0$.

Let $\alpha \in A$ be regular and let α^* be the successor of α in A. It is instructive to compare $\mathbb{M}_i \restriction \alpha$ and $\mathbb{M}_i \restriction \alpha^*$ for $i = 0$ and $i = 1$. In the case $i = 0$, the extension by $\mathbb{M}_0 \restriction \alpha^*$ is obtained from the extension by $\mathbb{M}_0 \restriction \alpha$ as follows: we are essentially forcing with the product of $\mathbb{P}_{\alpha^*}/G_\alpha^{\mathbb{P}}$, $\mathrm{Coll}(\kappa^+, \alpha)^{V[G_\alpha^{\mathbb{P}}]}$ and $\mathrm{Add}(\alpha, 1)^{V[G_\alpha^{\mathbb{M}_0}]}$. In the case $i = 1$ the "collapse" factor is missing. The point of adding a Cohen generic subset of α at stage α, which we do both for $i = 0$ and $i = 1$, is to ensure stationary reflection and the tree property in the extension by $\mathbb{M}_i * Add(\lambda, 1)$.

For the nonapproachability argument for $\mathbb{M}_1$ it will be helpful to make two further assumptions about the set A (easily obtained by thinning): We require that

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successor points of A are inaccessible (recall that we are assuming that λ is at least Mahlo and therefore a limit of inaccessibles) and also that for any α in A, $P(\kappa) \cap V[G^{\mathbb{P}}_{\beta}] \subsetneq P(\kappa) \cap V[G^{\mathbb{P}}_{\alpha}]$ for any ordinal $\beta < \alpha$. This will ensure that the forcings $\mathbb{M}_i \restriction \alpha^*$ preserve the regularity of α^* and force $2^\kappa = \alpha^*$ for successor elements α^* of A.

3.4. Common properties of $\mathbb{M}_0$ and $\mathbb{M}_1$. The forcing posets $\mathbb{M}_0$ and $\mathbb{M}_1$ are very similar to forcing posets defined by Abraham [1] and Cummings and Foreman [5]. Accordingly we omit some proofs and refer the reader to those articles

The following lemma is straightforward:

Lemma 3.3. *$|\mathbb{M}_i| = \lambda$, and $\mathbb{M}_i$ has the λ-Knaster property.* $\dashv$

We will use various projection maps in our arguments. Many of the facts about projections that we will need hinge on an easy general fact about two-step iterated forcing.

Lemma 3.4 (Laver). *Let $\mathbb{A} * \dot{\mathbb{B}}$ be a two-step iteration. If $(a_1, \dot{b}_1) \leq (a_0, \dot{b}_0)$ then there is an $\mathbb{A}$-name $\dot{b}_1^*$ such that $\Vdash_{\mathbb{A}} \dot{b}_1^* \leq \dot{b}_0$ and $a_1 \Vdash \dot{b}_1 = \dot{b}_1^*$ (so that the conditions $(a_1, \dot{b}_1)$ and $(a_1, \dot{b}_1^*)$ are equivalent in $\mathbb{A} * \dot{\mathbb{B}}$).*

Proof. By the Maximum Principle we find a name $\dot{b}_1^*$ for the condition in $\mathbb{B}$ which is the interpretation of $\dot{b}_1$ by $G^{\mathbb{A}}$ when $a_1 \in G^{\mathbb{A}}$, and is the interpretation of $\dot{b}_0$ if $a_1 \notin G^{\mathbb{A}}$ and $a_0 \in G^{\mathbb{A}}$. Otherwise, let $\dot{b}_1^*$ name the empty condition. $\dashv$

Laver defined a "term forcing" $\mathcal{A}(\mathbb{A}, \dot{\mathbb{B}})$, where the conditions are $\mathbb{A}$-names for elements of $\mathbb{B}$ ordered by $\dot{b}_1 \leq \dot{b}_0 \iff \Vdash_{\mathbb{A}} \dot{b}_1 \leq \dot{b}_0$. It follows from Lemma 3.4 that the identity map is a projection from $\mathbb{A} \times \mathcal{A}(\mathbb{A}, \dot{\mathbb{B}})$ to $\mathbb{A} * \dot{\mathbb{B}}$. Foreman [9] gave a detailed discussion of the properties of term forcing.

Following Abraham [1] we analyse the posets $\mathbb{M}_i$ using various term forcings and projections. For $i \in \{0, 1\}$ and $a \subseteq \{0, 1, 2\}$ let $\mathbb{M}_i^a$ be the set of conditions in $\mathbb{M}_i$ which are *trivial* at coordinates outside a, where the trivial value is $1_{\mathbb{B}}$ on coordinate zero and the empty function 0 on coordinates 1 and 2. We order $\mathbb{M}_i^a$ with the ordering inherited from $\mathbb{M}_i$. In an abuse of notation we omit parentheses and commas in the superscripts, so that $\mathbb{M}_0^{12}$ is shorthand for $\mathbb{M}_0^{\{1,2\}}$.

For example $\mathbb{M}_i^2$ can be viewed as an Easton support product of increasingly closed term forcings, while $\mathbb{M}_i^1$ is essentially a κ-support product of κ^+-closed term forcings. In particular $\mathbb{M}_i^2$ is $\min(A)$-closed and $\mathbb{M}_i^1$ is κ^+-closed. Note that $\mathbb{M}_i^0 \simeq \mathbb{P}$.

We summarise the various projection facts that we will need in a lemma.

Lemma 3.5. *Let $\mathbb{M}_i$ and $\mathbb{M}_i^a$ be as defined above. Then:*

(1) *The map $((b, q, 0), (1, 0, r)) \mapsto (b, q, r)$ is a projection from $\mathbb{M}_i^{01} \times \mathbb{M}_i^2$ to $\mathbb{M}_i$.*
(2) *The map $((b, 0, 0), (1, q, 0)) \mapsto (b, q, 0)$ is a projection from $\mathbb{M}_i^0 \times \mathbb{M}_i^1$ to $\mathbb{M}_i^{01}$.*
(3) *The map $((b, 0, 0), (1, q, 0), (1, 0, r)) \mapsto (b, q, r)$ is a projection from $\mathbb{M}_i^0 \times \mathbb{M}_i^1 \times \mathbb{M}_i^2$ to $\mathbb{M}_i$.*
(4) *The map $(b, q, r) \mapsto (b, q, 0)$ is a projection from $\mathbb{M}_i$ to $\mathbb{M}_i^{01}$.*
(5) *The map $(b, q, r) \mapsto (b, 0, 0)$ is a projection from $\mathbb{M}_i$ to $\mathbb{M}_i^0$.*
(6) *The map $(b, q, r) \mapsto (\pi_\alpha(b), q \restriction \alpha, r \restriction \alpha)$ is a projection from $\mathbb{M}_i$ to $\mathbb{M}_i \restriction \alpha$.*
(7) *The map $(b, q, r) \mapsto ((\pi_\alpha(b), q \restriction \alpha, r \restriction \alpha), r(\alpha))$ is a projection from $\mathbb{M}_i$ to $\mathbb{M}_i \restriction \alpha * \dot{\mathrm{Add}}(\alpha, 1)$.*

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(8) *For $i = 0$, the map $(b, q, r) \mapsto ((\pi_\alpha(b), q(\alpha))$ is a projection from $\mathbb{M}_i$ to $\mathbb{P}_\alpha * \mathrm{Coll}(\kappa^+, \alpha)^{V^{\mathbb{P}_\alpha}}$ for all $\alpha \in A$, while for $i = 1$ the same map is a projection for all $\alpha \in A^*$.*

