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First- and second-order accurate implicit difference schemes for the numerical resolution of the generalized Charney–Obukhov and Hasegawa–Mima equations

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Abstract. First- and second-order accurate implicit difference schemes for the numerical solution of the nonlinear generalized Charney–Obukhov and Hasegawa–Mima equations with scalar nonlinearity are constructed. On the basis of numerical calculations accomplished by means of these schemes, the dynamics of two-dimensional nonlinear solitary vortical structures are studied. The problem of stability for the first-order accurate semi-discrete scheme is investigated. The dynamic relation between solutions of the generalized Charney–Obukhov and Hasegawa–Mima equations is established. It is shown that, contrary to existing opinion, the scalar nonlinearity in the case of the generalized Hasegawa–Mima equation develops monopolar anticyclone, while in case of the generalized Charney–Obukhov equation it develops monopolar cyclone.

1. Statement of the problem and implicit difference schemes

Generalized Charney–Obukhov (GChO) and Hasegawa–Mima (GHM) equations describe the propagation dynamics of nonlinear solitary vortical structures in geophysical flows and magnetized plasmas, respectively. In the frame of reference moving with velocity v along the axis OX, these dimensionless equations can be written in the following form:

$$\frac{\partial(\Delta\psi-\gamma\psi)}{\partial t}+\beta\frac{\partial\psi}{\partial x}-v\frac{\partial(\Delta\psi-\gamma\psi)}{\partial x}+J(\psi,\Delta\psi)\pm\alpha\psi\frac{\partial\psi}{\partial x}=0, \tag{1}$$

where α, β and γ are positive constants defined through physical characteristics of the medium. The '+' at α defines the GChO equation when $\gamma=1, \alpha=\beta=v_{\mathrm{R}}$ is the dimensionless Rossby velocity, and ψ is the variable part of fluid depth. The '−' at α corresponds to the GHM equation when $\gamma=1, \beta=v_{\mathrm{d}}$ is the dimensionless drift velocity, and ψ is the perturbed potential. The Jacobian (Poisson bracket)

$$J(\psi,\Delta\psi)=\frac{\partial\psi}{\partial x}\frac{\partial\Delta\psi}{\partial y}-\frac{\partial\psi}{\partial y}\frac{\partial\Delta\psi}{\partial x}$$

describes the contribution of so-called vectorial nonlinearity, while the Korteweg–de Vries (KdV) type last term in (1) describes the contribution of so-called scalar nonlinearity.

If we introduce the generalized vorticity $W = \Delta\psi - \gamma\psi + \beta y$, (1) can be rewritten in terms of ψ and W:

$$\frac{\partial W}{\partial t} + J(\psi, W) - v\frac{\partial W}{\partial x} \pm \alpha\psi\frac{\partial \psi}{\partial x} = 0, \tag{2}$$

$$\Delta\psi - \gamma\psi = W - \beta y. \tag{3}$$

Our aim is to solve the system (2), (3) numerically in the cylindrical domain $Q_T = \Omega\times]0, T[$, where Ω is the rectangle, $\Omega =]-a_1, a_1[\times]-a_2, a_2[$. Space variables x, y vary in the domain Ω, and the variable t varies in the interval $]0, T[$. As an initial condition at time $t = 0$ we take the well-known solitary dipole solution $\psi(x, y, 0) = \psi_0(x, y)$ [1].

Let us introduce a time step $\tau = T/m$ $(m > 1)$ and approximate (2) at the point (x, y, t_k), where $t_k = k\tau$ $(k = 1, \ldots, m)$, by the following semi-discrete scheme:

$$\frac{W^k - W^{k-1}}{\tau} + \theta J(\psi^{k-1}, W^k) + (1-\theta)J(\psi^k, W^{k-1})$$
$$- v\left(\theta\frac{\partial W^k}{\partial x} + (1-\theta)\frac{\partial W^{k-1}}{\partial x}\right) \pm \alpha\left(\theta\psi^{k-1}\frac{\partial \psi^k}{\partial x} + (1-\theta)\psi^k\frac{\partial \psi^{k-1}}{\partial x}\right) = 0, \tag{4}$$

where the weight $\theta \in [0, 1]$. We assume that $W(t, x, y)$ and $\psi(t, x, y)$ are sufficiently smooth functions. Equation (4) approximates (2) at the point (x, y, t_k) with an accuracy $O((1-2\theta)\tau + \tau^2)$. When $\theta = 0.5$, we obtain the second-order accurate scheme, in all other cases we have the first-order accurate scheme.

