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Existence and multiplicity of solutions for discontinuous elliptic problems in $\mathbb{R}^N$

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This paper concerns with the existence of multiple solutions for a class of elliptic problems with discontinuous nonlinearity. By using dual variational methods, properties of the Nehari manifolds and Ekeland's variational principle, we show how the 'shape' of the graph of the function A affects the number of nontrivial solutions.

Keywords: Multiplicity of solutions; variational method; discontinuous nonlinearity

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1. Introduction

We consider the existence of multiple solutions for the following discontinuous problem

$$\begin{cases} -\Delta u + u = A(\epsilon x) f(u(x)) \quad \text{a.e.} \quad \text{in } \mathbb{R}^N, \\ u \in W^{2,p/p-1}_{loc}(\mathbb{R}^N) \cap W^{1,p}(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $N \geqslant 1$, ϵ is a positive parameter and $f : \mathbb{R} \to \mathbb{R}$ is the odd function given by

$$f(t) = \begin{cases} t|t|^{p-2}, & t \in [0, a], \\ (1+\delta)t|t|^{p-2}, & t > a, \end{cases}$$

with $a, \delta > 0$ and $p \in (2, 2^*)$,

$$2^* = \begin{cases} \frac{2N}{N-2}, & \textit{if } N \geqslant 3, \\ +\infty, & \textit{if } N = 1, 2. \end{cases}$$

Moreover, the function $A : \mathbb{R}^N \to \mathbb{R}$ is a continuous function satisfying the following conditions:

$(H1)$ $A : \mathbb{R}^N \to \mathbb{R}$ is a $C^2(\mathbb{R}, \mathbb{R})$ function such that $A, \partial A/\partial x_i$ and $\partial^2 A/\partial x_i x_j$ are bounded functions in whole $\mathbb{R}^N$ for all $i, j \in \{1, 2, \ldots, N\}$ and

$$\lim_{|x|\to\infty} A(x) = A_\infty,$$

with $0 < A_\infty < A(x)$ for any $x \in \mathbb{R}^N$.

$(H2)$ There exist l points $z_1, z_2, \ldots, z_l$ in $\mathbb{R}^N$ with $z_1 = 0$ such that

$$1 = A(z_i) = \max_{x\in\mathbb{R}^N} A(x) \quad \text{for } 1 \leqslant i \leqslant l.$$

In [**13**] Cao and Noussair have considered the existence and multiplicity of solutions for problem (1.1) with $\delta = 0$. In this case, $f(t) = |t|^{p-2}t$ for $t \in \mathbb{R}$ and the problem becomes

$$\begin{cases} -\Delta u + u = A(\epsilon x)|u|^{p-2}u & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N). \end{cases} \tag{1.2}$$

By using Ekeland's variational principle and concentration compactness principle of Lions [**23**], Cao and Noussair proved that if A has k equal maximum points, then problem (1.2) has at least k positive solutions and k nodal solutions if ϵ is small enough. Later, Wu in [**28**] has proved the existence of at least ℓ positive solutions for the perturbed problem

$$\begin{cases} -\Delta u + u = h(\varepsilon x)|u|^{r-2}u + \lambda g(\varepsilon x)|u|^{q-2}u & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{P_2}$$

where λ is a positive small parameter, $q \in [1, 2)$ and $g : \mathbb{R}^N \to \mathbb{R}$ is a nonnegative continuous function satisfying

$$g(x) \to 0 \quad \text{and} \quad |x| \to +\infty.$$

In [**19**] the authors have considered the following class of quasilinear problems

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = h(\varepsilon x)|u|^{r-2}u + \lambda g(\varepsilon x) & \text{in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N), \end{cases} \tag{P_3}$$

with $N \geqslant 3$ and $2 \leqslant p < N$. In that paper, the authors have proved the same type of results found in [13] and [28].

For the case $\delta > 0$, the function f is discontinuous and the study of the existence of solution for problem (1.1) is completely different from the case $\delta = 0$, since we cannot directly use the methods for C^1-functionals, as for example, there is no Nehari manifold associated with the energy functional. This fact brings us some difficulties to use traditional methods to obtain the multiplicity of solutions of problem (1.1). In order to overcome this difficulty, we will use a method, called Clark's dual action principle, which was employed in [6, 10] and [8].

As is well known, many free boundary problems and obstacle problems may be reduced to partial differential equations with nonsmooth potentials. The area of nonsmooth analysis is closely related with the development of critical points theory for nondifferentiable functions, in particular, for locally Lipschitz continuous functionals based on Clarke's generalized gradient [15]. It provides an appropriate mathematical framework to extend the classic critical point theory for C^1-functionals in a natural way, and to meet specific needs in applications, such as in nonsmooth mechanics and engineering. For a comprehensive understanding, we refer to the monographs of [17, 24, 25] and references [1–5, 9, 11, 14, 16, 18, 20, 21, 29].

Our main result is the following:

THEOREM 1.1. *There are $\delta^*, \epsilon^*, a^* > 0$ such that for each $\delta \in (0, \delta^*)$, $\epsilon \in (0, \epsilon^*)$ and $a \in (0, a^*)$, problem (1.1) has at least l nontrivial solutions.*

This paper is organized as follows. In § 2, we present an auxiliary problem and some necessary preliminary knowledge. We prove our main result in § 3.

2. An auxiliary problem

In this section, we discuss the energy function $I_\epsilon : H^1(\mathbb{R}^N) \to \mathbb{R}$ associated with (1.1) given by

$$I_\epsilon(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2)\,\mathrm{d}x - \int_{\mathbb{R}^N} A(\epsilon x)F(u)\,\mathrm{d}x,$$

where $F(t) = \int_0^t f(s)\,\mathrm{d}s$. Notice that I_ϵ is not a differentiable functional, because F is only a continuous function. Hence, there does not exist any Nehari Manifold associated with I_ϵ, which brings some difficulties to show the existence of multiplicity of solutions, and makes the problem more interesting from the mathematical point of view. To avoid this difficulty, we will adapt for our problem an approach developed in Ambrosetti and Badiale [8].

In what follows, we denote by $g : \mathbb{R} \to \mathbb{R}$ the odd function given by

$$g(s) = \begin{cases} |s|^{p'-2}s, & s \in [0, a^{p-1}], \\ a, & s \in [a^{p-1}, (1+\delta)a^{p-1}], \\ (1+\delta)^{-1/p-1}|s|^{p'-2}s, & s \in [(1+\delta)a^{p-1}, +\infty), \end{cases}$$

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where $p' = p/p-1$. The functions f and g are related each other of the following way:

$$f(g(s)) = \begin{cases} s, & s \notin [a^{p-1}, (1+\delta)a^{p-1}], \\ a^{p-1}, & s \in [a^{p-1}, (1+\delta)a^{p-1}], \end{cases} \tag{a}$$

and

$$g(f(t)) = t, \quad \forall t \in \mathbb{R}. \tag{b}$$

In the sequel G denotes the primitive function of g, i.e.,

$$G(s) := \int_0^s g(r)\,\mathrm{d}r.$$

By the definition of g, G is an even function with

$$G(s) = \begin{cases} \frac{1}{p'} s^{p'}, & s \in [0, a^{p-1}], \\ as - \frac{a^p}{p}, & s \in [a^{p-1}, (1+\delta)a^{p-1}], \\ \frac{\gamma_\delta}{p'} s^{p'} + \delta \frac{a^p}{p}, & s \in [(1+\delta)a^{p-1}, +\infty), \end{cases}$$

for $\gamma_\delta = (1+\delta)^{-1/(p-1)}$. Thus

$$\gamma_\delta |s|^{1/p-1} \leqslant |g(s)| \leqslant |s|^{1/p-1}, \quad \forall s \in \mathbb{R}, \tag{2.1}$$

and

$$\frac{\gamma_\delta}{p'} |s|^{p'} \leqslant G(s) \leqslant \frac{1}{p'} |s|^{p'}, \quad \forall s \in \mathbb{R}. \tag{2.2}$$

The next step is to define the dual functional associated with I_ϵ. By [**12**, theorem 9.32], we know that for each $u \in L^{p'}(\mathbb{R}^N)$ there is a unique solution $w \in W^{1,p'}(\mathbb{R}^N) \cap W^{2,p'}(\mathbb{R}^N)$ for the equation

$$-\Delta w + w = u, \quad \text{in } \mathbb{R}^N. \tag{2.3}$$

Moreover, there is a positive constant C independent of w such that

$$\|w\|_{W^{2,p'}(\mathbb{R}^N)} \leqslant C|u|_{p'}.$$

The above information permits to define a linear operator $K : L^{p'}(\mathbb{R}^N) \to W^{2,p'}(\mathbb{R}^N)$, such that for $u \in L^{p'}(\mathbb{R}^N)$, $K(u)$ is the unique solution of (2.3). Hence,

$$\|K(u)\|_{W^{2,p'}(\mathbb{R}^N)} \leqslant C|u|_{p'}, \quad \forall u \in L^{p'}(\mathbb{R}^N),$$

from which it follows that K is continuous. On the other hand, since the embedding

$$W^{2,p'}(\mathbb{R}^N) \hookrightarrow L^s(\mathbb{R}^N), \quad \forall s \in [p', (p')^*)$$

are continuous for

$$(p')^* = \begin{cases} \dfrac{Np'}{N-2p'}, & N > 2p', \\ +\infty, & 1 \leqslant N \leqslant 2p', \end{cases}$$