Proof. We give a proof only for the first projection fact, the second is similar and the remainder are completely straightforward. The given map is clearly order-preserving, and maps the weakest condition to the weakest condition. To verify that it is a projection, let $(b', q', r') \leq (b, q, r)$ in $\mathbb{M}_i$. Appealing to Lemma 3.4, define r^* such that $(1, 0, r^*) \in \mathbb{M}_i^2$ as follows: $\mathrm{dom}(r^*) = \mathrm{dom}(r')$, $r^*(\alpha)$ is a term such that $\Vdash r^*(\alpha) \leq r(\alpha)$ for all $\alpha \in \mathrm{dom}(r)$ and $(\pi_\alpha(b'), q' \restriction \alpha, r' \restriction \alpha) \Vdash r^*(\alpha) = r'(\alpha)$ for all $\alpha \in \mathrm{dom}(r')$. Now $(b', q', 0) \leq (b, q, 0)$ in $\mathbb{M}_i^{01}$, $(1, 0, r^*) \leq (1, 0, r)$ in $\mathbb{M}_i^2$, and $(b', q', r^*) \leq (b'q', r')$ in $\mathbb{M}_i$. ⊣

Using the various projection facts and Easton's Lemma we get an important conclusion:

Lemma 3.6. *All κ-sequences of ordinals from $V[G^{\mathbb{M}_i}]$ lie in $V[G^{\mathbb{P}}]$.*
In particular, (by the κ^+-cc of $\mathbb{P}$) every set of ordinals of size κ in $V[G^{\mathbb{M}_i}]$ is covered by a set of size κ in V. ⊣

As a consequence $\mathbb{M}_i$ preserves κ and κ^+. Since there is a projection from $\mathbb{M}_i$ to $\mathbb{P}_\alpha * \mathrm{Coll}(\kappa^+, \alpha)^{V^{\mathbb{P}_\alpha}}$ for all $\alpha \in A^*$, $\mathbb{M}_i$ collapses all cardinals between κ^+ and λ, while λ is preserved by the λ-cc. The upshot is that $2^\kappa = \kappa^{++} = \lambda$ after forcing with $\mathbb{M}_i$.

Suppose that $\alpha \in A$ is an inaccessible limit point of A. Then $\mathbb{M}_i \restriction \alpha$ preserves the regularity of α, as by the above, the forcing $\mathbb{M}_i \restriction \alpha$ is the projection of the product of $\mathbb{M}_i^2 \restriction \alpha$, an Easton product which preserves cofinalities and the inaccessibility of α, and $\mathbb{M}_i^{01} \restriction \alpha$, which is α-cc forcing after forcing with $\mathbb{M}_i^2 \restriction \alpha$. We also have $2^\kappa = \alpha$ in the extension by $\mathbb{M}_i \restriction \alpha$. If α^* is the least element of A greater than α, then $\mathbb{M}_0 \restriction \alpha^*$ forces $2^\kappa = \alpha^* = \kappa^{++}$ as it collapses α, but $\mathbb{M}_1 \restriction \alpha^*$ forces that $2^\kappa = \alpha^*$ is a regular cardinal greater than $\kappa^{++} = \alpha$, as it does not collapse α.

Let $G^{\mathbb{M}_i}$ be $\mathbb{M}_i$-generic and let $G_\alpha^{\mathbb{M}_i}$ be the induced generic object for $\mathbb{M}_i \restriction \alpha$. We wish to argue that $\mathbb{M}_i / G_\alpha^{\mathbb{M}_i}$ is equivalent to a forcing poset with a similar definition to $\mathbb{M}_i$, computed in $V[G_\alpha^{\mathbb{M}_i}]$. The argument is very similar to the standard argument that a final segment of an iterated forcing poset can itself be viewed as an iterated forcing poset [3], so we have omitted some details about translation of names.

Working in $V[G_\alpha^{\mathbb{M}_i}]$, we define a poset $\mathbb{N}_{i,\alpha}$ as follows: conditions are triples (b, q, r) where

(1) $b \in \mathbb{B}/G_\alpha^{\mathbb{P}}$, $b \neq 0$.
(2) $\mathrm{dom}(q) \subseteq A \setminus \alpha$ (if $i = 0$) or $\mathrm{dom}(q) \subseteq A^* \setminus \alpha$ (if $i = 1$), and $|\mathrm{dom}(q)| \leq \kappa$.
(3) $q(\eta)$ is a $\mathbb{P}_\eta / G_\alpha^{\mathbb{P}}$-name for a condition in $\mathrm{Coll}(\kappa^+, \eta)^{V^{\mathbb{P}_\eta}}$.
(4) $\mathrm{dom}(r) \subseteq A \setminus \alpha$, and $\mathrm{dom}(r)$ is an Easton set of regular cardinals.
(5) $r(\eta)$ is a $\mathbb{M}_i \restriction \eta / G_\alpha^{\mathbb{M}_i}$-name for a condition in $\mathrm{Add}(\eta, 1)^{V^{\mathbb{M}_i \restriction \eta}}$.

There is a natural order-preserving map from $\mathbb{M}_i$ to $\mathbb{M}_i \restriction \alpha * \dot{\mathbb{N}}_{i,\alpha}$; a condition (b, q, r) is mapped to the pair with first entry $(\pi_\alpha(b), q \restriction \alpha, r \restriction \alpha)$ and second entry $(\dot{b}_\alpha, q' \restriction [\alpha, \lambda), r' \restriction [\alpha, \lambda))$, where $\dot{b}_\alpha$ names the image of b in the canonical map from $\mathbb{B}$ to $\mathbb{B}/G_\alpha^{\mathbb{P}}$ and $q'(\eta), r'(\eta)$ are appropriately translated version of the names $q(\eta), r(\eta)$.

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The key point is that this order-preserving map has a dense image. The only potential issue is that in a $\mathbb{M}_i \upharpoonright \alpha$-name for a condition in $\mathbb{N}_{i,\alpha}$, the supports of the q-part and the r-part may not lie in V. However:

(1) Since every set of size κ in $V[G_\alpha^{\mathbb{M}_i}]$ is covered by a set of size κ in V, we may assume that the support of the q-part does lie in V.
(2) Since $\mathbb{M}_i \upharpoonright \alpha$ has size less than the least inaccessible limit point of $A \setminus \alpha$, every Easton subset of $A \setminus \alpha$ in $V[G_\alpha^{\mathbb{M}_i}]$ is covered by an Easton set in V, so we may also assume that the support of the r-part lies in V.

The rest of the argument is routine.

The poset $\mathbb{N}_{i,\alpha}$ is susceptible to the same kind of product analysis as $\mathbb{M}_i$. In particular, using the fact that $\mathbb{N}_{i,\alpha}^2$ is α-closed and $\alpha = \kappa^{++}$, we get that every κ^+-sequence of ordinals in $V[G^{\mathbb{M}_i}]$ is in the generic extension of $V[G_\alpha^{\mathbb{M}_i}]$ by $\mathbb{N}_{i,\alpha}^{01}$.

We record some technical facts for use later:

Lemma 3.7. *$\mathbb{P}/G_\alpha^{\mathbb{P}} \times \mathbb{P}/G_\alpha^{\mathbb{P}}$ is κ^+-cc in $V[G_\alpha^{\mathbb{M}_i}]$, and also in the extension of $V[G_\alpha^{\mathbb{M}_i}]$ by $\mathbb{N}_{i,\alpha}^1 \times \mathbb{N}_{i,\alpha}^2$.*

Proof. By the projection analysis, we can embed $V[G_\alpha^{\mathbb{M}_i}]$ in the extension of $V[G_\alpha^{\mathbb{P}}]$ by the poset $\mathbb{M}_{i,\alpha}^1 \times \mathbb{M}_{i,\alpha}^2$, which is κ^+-closed in V. By Remark 3.2, $\mathbb{P}/G_\alpha^{\mathbb{P}} \times \mathbb{P}/G_\alpha^{\mathbb{P}}$ is κ^+-cc in this extension, hence it is κ^+-cc in $V[G_\alpha^{\mathbb{M}_i}]$. By Easton's Lemma it is still κ^+-cc in the extension of $V[G_\alpha^{\mathbb{M}_i}]$ by the κ^+-closed forcing poset $\mathbb{N}_{i,\alpha}^1 \times \mathbb{N}_{i,\alpha}^2$. ⊣

We now analyse the status of AP, TP and RP in the extensions by $\mathbb{M}_i$ and $\mathbb{M}_i * \dot{\mathrm{Add}}(\lambda, 1)$. In the case of TP and RP, the main point is to show that (assuming that λ is weakly compact) TP and RP hold in the extension $\mathbb{M}_i * \dot{\mathrm{Add}}(\lambda, 1)$. Easy arguments will then show that these properties also hold in the extension by $\mathbb{M}_i$.

