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Topological mixing of Weyl chamber flows

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Abstract. In this paper we study topological properties of the right action by translation of the Weyl chamber flow on the space of Weyl chambers. We obtain a necessary and sufficient condition for topological mixing.

Key words: Weyl flow, infinite volume, Zariski dense discrete group, higher rank, topological mixing
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1. *Introduction*

Let G be a semisimple real, connected, Lie group of non-compact type. Let K be a maximal compact subgroup of G, and A a maximal torus of G for which there is a Cartan decomposition. Let M be the centralizer of A in K. We establish mixing properties for right action by translation of one-parameter subgroups of A on quotients $\Gamma\backslash G/M$ where Γ is a discrete, Zariski dense subgroup of G.

The particular case when G is of real rank one is well known. In this case, the symmetric space $X = G/K$ is a complete, connected, simply connected Riemannian manifold of negative curvature. The right action by translation of A on G/M coincides with the geodesic flow on T^1X. Dal'bo [**Dal00**] proved that it is mixing (on its non-wandering set) if and only if the length spectrum is non-arithmetic. The latter holds when Γ is a Zariski dense subgroup; see Benoist [**Ben00**], Kim [**Kim06**].

We are interested in cases where G is of higher real rank $k \geq 2$. When $\Gamma\backslash G/M$ is of finite volume, that is, when Γ is a lattice, it follows from the Howe–Moore theorem that the action of any non-compact subgroup of G is mixing.

We study the general situation of any discrete, Zariski dense subgroup, which of course includes the case of lattices.

If $\Gamma\backslash G/M$ has infinite volume, the known results are not as general.

In the particular case of so-called ping-pong subgroups of $\mathrm{PSL}(k+1, \mathbb{R})$, Thirion [**Thi07**, **Thi09**] proved mixing with respect to a natural measure on a natural closed A-invariant set $\Omega(X) \subset \Gamma\backslash G/M$, for a one-parameter flow associated to the 'maximal growth vector' introduced by Quint in [**Qui02**]. Sambarino [**Sam15**] did the same for hyperconvex representations.

Conze and Guivarc'h in [**CG02**] proved the topological transitivity (i.e. existence of dense orbits) of the right A-action on $\Omega(X)$ for any Zariski dense subgroup of $\mathrm{PSL}(k+1, \mathbb{R})$.

Let $\mathfrak{a} \simeq \mathbb{R}^k$ be the Cartan Lie subalgebra over A and $\mathfrak{a}^{++}$ the choice of a positive Weyl chamber. For any $\theta \in \mathfrak{a}^{++}$, the Weyl chamber flow (ϕ_t^θ) corresponds to the right action by translation of $\exp(t\theta)$. Benoist [**Ben97**] introduced a convex limit cone $\mathcal{C}(\Gamma) \subset \mathfrak{a}$ and proved that for Zariski dense semigroups, the limit cone is of non-empty interior. We prove topological mixing for any direction of the interior of $\mathcal{C}(\Gamma)$.

THEOREM 1.1. *Let G be a semisimple, connected, real linear Lie group, of non-compact type. Let Γ be a Zariski dense, discrete subgroup of G. Let $\theta \in \mathfrak{a}^{++}$. Then the dynamical system $(\Omega(X), \phi_t^\theta)$ is topologically mixing if and only if θ is in the interior of the limit cone $\mathcal{C}(\Gamma)$.*

Taking $\widetilde{\Omega} \subset G/M$ to be the universal cover of $\Omega(X)$, we remark that this theorem is a direct consequence of the following statement, where Γ is a Zariski dense semigroup of G. We insist that under this hypothesis, Γ is not necessarily a subgroup and can even be non-discrete.

THEOREM 1.2. *Let G be a semisimple, connected, real linear Lie group, of non-compact type. Let Γ be a Zariski dense semigroup of G. Let $\theta \in \mathfrak{a}^{++}$. Then θ is in the interior of the limit cone if and only if for all non-empty open subsets $\widetilde{U}, \widetilde{V} \subset \widetilde{\Omega}$, there exists $T > 0$ so that for any later time $t > T$, there exists $\gamma_t \in \Gamma$ with*

$$\gamma_t \widetilde{U} \cap \phi_t^\theta(\widetilde{V}) \neq \emptyset.$$

In §2 we give some background on globally symmetric spaces. We introduce the space of Weyl chambers, the Weyl chamber flow, give a bordification of the space of Weyl chambers and present a higher- rank generalization of the Hopf coordinates.

In §3 we introduce the main tools: Schottky semigroups and estimations on the spectrum of products of elements in G.

In §4 we introduce the *non-wandering Weyl chambers set*, a closed A-invariant subset $\Omega(X) \subset \Gamma\backslash G/M$. Then we study topological transitivity in Proposition 4.7. We prove that if the flow ϕ_t^θ is topologically transitive in $\Omega(X)$, where $\theta \in \mathfrak{a}^{++}$, then the direction θ must be in the interior of the limit cone. Since topological mixing implies topological transitivity, this provides one direction of the main Theorem 1.2.

In §5 we prove a key proposition, Proposition 5.4, using density results that come from non-arithmeticity of the length spectrum. Then we prove the main theorem.

In the Appendix we prove a density lemma of subgroups of $\mathbb{R}^n$ needed in the proof of Proposition 5.4.

Throughout this paper, G is a semisimple, connected, real linear Lie group, of non-compact type.

2. *Background on symmetric spaces*

Classical references for this section are [**Thi07**, Ch. 8, §§8.B, 8.D, 8.E, 8.G], [**GJT12**, Ch. III, §1–4] and [**Hel01**, Chs. IV– VI].

Let K be a maximal compact subgroup of G. Then $X = G/K$ is a globally symmetric space of non-compact type. The group G is the identity component of its isometry group. It acts transitively on X, by left multiplication. We fix a point $o = K \in X$. Then K is in the fixed point set of the involutive automorphism induced by the geodesic symmetry in o (cf. [**Hel01**, Ch. VI, Theorem 1.1]).

Denote by $\mathfrak{g}$ (respectively, $\mathfrak{k}$) the Lie algebra of G (respectively, K). The differential of the involutive automorphism induced by the geodesic symmetry in o is a *Cartan involution* of $\mathfrak{g}$. Then $\mathfrak{k}$ is the eigenspace of the eigenvalue 1 (for the Cartan involution) and we denote by $\mathfrak{p}$ the eigenspace of the eigenvalue -1. The decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a *Cartan decomposition*.

2.1. *Flats, Weyl chambers, classical decompositions.*

A *flat* of the symmetric space X is a totally geodesic, isometric embedding of a Euclidean space. We are interested in flats of maximal dimension in X, called *maximal flats*. One can construct the space of maximal flats following [**Thi07**, Ch. 8, §§8.D, 8.D] thanks to [**Hel01**, Ch. V, Proposition 6.1]. Let $\mathfrak{a} \subset \mathfrak{p}$ be a *Cartan subspace* of $\mathfrak{g}$, that is, a maximal abelian subspace such that the adjoint endomorphism of every element is semisimple. We denote by A the subgroup $\exp(\mathfrak{a})$. The *real rank* of the symmetric space X, denoted by r_G, is the dimension of the real vector space $\mathfrak{a}$.

Definition 2.1. A *parametrized flat* is an embedding of $\mathfrak{a}$ of the form gf_0, where $g \in G$ and f_0 is the map defined by

$$\begin{aligned} f_0 : \mathfrak{a} &\longrightarrow X \\ v &\longmapsto \exp(v)o. \end{aligned}$$

We denote by $\mathcal{W}(X)$ the set of parametrized flats of X.

By definition, the set of parametrized flats is the orbit of f_0 under the left action by multiplication of G. The stabilizer of f_0 is the centralizer of A in K, denoted by M. We deduce that the set of parametrized flats $\mathcal{W}(X)$ identifies with the homogeneous space G/M. For any parametrized flat $f \in \mathcal{W}(X)$, there is an element g_f in G such that $f = g_f f_0$. Hence, the map

$$\begin{aligned} \mathcal{W}(X) &\overset{\sim}{\longrightarrow} G/M \\ f &\longmapsto g_f M \end{aligned}$$

is a G-equivariant bijection.

For any linear form α on $\mathfrak{a}$, set $\mathfrak{g}_\alpha := \{v \in \mathfrak{g} \mid \forall u \in \mathfrak{a},\ [u, v] = \alpha(u)v\}$. The set of restricted roots is $\Sigma := \{\alpha \in \mathfrak{a}^* \setminus \{0\} \mid \mathfrak{g}_\alpha \neq \{0\}\}$. The kernel of each restricted root is a hyperplane of $\mathfrak{a}$. The *Weyl chambers* of $\mathfrak{a}$ are the connected components of $\mathfrak{a} \setminus \bigcup_{\alpha \in \Sigma} \ker(\alpha)$. We fix such a component, call it the *positive Weyl chamber* and denote it (respectively, its closure) by $\mathfrak{a}^{++}$ (respectively, $\mathfrak{a}^{+}$).