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we can ensure that $K : L^{p'}(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$ is a linear continuous operator, because $p \in (2, 2^*)$ if, and only if, $p \in (p', (p')^*)$. Moreover, it is easy to check that

$$\int_{\mathbb{R}^N} K(u)v \, \mathrm{d}x = \int_{\mathbb{R}^N} K(v)u \, \mathrm{d}x, \quad \forall u, v \in L^{p'}(\mathbb{R}^N). \tag{2.4}$$

By the results of compactness Sobolev embeddings, one can easily obtain the following lemma:

Lemma 2.1. *If $u_n \rightharpoonup u$ in $L^{p'}(\mathbb{R}^N)$, then for some subsequence,*

$$K(u_n) \rightharpoonup K(u) \quad \text{in } L^p(\mathbb{R}^N)$$

and

$$K(u_n) \to K(u) \quad \text{in } L^p(B_R(0)), \quad \forall R > 0.$$

Using the above notation, we set the functional $J_\epsilon : L^{p'}(\mathbb{R}^N) \to \mathbb{R}$ given by

$$J_\epsilon(u) = \int_{\mathbb{R}^N} A(\epsilon x)^{-1/(p-1)} G(u) \, \mathrm{d}x - \frac{1}{2} \int_{\mathbb{R}^N} K(u)u \, \mathrm{d}x.$$

J_ϵ is called the dual functional associated with I_ϵ. It is obvious that $J_\epsilon \in C^1(L^{p'}(\mathbb{R}^N), \mathbb{R})$ and

$$J'_\epsilon(u)v = \int_{\mathbb{R}^N} A(\epsilon x)^{-1/(p-1)} g(u)v \, \mathrm{d}x - \int_{\mathbb{R}^N} K(u)v \, \mathrm{d}x, \quad \forall u, v \in L^{p'}(\mathbb{R}^N).$$

From the above equality, we have the following lemma.

Lemma 2.2. *If u is a critical point of J_ϵ, then $v(x) = A(\epsilon x)^{-1/(p-1)} g(u(x))$ is a solution of problem (1.1) if a, ϵ are small enough.*

Proof. If u is a critical point of J_ϵ, then

$$v(x) = K(u(x)) \quad \text{a.e.} \quad \text{in } \mathbb{R}^N,$$

i.e.,

$$-\Delta v(x) + v(x) = u(x) \quad \text{a.e.} \quad \text{in } \mathbb{R}^N.$$

Hence, if $|v(x)| \neq aA(\epsilon x)^{-1/(p-1)}$, then

$$-\Delta v(x) + v(x) = u(x) = A(\epsilon x) f(v(x)) \quad \text{a.e.} \quad \text{in } \mathbb{R}^N.$$

If $v(x) = aA(\epsilon x)^{-1/(p-1)}$, we have that

$$u(x) \in [a^{p-1}, (1+\delta)a^{p-1}],$$

and so,

$$\begin{aligned} &- a\epsilon^2 \Delta(A^{-1/(p-1)})(\epsilon x) + aA(\epsilon x)^{-1/(p-1)} \in [a^{p-1}, (1+\delta)a^{p-1}] \\ &\text{a.e.} \quad \text{in } \Omega_{a,\epsilon} = \left\{ x \in \mathbb{R}^N : v(x) = aA(\epsilon x)^{-1/(p-1)} \right\}, \end{aligned}$$

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or equivalently,

$$-\epsilon^2\Delta(A^{-1/(p-1)})(\epsilon x)+A(\epsilon x)^{-1/(p-1)}\in[a^{p-2},(1+\delta)a^{p-2}]$$
$$\text{a.e.}\quad \text{in } \Omega_{a,\epsilon}=\left\{x\in\mathbb{R}^N : v(x)=aA(\epsilon x)^{-1/(p-1)}\right\}.$$

Since $p\in(2,2^*)$, $\inf_{z\in\mathbb{R}^N}A(z)>0$ and $\sup_{z\in\mathbb{R}^N}\Delta(A^{-1/(p-1)})(z)<+\infty$, if a,ϵ are small enough, we deduce that $|\Omega_{a,\epsilon}|=0$, uniformly for all $\delta>0$. A similar argument works to prove that $|\Omega_{-a,\epsilon}|=0$. This completes the proof. □

LEMMA 2.3. *Let $\{u_n\}\subset L^{p'}(\mathbb{R}^N)$ be a $(PS)_d$ sequence for J_ϵ. Then, there exist $a^*,\epsilon^*,\delta^*>0$ such that, $\{u_n\}$ is bounded in $L^{p'}(\mathbb{R}^N)$, and for some subsequence, there is $u\in L^{p'}(\mathbb{R}^N)$ such that*

$$u_n\rightharpoonup u\quad \text{in } L^{p'}(\mathbb{R}^N),$$
$$u_n(x)\to u(x)\quad \text{a.e.}\quad \text{in } \mathbb{R}^N,$$

and

$$J'_\epsilon(u)=0,$$

for all $a\in[0,a^*]$, $\epsilon\in[0,\epsilon^*]$ and $\delta\in[0,\delta^*]$.

Proof. For each $n\in\mathbb{N}$,

$$J_\epsilon(u_n)-\frac{1}{2}J'_\epsilon(u_n)u_n=\int_{\mathbb{R}^N}A(\epsilon x)^{-1/(p-1)}\left(G(u_n)-\frac{1}{2}g(u_n)u_n\right)\mathrm{d}x. \tag{2.5}$$

Note that G and g are even and odd functions respectively, then by (2.1) and (2.2),

$$G(t)-\frac{1}{2}g(t)t\geqslant\left(\frac{\gamma_\delta}{p'}-\frac{1}{2}\right)|t|^{p'},\quad \forall t\in\mathbb{R}.$$

Since $p'=p/p-1<2$ and $\gamma_\delta=(1+\delta)^{-1/(p-1)}$, there exists $\delta^*>0$ such that

$$\left(\frac{\gamma_\delta}{p'}-\frac{1}{2}\right)>0,\quad \forall\delta\in[0,\delta^*].$$

Therefore, from (2.5), $(H1)$ and $(H2)$,

$$J_\epsilon(u_n)-\frac{1}{2}J'_\epsilon(u_n)u_n\geqslant\left(\frac{\gamma_\delta}{p'}-\frac{1}{2}\right)|u_n|_{p'}^{p'}. \tag{2.6}$$

Using the fact that $\{u_n\}$ is a $(PS)_d$ sequence, there is $n_0\in\mathbb{N}$ such that

$$J_\epsilon(u_n)-\frac{1}{2}J'_\epsilon(u_n)u_n\leqslant d+1+|u_n|_{p'},\quad \forall n\geqslant n_0. \tag{2.7}$$

From (2.6) and (2.7), $\{u_n\}$ is a bounded sequence in $L^{p'}(\mathbb{R}^N)$. Therefore, there is $u\in L^{p'}(\mathbb{R}^N)$ and a subsequence of $\{u_n\}$, still denoted by itself, such that

$$u_n\rightharpoonup u\quad \text{in } L^{p'}(\mathbb{R}^N).$$

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Using again the fact that $\{u_n\}$ is a $(PS)_d$ sequence for J_ϵ, we derive that

$$\sup_{|v|_{L^{p'}(\mathbb{R}^N)} \leqslant 1} \int_{\mathbb{R}^N} (A(\epsilon x)^{-1/(p-1)} g(u_n) - K(u_n)) v \, \mathrm{d}x \to 0.$$

Then by Riesz Representation theorem,

$$|A(\epsilon x)^{-1/(p-1)} g(u_n) - K(u_n)|_{L^p(\mathbb{R}^N)} \to 0.$$

Setting $v_n = K(u_n)$, it follows from lemma 2.1 that

$$v_n \rightharpoonup v = K(u) \quad \text{in } L^{p'}(\mathbb{R}^N)$$

and

$$v_n(x) \to v(x) \quad \text{a.e.} \quad \text{in } \mathbb{R}^N.$$

Hence,

$$A(\epsilon x)^{-1/(p-1)} g(u_n) \rightharpoonup v \quad \text{in } L^{p'}(\mathbb{R}^N),$$

and

$$A(\epsilon x)^{-1/(p-1)} g(u_n(x)) \to v(x) \quad \text{a.e.} \quad \text{in } \mathbb{R}^N.$$

The above limits together with the growth conditions on g yield

$$u(x) = A(\epsilon x) f(v(x)) \quad \text{if } |v(x)| \neq a A^{-1/(p-1)}(\epsilon x),$$

and

$$|u(x)| \in [a^{p-1}, (1+\delta) a^{p-1}] \quad \text{a.e.} \quad \text{in } |v(x)| = a A^{-1/(p-1)}(\epsilon x).$$

Recalling that

$$-\Delta v + v = u, \quad \text{a.e.} \quad \text{in } \mathbb{R}^N,$$

we deduce that $\left|\{x \in \mathbb{R}^N : |v(x)| = a A^{-1/(p-1)}(\epsilon x)\}\right| = 0$ if a, ϵ are sufficiently small (see the proof of lemma 2.2). Therefore,

$$u_n(x) \to u(x) \quad \text{a.e.} \quad \text{in } \mathbb{R}^N.$$

Now, fixed $\phi \in L^{p'}(\mathbb{R}^N)$, the last limit combined with the $J'_\epsilon(u_n)\phi = o_n(1)$ gives $J'_\epsilon(u)\phi = 0$, which shows that $J'_\epsilon(u) = 0$. □

Hereafter, we assume that $a \in [0, a^*]$, $\epsilon \in [0, \epsilon^*]$ and $\delta \in [0, \delta^*]$. In § 3, we will make some adjustments in ϵ^*, decreasing it if necessary. For more details see lemmas 3.3 and 3.4.