3.5. Approachability. We are finally ready to analyse the approachability ideal $I[\lambda]$ in $V[G^{\mathbb{M}_i}]$. By a result of Shelah [26] $\kappa^{++} \cap \mathrm{cof}(\leq \kappa) \in I[\kappa^{++}]$, so the only relevant question is about $\lambda \cap \mathrm{cof}(\kappa^+)$. We note that since $2^\kappa = \lambda$ there is a maximal stationary subset S of $\lambda \cap \mathrm{cof}(\kappa^+)$ that lies in $I[\lambda]$, which can be obtained as follows: enumerate $[\lambda]^{\leq\kappa}$ as $\vec{a} = \langle a_\zeta : \zeta < \lambda \rangle$, and define $S = S(\vec{a})$ to be the stationary set of points $\alpha \in \lambda \cap \mathrm{cof}(\kappa^+)$ such that there is $E \subseteq \alpha$ a cofinal set of order type κ^+ with $E \cap \eta \in \{a_\zeta : \zeta < \alpha\}$ for all $\eta < \alpha$. The set S is well-defined modulo the club filter, and every subset of $\lambda \cap \mathrm{cof}(\kappa^+)$ in $I[\lambda]$ is contained in S modulo clubs.

It will be convenient to organise the definition of S differently. By Lemma 3.6, all elements of $[\lambda]^{\leq\kappa}$ lie in $V[G^{\mathbb{P}}]$, so we assume that the enumeration $\langle a_\zeta : \zeta < \lambda \rangle$ lies in $V[G^{\mathbb{P}}]$. By the κ^+-cc of $\mathbb{P}$, for almost all $\alpha \in A$ of cofinality κ^+, $\{a_\zeta : \zeta < \alpha\} = [\lambda]^{\leq\kappa} \cap V[G_\alpha^{\mathbb{P}}]$, and hence we may as well redefine S as a subset of A, namely S is the set of $\alpha \in A$ such that there is $E \subseteq \alpha$ a cofinal set of order type κ^+ with $E \cap \eta \in V[G_\alpha^{\mathbb{P}}]$ for all $\eta < \alpha$. This definition is equivalent modulo clubs to the previous one.

We recall that in the generic extension $V[G^{\mathbb{M}_0}]$, for every $\alpha \in A$ the forcing adds a $\mathrm{Coll}(\kappa^+, \alpha)^{V[G_\alpha^{\mathbb{P}}]}$-generic object. This gives a cofinal map from κ^+ to α with every initial segment in $V[G_\alpha^{\mathbb{P}}]$, whose range will serve as a witness that $\alpha \in S$. So $S = A$ and it follows easily that $\lambda \in I[\lambda]$.

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We now consider the situation in $V[G^{\mathbb{M}_1}]$, where we will argue (assuming that λ is Mahlo) that there is a stationary subset of A which is disjoint from S, and hence $\lambda \notin I[\lambda]$.

Let B be the set of inaccessibles in A which are limit points of A.

Recalling our background assumption that λ is Mahlo, we see that B is stationary in the ground model V, and since $\mathbb{M}_1$ is λ-cc we also have that B is stationary in $V[G^{\mathbb{M}_1}]$. Recall that for α in B, $2^\kappa = \kappa^{++} = \alpha$ in the extension by $\mathbb{M}_i \restriction \alpha$ but if α^* is the A-successor of α, then $2^\kappa = \alpha^*$ is a regular cardinal greater than $\alpha = \kappa^{++}$ in the extension by $\mathbb{M}_1 \restriction \alpha^*$.

Lemma 3.8. *The stationary set B is disjoint from S.*

Proof. If $\alpha \in B \cap S$ then by definition there is $E \subseteq \alpha$ a cofinal set of order type κ^+ with $E \cap \eta \in V[G^{\mathbb{P}}_\alpha]$ for all $\eta < \alpha$. Let g enumerate E in increasing order, so that g is a cofinal map from κ^+ to α such that each of its proper initial segments is in $V[G^{\mathbb{P}}_\alpha]$.

Note that α is regular in the model $V[G^{\mathbb{M}_1} \restriction \alpha^*]$, where α^* is the A-successor to α, and therefore the function g is not a member of $V[G^{\mathbb{M}_1} \restriction \alpha^*]$.

Moreover, in this model $2^\kappa = \alpha^*$ is a regular cardinal greater than α.

The function g is added over $V[G^{\mathbb{M}_1} \restriction \alpha^*]$ by the forcing poset $\mathbb{N}_{1,\alpha^*}$, which by the projection analysis from Section 3.4 is the projection of the product of a forcing whose square is κ^+-cc with a forcing which is κ^+-closed. But then by Remark 1.7, this product could not add g, else it would add a new branch to the tree ${}^{<\kappa^+}\alpha$ as computed in $V[G^{\mathbb{M}_1} \restriction \alpha]$. ⊣

As we mentioned above, for technical reasons we also need to understand the status of AP in the extension by $\mathbb{M}_i * \dot{\mathrm{Add}}(\lambda, 1)$. This is straightforward, because the enumeration of the bounded subsets of λ (which we used to compute the maximum element of $I[\lambda]$) is still an enumeration of all bounded subsets after forcing with $\mathrm{Add}(\lambda, 1)$, and $\mathrm{Add}(\lambda, 1)$ is λ-closed so that it preserves stationary subsets of λ.

It follows readily that AP holds in the extension by $\mathbb{M}_0 * \dot{\mathrm{Add}}(\lambda, 1)$ and fails in the extension by $\mathbb{M}_1 * \dot{\mathrm{Add}}(\lambda, 1)$.

3.6. The tree property. Let λ be weakly compact. Towards a contradiction, suppose that λ fails to have the tree property in $V^{\mathbb{M}_i * \mathrm{Add}(\lambda,1)}$ and let $\dot{T}$ be a name for a λ-Aronszajn tree in this extension. Since $\mathbb{M}_i$ is λ-cc forcing of size λ, we may view $\mathbb{M}_i * \mathrm{Add}(\lambda, 1)$ and $\dot{T}$ as subsets of V_λ. Now, since λ is Π^1_1-indescribable, there exists an inaccessible $\alpha \in A$ such that $\dot{T} \cap V_\alpha$ is a $(\mathbb{M}_i \restriction \alpha) * \mathrm{Add}(\alpha, 1)$-name for an α-Aronszajn tree. By the definition of $\mathbb{M}_i$, at stage α the third coordinate generates g_α which is $\mathrm{Add}(\alpha, 1)$-generic over $V[G^{\mathbb{M}_i}_\alpha]$. Consider the realisation T^* of $\dot{T} \cap V_\alpha$ via $G^{\mathbb{M}_i}_\alpha * g_\alpha$, and observe that it is an α-Aronszajn tree in $V[G^{\mathbb{M}_i}_\alpha * g_\alpha]$.