We denote by $N_K(A)$ the normalizer of A in K. The group $N_K(A)/M$ is called the *Weyl group*. The positive Weyl chamber of $\mathfrak{a}$ allows us to tessellate the maximal flats in the symmetric space X. Indeed, $f_0(\mathfrak{a}^+)$ is a fundamental domain for the action of the Weyl group on the maximal flat $f_0(\mathfrak{a})$ and G acts transitively on the space of parametrized flats. Finally, the orbit $G.f_0(\mathfrak{a}^+)$ identifies with the space of parametrized flats, and the image of $g.f_0(\mathfrak{a}^+)$ is a *geometric Weyl chamber*. This explains why the set of parametrized flats is also called the *space of Weyl chambers*. For any geometric Weyl chamber $f(\mathfrak{a}^+) \in G.f_0(\mathfrak{a}^+)$, the image of $0 \in \mathfrak{a}^+$ is the *origin*. Furthermore,

$$G/M \simeq \mathcal{W}(X) \simeq G.f_0(\mathfrak{a}^+).$$

Definition 2.2. The right action of $\mathfrak{a}$ on $\mathcal{W}(X)$ is defined by $\alpha \cdot f : v \mapsto f(v+\alpha)$ for all $\alpha \in \mathfrak{a}$ and $f \in \mathcal{W}(X)$. The *Weyl chamber flow* is defined for all $\theta \in \mathfrak{a}_1^{++}$ and $f \in \mathcal{W}(X)$ by

$$\begin{aligned}\phi^\theta(f) : \mathbb{R} &\longrightarrow \mathcal{W}(X)\\ t &\longmapsto \phi_t^\theta(f) = f(v+\theta t) = f(v)e^{\theta t}.\end{aligned}$$

Remark that the Weyl chamber flow ϕ_t^θ is also the right action of the one-parameter subgroup $\exp(t\theta)$ on the space of Weyl chambers.

The set of *positive roots*, denoted by Σ^+, is the subset of roots which take positive values in the positive Weyl chamber. The positive Weyl chamber also allows us to define two particular nilpotent subalgebras $\mathfrak{n} = \bigoplus_{\alpha\in\Sigma^+} \mathfrak{g}_\alpha$ and $\mathfrak{n}^- = \bigoplus_{\alpha\in\Sigma^+} \mathfrak{g}_{-\alpha}$. Finally, set $A^+ := \exp(\mathfrak{a}^+)$, $A^{++} := \exp(\mathfrak{a}^{++})$, $N := \exp(\mathfrak{n})$ and $N^- := \exp(\mathfrak{n}^-)$. For all $a \in A^{++}$, $h_+ \in N$, $h_- \in N^-$, notice that

$$a^{-n} h_\pm a^n \underset{\pm\infty}{\longrightarrow} id_G. \tag{1}$$

Definition 2.3. For any $g \in G$, we define, by Cartan decomposition, a unique element $\mu(g) \in \mathfrak{a}^+$ such that $g \in K \exp(\mu(g))K$. The map $\mu : G \to \mathfrak{a}^+$ is called the *Cartan projection*.

The Cartan projection allows us to define an $\mathfrak{a}^+$-valued function on $X \times X$, denoted by $d_{\mathfrak{a}^+}$, following Thirion [**Thi07**, Def-Thm 8.38]. For any $x, x' \in X$, there exists $g, g' \in G$ such that $x = gK$ and $x' = g'K$, and we set

$$d_{\mathfrak{a}^+}(x, x') := \mu(g'^{-1}g).$$

This function is independent of the choice of g and g' up to right multiplication by K. Recall [**Hel01**, Ch. V, Lemma 5.4] that $\mathfrak{a}$ is endowed with a scalar product coming from the Killing form on $\mathfrak{g}$, and the norm of $d_{\mathfrak{a}^+}(x, x')$ coincides with the distance between x and x' in the symmetric space X.

An element of G is *unipotent* if all its eigenvalues are equal to 1, or equivalently if it is the exponential of a nilpotent element. An element of G is *semisimple* if it is diagonalizable over $\mathbb{C}$, *elliptic* (respectively, *hyperbolic*) if it is semisimple with eigenvalues of modulus 1 (respectively, real eigenvalues). Equivalently, elliptic (respectively, hyperbolic, unipotent) elements are conjugated to elements in K (respectively, A, N).

Any element $g \in G$ admits a unique decomposition (in G) $g = g_e g_h g_u$, called the *Jordan decomposition*, where g_e, g_h and g_u commute and g_e (respectively, g_h, g_u) is

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elliptic (respectively, hyperbolic, unipotent). The element g_e (respectively, g_h, g_u) is called the *elliptic part* (respectively, *hyperbolic part*, *unipotent part*) of g.

Definition 2.4. For any element $g \in G$, there is a unique element $\lambda(g) \in \mathfrak{a}^+$ such that the hyperbolic part of g is conjugated to $\exp(\lambda(g)) \in A^+$. The map $\lambda : G \to \mathfrak{a}^+$ is called the *Jordan projection*.

An element $g \in G$ is *loxodromic* if $\lambda(g) \in \mathfrak{a}^{++}$. Since any element of N that commutes with $\mathfrak{a}^{++}$ is trivial, the unipotent part of loxodromic elements is trivial. Furthermore, the only elements of K that commute with $\mathfrak{a}^{++}$ are in M. We deduce that the elliptic part of loxodromic elements is conjugated to elements in M. Hence, for any loxodromic element $g \in G$, there exist $h_g \in G$ and $m(g) \in M$ so that we can write $g = h_g m(g) e^{\lambda(g)} h_g^{-1}$. For any $m \in M$ we can also write $g = (h_g m)(m^{-1} m(g) m) e^{\lambda(g)} (h_g m)^{-1}$. This allows us to associate to any loxodromic element $g \in G$ an *angular* part $m(g)$ which is defined up to conjugacy by M.

The spectral radius formula [**BQ16**, Corollary 5.34]

$$\lambda(g) = \lim_{n\to\infty} \frac{1}{n}\mu(g^n)$$

allows us to compute the Jordan projection thanks to the Cartan projection.

Recall that by *Iwasawa decomposition* (see, for example, [**Hel01**, Ch. IX, Theorem 1.3]) for any $g \in G$, there exists a unique triple $(k, v, n) \in K \times \mathfrak{a} \times N$ such that $g = k \exp(v) n$. Furthermore, the map

$$\begin{aligned} K \times \mathfrak{a} \times N &\longrightarrow G \\ (k, v, n) &\longmapsto k e^v n \end{aligned}$$

is a diffeomorphism.

2.2. *Asymptotic Weyl chambers, Busemann–Iwasawa cocycle.*

The main references for this subsection are [**Thi07**, Ch. 8, §8.D], [**GJT12**] and [**BQ16**].

We endow the space of geometric Weyl chambers with the equivalence relation

$$f_1(\mathfrak{a}^+) \sim f_2(\mathfrak{a}^+) \Leftrightarrow \sup_{u\in\mathfrak{a}^{++}} d(f_1(u), f_2(u)) < \infty.$$

Equivalently, $f_1(\mathfrak{a}^+) \sim f_2(\mathfrak{a}^+)$ if and only if for any $v \in \mathfrak{a}^{++}$, the geodesics $t \mapsto f_1(tv)$ and $t \mapsto f_2(tv)$ are at bounded distance when $t \to +\infty$. Equivalence classes for this relation are called *asymptotic Weyl chambers*. We denote by $\mathcal{F}(X)$ the set of asymptotic Weyl chambers and by η_0 the asymptotic class of the Weyl chamber $f_0(\mathfrak{a}^+)$.

FACT 2.5. *The set $\mathcal{F}(X)$ identifies with the Furstenberg boundary G/P, where $P = MAN$. Furthermore,*

$$G/P \simeq \mathcal{F}(X) \simeq K/M \simeq K.\eta_0.$$

Proof. Since G acts transitively on the space of Weyl chambers, it also acts transitively on the set of asymptotic Weyl chambers.

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We show that P is the stabilizer of η_0. For any $g \in G$ and $u \in \mathfrak{a}^{++}$, we compute the distance

$$d(gf_0(u),\, f_0(u)) = \|d_{\mathfrak{a}^+}(gf_0(u),\, f_0(u))\| = \|\mu(e^{-u}ge^u)\|.$$

By Bruhat decomposition (see [**Hel01**, Ch. IX, Theorem 1.4]), there exist an element w in the normalizer of A in K and elements $p_1,\, p_2 \in P = MAN$ such that $g = p_1wp_2$. Then

$$e^{-u}ge^u = (e^{-u}p_1e^u)e^{-u}(we^uw^{-1})w(e^{-u}p_2e^u).$$

Note that by equation (1), the sets $\{e^{-u}p_ie^u\}_{u\in\mathfrak{a}^{++},i=1,2}$ are bounded. Hence, the sets $\{e^{-u}ge^u\}_{u\in\mathfrak{a}^{++}}$ and $\{e^{-u}we^uw^{-1}\}_{u\in\mathfrak{a}^{++}}$ have the same behavior. Notice the simplification $e^{-u}we^uw^{-1} = e^{-u+Ad(w)u}$ which is bounded uniformly in $\mathfrak{a}^{++}$ only when $w \in M$. We deduce that $\{e^{-u}ge^u\}_{u\in\mathfrak{a}^{++}}$ is bounded only when $g \in P$. Hence the subgroup P is the stabilizer of the asymptotic class η_0.