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In the sequel, set

$$J_\infty(u) = \int_{\mathbb{R}^N} G(u)\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} K(u)u\,\mathrm{d}x$$

and

$$J_{A_\infty}(u) = A_\infty^{-1/(p-1)}\int_{\mathbb{R}^N} G(u)\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} K(u)u\,\mathrm{d}x.$$

Furthermore, we denote by c_ϵ and c_∞ the mountain pass levels of J_ϵ and J_∞ respectively, which can be characterized as

$$c_\epsilon = \inf_{u\in\mathcal{M}_\varepsilon} J_\epsilon(u) \quad \text{and} \quad c_\infty = \inf_{u\in\mathcal{M}_\infty} J_\infty(u), \tag{2.8}$$

where

$$\begin{aligned}\mathcal{M}_\epsilon &= \{u \in L^{p'}(\mathbb{R}^N)\setminus\{0\} : J'_\epsilon(u)u = 0\}\\ &= \left\{u \in L^{p'}(\mathbb{R}^N)\setminus\{0\} : \int_{\mathbb{R}^N} A(\epsilon x)^{-1/(p-1)} g(u)u\,\mathrm{d}x = \int_{\mathbb{R}^N} K(u)u\,\mathrm{d}x\right\},\end{aligned}$$

and

$$\begin{aligned}\mathcal{M}_\infty &= \{u \in L^{p'}(\mathbb{R}^N)\setminus\{0\} : J'_\infty(u)u = 0\}\\ &= \left\{u \in L^{p'}(\mathbb{R}^N)\setminus\{0\} : \int_{\mathbb{R}^N} g(u)u\,\mathrm{d}x = \int_{\mathbb{R}^N} K(u)u\,\mathrm{d}x\right\}.\end{aligned}$$

The sets $\mathcal{M}_\epsilon$ and $\mathcal{M}_\infty$ are called Nehari manifolds associated with J_ϵ and J_∞ respectively.

As g is not a C^1 function, we cannot say that $\mathcal{M}_\epsilon$ is a differentiable manifold, and this brings us some difficulties to employ Lagrange multiplier on $\mathcal{M}$. Here, we overcome this difficulty by using some arguments found in [**26**].

Lemma 2.4. *The functional J_ϵ is bounded from below on $\mathcal{M}_\epsilon$ for all $\epsilon > 0$ and $\delta \in [0, \delta^*]$. Moreover, J_ϵ is coercive on $\mathcal{M}_\epsilon$ for all $\epsilon > 0$.*

Proof. For each $u \in \mathcal{M}_\epsilon$,

$$J_\epsilon(u) = J_\epsilon(u) - \frac{1}{2}J'_\epsilon(u)u = \int_{\mathbb{R}^N} A(\epsilon x)^{-1/(p-1)}\left(G(u) - \frac{1}{2}g(u)u\right)\mathrm{d}x. \tag{2.9}$$

Note that G and g are even and odd functions respectively, then by (2.1) and (2.2), one has

$$G(t) - \frac{1}{2}g(t)t \geqslant \left(\frac{\gamma_\delta}{p'} - \frac{1}{2}\right)|t|^{p'}, \quad \forall t \in \mathbb{R}.$$

Since $p' = p/p-1 < 2$ and $\gamma_\delta = (1+\delta)^{-1/(p-1)}$, there exists $\delta_0^* > 0$ such that

$$\left(\frac{\gamma_\delta}{p'} - \frac{1}{2}\right) > 0, \quad \forall \delta \in [0, \delta_0^*].$$

Therefore, it follows from (2.9), $(H1)$ and $(H2)$ that

$$J_\epsilon(u) \geqslant \left(\frac{\gamma_\delta}{p'} - \frac{1}{2}\right) |u|_{p'}^{p'},$$

showing that J_ϵ is bounded from below and coercive on $\mathcal{M}_\epsilon$. □

It follows from the continuity of K and A, and the inequality $G(t) - \frac{1}{2}g(t)t \geqslant C|t|^{p'}$ for all $t \in \mathbb{R}$ that we can derive the next lemma.

LEMMA 2.5. *There exists* $\theta = \theta(p) > 0$ *such that*

$$|u|_{p'}, \; J_\epsilon(u) > \theta, \quad \forall u \in \mathcal{M}_\epsilon.$$

Since the function g is odd, $g(t)/t$ is decreasing for $t > 0$ and K is a linear operator, by using the same method in [**27**, Chapter 4], we have the following lemma.

LEMMA 2.6. *For each* $v \in L^{p'}(\mathbb{R}^N) \setminus \{0\}$, *there exists a unique* $t_v > 0$ *such that*

$$J'_\epsilon(t_v v)(t_v v) = 0. \tag{2.10}$$

Proof. Let $v \in L^{p'}(\mathbb{R}^N) \setminus \{0\}$ be fixed and define the function $H(t) = J_\epsilon(tv)$ on $[0, \infty)$. Clearly, we have

$$H'(t) = 0 \Leftrightarrow v \in \mathcal{M}_\epsilon \Leftrightarrow \int_{\mathbb{R}^N} K(v)v \,\mathrm{d}x = \frac{1}{t} \int_{\mathbb{R}^N} A(\epsilon x)^{-1/(p-1)} g(tv)v \,\mathrm{d}x.$$

Note that $g(tv)/t$ is decreasing on $[0, \infty)$, $H(0) = 0$, $H(t) > 0$ for t small and $H(t) < 0$ for t large. Hence $\max_{t \in [0,\infty)} H(t)$ is achieved at a unique $t = t_v$ so that $H'(t_v) = 0$ and $t_v v \in \mathcal{M}_\epsilon$. □

From the last lemma, combined with the definition of c_ϵ, one has the following corollary:

COROLLARY 2.1. *If u is a critical point of J_ϵ with $u^\pm \neq 0$, then $J_\epsilon(u) \geqslant 2c_\epsilon$.*

Proof. If u is a critical point of J, then $0 = J'(u)u^+ = J'(u^+)u^+$ and $0 = J'(u)u^- = J'(u^-)u^-$. Hence, if $u^\pm \neq 0$, we must have that $u^\pm \in \mathcal{M}_\epsilon$, and so, $J(u^\pm) \geqslant c_\epsilon$. Now, the corollary follows by using the equality $J(u) = J(u^+) + J(u^-)$. □

The following lemma is very important in our paper, because it ensures the continuity of the function $v \mapsto t_v$ in $L^{p'}(\mathbb{R}^N) \setminus \{0\}$.

LEMMA 2.7. *For* $\{u_n\} \subset L^{p'}(\mathbb{R}^N)$ *and* $u \in L^{p'}(\mathbb{R}^N) \setminus \{0\}$, *let* t_{u_n}, $t_u > 0$ *be as in* (2.10). *If* $u_n \to u$ *in* $L^{p'}(\mathbb{R}^N)$, *then* $t_{u_n} \to t_u$.

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Proof. Firstly, we prove $t_{u_n} \nrightarrow 0$. Set $v = v_n$ in (2.10). By virtue of Hölder inequality and the continuity of K, one has

$$\int_{\mathbb{R}^N} K(u)u \,\mathrm{d}x \leqslant C|u|_{p'}^2, \quad \forall u \in L^{p'}(\mathbb{R}^N). \tag{2.11}$$

It follows from (2.1) and (2.11) that

$$\begin{aligned}\gamma_\delta t_{u_n}^{p'}|u_n|_{p'}^{p'} &\leqslant \int_{\mathbb{R}^N} A(\epsilon x)^{-1/(p-1)} g(t_{u_n}u_n)t_{u_n}u_n \,\mathrm{d}x \\ &= \int_{\mathbb{R}^N} K(t_{u_n}u_n)(t_{u_n}u_n)\,\mathrm{d}x \leqslant Ct_{u_n}^2|u_n|_{p'}^2 \,\mathrm{d}x \leqslant Ct_{u_n}^2|u_n|_{p'}^2.\end{aligned}$$

Hence for some $c > 0$

$$c|u_n|_{p'}^{p'-2} \leqslant t_{u_n}^{2-p'}, \quad \forall n \in \mathbb{N}.$$

Since $p' \in (1, 2)$ and $\{u_n\}$ is a bounded sequence in $L^{p'}(\mathbb{R}^N)$, one can easily obtain $t_{u_n} \nrightarrow 0$.