Let us cover area Ω by a grid and denote by h_1 a grid spacing in the x-direction and by h_2 in the y-direction, $h_1 = 2a_1/N_1$, $h_2 = 2a_2/N_2$, where $N_1(> 1)$ and $N_2(> 1)$ are natural numbers. If in (4) we replace the first-order derivatives with respect to spatial variables by central differences, we obtain the following difference equation:

$$\frac{W_{i,j}^k - W_{i,j}^{k-1}}{\tau} + F\big(\psi_{i,j}^{k-1}, \psi_{i,j}^k, W_{i,j}^{k-1}, W_{i,j}^k\big) = 0, \tag{5}$$

where $i = 1, \ldots, N_1 - 1,\ j = 1, \ldots, N_2 - 1$,

$$F\big(\psi_{i,j}^{k-1}, \psi_{i,j}^k, W_{i,j}^{k-1}, W_{i,j}^k\big) = \theta\widehat{J}\big(\psi_{i,j}^{k-1}, W_{i,j}^k\big) + (1-\theta)\widehat{J}\big(\psi_{i,j}^k, W_{i,j}^{k-1}\big)$$
$$- v\big(\theta\delta_x W_{i,j}^k + (1-\theta)\delta_x W_{i,j}^{k-1}\big)$$
$$\pm \alpha\big(\theta\psi_{i,j}^{k-1}\delta_x\psi_{i,j}^k + (1-\theta)\psi_{i,j}^k\delta_x\psi_{i,j}^{k-1}\big)$$

and $\widehat{J}(\psi_{i,j}, W_{i,j}) = (\delta_x\psi_{i,j})(\delta_y W_{i,j}) - (\delta_y\psi_{i,j})(\delta_x W_{i,j})$; the operator δ_x is the central difference analogy of the first-order derivative with respect to x variable (analogously defined by δ_y). The difference equation (5) approximates (2) with accuracy $O((1-2\theta)\tau + \tau^2 + h_1^2 + h_2^2)$ at the point (x_i, y_j, t_k).

Reconstruction of the perturbed potential ψ by means of the generalized vorticity W can be accomplished from the standard difference equation corresponding to (3).

We solve the system of difference equations (2), (3) by the following iteration (in order to simplify writing we omit the index k in $W_{i,j}^k$ and $\psi_{i,j}^k$):

$$\overset{n}{W}_{i,j} = W_{i,j}^{k-1} + \tau F\big(\psi_{i,j}^{k-1}, \overset{n-1}{\psi}_{i,j}, W_{i,j}^{k-1}, \overset{n-1}{W}_{i,j}\big), \tag{6}$$

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$$\frac{\overset{n}{\psi}_{i+1,j} - 2\overset{n}{\psi}_{i,j} + \overset{n}{\psi}_{i-1,j}}{h_1^2} + \frac{\overset{n}{\psi}_{i,j+1} - 2\overset{n}{\psi}_{i,j} + \overset{n}{\psi}_{i,j-1}}{h_2^2} - \gamma\,\overset{n}{\psi}_{i,j} = \overset{n}{W}_{i,j} - \beta y_j, \qquad (7)$$

where n is an iteration number ($n = 1, 2, \ldots$), initial approximation is defined as $(\overset{0}{W}_{i,j}, \overset{0}{\psi}_{i,j}) = (W_{i,j}^{k-1}, \psi_{i,j}^{k-1})$, and the nth iteration $(\overset{n}{W}_{i,j}, \overset{n}{\psi}_{i,j})$ is calculated from (6), (7) by means of the previous $(\overset{n-1}{W}_{i,j}, \overset{n-1}{\psi}_{i,j})$ iteration. The transition step τ from one time level to the next is subject to the condition