The geometric Weyl chambers whose origin is $o \in X$ are in the orbit $K.f_0(\mathfrak{a}^+)$. Any equivalence class in $\mathcal{F}(X)$ admits, by Iwasawa decomposition, a unique representative in $K.f_0(\mathfrak{a}^+)$. Moreover, K/M identifies with the orbit $K.f_0(\mathfrak{a}^+)$ since M is the stabilizer of f_0 in K. □

For any asymptotic Weyl chamber $\eta \in \mathcal{F}(X)$ and $g \in G$, consider, by Iwasawa decomposition, the unique element $\sigma(g,\eta) \in \mathfrak{a}$, called the *Iwasawa cocycle*, such that if $k_\eta \in K$ satisfies $\eta = k_\eta\eta_0$, then

$$gk_\eta \in K\,\exp(\sigma(g,\eta))N.$$

The *cocycle relation* holds (cf. [**BQ16**, Lemma 5.29]), that is to say, for all $g_1,\, g_2 \in G$ and $\eta \in \mathcal{F}(X)$,

$$\sigma(g_1g_2,\eta) = \sigma(g_1,\, g_2\eta) + \sigma(g_2,\eta).$$

For any pair of points $x,\, y \in X$, any asymptotic Weyl chamber $\eta \in \mathcal{F}(X)$ and $u \in \mathfrak{a}^{++}$, we consider a representative $f_\eta(\mathfrak{a}^+)$ of η and define the *Busemann cocycle* by

$$\beta_{f_\eta,u}(x,\, y) = \lim_{t\to+\infty} d_{\mathfrak{a}^+}(f_\eta(tu),\, x) - d_{\mathfrak{a}^+}(f_\eta(tu),\, y).$$

Remark that the following equivariance relation holds for any $g \in G$, any pair of points $x,\, y \in X$, any asymptotic Weyl chamber $\eta \in \mathcal{F}(X)$ and $u \in \mathfrak{a}^{++}$:

$$\beta_{gf_\eta,u}(gx,\, gy) = \beta_{f_\eta,u}(x,\, y). \tag{2}$$

It turns out that the Busemann cocycle depends neither on the choice of the geometric Weyl chamber in the class η, nor on the choice of $u \in \mathfrak{a}^{++}$. We will write $\beta_{f_\eta,u}(x,\, y) = \beta_\eta(x,\, y)$. By [**BQ16**, Corollary 5.34], the Iwasawa and Busemann cocycles coincide in the sense that for all $g \in G$, $\eta \in \mathcal{F}(X)$ and $u \in \mathfrak{a}^{++}$,

$$\beta_{f_\eta,u}(g^{-1}o,\, o) = \sigma(g,\eta). \tag{3}$$

We associate attracting and repelling asymptotic geometric Weyl chambers to loxodromic elements of G as follows.

Recall that for any loxodromic element $g \in G$, there is an element $h_g \in G$ and an angular part $m(g) \in M$ such that $g = h_ge^{\lambda(g)}m(g)h_g^{-1}$. We set $g^+ := [h_g.f_0(\mathfrak{a}^+)]$ and $g^- := [h_g.f_0(-\mathfrak{a}^+)]$. Then $g^+ \in \mathcal{F}(X)$ (respectively, g^-) is called the *attracting* (respectively, *repelling*) asymptotic Weyl chamber.

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FACT 2.6. *Let $g \in G$ be a loxodromic element. Then $\sigma(g, g^+) = \lambda(g)$.*

Proof. Let $g \in G$ be a loxodromic element. Consider an element $h_g \in G$ and an angular part $m(g) \in M$ such that $g = h_g e^{\lambda(g)} m(g) h_g^{-1}$. Denote by f_g the parametrized flat $f_g : v \mapsto h_g e^v o$. Then the geometric Weyl chamber $f_g(\mathfrak{a}^+)$ (respectively, $f_g(-\mathfrak{a}^+)$) is a representative of the limit points g^+ (respectively, g^-).

Fix any $u \in \mathfrak{a}^{++}$. Then by equation (3) and by intercalating the point $f_g(0)$ we deduce

$$\sigma(g, g^+) = \beta_{f_g,u}(g^{-1}o, o) = \beta_{f_g,u}(g^{-1}o, g^{-1}f_g(0)) + \beta_{f_g,u}(g^{-1}f_g(0), f_g(0)) + \beta_{f_g,u}(f_g(0), o).$$

Using equation (2), the first term simplifies to $\beta_{f_g,u}(g^{-1}o, g^{-1}f_g(0)) = \beta_{g^{-1}f_g,u}(o, f_g(0))$. Because g^{-1} fixes g^+, the first and third terms cancel out. We deduce that

$$\begin{aligned}
\sigma(g, g^+) &= \beta_{f_g,u}(g^{-1}f_g(0), f_g(0)) \\
&= \lim_{t\to+\infty} d_{\mathfrak{a}^+}(f_g(tu), g^{-1}f_g(0)) - d_{\mathfrak{a}^+}(f_g(tu), f_g(0)) \\
&= \lim_{t\to+\infty} d_{\mathfrak{a}^+}(gf_g(tu), f_g(0)) - tu \\
&= \lim_{t\to+\infty} d_{\mathfrak{a}^+}(f_g(tu + \lambda(g)), f_g(0)) - tu \\
&= \lambda(g).
\end{aligned}$$

□

2.3. *Hopf parametrization.* Our main reference for this subsection is [**Thi07**, Ch. 8, §8.G.2].

In the geometric compactification of the hyperbolic plane, any bi-infinite geodesic defines opposite points in the geometric boundary. In a similar way, we introduce asymptotic Weyl chambers in general position.

We endow the product $\mathcal{F}(X) \times \mathcal{F}(X)$ with the diagonal left G-action. For any pair of elements $(\xi, \eta) \in \mathcal{F}(X) \times \mathcal{F}(X)$ and $g \in G$, we set $g.(\xi, \eta) := (g.\xi, g.\eta)$. For any parametrized flat $f \in \mathcal{W}(X)$, denote by f_+ (respectively, f_-) the asymptotic class of the geometric Weyl chamber $f(\mathfrak{a}^+)$ (respectively, $f(-\mathfrak{a}^+)$). Then the map

$$\begin{aligned}
\mathcal{H}^{(2)} : \mathcal{W}(X) &\longrightarrow \mathcal{F}(X) \times \mathcal{F}(X) \\
f &\longmapsto (f_+, f_-)
\end{aligned}$$

is G-equivariant.

Two asymptotic Weyl chambers $\xi, \eta \in \mathcal{F}(X)$ are in *general position* or *opposite*, if they are in the image $\mathcal{H}^{(2)}(\mathcal{W}(X))$, that is, if there exists a parametrized flat $f \in \mathcal{W}(X)$ such that the geometric Weyl chamber f_+ (respectively, f_-) is a representative of ξ (respectively, η).

We denote by $\mathcal{F}^{(2)}(X)$ the set of asymptotic Weyl chambers in general position. The product topology on the product space $\mathcal{F}(X) \times \mathcal{F}(X)$ (where $\mathcal{F}(X)$ is identified with G/MAN) induces a natural topology on $\mathcal{F}^{(2)}(X)$.

FACT 2.7. [**Thi09**, §3.2] *The set $\mathcal{F}^{(2)}(X)$ identifies with the homogeneous space G/AM. Furthermore, if we denote by η_0 (respectively, $\check{\eta}_0$) the asymptotic class of the Weyl chamber $f_0(\mathfrak{a}^+)$ (respectively, $f_0(-\mathfrak{a}^+)$), then*

$$G.(\eta_0, \check{\eta}_0) \simeq \mathcal{F}^{(2)}(X) \simeq G/AM.$$

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The *Hopf coordinates map* is defined by

$$\mathcal{H}: \begin{array}{rcl} \mathcal{W}(X) & \longrightarrow & \mathcal{F}^{(2)}(X) \times \mathfrak{a} \\ f & \longmapsto & (f_+, f_-; \beta_{f_+}(f(0), o)). \end{array}$$

Using the identifications $\mathcal{W}(X) \simeq G/M$ and $\mathcal{F}^{(2)}(X) \simeq G.(\eta_0, \check{\eta}_0)$, it reads

$$\begin{array}{rcl} G/M & \longrightarrow & G.(\eta_0, \check{\eta}_0) \times \mathfrak{a} \\ gM & \longmapsto & (g\eta_0, g\check{\eta}_0; \sigma(g, \eta_0)). \end{array}$$

We define the left G-action on the skew product $\mathcal{F}^{(2)}(X) \times \mathfrak{a}$ as follows. For any $g \in G$ and $(\xi, \eta; v) \in \mathcal{F}^{(2)}(X) \times \mathfrak{a}$, we set

$$g.(\xi, \eta; v) = (g.\xi, g.\eta; v + \beta_{g.\xi}(g.o, o)).$$

The right $\mathfrak{a}$-action defined for any $\alpha \in \mathfrak{a}$ and $(\xi, \eta; v) \in \mathcal{F}^{(2)}(X) \times \mathfrak{a}$ by

$$\alpha \cdot (\xi, \eta; v) = (\xi, \eta; v + \alpha)$$

is called the right $\mathfrak{a}$-action by *translation*.