Let $\mathbb{N}_{i,\alpha}$ be the quotient forcing from Section 3.4, so that we may view $V[G^{\mathbb{M}_i}]$ as an $\mathbb{N}_{i,\alpha}$-generic extension of $V[G^{\mathbb{M}_i}_\alpha]$. Note that $\mathrm{Add}(\alpha, 1)$ appears as part of the first stage of $\mathbb{N}_{i,\alpha}$, so we may view $V[G^{\mathbb{M}_i}]$ as an $\mathbb{N}_{i,\alpha}/g_\alpha$-generic extension of $V[G^{\mathbb{M}_i}_\alpha * g_\alpha]$.

By the projection argument from Section 3.4, $\mathbb{N}_{i,\alpha}$ can be viewed as the projection of a product $\mathbb{N}^0_{i,\alpha} \times \mathbb{N}^1_{i,\alpha} \times \mathbb{N}^2_{i,\alpha}$. Now $\mathbb{N}^2_{i,\alpha}$ is a product whose first component is $\mathrm{Add}(\alpha, 1)$, so easily $\mathbb{N}_{i,\alpha}/g \restriction \alpha$ can be viewed as the projection of $\mathbb{N}^0_{i,\alpha} \times \mathbb{N}^1_{i,\alpha} \times \mathbb{N}^2_{i,\alpha}/g_\alpha$.

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We work in $V[G^{\mathbb{M}_i}_\alpha * g_\alpha]$. Note that $\mathbb{N}^1_{i,\alpha}$ is κ^+-closed and $\mathbb{N}^2_{i,\alpha}/g_\alpha$ is α^*-closed where α^* is the successor of α in A. What is more $\mathbb{N}^0_{i,\alpha} \simeq \mathbb{P}/G^{\mathbb{P}}_\alpha$ and the square of this poset is κ^+-cc in the extension by $\mathbb{N}^1_{i,\alpha} \times \mathbb{N}^2_{i,\alpha}$. Thus it follows from Fact 1.6 that $\mathbb{N}^0_{i,\alpha} \times \mathbb{N}^1_{i,\alpha} \times \mathbb{N}^2_{i,\alpha}/g_\alpha$ cannot add a branch through T^*, so that T^* has no cofinal branch in $V[G^{\mathbb{M}_i}]$.

We may view g_α as a condition in $\mathrm{Add}(\lambda, 1)$, and force below it to get g which is $\mathrm{Add}(\lambda, 1)$-generic and has g_α as an initial segment. By the reflection properties of α, if T is the realisation of $\dot{T}$ by $G^{\mathbb{M}_i} * g$ then $T^* = T \restriction \alpha$. But $\mathrm{Add}(\lambda, 1)$ is λ-closed and so T^* has no cofinal branch in $V[G^{\mathbb{M}_i} * g]$, an immediate contradiction since T has points on level α.

Since $\mathrm{Add}(\lambda, 1)$ is λ-closed, it follows from Fact 1.6 that TP also holds in $V[G^{\mathbb{M}_i}]$.

3.7. Reflection. The arguments for RP are quite similar to those for TP. Let λ be weakly compact and let $\dot{S}$ name a nonreflecting stationary subset of $\lambda \cap \mathrm{cof}(\leq \kappa)$. Since λ is Π^1_1-indescribable there is an inaccessible cardinal $\alpha \in A$ such that $\dot{S} \cap V_\alpha$ is a $(\mathbb{M}_i \restriction \alpha) * \mathrm{Add}(\alpha, 1)$-name for a stationary subset of $\alpha \cap \mathrm{cof}(\leq \kappa)$. Let S^* be the realisation of $\dot{S} \cap V_\alpha$ in $V[G^{\mathbb{M}_i}_\alpha * g_\alpha]$.

Arguing as in the last subsection, it will suffice to show that S^* remains stationary in the extension by $\mathbb{N}^0_{i,\alpha} \times \mathbb{N}^1_{i,\alpha} \times \mathbb{N}^2_{i,\alpha}/g_\alpha$. We start by noting that since $\alpha = \kappa^{++}$, $S^* \in I[\alpha]$ and so its stationarity is preserved by κ^+-closed forcing. In particular S^* is still stationary after forcing with the κ^+-closed forcing poset $\mathbb{N}^1_{i,\alpha} \times \mathbb{N}^2_{i,\alpha}/g_\alpha$. After this forcing $\mathrm{cf}(\alpha) = \kappa^+$, so that there is a stationary subset $S^*_0 \subseteq \kappa^+$ which is the "collapsed" version of S^*.

By Easton's Lemma $\mathbb{N}^0_{i,\alpha}$ is still κ^+-cc in the extension by $\mathbb{N}^1_{i,\alpha} \times \mathbb{N}^2_{i,\alpha}/g_\alpha$, so that S^*_0 (and hence S^*) is stationary in $V[G^{\mathbb{M}_i} * g]$. As in the last subsection we force to get g which is $\mathrm{Add}(\lambda, 1)$-generic and has g_α as an initial segment, and realise $\dot{S}$ in $V[G^{\mathbb{M}_i} * g]$ as a stationary set S with $S \cap \alpha = S^*$. Since $\mathrm{Add}(\lambda, 1)$ is λ-closed we see that S^* is a stationary initial segment of S in $V[G^{\mathbb{M}_i} * g]$, contradicting the choice of $\dot{S}$ as a name for a nonreflecting stationary set.

To see that RP also holds in $V[G^{\mathbb{M}_i}]$, let $S \subseteq \lambda$ be stationary in this model. Then S is still stationary in $V[G^{\mathbb{M}_i} * g]$, it reflects there and hence easily it reflects in $V[G^{\mathbb{M}_i}]$.

§4. Down to the first singular cardinal. In this section we prove that Theorem 2.2 can be obtained for $\kappa = \aleph_\omega$. For the not-TP cases the argument is the same as before. For the rest we will use the variants of Mitchell forcing from the last section with $\mathbb{P} = \mathrm{Add}(\kappa, \lambda)$. Then we will force with a Prikry poset with interleaved collapses with guiding generics to singularise κ and make it become $\aleph_\omega$.

In V, suppose that κ is indestructibly supercompact and $\lambda > \kappa$ is a weakly compact cardinal. Let $\mathbb{M}_i$ for $i \in \{0, 1\}$ be the Mitchell type posets defined in the previous section with respect to $\mathbb{P} = \mathrm{Add}(\kappa, \lambda)$, i.e., the forcing used for the proof of Theorem 2.1. Now let G be $\mathbb{M}_i$-generic. Then in $V[G]$, $\lambda = \kappa^{++} = 2^\kappa$ and κ is supercompact. In $V[G]$, let U^* be a normal measure on $P_\kappa(\kappa^+)$. Using arguments as in [16], we can arrange that if $j^* = j_{U^*}$, then for every $\alpha < j^*(\kappa)$, there is a function $f : \kappa \to \kappa$, such that $j^*(f)(\kappa) = \alpha$. The key to the proof is that $|j^*(\kappa)| = 2^\kappa$. For a detailed presentation of the preparation, which is analogous to the case here, see [29, Section 2].

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Now let U be the normal measure on κ obtained from U^* and set $j = j_U$. Note that $\mathrm{Ult}(V[G], U^*)$, and $\mathrm{Ult}(V[G], U)$ compute cardinals correctly up to and including λ. Denote $\mathbb{C} = [\alpha \mapsto \mathrm{Coll}(\alpha^{++}, < \kappa)]_U$. For $x \in P_\kappa(\kappa^+)$, we use κ_x to denote $\kappa \cap x$.

Lemma 4.1. *There is a generic filter for* $\mathbb{C}$ *over* $\mathrm{Ult}(V[G], U)$ *in* $V[G]$.