$$\frac{1}{\gamma}(1-\theta)\left(\frac{\tau}{h_1}c_4 + \frac{\tau}{h_2}c_3 + \tau\alpha c_1\right) + \theta\left(\frac{\tau}{h_2}c_1 + \frac{\tau}{h_1}\left(c_2 + v + \frac{1}{\gamma}c_0\alpha\right)\right) < 1, \qquad (8)$$

where c_0, c_1, c_2, c_3 and c_4 are maximums of $|\psi_{i,j}^{k-1}|, |\delta_x\psi_{i,j}^{k-1}|, |\delta_y\psi_{i,j}^{k-1}|, |\delta_x W_{i,j}^{k-1}|$ and $|\delta_y W_{i,j}^{k-1}|$ ($i = 1, \ldots, N_1 - 1$, $j = 1, \ldots, N_2 - 1$), respectively. Inequality (7) represents a sufficient condition of convergence of the iteration process (6), (7). We solve the system of difference equations (6) with respect to the variable x by means of factorization method, and with respect to the variable y by means of iteration.

For problem (2),(3), a uniqueness of the solution in the case of periodic boundary conditions may be proved. It may be also proved that, in the case when $\theta = 1$, the solutions obtained by the semi-discrete scheme (4) are uniformly bounded by the norm L_2, which gives us enough information about the stability of the scheme. That is also confirmed by the results of numerical calculations.

2. Discussions and conclusions

On the basis of implicit difference schemes constructed in the given paper we carried out the numerical calculations of GChO and GHM (1) for various parameters γ, β, v and α. Numerical calculations accomplished for the domain $a_1 = a_2 = 5$. The influence of scalar nonlinearity (of KdV type) on the dynamics of propagation of solitary vortical structures is investigated numerically. It was shown that even for very small α these equations have a qualitatively different behavior in the long time limit due to the existence of new solitary waves that have peak amplitude varying as $1/\alpha$. We may conclude that scalar nonlinearity in case of the GHM equation stimulates the amplification of anticyclone ($\psi > 0$, see Fig. 1(a_1), (a_2), (a_3)). In case of the GChO equation the result changes symmetrically and a cyclone ($\psi < 0$, see Fig. 1(b_1), (b_2), (b_3)) will dominate. Indeed the transformation $x \to x$, $y \to -y$, $\psi(x, y, t) \to -\psi(x, y, t)$ changes sign at the α in (1). So, if the solution of the one equation is such a function $\psi(x, y, t)$ whose initial value at the $t = 0$ moment is an odd function $\psi(x, y, 0)$ with respect to the spatial variable y, then the solution of the other equation with the same initial condition will be the $-\psi(x, -y, t)$ function defined on the whole (x, y) plane. From this, follows the important dynamic relation between the solutions of GChO and GHM equations: if for one equation at a certain moment a cyclone (anticyclone) was formed, then for the second equation an anticyclone (cyclone) will be formed automatically. This conclusion is confirmed by numerous numerical calculations of our work.

It should be noted that a decrease of steps h and τ, beginning from the limiting value, has no real effect on the results, but the calculation time increases substantially. The fact that a decrease of steps does not spoil the results confirms practically the stability of the presented scheme.