Secondly, we show that $\{t_{u_n}\}$ is a bounded sequence. According to (2.1), (2.10) and the continuity of K, we have

$$\begin{aligned}cA_0^{-1/(p-1)}t_{u_n}^{p'-2}|u_n|_{p'}^{p'} &\geqslant \frac{1}{t_{u_n}^2}\int_{\mathbb{R}^N} A(\epsilon x)^{-1/(p-1)} g(t_{u_n}u_n)t_{u_n}u_n \,\mathrm{d}x \\ &= \int_{\mathbb{R}^N} K(u_n)u_n \,\mathrm{d}x \to \int_{\mathbb{R}^N} K(u)u \,\mathrm{d}x > 0,\end{aligned}$$

where $A_0 = \inf_{x\in\mathbb{R}^N} A(x)$, and thus

$$t_{u_n}^{2-p'} < c_1 A_0^{-1/(p-1)}|u_n|_{p'}^{p'},$$

for some $c_1 > 0$, which means the boundedness of $\{t_{u_n}\}$. Finally, from Lebesgue's theorem, passing to a subsequence, we have $t_{u_n} \to t_0$, and so,

$$\begin{aligned}\int_{\mathbb{R}^N} A(\epsilon x)^{-1/(p-1)} g(t_0 u)t_0 u \,\mathrm{d}x &= \lim_{n\to\infty}\int_{\mathbb{R}^N} A(\epsilon x)^{-1/(p-1)} g(t_{u_n}u_n)t_{u_n}u_n \,\mathrm{d}x \\ &= \lim_{n\to\infty}\int_{\mathbb{R}^N} K(t_{u_n}u_n)t_{u_n}u_n \,\mathrm{d}x = \int_{\mathbb{R}^N} K(t_0 u)t_0 u \,\mathrm{d}x.\end{aligned}$$

Now, the uniqueness of t_u guarantees that $t_u = t_0 = \lim_{n\to+\infty} t_{u_n}$. □

Similar as in [**6**], one has the following two lemmas.

LEMMA 2.8. *If $\{u_n\} \subset L^{p'}(\mathbb{R}^N)$ is a $(PS)_d$ sequence for J_ϵ, then there exists $t_n > 0$ such that $t_n u_n \in \mathcal{M}_\epsilon$, i.e., $J'_\epsilon(t_n u_n)t_n u_n = 0$ and $t_n \to 1$ as $n \to \infty$.*

LEMMA 2.9. *If A satisfies hypotheses $(H1)$ and $(H2)$, then*

(i) *$J_\epsilon(0) = 0$ and there exists $\rho > 0$ such that $\inf_{|u|_{p'}=\rho} J_\epsilon(u) > 0$ and $J_\epsilon(u) \geqslant 0$ for $u \in L^{p'}(\mathbb{R}^N)$ with $|u|_{p'} \leqslant \rho$;*

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(ii) *There exists* $e \in L^{p'}(\mathbb{R}^N)$ *such that* $|e|_{p'} > \rho$ *and* $J_\epsilon(e) < 0$.

The next lemma is a crucial result to prove that the weak limit of some (PS) sequences can be chosen nontrivial.

LEMMA 2.10. *Let* $\{u_n\} \subset L^{p'}(\mathbb{R}^N)$ *be a* $(PS)_d$ *sequence for* J_ϵ. *Then, for some subsequence either*

(i) $u_n \to 0$ *in* $L^{p'}(\mathbb{R}^N)$, *or*

(ii) *there are* $r, \beta > 0$ *and* $(y_n) \subset \mathbb{R}^N$ *such that*

$$\int_{B_r(y_n)} |u_n|^{p'} \, \mathrm{d}x \geqslant \beta, \quad \forall n \in \mathbb{N}.$$

Proof. Assume that (ii) does not hold for $r > 0$. Then,

$$\lim_{n \to +\infty} \sup_{y \in \mathbb{R}^N} \int_{B_r(y)} |u_n|^{p'} = 0, \quad \forall n \in \mathbb{N}. \tag{2.12}$$

It follows from the fact that $\{u_n\}$ is a $(PS)_d$ sequence that

$$\sup_{|v|_{L^{p'}(\mathbb{R}^N)} \leqslant 1} \int_{\mathbb{R}^N} (A(\epsilon x)^{-1/(p-1)} g(u_n) - K(u_n)) v \, \mathrm{d}x \to 0. \tag{2.13}$$

Then by Riesz Representation theorem, we have

$$|A^{-1/(p-1)}(\epsilon x) g(u_n) - K(u_n)|_{L^p(\mathbb{R}^N)} \to 0. \tag{2.14}$$

By the growth condition of g, it is possible to find some $c > 0$ such that

$$|A^{-1/(p-1)}(\epsilon x) g(t)|^p \leqslant c|t|^{p'}, \quad \forall t \in \mathbb{R} \quad \text{and} \quad x \in \mathbb{R}^N. \tag{2.15}$$

Combining (2.12) and (2.15), we derive that

$$\limsup_{n \to +\infty} \int_{B_R(y_n)} |A^{-1/(p-1)}(\epsilon x) g(u_n)|^p \, \mathrm{d}x = 0.$$

Therefore, from (2.14),

$$\lim_{n \to +\infty} \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |K(u_n)|^p \, \mathrm{d}x = 0.$$

Thus, by a well-known result due to Lions [**22**],

$$K(u_n) \to 0 \quad \text{in } L^p(\mathbb{R}^N). \tag{2.16}$$

According to (2.14) and (2.16) we obtain that

$$A^{-1/(p-1)}(\epsilon x) g(u_n) \to 0 \quad \text{in } L^p(\mathbb{R}^N),$$

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which implies

$$g(u_n) \to 0 \quad \text{in } L^p(\mathbb{R}^N). \tag{2.17}$$

Since $\{u_n\}$ is a bounded sequence in $L^{p'}(\mathbb{R}^N)$, we derive that

$$\int_{\mathbb{R}^N} g(u_n)u_n \, \mathrm{d}x \to 0.$$

Recall that there exists $c > 0$ such that

$$c|t|^{p'} \leqslant |g(t)t|, \quad \forall t \in \mathbb{R},$$

then the last limit implies that $u_n \to 0$ in $L^{p'}(\mathbb{R}^N)$, which proves (i). □

REMARK 2.1. *The reader is invited to observe that a version of lemma 2.10 is also true, if we replace J_ϵ by J_∞.*

As a byproduct of the above lemma, we have the following corollary:

COROLLARY 2.2. *Let $\{u_n\}$ be a $(PS)_d$ sequence for J_ϵ with $d > 0$ and $u_n \rightharpoonup 0$ in $L^{p'}(\mathbb{R}^N)$. Then, for each $r > 0$, there are $\{y_n\} \subset \mathbb{R}^N$ and $\beta > 0$ such that*

$$\int_{B_r(y_n)} |u_n|^{p'} \, \mathrm{d}x \geqslant \beta, \quad \forall n \in \mathbb{N}.$$

Proof. If the corollary does not hold, for some subsequence, we must have

$$\lim_{n\to+\infty} \sup_{y\in\mathbb{R}^N} \int_{B_r(y)} |u_n|^{p'} \, \mathrm{d}x = 0.$$

Then by lemma 2.10,

$$u_n \to 0 \quad \text{in } L^{p'}(\mathbb{R}^N),$$

which is absurd, because $d > 0$. □

COROLLARY 2.3. *Let $\{u_n\}$ be a $(PS)_d$ sequence for J_∞ with $d > 0$ and $u_n \rightharpoonup 0$ in $L^{p'}(\mathbb{R}^N)$. Then, there is $\{y_n\} \subset \mathbb{R}^N$ such that $w_n(x) = u_n(x + y_n)$ is a $(PS)_d$ sequence for J_ϵ with $w_n \rightharpoonup w$ in $L^{p'}(\mathbb{R}^N)$ and $w \neq 0$.*

Proof. Firstly, we would like to point out that a version of corollary 2.2 holds with J_ϵ replaced by J_∞. Thus, fixed $r > 0$, there are $\{y_n\} \subset \mathbb{R}^N$ and $\beta > 0$ such that

$$\int_{B_r(y_n)} |u_n|^{p'} \, \mathrm{d}x \geqslant \beta, \quad \forall n \in \mathbb{N}.$$

Since J_∞ is invariant by translation, it is easy to check that $\{w_n\}$ is a $(PS)_d$ sequence for J_∞ and

$$\int_{B_r(0)} |w_n|^{p'} \, \mathrm{d}x \geqslant \beta, \quad \forall n \in \mathbb{N}.$$

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Consequently, by assumptions on g, we derive

$$\int_{B_r(0)} g(w_n)w_n \, \mathrm{d}x \geqslant \beta_1, \quad \forall n \in \mathbb{N},$$

for some $\beta_1 > 0$. Arguing as in the proof of lemma 2.10, we get

$$\int_{B_r(0)} K(w_n)w_n \, \mathrm{d}x \geqslant \beta_2, \quad \forall n \in \mathbb{N},$$

for some $\beta_2 > 0$. Since $w_n \rightharpoonup w$ in $L^p(\mathbb{R}^N)$, by lemma 2.1 we have $K(w_n) \to K(w)$ in $L^p(B_r(0))$, and so,

$$\int_{B_r(0)} K(w)w \, \mathrm{d}x \geqslant \beta_2, \quad \forall n \in \mathbb{N},$$

from which it follows that $w \neq 0$. □

Since the functional J_∞ is invariant by translation in $\mathbb{R}^N$, corollary 2.3 permits us to use a standard argument to show the following result:

Proposition 2.1. *There is $u_* \in \mathcal{M}_\infty$ such that*

$$J_\infty(u_*) = c_\infty \quad \text{and} \quad J'_\infty(u_*) = 0.$$

The following result is a key point in our arguments, because in some sense it establishes a compactness result for J_∞ on $\mathcal{M}_\infty$.