Proof. Define $k : \mathrm{Ult}(V[G], U) \to \mathrm{Ult}(V[G], U^*)$ by stipulating $k([f]) = j^*(f)(\kappa)$, so that $j^* = k \circ j$. Since we arranged that every $\alpha < j^*(\kappa)$ can be represented by a function from κ to κ, we have that the range of k contains $j^*(\kappa)$, and so the critical point of k is above $j^*(\kappa)$.

Denote $\mathbb{C}^* = [x \mapsto \mathrm{Coll}(\kappa_x^{+2}, < \kappa)]_{U^*}$. Note that $k(\mathbb{C}) = \mathbb{C}^*$. By standard arguments, in $V[G]$ there is a generic filter K^* for $\mathbb{C}^*$ over $\mathrm{Ult}(V[G], U^*)$. Here we use the fact that there are κ^{++}-many antichains to meet, and the poset is κ^{++}-closed. Clearly, $K = k^{-1}$"K^* is a filter for $\mathbb{C}$. Also, since $\mathbb{C}$ has the $j(\kappa)$-cc and the critical point of k is high enough, we have that for every maximal antichain $\mathcal{A} \subseteq \mathbb{C}$, $k(\mathcal{A}) = k$"$\mathcal{A}$. It follows that K is $\mathbb{C}$-generic over $\mathrm{Ult}(V[G], U)$. ⊣

Let K in $V[G]$ be a generic filter for $\mathbb{C}$ over $\mathrm{Ult}(V[G], U)$. Let $X = \{\alpha < \kappa : \alpha \text{ is inaccessible}\}$, and note that $X \in U$. Then define a Prikry-type forcing poset $\mathbb{R}$ to have conditions $\langle d, \alpha_0, c_0, \ldots, \alpha_{n-1}, c_{n-1}, A, C\rangle$, where

- $\langle \alpha_i \mid i < n\rangle$ is a finite increasing sequence of inaccessibles;
- if $n > 0$, then $d \in \mathrm{Coll}(\omega, \alpha_0)$, otherwise $d \in \mathrm{Coll}(\omega, \kappa)$;
- for $i < n - 1$, $c_i \in \mathrm{Coll}(\alpha_i^{++}, < \alpha_{i+1})$ and $c_{n-1} \in \mathrm{Coll}(\alpha_{n-1}^{++}, < \kappa)$;
- $A \in U$, $A \subseteq X$; and
- $\mathrm{dom}(C) = A$, for each $\alpha \in A$, $C(\alpha) \in \mathrm{Coll}(\alpha^{++}, < \kappa)$, and $[C]_U \in K$.

The ordering is as follows:

$\langle d', \alpha_0, c_0', \ldots, \alpha_{m-1}, c_{m-1}', A', C'\rangle \leq \langle d, \alpha_0, c_0, \ldots, \alpha_{n-1}, c_{n-1}, A, C\rangle$ iff:

- $m \geq n$, $d' \leq d$, for all $i < n$, $c_i' \leq c_i$;
- for all $n \leq i < m$, $\alpha_i \in A$ and $c_i' \leq C(\alpha_i')$;
- $A' \subseteq A$ and for all $\alpha \in A'$, $C'(\alpha) \leq C(\alpha)$.

For a condition p, we denote the stem of p by $s(p) = \langle d, \alpha_0, c_0, \ldots, \alpha_{n-1}, c_{n-1}\rangle$ and the length of p by $\ell(p) = n$. We also denote the length of a stem $s = s(p)$, by $\ell(s) = n$.

We refer the reader to Gitik's survey [14] for a detailed account on this type of forcing and its properties. We will use the following facts about $\mathbb{R}$:

- It has the κ^+-cc.
- It has the Prikry property, that is to say for every sentence in the forcing language ϕ and condition p there is $p' \leq p$ with the same length deciding ϕ. As a consequence, the only collapsing of cardinals occurs below κ and is done by the Lévy collapses.
- By similar arguments to those for the Prikry property, for every dense open set D and every condition p there exist an extension q of p with $\ell(q) = \ell(p)$ and an integer n such that every n-step extension of q lies in D. That is, every $r \leq q$ with $\ell(r) = \ell(q) + n$ lies in D.
- The generic object is a sequence

$$g, \alpha_0, g_0, \alpha_1, g_1, \ldots,$$

https://doi.org/10.1017/jsl.2017.69 Published online by Cambridge University Press

where the α_n form an increasing sequence of inaccessible cardinals, g is $\mathrm{Coll}(\omega, \alpha_0)$-generic and g_i is $\mathrm{Coll}(\alpha_i^{++}, < \alpha_{i+1})$-generic. Genericity has a simple and absolute characterisation in terms of U and K; for every C such that $[C]_U \in K$, $\alpha_i \in \mathrm{dom}(C)$ and $C(\alpha_i) \in g_i$ for all large i.

Let $\mathcal{R}$ be $\mathbb{R}$-generic over $V[G]$. Using the properties we just listed, in $V[G][\mathcal{R}]$ we have that $\kappa = \aleph_\omega$ and $\lambda = \aleph_{\omega+2}$.

Fix $i \in \{0, 1\}$. Let $\mathbb{Q}$ be the term forcing $\mathbb{M}_i^{12}$. Then $\mathbb{Q}$ is κ^+-directed closed and $\mathbb{P} \times \mathbb{Q}$ projects to $\mathbb{M}_i$. Note that by the κ^+-directed closure of $\mathbb{Q}$, in $V^{\mathbb{P}\times\mathbb{Q}}$, U is still a normal measure and K is a still a guiding generic over the ultrapower of $V^{\mathbb{P}\times\mathbb{Q}}$ by U. So in that extension $\mathbb{R}$ has the same definition and retains the same properties.

Our next task is analysing TP, RP and AP in $V^{\mathbb{M}_i * \mathbb{R}}$.

4.1. The tree property. In this subsection, we show a *branch lemma* to ensure that forcing with $\mathbb{R}$ preserves the tree property. We will follow the argument in Section 4 of a recent by Sinapova and Unger [29]. In V let $\dot{T}$ be an $\mathbb{M}_i * \mathbb{R}$ name for a λ-tree. Using the weak compactness of λ in V, fix an embedding k with critical point λ on a transitive model of size λ containing $\dot{T}$. Since $\mathbb{M}_i * \dot{\mathbb{R}}$ has the λ-cc, we can lift k to this extension by forcing with $k(\mathbb{M}_i * \mathbb{R})/G * \mathcal{R}$. Since the lifted embedding determines a branch through the tree, it is enough to show that forcing with $k(\mathbb{M}_i * \mathbb{R})/G * \mathcal{R}$ does not add new branches through T.

Note that $U \subseteq k(U)$, $K \subseteq k(K)$, and $\mathbb{R} \subseteq k(\mathbb{R})$. Of course, there are more subsets in $k(U)$, but by the characterization of genericity for Prikry posets, $k(\mathbb{R})$ induces a generic for $\mathbb{R}$. Let us give some definitions. If r, r' are Prikry conditions we say that r' is a *direct extension* of r if $r' \leq r$ and they have the same length, and we write $r' \leq^* r$. For r in $\mathbb{R}$ or in $k(\mathbb{R})$, we let $s(r)$ denote its stem. Note that the stem is always in $V[G]$. For stems s, s', we say that s' *extends* s, if there are Prikry conditions $r' \leq r$ with stems s' and s, respectively. If $\ell(s') \geq \ell(s)$ and r is a condition with stem s, we say that "points in s' above s are compatible with r" if there is $r' \leq r$ with stem extending s'. Also, for a stem s, we write "$s \Vdash^* \phi$" if there is $r \in \mathbb{R}$ with stem s, such that $r \Vdash \phi$.

The next lemma comes from the work of Cummings and Foreman [5], adapted to our case.