Similarly, for any $\theta \in \mathfrak{a}_1^{++}$, we define the Weyl chamber flow ϕ^θ on the skew product: for all $(\xi, \eta; v) \in \mathcal{F}^{(2)}(X) \times \mathfrak{a}$ and $t \in \mathbb{R}_+$,

$$\phi_t^\theta(\xi, \eta; v) = (\xi, \eta; v + \theta t).$$

PROPOSITION 2.8. [**Thi07**, Proposition 8.54] *The Hopf coordinates map is a* $(G, \mathfrak{a})$-*equivariant homeomorphism in the sense that:*

(i) *the left action of G on $\mathcal{W}(X)$ reads in the Hopf coordinates as the left G-action on the skew product $\mathcal{F}^{(2)}(X) \times \mathfrak{a}$;*

(ii) *the right action of $\mathfrak{a}$ on $\mathcal{W}(X)$ reads in the Hopf coordinates map as the right $\mathfrak{a}$-action by translation on the skew product $\mathcal{F}^{(2)}(X) \times \mathfrak{a}$.*

Furthermore, for any $\theta \in \mathfrak{a}_1^{++}$ and $t \in \mathbb{R}_+$, for all $f \in \mathcal{W}(X)$, we obtain

$$\mathcal{H}(\phi_t^\theta(f)) = \phi_t^\theta(\mathcal{H}(f)).$$

3. *Loxodromic elements*

We first study loxodromic elements in $GL(V)$ for a real vector space V of finite dimension endowed with a Euclidean norm $\|\cdot\|$. Then we give some background on representations of semisimple Lie groups. Finally, we study the dynamical properties of the representations of G acting on the projective space of those representations.

3.1. *Proximal elements of* GL(V).

Denote by $X = \mathbb{P}(V)$ the projective space of V. We endow X with the distance

$$d(\mathbb{R}x, \mathbb{R}y) = \inf\{\|v_x - v_y\| \mid \|v_x\| = \|v_y\| = 1,\ v_x \in \mathbb{R}x,\ v_y \in \mathbb{R}y\}.$$

For $g \in \mathrm{End}(V)$, denote by $\lambda_1(g)$ its *spectral radius*.

Definition 3.1. An element $g \in \mathrm{End}(V) \setminus \{0\}$ is *proximal* on X if it has a unique eigenvalue $\alpha \in \mathbb{C}$ such that $|\alpha| = \lambda_1(g)$ and this eigenvalue is simple (therefore α is a real number). Denote by $V_+(g)$ the one-dimensional eigenspace corresponding to α and $V_-(g)$ the supplementary g-invariant hyperplane. In the projective space, denote by $x_+(g) = \mathbb{P}(V_+(g))$ (respectively, $X_-(g) = \mathbb{P}(V_-(g))$) the *attracting point* (respectively, the *repelling hyperplane*).

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The open ball centered in $x \in X$ of radius $\varepsilon > 0$ is denoted by $B(x, \varepsilon)$. For every subset $Y \subset X$, we denote by $\mathcal{V}_\varepsilon(Y)$ the open ε-neighborhood of Y. The following definition gives uniform control over the geometry of proximal elements (parametrized by r) and their contracting dynamics (parametrized by ε).

Definition 3.2. Let $0 < \varepsilon \leq r$. A proximal element g is (r, ε)-proximal if $d(x_+(g), X_-(g)) \geq 2r$, g maps $\mathcal{V}_\varepsilon(X_-(g))^c$ into the ball $B(x_+(g), \varepsilon)$ and its restriction to the subset $\mathcal{V}_\varepsilon(X_-(g))^c$ is an ε-Lipschitz map.

We give three remarks that follow from the definition.

(1) If an element is (r, ε)-proximal, then it is (r', ε)-proximal for $\varepsilon \leq r' \leq r$.
(2) If an element is (r, ε)-proximal, then it is (r, ε')-proximal for $r \geq \epsilon' \geq \varepsilon$.
(3) If g is (r, ε)-proximal, then g^n is also (r, ε)-proximal for $n \geq 1$.

The numbers r and ε depend on the metric of the projective space, which, in our case, depends on the choice of norm on the finite-dimensional vector space. However, in [**Ser16**, Remark 2.3] Sert claims the following statement. We provide a proof for completeness.

LEMMA 3.3. *For every proximal transformation g, there exist $r > 0$, an integer $n_0 \in \mathbb{N}$ and a sequence of non-increasing positive numbers $(\varepsilon_n)_{n \geq n_0}$ that converge to 0 such that for all $n \geq n_0$ large enough, g^n is (r, ε_n)-proximal.*

Since $\mathrm{GL}(V)$ is endowed with a Euclidean norm, it admits a canonical basis $(e_j)_{1 \leq j \leq \dim(V)}$. We set $x_0 := \mathbb{P}(e_1)$ and $H_0 := \mathbb{P}(\oplus_{j=2}^{\dim(V)} \mathbb{R} e_j)$. Recall that $\mathrm{GL}(V)$ admits a polar decomposition, that is, for any $g \in \mathrm{GL}(V)$, there exist orthogonal endomorphisms $k_g, l_g \in \mathrm{O}(V)$ and a unique symmetric endomorphism a_g of eigenvalues $(a_g(j))_{1 \leq j \leq \dim(V)}$ with $a_g(1) \geq a_g(2) \geq \cdots \geq a_g(\dim(V))$ such that $g = k_g a_g l_g$. Let us introduce a key [**BG03**, Lemma 3.4], due to Breuillard and Gelander, which is needed to obtain the Lipschitz properties.

LEMMA 3.4. [**BG03**] *Let $r, \delta \in (0, 1]$. Let $g \in \mathrm{GL}(V)$. If $|a_g(2)/a_g(1)| \leq \delta$, then g is δ/r^2-Lipschitz on $\mathcal{V}_r(l_g^{-1} H_0)^c$.*

Proof of Lemma 3.3. Let $g \in \mathrm{GL}(V)$ be a proximal element and assume without loss of generality that its first eigenvalue is positive. Set $r := \frac{1}{2} d(x_+(g), X_-(g))$. By proximality, r is positive. Let us prove that for all $0 < \varepsilon \leq r$, there exists n_0 such that g^n is (r, ε)-proximal for all $n \geq n_0$.

Denote by π_g the projector of kernel $V_-(g)$ and of image $V_+(g)$. Then

$$\frac{g^n}{\lambda_1(g)^n} = \pi_g + \frac{g^n_{|V_-(g)}}{\lambda_1(g)^n}.$$

By proximality, the spectral radius of $g_{|V_-(g)}$ is strictly smaller than $\lambda_1(g)$. It follows immediately by the spectral radius formula that $g^n/(\lambda_1(g)^n) \underset{n \to +\infty}{\longrightarrow} \pi_g$. Hence for any $y \in X \setminus X_-(g)$, uniformly on any compact subset of $X \setminus X_-(g)$,

$$g^n . y \underset{n \to +\infty}{\longrightarrow} x_+(g).$$

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It remains to show the Lipschitz properties of g^n, for n big enough. For all $n \in \mathbb{N}$, we denote by k_n, l_n (respectively, a_n) the orthogonal (respectively, symmetric) components of g^n so that $g_n = k_n a_n l_n$. We also set $x_n := k_n x_0$ and $H_n := l_n^{-1} H_0$.

For any $n \geq 1$, choose an endomorphism of norm 1 p_{x_n,H_n} such that $\mathbb{P}(\mathrm{im}(p_{x_n,H_n})) = x_n$ and $\mathbb{P}(\ker(p_{x_n,H_n})) = H_n$ and (by polar decomposition)

$$\frac{g^n}{a_n(1)} = p_{x_n,H_n} + O\left(\frac{a_n(2)}{a_n(1)}\right).$$

By the spectral radius formula, $|a_n(2)/a_n(1)|^{1/n} \underset{n\to\infty}{\longrightarrow} (\lambda_1(g_{|V_-(g)}))/\lambda_1(g) < 1$. Hence

$$\lim_{n\to\infty} \frac{a_n(2)}{a_n(1)} = 0.$$

Let (x, H) be an accumulating point of the sequence $(x_n, H_n)_{n\geq 1}$. Then there is a converging subsequence $x_{\varphi(n)}, H_{\varphi(n)} \underset{n\to+\infty}{\longrightarrow} x, H$. Denote by $p_{x,H}$ the endomorphism of norm 1 such that $\mathbb{P}(\mathrm{im}(p_{x,H})) = x$ and $\mathbb{P}(\ker(p_{x,H})) = H$. Then

$$\frac{g^{\varphi(n)}}{a_{\varphi(n)}(1)} \underset{n\to+\infty}{\longrightarrow} p_{x,H}.$$

This allows us to deduce, in particular, that for any $y \in X \setminus \{H, X_-(g)\}$,

$$g^{\varphi(n)}.y \underset{n\to+\infty}{\longrightarrow} x.$$

However, by proximality of g and uniqueness of the limit, we obtain that $x = x_+(g)$.