Theorem 2.1 Compactness theorem on Nehari manifold. *Let $\{u_n\} \subset L^{p'}(\mathbb{R}^N)$ be a sequence satisfying*

$$J_\infty(u_n) \to c_\infty, \quad J'_\epsilon(u_n) \to 0 \quad \textit{and} \quad u_n \in \mathcal{M}_\infty.$$

Then, for some subsequence either

(i) *$\{u_n\}$ is strongly convergent,*
or

(ii) *there exists $\{y_n\} \subset \mathbb{R}^N$ with $|y_n| \to \infty$ such that the sequence $v_n(x) = u_n(x + y_n)$ is strongly convergent to a function $v \in L^{p'}(\mathbb{R}^N)$ with*

$$J_\infty(v) = c_\infty, \quad J'_\infty(v) = 0 \quad \textit{and} \quad v \in \mathcal{M}_\infty.$$

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Proof. First of all, arguing as in lemma 2.3, we can assume that $\{u_n\}$ is bounded in $L^{p'}(\mathbb{R}^N)$, and that there is $u \in L^{p'}(\mathbb{R}^N)$ such that

$$u_n \rightharpoonup u \quad \text{in } L^{p'}(\mathbb{R}^N),$$

$$u_n(x) \to u(x) \quad \text{a.e.} \quad \text{in } \mathbb{R}^N,$$

and

$$J'_\infty(u) = 0.$$

If $u \neq 0$, we see that $u \in \mathcal{M}_\infty$ and

$$\begin{aligned} c_\infty = J_\infty(u) = J_\infty(u) - \frac{1}{2}J'_\infty(u)u &= \int_{\mathbb{R}^N} \left(G(u) - \frac{1}{2}g(u)u\right) \mathrm{d}x \\ &\leqslant \liminf_{n\to\infty} \int_{\mathbb{R}^N} \left(G(u_n) - \frac{1}{2}g(u_n)u_n\right) \mathrm{d}x \\ &\leqslant \limsup_{n\to\infty} \int_{\mathbb{R}^N} \left(G(u_n) - \frac{1}{2}g(u_n)u_n\right) \mathrm{d}x = \lim_{n\to\infty} (J_\infty(u_n) - \frac{1}{2}J'_\infty(u_n)u_n) = c_\infty, \end{aligned}$$

and so,

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} \left(G(u_n) - \frac{1}{2}g(u_n)u_n\right) \mathrm{d}x = \int_{\mathbb{R}^N} \left(G(u) - \frac{1}{2}g(u)u\right) \mathrm{d}x.$$

Recalling that $G(t) - \frac{1}{2}g(t)t \geqslant 0$ for all $t \in \mathbb{R}$, the above limit yields

$$G(u_n) - \frac{1}{2}g(u_n)u_n \to G(u) - \frac{1}{2}g(u)u \quad \text{in } L^1(\mathbb{R}^N).$$

From the above limit and the growth conditions on g we can easily have

$$u_n \to u \quad \text{in } L^{p'}(\mathbb{R}^N).$$

Therefore, $u \in \mathcal{M}_\infty$, $J_\infty(u) = c_\infty$ and $J'_\infty(u) = 0$.

If $u = 0$, we can use corollary 2.3 to find a sequence $\{y_n\} \subset \mathbb{R}^N$ such that $w_n(x) = u_n(x + y_n)$ is a $(PS)_{c_\infty}$ sequence with $w_n \rightharpoonup w$ in $L^{p'}(\mathbb{R}^N)$ and $w \neq 0$. Then, by previous arguments $w_n \to w$ in $L^{p'}(\mathbb{R}^N)$, and so, $w \in \mathcal{M}_\infty$, $J_\infty(w) = c_\infty$ and $J'_\infty(w) = 0$. □

The following step would be to check whether or not the restriction of J_ϵ to $\mathcal{M}_\epsilon$ satisfies the (PS)-condition. A standard method would lead us to deal with the second derivative of J_ϵ, which does not adapt our situation, because the functional is not twice differentiable. With this aim in mind, we will adopt some ideas applied in [**26**].

Consider the application

$$\hat{m}_\epsilon : L^{p'}(\mathbb{R}^N) \setminus \{0\} \to \mathcal{M}_\epsilon,$$

given by $\hat{m}_\epsilon(u) = t_u u$, where t_u is defined by (2.10). Using the above notations it is possible to show that

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(i) $\hat{m}_\epsilon$ is a continuous application;

(ii) There exists $\tau > 0$ such that $t_u > \tau$, $\forall u \in \mathcal{S}_{p'} = \{u \in L^{p'}(\mathbb{R}^N) : |u|_{p'} = 1\}$. Indeed, if $\{u_n\} \subset L^{p'}(\mathbb{R}^N)$ and $t_{u_n} \to 0$ as $n \to \infty$, then $t_{u_n} u_n \to 0$ as $n \to \infty$, and this contradicts lemma 2.5 for $t_{u_n} u_n \in \mathcal{M}_\epsilon$.

(iii) Given $\mathcal{W} \subset \mathcal{S}_{p'}$ compact, there is $C_\mathcal{W} > t_u$, $\forall u \in \mathcal{W}$.

In the sequel, we consider the application $m_\epsilon : \mathcal{S}_{p'} \to \mathcal{M}_\epsilon$, the restriction of $\hat{m}_\epsilon$ to the sphere $\mathcal{S}_{p'}$. Observe that m_ϵ is a homeomorphism, with its inverse given by

$$m_\epsilon^{-1}(u) = \frac{u}{|u|_{p'}}, \quad \forall u \in \mathcal{M}_\epsilon.$$

Let us also consider the application $\hat{\Psi}_\epsilon : L^{p'}(\mathbb{R}^N) \setminus \{0\} \to \mathbb{R}$ given by

$$\hat{\Psi}_\epsilon(u) := J_\epsilon(\hat{m}_\epsilon(u)),$$

and its restriction to the sphere, $\Psi_\epsilon(u) : \mathcal{S}_{p'} \to \mathbb{R}$. Note that both $\hat{\Psi}_\epsilon$ and Ψ_ϵ are continuous. The following lemma is very important in our approach, and the proof can be found in [**26**, Chapter 3].

Lemma 2.11 [**26**]. *The applications defined above satisfy*:

(i) $\hat{\Psi}_\epsilon \in C^1(L^{p'}(\mathbb{R}^N) \setminus \{0\}, \mathbb{R})$ *and, for* $u \in L^{p'}(\mathbb{R}^N) \setminus \{0\}$,

$$\begin{aligned}\hat{\Psi}'_\epsilon(u)v &= \frac{|\hat{m}_\epsilon(u)|_{p'}}{|u|_{p'}} J'_\epsilon(\hat{m}_\epsilon(u))v \\ &= t_u J'_\epsilon(\hat{m}_\epsilon(u))v, \quad \forall v \in L^{p'}(\mathbb{R}^N);\end{aligned}$$

(ii) $\Psi_\epsilon \in C^1(\mathcal{S}_{p'}, \mathbb{R})$ *and, for* $u \in \mathcal{S}_{p'}$,

$$\Psi_\epsilon(u)v = |m_\epsilon(u)|_{p'} J'_\epsilon(m_\epsilon(u))v, \quad \forall v \in T_u\mathcal{S}_{p'},$$

where $T_u\mathcal{S}_{p'}$ *denotes the tangent space of* $\mathcal{S}_{p'}$ *at* u;

(iii) *If* $\{u_n\} \subset \mathcal{S}_{p'}$ *is a (PS)-sequence for* Ψ_ϵ, *then* $\{m_\epsilon(u_n)\}$ *is a (PS)-sequence for* J_ϵ. *Reciprocally, if* $\{m_\epsilon(u_n)\}$ *is a (PS)-sequence for* J_ϵ, *then* $\{m_\epsilon^{-1}(u_n)\}$ *is a (PS)-sequence for* Ψ_ϵ;

(iv) $u \in \mathcal{S}_{p'}$ *is a critical point of* Ψ_ϵ *if, and only if* m_ϵ *is a (nonzero) critical of* J_ϵ. *Moreover,*

$$\inf_{\mathcal{S}_{p'}} \Psi_\epsilon = \inf_{\mathcal{M}_\epsilon} J_\epsilon.$$

We would like to point out that a version of lemma 2.11 also holds for the functional J_∞. Hereafter, $\hat{\Psi}_\infty$ and Ψ_∞ are the functionals given in lemma 2.11 with J_ϵ replaced by J_∞. Combining lemma 2.11 and proposition 2.1, we have the following result:

Corollary 2.4. *There is* $\omega_* \in \mathcal{S}_{p'}$ *such that*

$$\Psi_\infty(\omega_*) = c_\infty \quad \textit{and} \quad \Psi'_\infty(\omega_*) = 0.$$

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3. Estimates involving the minimax levels

In this section, we will prove some estimates involving the minimax levels c_ϵ and c_∞. From

$$J_\infty(u) \leqslant J_\epsilon(u) \quad \forall u \in L^{p'}(\mathbb{R}^N),$$

we have

$$c_\infty \leqslant c_\epsilon.$$

Lemma 3.1. *The minimax levels c_ϵ and c_{A_∞} satisfy the inequality $c_\epsilon < c_{A_\infty}$. Then $c_\infty < c_{A_\infty}$.*

Proof. Arguing as in the proof of theorem 2.1, there exists $u \in L^{p'}(\mathbb{R}^N)$ such that

$$J_{A_\infty}(u_0) = c_{A_\infty} \quad \text{and} \quad J'_{A_\infty}(u_0) = 0,$$

where $J_{A_\infty} : L^{p'}(\mathbb{R}^N) \to \mathbb{R}$ denotes the dual energy functional associated with the problem

$$\begin{cases} -\Delta u + u = A_\infty f(u(x)) \text{ a.e.} \quad \text{in } \mathbb{R}^N, \\ u \in W^{2,p/p-1}(\mathbb{R}^N), \end{cases} \tag{3.1}$$

and c_{A_∞} denotes the mountain pass level of J_{A_∞}.