Lemma 4.2. *Work in $V[G]$. Let $\bar{r} \in \mathbb{R}$, $m \in k(\mathbb{M}_i)$ and $\dot{r}$ be a $k(\mathbb{M}_i)$-name for an element of $k(\mathbb{R})$, such that m decides the value of the stem of $\dot{r}$. Then $\bar{r}$ forces $(m, \dot{r}) \notin k(\mathbb{M}_i * \mathbb{R})/(G * \dot{\mathcal{R}})$ if and only if one of the following holds*:

(1) *$m \notin k(\mathbb{M}_i)/G$.*
(2) *$s(\bar{r})$ and $s(\dot{r})$ have no common extension.*
(3) *$\ell(\dot{r}) \geq \ell(\bar{r})$ and points in $s(\dot{r})$ above $s(\bar{r})$ are not compatible with $\bar{r}$.*
(4) *$\ell(\bar{r}) \geq \ell(\dot{r})$ and m forces that points in $s(\bar{r})$ above $s(\dot{r})$ are not compatible with $\dot{r}$.*

Remark 4.3. A key point in the proof of Lemma 4.2 is that due to the guiding generics, conditions in $\mathbb{R}$ with the same stem are compatible.

Lemma 4.4. *Working in $V[G]$, let $\bar{r} \in \mathbb{R}$, $m \in k(\mathbb{M}_i)/G$ and let $\dot{r}$ be a $k(\mathbb{M}_i)/G$-name for a condition in $k(\mathbb{R})$ such that*

(1) *m decides the value of $s(\dot{r})$,*

(2) *$s(\bar{r})$ extends $s(\dot{r})$, and*
(3) *m forces that points in $s(\bar{r})$ above $s(\dot{r})$ are compatible with $\dot{r}$,*

then there is a direct extension of $\bar{r}$ which forces that $(m, \dot{r}) \in k(\mathbb{M}_i * \mathbb{R})/(G * \dot{\mathcal{R}})$.

Proof. Let $\bar{r}_0$ be a direct extension of $\bar{r}$ which decides the statement $(m, \dot{r}) \in k(\mathbb{M}_i * \mathbb{R})/(G * \mathcal{R})$. It is straightforward to check that we are not in any of the cases of Lemma 4.2, so it is not the case that $\bar{r}_0 \Vdash (m, \dot{r}) \notin k(\mathbb{M}_i * \mathbb{R})/(G * \dot{\mathcal{R}})$. It follows that r_0 forces $(m, \dot{r})$ into the quotient. ⊣

Let $\mathbb{N} = k(\mathbb{M}_i * \mathbb{R})/G * \mathcal{R}$. We will write conditions in $\mathbb{N}$ as triples $(p, f, \dot{r})$, where $p \in k(\mathbb{P})$, $f \in k(\mathbb{Q})$ and $\dot{r}$ is a $k(\mathbb{M}_i)$-name for a condition in $k(\mathbb{R})$. Here we identify $k(\mathbb{Q})$ with its nontrivial coordinates. We will also refer to $\leq_{k(\mathbb{Q})}$ as the "term ordering".

Let $\tau : \lambda \to \kappa^+$ be a branch in the extension by $\mathbb{N}$. Suppose for contradiction that τ is a new branch. Note that for all $\alpha < \lambda$, $\tau \restriction \alpha \in V[G][\mathcal{R}]$.

Lemma 4.5. *In $V[G][\mathcal{R}]$, there is a condition $(p, f, \dot{r}) \in \mathbb{N}$, such that for each x, $\alpha < \lambda$ and $(p', f', \dot{r}') \leq_{\mathbb{N}} (p, f, \dot{r})$, if $f' \leq_{k(\mathbb{Q})} f$ and $(p', f', \dot{r}') \Vdash \dot{\tau} \restriction \alpha = x$, then $(p, f', \dot{r}) \Vdash \dot{\tau} \restriction \alpha = x$.*

Proof. The proof is essentially the same as in [29], but we will go over the main steps for completeness. Suppose otherwise. Then in $V[G]$, let $\bar{r} \in \mathbb{R}$ force the negation of the conclusion. Then whenever $\bar{r} \Vdash (p, f, \dot{r}) \in \dot{\mathbb{N}}$, the following set D is dense below $\bar{r}$ in $\mathbb{R}$, where D consists of conditions $\bar{r}' \in \mathbb{R}$, such that there are $p_0, p_1 \in k(\mathbb{P}), f^* \leq_{k(\mathbb{Q})} f, k(\mathbb{M}_i)/G$-names $\dot{r}_0, \dot{r}_1$ for elements in $k(\mathbb{R}), \alpha < \lambda$, and $\mathbb{R}$-names x_0, x_1 such that

- for $i \in \{0, 1\}, \bar{r}' \Vdash (p_i, f^*, \dot{r}_i) \leq_{\mathbb{N}} (p, f, \dot{r})$;
- for $i \in \{0, 1\}, \bar{r}' \Vdash$ "$(p_i, f^*, \dot{r}_i) \Vdash_{\mathbb{N}} \dot{\tau} \restriction \alpha = x_i$";
- x_0, x_1 are forced to be distinct.

Now, by recursion over $\alpha < \kappa^+$, construct $p^i_\alpha, x^i_\alpha, x_\alpha, f_\alpha, \dot{r}^i_\alpha, \bar{r}_\alpha$ and γ_α for $i \in 2$, such that $\langle \gamma_\alpha : \alpha < \kappa^+ \rangle$ is increasing, $\langle f_\alpha : \alpha < \kappa^+ \rangle$ is $\leq_{k(\mathbb{Q})}$-decreasing, and for each α, i, $\bar{r}_\alpha \in \mathbb{R}$ forces that:

- $(p^i_\alpha, f_\alpha, \dot{r}^i_\alpha) \in \mathbb{N}$;
- $(p^i_\alpha, f_\alpha, \dot{r}^i_\alpha) \Vdash \dot{\tau} \restriction \sup_{\beta<\alpha} \gamma_\beta = x_\alpha$;
- $x^0_\alpha \neq x^1_\alpha$;
- $(p^i_\alpha, f_\alpha, \dot{r}^i_\alpha) \Vdash \dot{\tau} \restriction \gamma_\alpha = x^i_\alpha$; and
- (p^i_α, f_α) decides $s(\dot{r}^i_\alpha)$ and $s(\bar{r}_\alpha)$ extends it.

Using that there are only κ many possible stems and that $\mathbb{P} \times \mathbb{P}$ has the κ^+-cc, we find $\beta < \beta' < \kappa^+$, such that $s(\bar{r}_\beta) = s(\bar{r}_{\beta'})$, and for $i \in 2$, $s(\dot{r}^i_\beta) = s(\dot{r}^i_{\beta'})$, and p^i_β is compatible with $p^i_{\beta'}$. Then for $i \in 2$, let p^i be the weakest lower bound for p^i_β and $p^i_{\beta'}$ and let $\dot{r}^i$ be a name for a common extension of $\dot{r}^i_\beta$ and $\dot{r}^i_{\beta'}$ in $k(\mathbb{R})$ with the same stem.