Similarly, by duality, we obtain that $H = X_-(g)$. Hence $(x_n, H_n)_{n\geq 1}$ converges toward $(x_+(g), X_-(g))$.

Fix $0 < \varepsilon \leq r$. Then for n large enough, the inclusion $\mathcal{V}_\varepsilon(X_-(g)) \supset \mathcal{V}_{\varepsilon/2}(H_n)$ holds. By Lemma 3.4, the restriction of g^n to $\mathcal{V}_\varepsilon(X_-(g))^c \subset \mathcal{V}_{\varepsilon/2}(H_n)^c$ is then a $|a_n(2)/a_n(1)|4/\varepsilon^2$-Lipschitz map. Finally, for n large enough so that $|a_n(2)/a_n(1)|4/\varepsilon^2 < \varepsilon$, the restriction of g^n to $\mathcal{V}_\varepsilon(X_-(g))^c$ is ε-Lipschitz. □

The following proximality criterion is due to Tits [**Tit71**], and one can find the statement in this form in [**Ben00**].

LEMMA 3.5. *Fix $0 < \varepsilon \leq r$. Let $x \in \mathbb{P}(V)$ and a hyperplane $Y \subset \mathbb{P}(V)$ such that $d(x, Y) \geq 6r$. Let $g \in \mathrm{GL}(V)$. If*
(i) *$g\mathcal{V}_\varepsilon(Y)^c \subset B(x, \varepsilon)$,*
(ii) *g restricted to $\mathcal{V}_\varepsilon(Y)^c$ is ε-Lipschitz,*
then g is $(2r, 2\varepsilon)$-proximal. Furthermore, the attracting point $x_+(g)$ is in $B(x, \varepsilon)$ and the repelling hyperplane $X_-(g)$ in a ε-neighborhood of Y.

COROLLARY 3.6. *Fix $0 < \varepsilon \leq r$. Let $g \in \mathrm{GL}(V)$ be an $(r, \varepsilon/2)$-proximal element such that $d(x_+(g), X_-(g)) \geq 7r$. Then for any $h \in \mathrm{GL}(V)$ such that $\|h - id_V\| \leq \varepsilon/2$, the product gh is $(2r, 2\varepsilon)$-proximal, with $x_+(gh) \in B(x_+(g), \varepsilon)$.*

Proof. Consider an $(r, \varepsilon/2)$-proximal element g and $h \in \mathrm{GL}(V)$ as in the hypothesis.

Remark that gh maps $h^{-1}\mathcal{V}_{\varepsilon/2}(X_-(g))^c$ to the open ball $B(x_+(g), \varepsilon/2)$. Furthermore, by proximality of g, the restriction of gh to $h^{-1}\mathcal{V}_{\varepsilon/2}(X_-(g))^c$ is $\varepsilon/2$-Lipschitz.

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Since h is close to id_V, we have that $\mathcal{V}_\varepsilon(h^{-1}X_-(g))^c \subset h^{-1}\mathcal{V}_{\varepsilon/2}(X_-(g))^c$. Hence gh restricted to $\mathcal{V}_\varepsilon(h^{-1}X_-(g))^c$ is ε-Lipschitz of image in the open ball $B(x_+(g), \varepsilon)$. Furthermore, $d(x_+(g), h^{-1}X_-(g)) \geq d(x_+(g), X_-(g)) - \varepsilon > 7r - \varepsilon \geq 6r$.

Finally, by Lemma 3.5, we deduce that gh is $(2r, 2\varepsilon)$-proximal, with $x_+(gh) \in B(x_+(g), \varepsilon)$. □

For all proximal elements g, h of End(V) such that $x_+(h) \notin X_-(g)$, we consider two unit eigenvectors $v_+(h) \in x_+(h)$ and $v_+(g) \in x_+(g)$ and denote by $c(g, h)$ the unique real number such that $v_1 - c(g, h)v_2 \in H$. *A priori*, $c(g, h)$ depends on the choice of the unit vectors, but its absolute value does not.

Given $g_1, \ldots, g_l$ of $End(V)$, set $g_0 = g_l$ and assume $x_+(g_{i-1}) \notin X_-(g_i)$ for all $1 \leq i \leq l$. We set

$$\nu_1(g_l, \ldots, g_1) = \sum_{1\leq j\leq l} \log |c(g_j, g_{j-1})|.$$

The following proposition explains how to control the spectral radius $\lambda_1(\gamma)$ when γ is a product of (r, ϵ)-proximal elements.

PROPOSITION 3.7. [**Ben00**] *For all $0 < \varepsilon \leq r$, there exist positive constants $C_{r,\varepsilon}$ such that for all $r > 0$, $\lim_{\epsilon\to 0} C_{r,\varepsilon} = 0$ and such that the following holds. If $\gamma_1, \ldots, \gamma_l$ are (r, ε)-proximal elements, such that $d(x_+(\gamma_{i-1}), X_-(\gamma_i)) \geq 6r$ for all $1 \leq i \leq l$ with $\gamma_0 = \gamma_l$, then for all $n_1, \ldots, n_l \geq 1$,*

$$\left|\log(\lambda_1(\gamma_l^{n_l} \ldots \gamma_1^{n_1})) - \sum_{i=1}^{l} n_i \log(\lambda_1(\gamma_i)) - \nu_1(\gamma_l, \ldots, \gamma_1)\right| \leq lC_{r,\varepsilon}.$$

Furthermore, the map $\gamma_l^{n_l} \ldots \gamma_1^{n_1}$ is $(2r, 2\varepsilon)$-proximal with $x_+(\gamma_l^{n_l} \ldots \gamma_1^{n_1}) \in B(x_+(\gamma_l), \varepsilon)$ and $X_-(\gamma_l^{n_l} \ldots \gamma_1^{n_1}) \subset \mathcal{V}_\varepsilon(X_-(\gamma_1))$.

Proof. Taking the logarithm in Benoist's [**Ben00**, Lemma 1.4] gives us the first part of the statement (the estimates). We only give a proof of the proximality and the localization of the attracting points and repelling hyperplane.

Let $n_1, \ldots, n_l \geq 1$ and assume that $0 < \varepsilon \leq r$ and $\varepsilon < 1$. Let us prove that $g_n := \gamma_l^{n_l} \ldots \gamma_1^{n_1}$ satisfies assumptions (i) and (ii) of the proximality criterion Lemma 3.5. More precisely, we prove by induction on l that g_n restricted to $\mathcal{V}_\varepsilon(X_-(\gamma_1))^c$ is ε-Lipschitz and $g_n\mathcal{V}_\varepsilon(X_-(\gamma_1))^c \subset B(x_+(\gamma_l), \varepsilon)$.

By (r, ε)-proximality of $\gamma_1^{n_1}$, the restriction of $\gamma_1^{n_1}$ to $\mathcal{V}_\varepsilon(\gamma_1)^c$ is an ε-Lipschitz map and $\gamma_1^{n_1}\mathcal{V}_\varepsilon(\gamma_1)^c \subset B(x_+(\gamma_1), \varepsilon)$.

Assume for some $1 \leq i \leq l$ that $\gamma_i^{n_i} \ldots \gamma_1^{n_1}$ restricted to $\mathcal{V}_\varepsilon(X_-(\gamma_1))^c$ is ε-Lipschitz and $\gamma_i^{n_i} \ldots \gamma_1^{n_1}\mathcal{V}_\varepsilon(X_-(\gamma_1))^c \subset B(x_+(\gamma_i), \varepsilon)$. Since $d(x_+(\gamma_i), X_-(\gamma_{i+1})) \geq 6r$ and $0 < \varepsilon \leq r$ we obtain $B(x_+(\gamma_i), \varepsilon) \subset \mathcal{V}_\varepsilon(X_-(\gamma_{i+1}))^c$. Then, using (r, ε)-proximality of γ_{i+1}, its restriction to $B(x_+(\gamma_i), \varepsilon)$ is ε-Lipschitz and $\gamma_{i+1}^{n_{i+1}} B(x_+(\gamma_i), \varepsilon) \subset B(x_+(\gamma_{i+1}), \varepsilon)$. Hence by the induction hypothesis and using $\varepsilon < 1$, the map $\gamma_{i+1}^{n_{i+1}} \ldots \gamma_1^{n_1}$ restricted to $\mathcal{V}_\varepsilon(X_-(\gamma_1))^c$ is ε-Lipschitz and $\gamma_{i+1}^{n_{i+1}} \ldots \gamma_1^{n_1}\mathcal{V}_\varepsilon(X_-(\gamma_1))^c \subset B(x_+(\gamma_{i+1}), \varepsilon)$.