By lemma 2.6, there exists $t > 0$ such that $tu_0 \in \mathcal{M}_\epsilon$, i.e.,

$$c_\epsilon \leqslant J_\epsilon(tu_0) = \int_{\mathbb{R}^N} A(\epsilon x)^{-1/(p-1)} G(tu_0)\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} (tu_0)K(tu_0)\,\mathrm{d}x.$$

Since $A(\epsilon x) > A_\infty$ for all $x \in \mathbb{R}^N$, we have

$$c_\epsilon \leqslant J_{A_\infty}(tu_0) \leqslant \max_{s \geqslant 0} J_{A_\infty}(su_0) = J_{A_\infty}(u_0) = c_{A_\infty},$$

which finishes the proof. □

The next lemma is very useful in verifying $(PS)_c$ condition for some values of c.

Lemma 3.2. *The functional Ψ_ϵ satisfies the $(PS)_c$ condition for $c \leqslant c_\infty + \gamma$, where $\gamma = \frac{1}{2}(c_{A_\infty} - c_\infty)$.*

Proof. Let $(\omega_n) \subset \mathcal{S}_{p'}$ be a $(PS)_c$ sequence for Ψ_ϵ. From lemma 2.11 $u_n = m_\epsilon(\omega_n)$ is also a $(PS)_c$ sequence for functional J_ϵ. Then by lemma 2.3, $\{u_n\}$ is a bounded sequence in $L^{p'}(\mathbb{R}^N)$, passing to a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, there

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exists $u \in L^{p'}(\mathbb{R}^N)$ such that

$$u_n \rightharpoonup u \quad \text{in } L^{p'}(\mathbb{R}^N) \quad \text{and} \quad u_n(x) \to u(x) \quad \text{a.e.} \quad \text{in } \mathbb{R}^N.$$

As $G \in C^1(\mathbb{R})$, the same argument found in [7, propositions 2.1 and 4.1] works to show that

$$J_\epsilon(u_n) - J_\epsilon(w_n) - J_\epsilon(u) = o_n(1), \tag{3.2}$$

where $w_n = u_n - u$. Now, we will show that

$$\|J'_\epsilon(u_n) - J'_\epsilon(w_n) - J'_\epsilon(u)\| = o_n(1). \tag{3.3}$$

Unfortunately, we cannot use [7], because g is not $C^1(\mathbb{R})$. Here the idea is the following: For each $v \in L^{p'}(\mathbb{R}^N)$ with $|v|_{p'} \leqslant 1$, as $\{u_n\}$ and $\{w_n\}$ are bounded sequences in $L^{p'}(\mathbb{R}^N)$, combining the growth conditions on g and Hölder inequality, we can ensure that, given $\tau > 0$, there exists $R > 0$, which is independent of n and v, such that

$$\sup_{|v|_{p'} \leqslant 1} \int_{B_R^c(0)} |g(u_n) - g(w_n) - g(u)||v| \, \mathrm{d}x \leqslant \frac{\tau}{2}, \quad \forall n \in \mathbb{N}. \tag{3.4}$$

On the other hand, arguing as in the proof of lemma 2.3, and using lemma 2.1, we know that

$$|g(u_n) - K(u_n)|_p \to 0,$$

$$K(u_n) \rightharpoonup K(u) \quad \text{in } L^p(\mathbb{R}^N),$$

and

$$K(u_n) \to K(u) \quad \text{in } L^p(B_R(0)), \quad \forall R > 0.$$

Then, for $R > 0$ fixed, there is $h_1 \in L^p(B_R(0))$ such that

$$|K(u_n)| \leqslant h_1 \quad \text{a.e.} \quad \text{in } B_R(0).$$

From this, there is $h_2 \in L^p(B_R(0))$ such that

$$|g(u_n)| \leqslant h_2 \quad \text{a.e.} \quad \text{in } B_R(0).$$

By growth condition on g, there exists $h_3 \in L^{p'}(B_R(0))$ such that

$$|u_n| \leqslant h_3 \quad \text{a.e.} \quad \text{in } B_R(0).$$

Recalling that $u_n(x) \to u(x)$ a.e in $\mathbb{R}^N$, the Lebesgue theorem gives

$$g(u_n) - g(w_n) - g(u) \to 0 \quad \text{in } L^p(B_R(0)).$$

From this, we derive that there is $n_0 \in \mathbb{N}$ such that

$$\sup_{|v|_{p'} \leqslant 1} \int_{B_R(0)} |g(u_n) - g(w_n) - g(u)||v| \, \mathrm{d}x \leqslant \frac{\tau}{2}, \quad \forall n \geqslant n_0. \tag{3.5}$$

Then (3.3) follows from (3.4) and (3.5).

Recalling that $J'_\epsilon(u) = 0$ and $J_\epsilon(u) \geqslant 0$, it follows from (3.2) and (3.3) that

$$\|J'_\epsilon(w_n)\| = o_n(1) \quad \text{and} \quad J_\epsilon(w_n) \to c^* = c - J_\epsilon(u). \tag{3.6}$$

This implies that $\{w_n\}$ is a $(PS)_{c^*}$ sequence for J_ϵ with $c^* \leqslant c_\infty + \gamma$.

Claim 1. For each $R > 0$ fixed,

$$\lim_{n\to+\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |w_n|^{p'} \, \mathrm{d}x = 0. \tag{3.7}$$

Indeed, we can apply lemma 2.10 to deduce that $w_n \to 0$ in $L^{p'}(\mathbb{R}^N)$, or equivalently, $u_n \to u$ in $L^{p'}(\mathbb{R}^N)$. From this we have $\omega_n = m_\epsilon^{-1}(u_n) \to m_\epsilon^{-1}(u) = \omega$ in $L^{p'}(\mathbb{R}^N)$. Then the lemma is proved.

Now, we are going to prove Claim 1. If the claim is false, by lemma 2.10, there are $\xi > 0$ and $\{y_n\} \subset \mathbb{R}^N$ such that

$$\limsup_{n\to+\infty} \int_{B_R(y_n)} |w_n|^{p'} \, \mathrm{d}x \geqslant \xi > 0.$$

Hence, there is $c > 0$ such that

$$\limsup_{n\to+\infty} \int_{B_R(y_n)} g(w_n)w_n \, \mathrm{d}x \geqslant \xi > 0.$$

Due to the fact that $w_n \rightharpoonup 0$ in $L^{p'}(\mathbb{R}^N)$ and $\{g(w_n)\}$ is convergent in $L^p(B_R(0))$, it means that $\{y_n\}$ is an unbounded sequence. Setting

$$\tilde{w}_n = w_n(. + y_n),$$

we have that $\{\tilde{w}_n\}$ is a bounded sequence in $L^{p'}(\mathbb{R}^N)$. Hence, there exist $\tilde{w} \in L^{p'}(\mathbb{R}^N) \setminus \{0\}$ and a subsequence of $\{\tilde{w}_n\}$, still denoted by itself, such that

$$\tilde{w}_n \rightharpoonup \tilde{w} \quad \text{in } L^{p'}(\mathbb{R}^N).$$

Moreover, the same arguments used in the proof of lemma 2.3 give

$$\tilde{w}_n(x) \to \tilde{w}(x) \quad \text{a.e.} \quad \text{in } \mathbb{R}^N.$$

Since $J'_\epsilon(w_n)\psi(. - y_n) = o_n(1)$ for $\forall \psi \in L^{p'}(\mathbb{R}^N)$, the above limits ensure that

$$\int_{\mathbb{R}^N} A_\infty^{-1/(p-1)} g(\tilde{w})\psi \, \mathrm{d}x = \int_{\mathbb{R}^N} K(\tilde{w})\psi \, \mathrm{d}x,$$

implying that $\tilde{w}$ is a nontrivial critical point of solution of J_{A_∞}. As a consequence

$$\begin{aligned}
c_{A_\infty} &\leqslant J_{A_\infty}(\bar{w}) \\
&= J_{A_\infty}(\bar{w}) - \frac{1}{2} J'_{A_\infty}(\bar{w})\bar{w} \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^N} A(\epsilon(x + y_n))^{-1/(p-1)} \left(G(\tilde{w}_n) - \frac{1}{2} g(\tilde{w}_n)\tilde{w}_n \right) \mathrm{d}x \\
&\leqslant \liminf_{n\to\infty} \left[J_\epsilon(w_n) - \frac{1}{2} J'_\epsilon(w_n)w_n \right] \\
&= c_* \leqslant c_\infty + \gamma,
\end{aligned}$$

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which contradicts to the fact that $\gamma < c_{A_\infty} - c_\infty$. Hence Ψ_ϵ satisfies the $(PS)_c$ condition for $c \leqslant c_\infty + \gamma$. □

In the following, fix $\rho_0, r_0 > 0$ satisfying $\overline{B_{\rho_0}(z_i)} \cap \overline{B_{\rho_0}(z_j)} = \emptyset$ for $i \neq j$ and $i, j \in \{1, \ldots, l\}$, $\bigcup_{i=1}^{l} B_{\rho_0}(z_i) \subset B_{r_0}(0)$ and $K_{\rho_0/2} = \bigcup_{i=1}^{l} \overline{B_{\rho_0/2}(z_i)}$. Moreover, we also set the function $Q_\varepsilon : L^{p'}(\mathbb{R}^N) \to \mathbb{R}^N$ by

$$Q_\varepsilon(u) = \frac{\int_{\mathbb{R}^N} \chi(\varepsilon x)|u|^{p'} \, \mathrm{d}x}{\int_{\mathbb{R}^N} |u|^{p'} \, \mathrm{d}x},$$

where $\chi : \mathbb{R}^N \to \mathbb{R}^N$ is given by

$$\chi(x) = \begin{cases} x, & \text{if } |x| \leqslant r_0, \\ r_0 \frac{x}{|x|}, & \text{if } |x| > r_0. \end{cases}$$

The following lemma is very useful to obtain $(PS)_c$ sequences associated with Ψ_ϵ.