By Lemma 4.4, there is a direct extension r of $\bar{r}_\beta$ and $\bar{r}_{\beta'}$ which forces that each $(p^i, f_{\beta'}, \dot{r}^i)$ is in $\mathbb{N}$. We choose a generic $\mathcal{R}'$ containing r. Then in $V[G][\mathcal{R}']$, we have that $(p_i, f_{\alpha'}, \dot{r}^i)$ is in $\mathbb{N}$, and so for $i \in 2$, $x_{\alpha'} \restriction \gamma_\alpha = x^i_\alpha$. This implies that $x^0_\alpha = x^1_\alpha$, a contradiction. ⊣

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Work in $V[G]$. Let $r^* \in \mathbb{R}$ force that $(p, f, \dot{r})$ is as in the conclusion of Lemma 4.5. We construct sequences

$$\langle f_s : s \in 2^{<\kappa}\rangle, \langle \alpha_s^h, x_s^h, \gamma_{s^\frown i}^h : s \in 2^{<\kappa}, i \in 2, h \text{ is a stem extending } s(r^*)\rangle,$$

such that:

(1) if $s' \supseteq s$, then $f_{s'} \leq_{k(\mathbb{Q})} f_s$;
(2) for all s, h, i, $h \Vdash^* (p, f_{s^\frown i}, \dot{r}) \Vdash$ "$\dot{\tau}(\alpha_s^h) = \gamma_{s^\frown i}^h, \dot{\tau} \restriction \alpha_s^h = x_s^h$"; and
(3) for all s, h, $\gamma_{s^\frown 0}^h \neq \gamma_{s^\frown 1}^h$.

Let $\alpha^* = \sup_{h,s} \alpha_s^h < \lambda$. For every $g \in 2^\kappa$, let $f_g \leq_{k(\mathbb{Q})} f_{g \restriction \eta}$ for all $\eta < \kappa$ and let $r_g \in \mathbb{R}$ be such that $r_g \leq r^*$ and for some $\gamma_g < \kappa^+$,

$$r_g \Vdash_{\mathbb{R}} (p, f_g, \dot{r}) \Vdash \dot{\tau}(\alpha^*) = \gamma_g.$$

Take $g_1 \neq g_2$ to be such that for some h, γ, $h = s(r_{g_1}) = s(r_{g_2})$, $\gamma = \gamma_{g_1} = \gamma_{g_2}$. Let $\bar{r}$ be a common extension of r_{g_1} and r_{g_2} with stem h. Let $\eta < \kappa$ be such that $g_1 \restriction \eta = g_2 \restriction \eta = s$ and $g_1(\eta) \neq g_2(\eta)$. Finally, let $q \leq \bar{r}$ witness item (2) above for $s^\frown 0, s^\frown 1$ and h.

We choose a generic $\mathcal{R}'$ containing q. Then in $V[G][\mathcal{R}']$, we have that both $(p_i, f_{g_1}, \dot{r})$ and $(p_i, f_{g_2}, \dot{r})$ force that $\dot{\tau}(\alpha^*) = \gamma$, but they force different values for $\dot{\tau}(\alpha_s^h)$. This is a contradiction.

We have proven the following lemma, which takes care of preserving the tree property for the cases TP + RP.

Lemma 4.6. *$V[G][\mathcal{R}]$ satisfies the tree property at $\lambda = \aleph_{\omega+2}$.* ⊣

Next we look at the cases TP + not-RP. Namely, we want to modify the above argument to show that after forcing with $\mathbb{R}$ over $V^{\mathbb{M}_i * \mathbb{P}_{\mathrm{NRSS}}}$, the tree property is preserved.

Note that there is a projection from $\mathbb{M}_i * \mathrm{Add}(\lambda, 1) * \mathbb{R}$ to $\mathbb{M}_i * (\mathbb{P}_{\mathrm{NRSS}} \times \mathbb{R})$. Let G be a generic for $\mathbb{M}_i$ and $\mathcal{S} \times \mathcal{R}$ be generic for $\mathbb{P}_{\mathrm{NRSS}} \times \mathbb{R}$ over $V[G]$. As before, we lift an embedding k witnessing the weak compactness of λ by forcing with $k(\mathbb{M}_i * (\dot{\mathbb{P}}_{\mathrm{NRSS}} \times \dot{\mathbb{R}}))/(G * (\mathcal{R} \times \mathcal{S}))$, and we get a branch in that extension. By the distributivity of $k(\mathbb{P}_{\mathrm{NRSS}})$, it is enough to show that $\mathbb{N}' = k(\mathbb{M}_i * \dot{\mathbb{R}})/(G * (\mathcal{R} \times \mathcal{S}))$ does not add a new branch.

In $V[G * \mathcal{R}]$ define $\mathbb{N} = \mathrm{Add}(\lambda, 1) * \dot{\mathbb{N}}'$. We write conditions in $\mathbb{N}$ of the form $(a, p, f, \dot{r})$ where $a = (a_0, a_1) \in \mathrm{Add}(\lambda, 1)$, $a_0 \in \mathbb{P}_{\mathrm{NRSS}}$, $p \in k(\mathbb{P})$, $f \in k(\mathbb{Q})$, and $\dot{r}$ is a name for an element in $k(\mathbb{R})$. We will show that $\mathbb{N}$ does not add a branch.

Let $\tau : \lambda \to \kappa^+$ be a branch in the extension by $\mathbb{N}$. Suppose for contradiction that τ is a new branch. Note that as before, for all $\alpha < \lambda$, $\tau \restriction \alpha \in V[G][\mathcal{R}]$, so that Lemma 4.5 applies. In particular we can decide different values for initial segments of $\dot{\tau}$ only by extending the f and the a-parts.

Work in $V[G]$ and as before apply Lemma 4.5 to construct sequences $\langle a_0^\eta : \eta < \kappa\rangle$, $\langle a_1^s, f_s : s \in 2^{<\kappa}\rangle$, $\langle \alpha_s^h, x_s^h, \gamma_{s^\frown i}^h : s \in 2^{<\kappa}, i \in 2, h \text{ is a stem}\rangle$, such that:

(1) if $s' \supseteq s$, then $f_{s'} \leq_{k(\mathbb{Q})} f_s$, $(a_0^{|s'|-1}, a_1^{s'}) \leq (a_0^{|s|-1}, a_1^s)$;
(2) for all s, h, i, $h \Vdash^* (a_0^{|s|}, a_1^{s^\frown i}, p, f_{s^\frown i}, \dot{r}) \Vdash \dot{\tau}(\alpha_s^h) = \gamma_{s^\frown i}^h, \dot{\tau} \restriction \alpha_s^h = s_s^h$; and
(3) for all s, h, $\gamma_{s^\frown 0}^h \neq \gamma_{s^\frown 1}^h$.

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Let $\alpha^* = \sup_{h,s} \alpha_s^h < \lambda$. For every $g \in 2^\kappa$, let $f_g \leq_{k(\mathbb{Q})} f_{g\restriction\eta}$ for all $\eta < \kappa$, and let $(a_0^g, a_1^g) \leq (a_0^\eta, a_1^{g\restriction\eta})$ for all $\eta < \kappa$. Then let $r_g \in \mathbb{R}$ be such that

$$r_g \Vdash_{\mathbb{R}} (a_0^g, a_1^g, p, f_g, \dot{r}) \Vdash \dot{\tau}(\alpha^*) = \gamma_g.$$

Take $g_1 \neq g_2$ to be such that for some h, γ, $h = s(r_{g_1}) = s(r_{g_2})$, $\gamma = \gamma_{g_1} = \gamma_{g_2}$. Let $\bar{r}$ be a common extension of r_{g_1} and r_{g_2} with stem h. Let $\eta < \kappa$ be such that $g_1 \restriction \eta = g_2 \restriction \eta = s$ and $g_1(\eta) \neq g_2(\eta)$. Finally, let $q \leq \bar{r}$ witness item (2) above for $s^\frown 0, s^\frown 1$. Then force below q to get a contradiction.

Then we have proven the following lemma:

Lemma 4.7. *Let $G * \mathcal{S}$ be $\mathbb{M}_i * \mathbb{P}_{\mathrm{NRSS}}$-generic and $\mathcal{R}$ be $\mathbb{R}$-generic over $V[G * \mathcal{S}]$. Then in $V[G][\mathcal{S}][\mathcal{R}]$ we have the tree property at $\lambda = \aleph_{\omega+2}$.* ⊣

As explained before, we use the preceding lemma in the cases TP + not-RP.