We conclude the proof. By assumption, $d(x_+(\gamma_l), X_-(\gamma_1)) \geq 6r$. Finally, by Lemma 3.5 we deduce $(2r, 2\varepsilon)$-proximality of g_n with $x_+(g_n) \in B(x_+(\gamma_l), \varepsilon)$ and $X_-(g_n) \subset \mathcal{V}_\varepsilon(X_-(\gamma_1))$. □

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The previous proposition motivates the next definition.

Definition 3.8. Let $0 < \varepsilon \leq r$. A semigroup $\Gamma \subset \mathrm{GL}(V)$ is *strongly (r, ε)-Schottky* if:
(i) every $h \in \Gamma$ is (r, ε)-proximal;
(ii) $d(x_+(h), X_-(h')) \geq 6r$ for all $h, h' \in \Gamma$.
We also write that Γ is a *strong (r, ε)-Schottky semigroup*.

3.2. *Representations of a semisimple Lie group G.* Let (V, ρ) be a representation of G in a real vector space of finite dimension. For every character χ of $\mathfrak{a}$, denote the associated eigenspace by $V_\chi := \{v \in V \mid \forall a \in \mathfrak{a},\ \rho(a)v = \chi(a)v\}$. The set of *restricted weights* of V is the set $\Sigma(\rho) := \{\chi | V_\chi \neq 0\}$. Simultaneous diagonalization leads to the decomposition $V = \bigoplus_{\chi \in \Sigma(\rho)} V_\chi$. The set of weights is partially ordered as follows:

$$(\chi_1 \leq \chi_2) \Leftrightarrow (\forall a \in A^+,\ \chi_1(a) \leq \chi_2(a)).$$

Whenever ρ is irreducible, the set $\Sigma(\rho)$ has a highest element $\chi_{\rho,\max}$ which is the *highest restricted weight* of V. Denote by $V_{\chi_{\rho,\max}}$ the eigenspace of the highest restricted weight, and by Y_ρ the $\mathfrak{a}$-invariant supplementary subspace of V_ρ, that is,

$$Y_\rho := \ker(V^*_{\chi_{\rho,\max}}) = \bigoplus_{\chi \in \Sigma(\rho) \setminus \{\chi_{\max}\}} V_\chi.$$

The irreducible representation ρ is *proximal* when $\dim(V_{\chi_{\rho,\max}}) = 1$. The following lemma can be found in [**BQ16**, Lemma 5.32]. It is due to Tits [**Tit71**].

Denote by $\Pi \subset \Sigma^+$ the subset of *simple roots* of the set of positive roots for the adjoint representation of G.

LEMMA 3.9. [**Tit71**] *For every simple root $\alpha \in \Pi$, there exists a proximal irreducible algebraic representation (ρ_α, V_α) of G whose highest weight $\chi_{\rho_\alpha,\max}$ is orthogonal to β for every simple root $\beta \neq \alpha$. These weights $(\chi_{\rho_\alpha,\max})_{\alpha \in \Pi}$ form a basis of the dual space $\mathfrak{a}^*$.*

Moreover, the map

$$\mathcal{F}(X) \xrightarrow{y} \prod_{\alpha \in \Pi} \mathbb{P}(V_\alpha)$$
$$\eta := k_\eta \eta_0 \longmapsto (y_\alpha(\eta) := \rho_\alpha(k_\eta) V_{\chi_{\rho_\alpha,\max}})_{\alpha \in \Pi}$$

is an embedding of the set of asymptotic Weyl chambers in this product of projective spaces.

We also define a dual map $H : \mathcal{F}(X) \to \prod_{\alpha \in \Pi} Gr_{\dim(V_\alpha)-1}(V_\alpha)$ as follows. For every $\xi \in \mathcal{F}(X)$, let $k_\xi \in K$ be an element so that $\xi = k_\xi \check{\eta}_0$. Then

$$\mathcal{F}(X) \xrightarrow{Y} \prod_{\alpha \in \Pi} Gr_{\dim(V_\alpha)-1}(V_\alpha)$$
$$\xi := k_\xi \check{\eta}_0 \longmapsto (Y_\alpha(\xi) := \rho_\alpha(k_\xi) Y_{\rho_\alpha})_{\alpha \in \Pi}.$$

The maps y and Y provide us two ways to embed the space of asymptotic Weyl chambers $\mathcal{F}(X)$.

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COROLLARY 3.10. *The map*

$$\mathcal{F}^{(2)}(X) \longrightarrow \prod_{\alpha\in\Pi} \mathbb{P}(V_\alpha) \times Gr_{\dim(V)-1}(V_\alpha)$$
$$(f_+, f_-) \longmapsto (y_\alpha(f_+), Y_\alpha(f_-))_{\alpha\in\Pi}$$

is a G-equivariant injective map of the space of flags in general position into this product of projective spaces in general position, that is, the associated subspaces are in direct sum.

We now give an interpretation of the Cartan projection, the Iwasawa cocycle and the Jordan projection in terms of representations of G. The complete proof can be found in [**BQ16**].

LEMMA 3.11. [**BQ16**, Lemma 5.33] *Let $\alpha \in \Pi$ be a simple root and consider (V_α, ρ_α) the proximal representation of G given by Lemma 3.9. Then:*
(a) *there exists a $\rho_\alpha(K)$-invariant Euclidean norm on V_α such that, for all $a \in A$, the endomorphism $\rho_\alpha(a)$ is symmetric;*
(b) *for such a norm and the corresponding subordinate norm on* $\mathrm{End}(V_\alpha)$, *for all $g \in G$, $\eta \in \mathcal{F}(X)$ and $v_\eta \in y_\alpha(\eta)$,*
  (i) $\chi_{\rho_\alpha,\max}(\mu(g)) = \log(\|\rho_\alpha(g)\|)$*;*
  (ii) $\chi_{\rho_\alpha,\max}(\lambda(g)) = \log(\lambda_1(\rho_\alpha(g)))$*;*
  (iii) $\chi_{\rho_\alpha,\max}(\sigma(g, \eta)) = \log(\|\rho_\alpha(g)v_\eta\|/\|v_\eta\|)$.

The following lemma gives estimations on the Cartan projection of products of any pair of elements in G.

LEMMA 3.12. *There exists a continuous, left and right K-invariant, function $h \in G \mapsto C_h \in \mathbb{R}_+$ such that:*
(i) *for any $g \in G$, the Cartan projections $\mu(gh) - \mu(g)$ and $\mu(hg) - \mu(g)$ are in the ball $\overline{B_{\mathfrak{a}}(0, C_h)}$;*
(ii) *for any $\eta \in \mathcal{F}(X)$, the Iwasawa cocycle $\sigma(h, \eta) \in \overline{B_{\mathfrak{a}}(0, C_h)}$.*

Proof. In an abuse of terminology, we say that a function is K-invariant when it is K-invariant for both left and right action.

Let us prove the first point. For any $\alpha \in \Pi$, we consider the proximal irreducible representation (ρ_α, V_α) of G given by Lemma 3.9.

Using Lemma 3.11, we endow each vector space V_α with the $\rho_\alpha(K)$-invariant Euclidean norm. Classical properties of the norm lead, for all $\alpha \in \Pi$ and every $g, h \in G$, to

$$\frac{\|\rho_\alpha(g)\|}{\|\rho_\alpha(h^{-1})\|} \le \|\rho_\alpha(gh)\| \le \|\rho_\alpha(g)\|\|\rho_\alpha(h)\|,$$
$$\frac{1}{\|\rho_\alpha(h^{-1})\|} \le \frac{\|\rho_\alpha(gh)\|}{\|\rho_\alpha(g)\|} \le \|\rho_\alpha(h)\|.$$

Note that we obtain the same inequalities for hg. By Lemma 3.11, we deduce

$$-\chi_{\rho_\alpha,\max}(\mu(h^{-1})) \le \chi_{\rho_\alpha,\max}(\mu(gh) - \mu(g)) \le \chi_{\rho_\alpha,\max}(\mu(h)). \tag{4}$$

For any $\alpha \in \Pi$, set $h_\alpha := \max(\chi_{\rho_\alpha,\max}(\mu(h)), \chi_{\rho_\alpha,\max}(\mu(h^{-1})))$. Furthermore, by Lemma 3.9, the weights $(\chi_{\rho_\alpha,\max})_{\alpha\in\Pi}$ form a basis of the dual space $\mathfrak{a}^*$. In other words, they admit a dual basis in $\mathfrak{a}$. Denote by $C_h > 0$ the real number such that $\overline{B_{\mathfrak{a}}(0, C_h)}$ is the

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smallest closed ball containing any point of dual coordinates in $([-h_\alpha, h_\alpha])_{\alpha\in\Pi}$ for the dual basis of $(\chi_{\max,\alpha})_{\alpha\in\Pi}$. Hence $\overline{B_{\mathfrak{a}}(0, C_h)}$ is compact and contains $\mu(gh) - \mu(g)$ and $\mu(hg) - \mu(g)$.