Lemma 3.3. *There exist $\alpha_0 > 0$ and $\epsilon_1 > 0$ such that if $u \in \mathcal{S}_{p'}$ and $\Psi_\epsilon \leqslant c_\infty + \alpha_0$, then $Q_\epsilon(u) \in K_{\rho_0/2}$, $\quad \forall \epsilon \in (0, \epsilon_1)$.*

Proof. If the lemma is not true, then there exist $\alpha_n \to 0$, $\epsilon_n \to 0$ and $\omega_n \in \mathcal{S}_{p'}$ such that

$$\Psi_{\epsilon_n}(\omega_n) \leqslant c_\infty + \alpha_n$$

and

$$Q_{\epsilon_n}(\omega_n) \notin K_{\rho_0/2}.$$

From the definitions of Ψ_∞ and Ψ_ϵ we have

$$c_\infty \leqslant \Psi_\infty(\omega_n) \leqslant \Psi_{\epsilon_n}(\omega_n) \leqslant c_\infty + \alpha_n, \quad \forall n \in \mathbb{N}.$$

Then

$$\{\omega_n\} \subset \mathcal{S}_{p'} \quad \text{and} \quad \Psi_\infty(\omega_n) \to c_\infty.$$

By Ekeland's Variational principle, we can assume that $\Psi'_\infty(\omega_n) \to 0$. Hence, $u_n = m_\infty(\omega_n)$ verifies

$$\{u_n\} \subset \mathcal{M}_\infty, \quad J_\infty(u_n) \to c_\infty \quad \text{and} \quad J'_\epsilon(u_n) \to 0.$$

By virtue of theorem 2.1, we need to consider the following two cases:

(i) $u_n \to u \neq 0$ in $L^{p'}(\mathbb{R}^N)$,
or

(ii) there exists $\{y_n\} \subset \mathbb{R}^N$ with $|y_n| \to \infty$ such that the sequence $v_n = u_n(. + y_n)$ is convergent in $L^{p'}(\mathbb{R}^N)$ for some $v \in L^{p'}(\mathbb{R}^N) \setminus \{0\}$.

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Analysis of (i): Applying Lebesgue's dominated convergence theorem, we have

$$Q_{\epsilon_n}(u_n) = \frac{\int_{\mathbb{R}^N} \chi(\epsilon_n x)|u_n|^{p'}\,\mathrm{d}x}{\int_{\mathbb{R}^N} |u_n|^{p'}\,\mathrm{d}x} \to \frac{\int_{\mathbb{R}^N} \chi(0)|u|^{p'}\,\mathrm{d}x}{\int_{\mathbb{R}^N} |u|^{p'}\,\mathrm{d}x} = 0 \in K_{\rho_0/2}.$$

Then $Q_{\epsilon_n}(\omega_n) = Q_{\epsilon_n}(u_n) \in K_{\rho_0/2}$ for n large enough, which is a contradiction.

Analysis of (ii): By $J'_\epsilon(u_n)u_n = 0$, we have

$$\int_{\mathbb{R}^N} A(\epsilon_n x + \epsilon_n y_n)^{-1/p-1} g(v_n)v_n\,\mathrm{d}x = \int_{\mathbb{R}^N} v_n K(v_n)\,\mathrm{d}x. \tag{3.8}$$

Next, we consider two cases:

(I) $\{\epsilon_n y_n\} \to +\infty$, and

(II) $\epsilon_n y_n \to y$ for some $y \in \mathbb{R}^N$, for some subsequence.

If (I) holds, it follows from $v_n \to v$ in $L^{p'}(\mathbb{R}^N)$ and (3.8) that

$$\int_{\mathbb{R}^N} A_\infty^{-1/p-1} g(v)v\,\mathrm{d}x = \int_{\mathbb{R}^N} vK(v)\,\mathrm{d}x,$$

and so, $v \in \mathcal{M}_\infty$. Thereby

$$\begin{aligned} c_{A_\infty} &\leqslant J_{A_\infty}(v) \\ &= J_{A_\infty}(v) - \frac{1}{2}J'_{A_\infty}(v)v \\ &= \lim_{n\to+\infty} \int_{\mathbb{R}^N} A(\epsilon_n x + \epsilon_n y_n)^{-1/(p-1)} \left(G(v_n) - \frac{1}{2}g(v_n)v_n\right)\mathrm{d}x. \end{aligned}$$

Since

$$\begin{aligned} &\int_{\mathbb{R}^N} A(\epsilon_n x + \epsilon_n y_n)^{-1/(p-1)} \left(G(v_n) - \frac{1}{2}g(v_n)v_n\right)\mathrm{d}x \\ &\quad = \int_{\mathbb{R}^N} A^{-1/(p-1)}(x) \left(G(u_n) - \frac{1}{2}g(u_n)u_n\right)\mathrm{d}x \\ &\quad = J_\infty(u_n) - \frac{1}{2}J'_\infty(u_n)u_n, \end{aligned}$$

we obtain

$$c_{A_\infty} \leqslant \lim_{n\to\infty}\left(J_\infty(u_n) - \frac{1}{2}J'_\infty(u_n)u_n\right) = \lim_{n\to\infty} J_\infty(u_n) = c_\infty,$$

which contradicts to lemma 3.1.

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Now, if (II) holds, the previous argument yields

$$c_{A(y)} \leqslant c_\infty, \tag{3.9}$$

where $c_{A(y)}$ is the mountain pass level of the functional $J_{A(y)}(u) : L^{p'}(\mathbb{R}^N) \to \mathbb{R}$ given by

$$J_{A(y)}(u) = A(y)^{-1/(p-1)} \int_{\mathbb{R}^N} G(u)\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} uK(u)\,\mathrm{d}x.$$

One can see that

$$c_{A(y)} = \inf_{u\in\mathcal{M}_{A(y)}} J_{A(y)}(u),$$

where

$$\mathcal{M}_{A(y)} = \{u \in L^{p'}(\mathbb{R}^N) \setminus \{0\} : J'_{A(y)}(u)u = 0\}.$$

If $A(y) < 1$, it is possible to prove that $c_{A(y)} > c_\infty$, which contradicts (3.9). Then $A(y) = 1$ and $y = z_i$ for some $i = 1, \ldots, l$. Hence

$$\begin{aligned} Q_{\epsilon_n}(u_n) &= \frac{\int_{\mathbb{R}^N} \chi(\epsilon_n x)|u_n|^{p'}\,\mathrm{d}x}{\int_{\mathbb{R}^N} |u_n|^{p'}\,\mathrm{d}x} = \frac{\int_{\mathbb{R}^N} \chi(\epsilon_n x + \epsilon_n y_n)|v_n|^{p'}\,\mathrm{d}x}{\int_{\mathbb{R}^N} |v_n|^{p'}\,\mathrm{d}x} \to \frac{\int_{\mathbb{R}^N} \chi(y)|v|^{p'}\,\mathrm{d}x}{\int_{\mathbb{R}^N} |v|^{p'}\,\mathrm{d}x} \\ &= a_i \in K_{\rho_0/2} \end{aligned}$$

from which it follows that $Q_{\epsilon_n}(\omega_n) = Q_{\epsilon_n}(u_n) \in K_{\rho_0/2}$ for n large, which is absurd, because we are assuming that $Q_{\epsilon_n}(\omega_n) \notin K_{\rho_0/2}$. Then the proof is completed. □

Next, we give the following symbols.

$$\begin{aligned} \Omega^i_\epsilon &= \{u \in S_{p'} : |Q_\epsilon(u) - z_i| < \rho_0\}, \\ \partial\Omega^i_\epsilon &= \{u \in S_{p'} : |Q_\epsilon(u) - z_i| = \rho_0\}, \\ \alpha^i_\epsilon &= \inf_{u\in\Omega^i_\epsilon} \Psi_\epsilon(u), \\ \tilde{\alpha}^i_\epsilon &= \inf_{u\in\partial\Omega^i_\epsilon} \Psi_\epsilon(u). \end{aligned}$$

LEMMA 3.4. *There exists $\epsilon_2 > 0$ such that*

$$\alpha^i_\epsilon < c_\infty + \gamma \quad \textit{and} \quad \alpha^i_\epsilon < \tilde{\alpha}^i_\epsilon,$$

for all $\epsilon \in (0, \epsilon_2)$, where $\gamma = \frac{1}{2}(c_{A_\infty} - c_\infty) > 0$.