4.2. Reflection. First, let us point out that $\mathbb{R}$ preserves failure of reflection at λ, simply because $\mathbb{R}$ preserves stationary sets of λ. This takes care of preserving not-RP for the cases of TP $\pm$ AP + not-RP.

Next, we show that $\mathbb{R}$ also preserves reflection at λ.

Lemma 4.8. *Suppose that W is a model where κ is regular, $\lambda = \kappa^{++}$,* RP *holds at λ. Then in $W^{\mathbb{R}}$,* RP *holds at λ.*

Proof. Given $\dot{S}$ that is forced to be a stationary subset of $\lambda = \kappa^{++}$, suppose for contradiction that it is forced to be nonreflecting at points of cofinality κ^+ by some p in $\mathbb{R}$. Since there are only κ-many stems, there is a stem h, extending the stem of p, such that $T = \{\beta < \lambda : h \Vdash^* \beta \in \dot{S}\}$ is stationary. By strengthening if necessary, we may assume p has a stem h. For each $\beta \in T$, let r_β be with stem h, such that $r_\beta \Vdash \beta \in \dot{S}$. By RP in W, there is $\bar{\lambda}$ with $\mathrm{cf}(\bar{\lambda}) = \kappa^+$, such that $T \cap \bar{\lambda}$ is stationary.

Let $D := \{q : (\exists C)(C \subseteq \bar{\lambda}$ is club and $q \Vdash C \cap \dot{S} = \emptyset)\}$. By our assumption and since $\mathbb{P}$ has the κ^+-cc, D is open and dense below p, so there exist $p' \leq^* p$ and n such that every n-step extension of p' is in D. Write $p' = \langle h, A', C' \rangle$. Also, for each $\beta \in T$, denote $r_\beta = \langle h, A_\beta, C_\beta \rangle$. By strengthening if necessary, we assume that for each β, $r_\beta \leq p'$.

For simplicity, suppose that $n = 1$ (the argument for the general case in similar). Define $\phi : T \cap \bar{\lambda} \to \kappa$ by stipulating $\phi(\beta) = \min(A_\beta)$. This function is constant on a stationary set, so let $\alpha < \kappa$ be such that $T_\alpha = \phi^{-1}\{\alpha\}$ is stationary in $\bar{\lambda}$. Now let

$$q = \langle h^\frown \langle \alpha, C'(\alpha) \rangle, A' \setminus \alpha + 1, C' \restriction (A' \setminus \alpha + 1) \rangle.$$

By our choice of p', we have that for some club $C \subseteq \bar{\lambda}$, $q \Vdash C \cap \dot{S} = \emptyset$. But now let $\beta \in T_\alpha \cap C$ and let $r = \langle h^\frown \langle \alpha, C_\beta(\alpha) \rangle, A'_\beta, C'_\beta \rangle$, where $A'_\beta = A_\beta \setminus \alpha + 1$ and $C'_\beta = C_\beta \restriction A'_\beta$. Then $r \leq r_\beta$, and so $r \Vdash \beta \in \dot{S}$, but also $r \leq q$. This is a contradiction. ⊣

4.3. Approachability. At this stage, we are left with showing that forcing with $\mathbb{R}$ preserves the properties of the approachability ideal at λ as arranged by the choice of $\mathbb{M}_0$ or $\mathbb{M}_1$. As we have verified that AP holds at λ after forcing with $\mathbb{M}_0$ and approachability is persistent for models with the same cardinals, we have:

Lemma 4.9. *Suppose that G is $\mathbb{M}_0$ or $\mathbb{M}_0 * \mathbb{P}_{\mathrm{NRSS}}$-generic. Then in $V[G][\mathcal{R}]$,* AP *at λ holds.* ⊣

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For not-AP we use a theorem of Gitik and Krueger from [15], who showed that for $\lambda = \kappa^{++}$, after forcing with a κ-centered poset the approachability ideal of λ in the generic extension is generated by the ground model ideal $I[\lambda]$, yielding:

LEMMA 4.10. *Suppose that G is $\mathbb{M}_1$ or $\mathbb{M}_1 * \mathbb{P}_{\mathrm{NRSS}}$-generic. Then in $V[G][\mathcal{R}]$,* AP *at λ fails.* ⊣

Thus, we have established the third theorem of this article (overstating the large cardinal assumption).

THEOREM 4.11. *Let κ be an indestructible supercompact cardinal and let $\mu = \kappa^+$. Then, assuming the existence of a weakly compact cardinal above κ, for each Boolean combination Φ of* TP, AP *and* RP *there exists a generic extension in which $\kappa = \aleph_\omega$, $\mu = \aleph_{\omega+1}$, $\lambda = \aleph_{\omega+2}$, and Φ holds.* ⊣

REMARK 4.12. It is also possible to prove Theorem 4.11 using an approach analogous to that used in the proof of Theorems 2.1 and 2.2, writing the relevant quotients as the projection of a product of a κ^+-closed forcing with a forcing whose square is κ^+-cc. This is the method used in [12].

§5. Open questions.

(1) In several of our cases we have assumed the existence of a weakly compact cardinal λ above κ to get a Boolean combination of AP, TP and RP to hold at κ^{++}. For the TP cases this is necessary, as the tree property demands a weakly compact. However for some of the not-TP cases we can use less: Our argument for not-TP + AP + not-RP used no large cardinal, and as Harrington and Shelah [17] obtained the RP from just a Mahlo cardinal, the case not-TP + AP + RP can be handled with just a Mahlo cardinal, and in fact, nonTP is moreover witnessed by a Souslin tree (cf. [23]). Also, our argument for the case not-TP + not-AP + not-RP only used a Mahlo cardinal and it can be shown that these uses are necessary. This leaves one open case: not-TP + not-AP + RP; can this also be done assuming just a Mahlo cardinal?

(2) In the cases of κ singular we used a measurable cardinal κ that remains measurable after forcing with $\mathrm{Add}(\kappa, \lambda)$, where $\lambda > \kappa$ is weakly compact. This has the consistency strength of a weakly compact hypermeasurable cardinal. But it is conceivable that much less strength is needed. For example, although the TP at κ^{++} with κ measurable is equiconsistent with a weakly compact hypermeasurable cardinal (see [6]) and this was used in [11] to get the TP at $\aleph_{\omega+2}$, Gitik [13] showed that indeed much less strength is needed for the latter result. Does Gitik's result extend to the entire eightfold way?

(3) This article looks at the eightfold way just for a single cardinal μ. Can it be carried out for many cardinals, such as all of the $\aleph_n$'s $(n > 1)$, simultaneously?

(4) What is the status of the eightfold way at successors of singular cardinals? Note that the situation here is known to be more complicated. For instance, Fontanella and Magidor [8] constructed a model in which $\aleph_{\omega^2+1}$ is a strong limit, every stationary subset of $\aleph_{\omega^2+1}$ reflects, but the approachability property fails at $\aleph_{\omega^2+1}$, whereas the same combination at the level of $\aleph_{\omega+1}$ is inconsistent (see Corollary 3.41 of Eisworth's survey [7]).

https://doi.org/10.1017/jsl.2017.69 Published online by Cambridge University Press

§6. Acknowledgements. Cummings was partially supported by the National Science Foundation, grants DMS-1101156 and DMS-1500790. Friedman would like to thank the Austrian Science Fund for its generous support through the research project P25748. Rinot was partially supported by the Israel Science Foundation, grant 1630/14. Sinapova was partially supported by the National Science Foundation, grants DMS-1362485 and Career-1454945. The results in this article were conceived and proved during three visits to the American Institute of Mathematics, as part of the Institute's SQuaRE collaborative research program. The authors are deeply grateful to Brian Conrey, Estelle Basor and all the staff at AIM for providing an ideal working environment.

We thank the referee for the useful comments.

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