It remains to show that the function $h \mapsto C_h$ is continuous and K-invariant. This is due to the fact that the Cartan projection and the map $h \mapsto \mu(h^{-1})$ are both continuous and K-invariant. Hence, by taking the supremum in each coordinate, the map $h \mapsto (h_\alpha)_{\alpha\in\Pi}$ is continuous and K-invariant. Furthermore, by definition of C_h, we obtain K-invariance and continuity of $h \mapsto C_h$.

Similarly, the second point is a direct consequence of Lemma 3.11, (i) and (iii) and of the inequality

$$\frac{1}{\|\rho_\alpha(h^{-1})\|} \le \frac{\|\rho_\alpha(h)(v_\eta)\|}{\|v_\eta\|} \le \|\rho_\alpha(h)\|, \tag{5}$$

where $\eta \in \mathcal{F}(X)$ and $v_\eta \in V_\alpha$ is the associated non-trivial vector. □

3.3. *Loxodromic elements.* Let us now study the dynamical properties of loxodromic elements in the representations of the previous paragraph. [**BQ16**, Lemma 5.37] states that any element of G is loxodromic if and only if its image is proximal for every representation given by Lemma 3.9. This allows us to extend the notions and results on proximal elements to loxodromic elements in G.

Definition 3.13. An element $g \in G$ is *loxodromic* if its Jordan projection $\lambda(g)$ is in the interior of the Weyl chamber $\mathfrak{a}^{++}$ or (equivalently) if for every $\alpha \in \Pi$ the endomorphism $\rho_\alpha(g)$ is proximal.

Let $0 < \varepsilon \le r$. An element $g \in G$ is (r, ε)-*loxodromic* if for every $\alpha \in \Pi$ the endomorphism $\rho_\alpha(g)$ is (r, ε)-proximal.

Finally, a semigroup Γ of G is said to be *strongly* (r, ε)-*Schottky* if for every $\alpha \in \Pi$ the semigroup $\rho_\alpha(\Gamma) \subset \mathrm{End}(V_\alpha)$ is strongly (r, ε)-Schottky.

Attracting and repelling asymptotic Weyl chambers of loxodromic elements were defined in §2.2 as follows. For any loxodromic element $g \in G$, we have $(g^+, g^-) := h_g(\eta_0, \check{\eta}_0) \in \mathcal{F}^{(2)}(X)$, where $h_g \in G$ is an element such that there is an angular part $m(g) \in M$ with $g = h_g e^{\lambda(g)} m(g) h_g^{-1}$.

The G-equivariant map $(f_+, f_-) \in \mathcal{F}^{(2)}(X) \to (y_\alpha(f_+), Y_\alpha(g_-))_{\alpha\in\Pi}$ given by Corollary 3.10 allows us to characterize attracting and repelling points in $\mathcal{F}(X)$ for loxodromic elements.

LEMMA 3.14. *For any loxodromic element $g \in G$, the following statements are true.*

(i) *g^{-1} is loxodromic, of attracting point g^- and repelling point g^+.*

(ii) *The image of $(g^+, g^-) \in \mathcal{F}^{(2)}(X)$ by the above map is the family of attracting points and repelling hyperplanes in general position $(x_+(\rho_\alpha(g)), X_-(\rho_\alpha(g)))_{\alpha\in\Pi}$.*

(iii) *Any point $\eta \in \mathcal{F}(X)$ in general position with g^- is attracted to g^+ that is, $\lim_{n\to+\infty} g^n\eta = g^+$.*

(iv) *For any non-empty open set $O_- \subset \mathcal{F}(X)$ in general position with g^+, for any non-empty open neighborhood $U_- \subset \mathcal{F}(X)$ of g^-, there exists $N \in \mathbb{N}$ such that for any $n \ge N$, we have $O_- \cap g^n U_- \ne \emptyset$.*

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Proof. Let $g \in G$ be a loxodromic element. Consider an element $h_g \in G$ and an angular part $m(g) \in M$ such that $g = h_g m(g) e^{\lambda(g)} h_g^{-1}$. Then $g^{-1} = h_g m(g)^{-1} e^{-\lambda(g)} h_g^{-1}$. Remark that $-\lambda(g)$ is in the interior of the Weyl chamber $-\mathfrak{a}^+$. Consider the element of the Weyl group $N_K(A)/M$ whose adjoint action on $\mathfrak{a}$ sends $\mathfrak{a}^+$ onto $-\mathfrak{a}^+$. Denote one representative by $k_\iota \in N_K(A)$. Then $-Ad(k_\iota)(\lambda(g)) \in \mathfrak{a}^{++}$, hence

$$g^{-1} = h_g k_\iota (k_\iota^{-1} m(g) k_\iota)^{-1} e^{-Ad(k_\iota)(\lambda(g))} (h_g k_\iota)^{-1}.$$

Remark that $k_\iota^{-1} M k_\iota$ is in the centralizer of $k_\iota^{-1} A k_\iota = A$, hence $k_\iota^{-1} m(g) k_\iota \in k_\iota^{-1} M k_\iota = M$. We deduce that $\lambda(g^{-1}) = -Ad(k_\iota)(\lambda(g))$ and set $h_{g^{-1}} = h_g k_\iota$ with angular part $m(g^{-1}) = (k_\iota^{-1} m(g) k_\iota)^{-1}$. Then the pair of attracting and repelling points of g^{-1} in $\mathcal{F}(X)$ is $(h_g k_\iota \eta_0, h_g k_\iota \check{\eta}_0)$. Since $k_\iota \eta_0 = \check{\eta}_0$ and $k_\iota \check{\eta}_0 = \eta_0$ we obtain the first statement, that is, that g^- (respectively, g^+) is the attracting (respectively, repelling) point of g^{-1}.

For the second point, it suffices to prove that for any loxodromic element $g \in G$, for every $\alpha \in \Pi$, the vector space $\rho_\alpha(h_g) V_{\rho_\alpha} = y_\alpha(g^+)$ is the eigenspace associated to the spectral radius of $\rho_\alpha(g)$ and that $\rho_\alpha(h_g) Y_{\rho_\alpha} = Y_\alpha(g^-)$ is the direct sum of the other eigenspaces.

Let $g \in G$ be a loxodromic element and let $\alpha \in \Pi$. By Lemma 3.11, the spectral radius of $\rho_\alpha(g)$ is $\exp(\chi_{\rho_\alpha,\max}(\lambda(g)))$. We deduce that the eigenspace of the highest eigenvalue is $\rho_\alpha(h_g) V_{\rho_\alpha}$. Furthermore, by definition of proximality, $x_+(\rho_\alpha(g)) = \mathbb{P}(\rho_\alpha(h_g) V_{\rho_\alpha}) = y_\alpha(g^+)$.

Remark that the other eigenvalues of $\rho_\alpha(g)$ are given by the other non-maximal restricted weights of the representation (ρ_α, V_α). Hence $\rho_\alpha(h_g) Y_{\rho_\alpha}$ is the direct sum of the other eigenspaces of $\rho_\alpha(h_g)$. The projective space $\mathbb{P}(\rho_\alpha(h_g) Y_{\rho_\alpha})$ is thus the repelling hyperplane of $\rho_\alpha(g)$. Hence the second statement is true.

For any point $\eta \in \mathcal{F}(X)$ in general position with g^- and for any $\alpha \in \Pi$, the point $y_\alpha(\eta)$ is then in general position with the hyperplane $Y_\alpha(g^-)$. Hence $\lim_{n \to +\infty} \rho_\alpha(g^n) y_\alpha(\eta) = x_+(\rho_\alpha(g))$. This gives the third statement.

For the last statement, we apply the third statement to g^{-1}. This means that, for any non-empty open set $O_- \subset \mathcal{F}(X)$ in general position with g^+ and for any non-empty open neighborhood $U_- \subset \mathcal{F}(X)$ of g^-, there exists $N \in \mathbb{N}$ such that, for any $n \geq N$,

$$(g^{-1})^n O_- \cap U_- \neq \emptyset.$$

Hence, for any $n \geq N$,

$$g^n (g^{-n} O_- \cap U_-) \neq \emptyset,$$

and finally,

$$O_- \cap g^n U_- \neq \emptyset. \qquad \square$$

Lemma 3.3 and Corollary 3.6 extend to loxodromic elements.

LEMMA 3.15. *For every loxodromic element $g \in G$, there exist $r > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ large enough, g^n is (r, ε_n)-loxodromic with $\varepsilon_n \underset{n \to \infty}{\to} 0$.*

COROLLARY 3.16. *Fix $0 < \varepsilon \leq r$. Let $g \in G$ be a $(r, \varepsilon/2)$-loxodromic element such that $d(g^+, g^-) \geq 7r$. Then for any $h \in G$ such that $\|h - id_G\| \leq \varepsilon/2$, the product gh is $(2r, 2\varepsilon)$-loxodromic, with $(gh)^+ \in B(g^+, \varepsilon)$.*

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Likewise, we generalize estimates of Proposition 3.7 to products of loxodromic elements of G in general configuration.