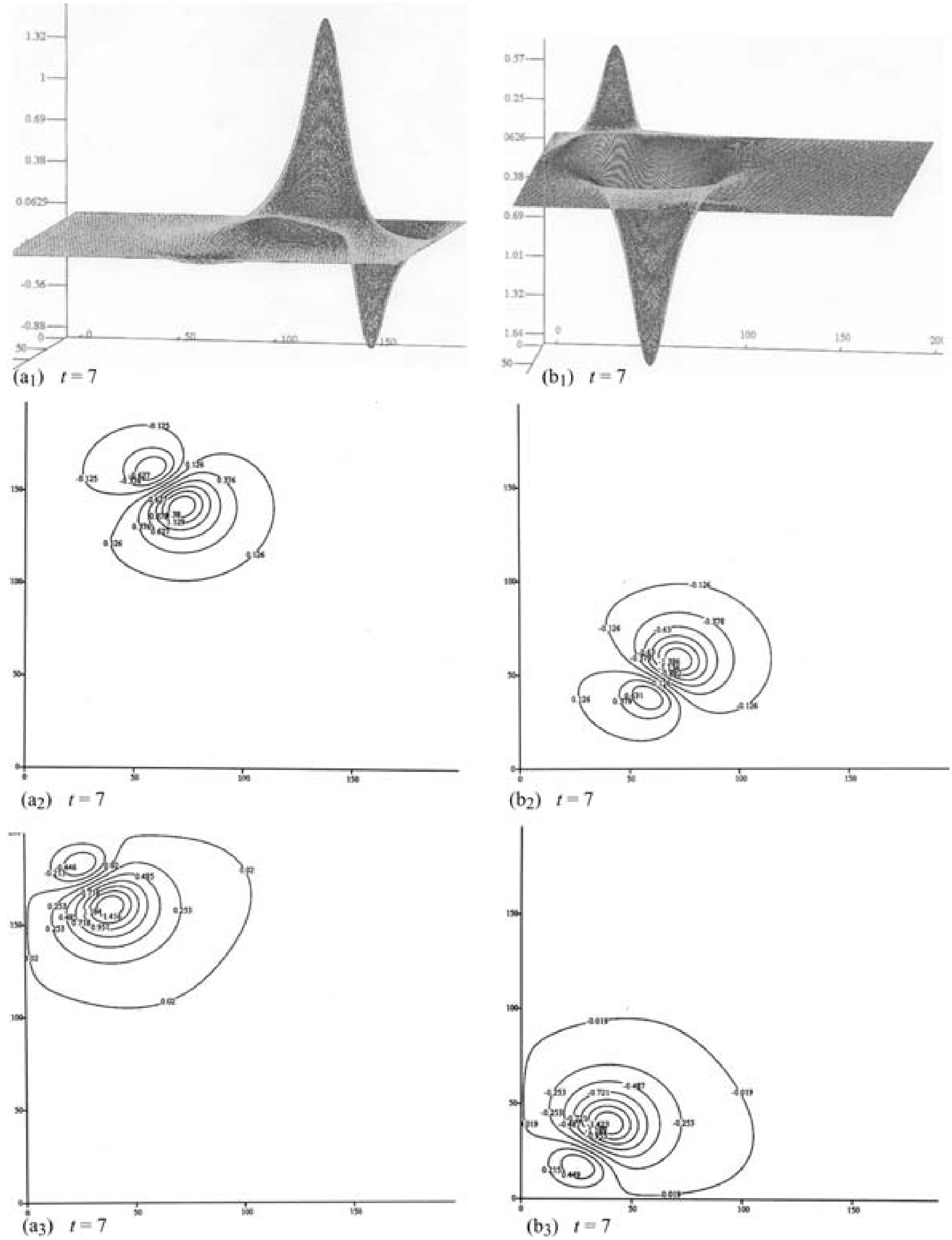

Figure 1. Dynamics of variation of the numerical solution of (1). (a_1), (a_2) and (a_3) correspond to the case of GHM equation and (b_1), (b_2) and (b_3) to the case of GChO equation. On (a_1) and (b_1) the surfaces are given and on (a_2), (a_3), (b_2) and (b_3) the isolines are given. $\theta = 1$, $\alpha = 1$, $h_1 = h_2 = 0.05$, $\tau = 0.00625$ and the accurate parameter of iterative process $\varepsilon = 0.001$.

Reference

Kaladze, T., Rogava, J., Tsamalashvili, L. and Tsiklauri, M. 2004 First- and second order accurate implicite difference schemes for the Charney–Obukhov equation. *Phys. Lett. A* **328**, 51–64.