Proof. Let $u \in L^{p'}(\mathbb{R}^N)$ be a ground state critical of Ψ_∞, i.e.,

$$u \in \mathcal{S}_{p'}, \quad \Psi_\infty(u) = c_\infty \quad \text{and} \quad \Psi'_\infty(u) = 0 \quad \text{(See corollary 2.4).}$$

For $1 \leqslant i \leqslant l$ and $\epsilon > 0$, we define the function $\tilde{u}^i_\epsilon : \mathbb{R}^N \to \mathbb{R}$ by $\tilde{u}^i_\epsilon(x) = u(x - z_i/\epsilon)$. Clearly $\tilde{u}^i \in \mathcal{S}_{p'}$ and the following claim holds:

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Claim 2. For all $1 \leqslant i \leqslant l$, we have

$$\limsup_{\epsilon \to 0} \left(\sup_{t \geqslant 0} J_\epsilon(t\tilde{u}^i_\epsilon) \right) \leqslant c_\infty.$$

In fact, by a simple change of variable,

$$J_\epsilon(t\tilde{u}^i_\epsilon) = \int_{\mathbb{R}^N} A(\epsilon x + z_i)^{-1/p-1} G(tu)\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} tuK(tu)\,\mathrm{d}x.$$

By virtue of a direct computation there exists $s = s(\epsilon) > 0$ such that

$$\max_{t \geqslant 0} J_\epsilon(t\tilde{u}^i_\epsilon) = J_\epsilon(s\tilde{u}^i_\epsilon).$$

Furthermore, it is possible to show that $s(\epsilon) \to s_0 > 0$ as $\epsilon \to 0$. Hence

$$\begin{aligned}\limsup_{\epsilon \to 0} \left(\max_{t \geqslant 0} J_\epsilon(t\tilde{u}^i_\epsilon) \right) &= \int_{\mathbb{R}^N} A(z_i)^{-1/p-1} G(s_0 u)\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} s_0 uK(s_0 u)\,\mathrm{d}x \\ &\leqslant J_\infty(s_0 u) \leqslant \max_{s \geqslant 0} J_\infty(u) = c_\infty.\end{aligned}$$

Then

$$\limsup_{\epsilon \to 0} \left(\sup_{t \geqslant 0} J_\epsilon(t\tilde{u}^i_\epsilon) \right) \leqslant c_\infty,$$

for $i \in \{1, \ldots, l\}$ and Claim 2 is proved.

Once $Q_\epsilon(\tilde{u}^i_\epsilon) \to z_i$ as $\epsilon \to 0$, it means that $\tilde{u}^i_\epsilon \in \Omega^i_\epsilon$ for ϵ sufficiently small. On the other hand, from Claim 2 one has

$$\limsup_{\epsilon \to 0} \Psi_\epsilon(\tilde{u}^i_\epsilon) < c_\infty + \frac{\alpha_0}{4}.$$

Hence, there is $\epsilon_* > 0$ such that

$$\alpha^i_\epsilon < c_\infty + \frac{\alpha_0}{4}, \quad \forall \epsilon \in (0, \epsilon_*). \tag{3.10}$$

Then, decreasing α_0 if necessary, we derive

$$\alpha^i_\epsilon < c_\infty + \gamma, \quad \forall \epsilon \in (0, \epsilon_*),$$

which proves the first inequality. We now show the second one. Note that if $u \in \partial\Omega^i_\epsilon$, then

$$u \in \mathcal{S}_{p'} \quad \text{and} \quad |Q_\epsilon(u) - z_i| = \rho_0 > \frac{\rho_0}{2},$$

that is, $Q_\epsilon(u) \notin K_{\rho_0/2}$. Thus, from lemma 3.3 we obtain that

$$\Psi_\epsilon(u) > c_\infty + \alpha_0 \,\text{forall}\, u \in \partial\Omega^i_\epsilon \,\text{and}\, \epsilon \in (0, \epsilon_1),$$

and so

$$\tilde{\alpha}^i_\epsilon = \inf_{u \in \partial\Omega^i_\epsilon} \Psi_\epsilon(u) \geqslant c_\infty + \alpha_0, \quad \forall \epsilon \in (0, \epsilon_1). \tag{3.11}$$

Consequently, it follows from (3.10) and (3.11) that

$$\alpha_\epsilon^i < \tilde{\alpha}_\epsilon^i \quad \text{for } \forall \epsilon \in (0, \epsilon_1).$$

Then the results are proved by fixing $\epsilon_2 = \min\{\epsilon_1, \epsilon_*\}$. □

Proof of theorem 1.1. From lemma 3.4 there exists $\epsilon_2 > 0$ such that

$$\alpha_\epsilon^i < \tilde{\alpha}_\epsilon^i \, \text{for} \, \forall \epsilon \in (0, \epsilon_1).$$

It follows from lemma 2.4 that Ψ_ϵ is bounded from on $S_{p'}$. By Ekeland's variational principle there exists a minimizing sequence $(u_n^i) \subset \Omega_\epsilon^i$ such that

$$\Psi_\epsilon(u_n^i) \to \alpha_\epsilon^i \tag{3.12}$$

and

$$\Psi_\epsilon(u_n^i) < \Psi_\epsilon(u) + \frac{1}{n}|u - u_n^i|_{p'}, \quad \forall u \in \Omega_\epsilon^i, \quad u \neq u_n^i. \tag{3.13}$$

Fixing $v \in L^{p'}(\mathbb{R}^N)$ and τ_n^i small enough, we set the path $\gamma_n^i : (-\tau_n^i, \tau_n^i) \to \Omega_\epsilon^i$ by

$$\gamma_n^i(s) = \frac{u_n^i + sv}{|u_n^i + sv|_{p'}}.$$

A simple computation gives that $\gamma_n^i \in C^1((-\tau_n^i, \tau_n^i), \Omega_\epsilon^i)$ with

$$\gamma_n^i(0) = u_n^i \quad \text{and} \quad (\gamma_n^i)'(0) = v - u_n^i \int_{\mathbb{R}^N} |u_n^i|^{p'-2} u_n^i v \, \mathrm{d}x. \tag{3.14}$$

From (3.13),

$$\Psi_\epsilon(u_n^i) < \Psi_\epsilon(\gamma_n^i(s)) + \frac{1}{n}|\gamma_n^i(s) - u_n^i|_{p'}, \quad \forall s \in (-\tau_n^i, \tau_n^i),$$

leading to

$$-\frac{1}{n}\left|\frac{\gamma_n^i(s) - u_n^i}{s}\right|_{p'} < \frac{\Psi_\epsilon(\gamma_n^i(s)) - \Psi_\epsilon(u_n^i)}{s}, \quad \forall s \in (0, \tau_n^i).$$

Taking the limit of $s \to 0^+$, we get

$$-\frac{1}{n}|(\gamma_n^i)'(0)| \leqslant \Psi_\epsilon'(\gamma_n^i(0))((\gamma_n^i)'(0)). \tag{3.15}$$

Since $|u_n^i|_{p'} = 1$, by Hölder inequality there is $C > 0$ such that

$$|(\gamma_n^i)'(0)| \leqslant C|v|_{p'}. \tag{3.16}$$

On the other hand, by (3.14),

$$\begin{aligned}\Psi_\epsilon'(\gamma_n^i(0))((\gamma_n^i)'(0)) &= \Psi_\epsilon'(u_n^i)((\gamma_n^i)'(0)) \\ &= \Psi_\epsilon'(u_n^i)(v) - \left(\int_{\mathbb{R}^N} |u_n^i|^{p'-2} u_n^i v \, \mathrm{d}x\right) \Psi_\epsilon'(u_n^i)(u_n^i).\end{aligned}$$

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Using the fact that $\Psi'_\epsilon(u_n^i)(u_n^i) = 0$, it follows that

$$\Psi'_\epsilon(\gamma_n^i(0))((\gamma_n^i)'(0)) = \Psi'_\epsilon(u_n^i)(v). \tag{3.17}$$

From (3.15)–(3.17),

$$-\frac{C}{n}|v| \leqslant \Psi'_\epsilon(u_n^i)(v), \quad \forall v \in L^{p'}(\mathbb{R}^N),$$

and so,

$$\|\Psi'_\epsilon(u_n^i)\| \leqslant \frac{1}{n}, \quad \forall n \in \mathbb{N}.$$

The above study ensures that (u_n^i) is a $(PS)_{\alpha_\epsilon^i}$ sequence for Ψ_ϵ. Noting that $\alpha_\epsilon^i < c_\infty + \rho$, from lemma 3.2 there exists u^i such that $u_n^i \to u^i$ in $L^{p'}(\mathbb{R}^N)$. So

$$u^i \in \Omega_\epsilon^i, \quad \Psi_\epsilon(u^i) = \alpha_\epsilon^i \quad \text{and} \quad \Psi'_\epsilon(u^i) = 0.$$

Since

$$Q_\epsilon(u^i) \in \overline{B_{\rho_0}(z_i)}, \quad Q_\epsilon(u^j) \in \overline{B_{\rho_0}(z_j)},$$
$$\overline{B_{\rho_0}(z_i)} \cap \overline{B_{\rho_0}(z_j)} = \emptyset \quad \text{for } i \neq j,$$

we deduce that $u^i \neq u^j$ for $i \neq j$ for $1 \leqslant i, j \leqslant l$. Hence Ψ_ϵ possess at least l nontrivial critical points for all $\epsilon \in (0, \epsilon^*)$ on $\mathcal{S}_{p'}$, with $\epsilon^* \in (0, \epsilon_2)$. From lemma 2.11, J_ϵ possess at least l nontrivial critical points for all $\epsilon \in (0, \epsilon_2)$ in $L^{p'}(\mathbb{R}^N)$. Hence, by lemma 2.2, problem (1.1) has at least l nontrivial solutions. □

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