Given l loxodromic elements $g_1, \ldots, g_l$ of G, set $g_0 = g_l$ and assume that the asymptotic points g_{i-1}^+ and g_i^- are opposite for all $1 \le i \le l$. Thanks to Lemma 3.9, there exists a unique element $\nu = \nu(g_1, \ldots, g_l) \in \mathfrak{a}$ whose coordinates in the dual basis of $(\chi_{\rho_\alpha,\max})_{\alpha\in\Pi}$ are

$$(\chi_{\rho_\alpha,\max}(\nu))_{\alpha\in\Pi} := (\nu_1(\rho_\alpha(g_1), \ldots, \rho_\alpha(g_l)))_{\alpha\in\Pi}.$$

The product of projective spaces $\prod_{\alpha\in\Pi} \mathbb{P}(V_\alpha)$ is endowed with the natural distance.

PROPOSITION 3.17. [**Ben00**, Benoist] *For all $0 < \varepsilon \le r$, there exist positive constants $C_{r,\varepsilon}$ such that for all $r > 0$, $\lim_{\varepsilon\to 0} C_{r,\varepsilon} = 0$ and such that the following holds. If $\gamma_1, \ldots, \gamma_l$ are (r, ε)-loxodromic elements, such that for all $1 \le i \le l$ with $\gamma_0 = \gamma_l$ we have $d(y(\gamma_{i-1}^+), Y(\gamma_i^-)) \ge 6r$, then for all $n_1, \ldots, n_l \ge 1$,*

$$\lambda(\gamma_l^{n_l} \ldots \gamma_1^{n_1}) - \sum_{i=1}^{l} n_i\lambda(\gamma_i) - \nu(\gamma_l, \ldots, \gamma_1) \in B_{\mathfrak{a}}(0, lC_{r,\epsilon}).$$

Furthermore, the map $g := \gamma_l^{n_l} \ldots \gamma_1^{n_1}$ is $(2r, 2\varepsilon)$-loxodromic with $y(g^+) \in B(y(\gamma_l^+), \varepsilon)$ and repelling hyperplanes $Y(g^-) \in \mathcal{V}_\varepsilon(Y(\gamma_1^-))$.

Using Proposition 3.17, one can construct finitely generated, strong (r, ε)-Schottky semigroups as follows. Let $0 < \varepsilon \le r$. Let $S \subset G$ be a family of $(r/2, \varepsilon/2)$-loxodromic elements such that $d(y(h^+), Y(h'^-)) \ge 7r$ for all $h, h' \in S$. Denote by Γ' the semigroup generated by S. Then every element $g \in \Gamma$ is a non-commuting product of proximal elements of the form $g_l^{n_l} \ldots g_1^{n_1}$ with $n_1, \ldots, n_l \ge 1$ and $g_i \ne g_{i+1} \in S$ for all $1 \le i < l$. By Proposition 3.17, we deduce $d(y(g^+), Y(g^-)) \ge d(y(g_l^+), Y(g_1^-)) - \varepsilon \ge 6r$ and that g is (r, ε)-loxodromic. Thus, Γ' is strongly (r, ε)-Schottky.

4. *Topological transitivity*

Recall the definition of topological transitivity. We denote by $\mathfrak{a}_1^+$ (respectively, $\mathfrak{a}_1^{++}$) the intersection of the unit sphere in $\mathfrak{a}$ with $\mathfrak{a}^+$ (respectively, $\mathfrak{a}^{++}$).

Definition 4.1. Let $\widetilde{\Omega} \subset \mathcal{W}(X)$ be a Γ-invariant and $\mathfrak{a}$-invariant subset of parametric flats. Let $\Omega := \Gamma\backslash\widetilde{\Omega}$. Fix a direction $\theta \in \mathfrak{a}_1^{++}$. The Weyl chamber flow $\phi_{\mathbb{R}}^\theta$ is *topologically transitive* on Ω if for all open non-empty subsets $U, V \subset \Omega$, there exists $t_n \to +\infty$ such that for every $n \ge 1$, we have $U \cap \phi_{t_n}^\theta(V) \ne \emptyset$.

It is a standard fact that this is equivalent to one the following properties.

(1) There is a $\phi_{\mathbb{R}}^\theta$-dense orbit in Ω.
(2) For all open non-empty subsets $\widetilde{U}, \widetilde{V} \subset \widetilde{\Omega}$, there exists $t_n \to +\infty$ such that for every $n \ge 1$, $\Gamma\widetilde{U} \cap \phi_{t_n}^\theta(\widetilde{V}) \ne \emptyset$.
(3) For all open non-empty subsets $\widetilde{U}, \widetilde{V} \subset \widetilde{\Omega}$, there exists $t_n \to +\infty$ such that for every $n \ge 1$, there exists $\gamma_n \in \Gamma$ with $\gamma_n\widetilde{U} \cap \phi_{t_n}^\theta(\widetilde{V}) \ne \emptyset$.

The equivalence between the definition and property (1) can be found in Eberlein [**Ebe72**, Proposition 3.5]. The other equivalences are straightforward.

4.1. *Limit set, limit cone of Zariski dense subgroup.* In the remaining parts of this paper, $\Gamma \subset G$ is a Zariski dense semigroup of G.

Definition 4.2. A point $\eta \in \mathcal{F}(X)$ is a *limit point* if there exists a sequence $(\gamma_n)_{n\geq 1}$ in Γ such that $((\gamma_n)_*\mathrm{Haar}_{G/MAN})_{n\geq 1}$ converges weakly toward the Dirac measure in η.

The *limit set* of Γ, denoted by $L_+(\Gamma)$, is the set of limit points of Γ. It is a closed subset of $\mathcal{F}(X)$. Denote by $L_-(\Gamma)$ the limit set of Γ^{-1} and, finally, let $L^{(2)}(\Gamma) = (L_+(\Gamma) \times L_-(\Gamma)) \cap \mathcal{F}^{(2)}(X)$.

Note that when Γ is a subgroup, $L_+(\Gamma) = L_-(\Gamma)$ and $L^{(2)}(\Gamma)$ is the subset of pair of points of $L_+(\Gamma)$ in general position. For the hyperbolic plane, we get the product of the usual limit set minus the diagonal.

LEMMA 4.3. [**Ben97**, Lemma 3.6] *The set of pairs of attracting and repelling points of loxodromic elements of* Γ *is dense in* $L_+(\Gamma) \times L_-(\Gamma)$.

Definition 4.4. We denote by $\widetilde{\Omega}(X)$ the subset of *non-wandering Weyl chambers*, defined through the Hopf parametrization by

$$\widetilde{\Omega}(X) := \mathcal{H}^{-1}(L^{(2)}(\Gamma) \times \mathfrak{a}).$$

This is a Γ-invariant subset of $\mathcal{W}(X)$. When Γ is a subgroup, we denote by $\Omega(X) := \Gamma\backslash\widetilde{\Omega}(X)$ the quotient space.

Conze and Guivarc'h proved in [**CG02**, Theorem 6.4] the existence of dense $\mathfrak{a}$-orbits in $\widetilde{\Omega}(X)$ for $G = \mathrm{SL}(n, \mathbb{R})$. By duality, this is equivalent to topological transitivity of left Γ-action on $\widetilde{\Omega}(X)/AM \simeq L^{(2)}(\Gamma)$. We propose a new simpler proof of this result, adapting that for negatively curved manifolds in Eberlein [**Ebe72**].

THEOREM 4.5. [**CG02**] *For any open non-empty subsets* $\mathcal{U}^{(2)}, \mathcal{V}^{(2)} \subset L^{(2)}(\Gamma)$ *there exists* $g \in \Gamma$ *such that* $g\mathcal{U}^{(2)} \cap \mathcal{V}^{(2)} \neq \emptyset$.

Proof. Without loss of generality, we assume that $\mathcal{U}^{(2)} = \mathcal{U}_+ \times \mathcal{U}_-$ and $\mathcal{V}^{(2)} = \mathcal{V}_+ \times \mathcal{V}_-$, where $\mathcal{U}_+, \mathcal{V}_+$ (respectively, $\mathcal{U}_-, \mathcal{V}_-$) are open non-empty subsets of $L_+(\Gamma)$ (respectively, $L_-(\Gamma)$).

We choose an open set $W^{(2)} = W_+ \times W_- \subset L^{(2)}(\Gamma)$ so that $\mathcal{V}_+$ and W_- (respectively, W_+ and $\mathcal{U}_-$) are opposite. Such a choice is always possible. If $\mathcal{V}_+$ and $\mathcal{U}_-$ are opposite, we can take $W^{(2)} = \mathcal{V}^{(2)}$. Otherwise, by taking $\mathcal{U}^{(2)}$ and $\mathcal{V}^{(2)}$ smaller, we can always assume that the subset of points in $L_+(\Gamma)$ (respectively, $L_-(\Gamma)$) in general position with $\mathcal{U}_-$ (respectively, $\mathcal{V}_+$) is non-empty. Then we choose a suitable opposite pair of open non-empty subsets $W_+ \times W_- \subset L_+(\Gamma) \times L_-(\Gamma)$.

Since $W_+ \times \mathcal{U}_- \subset L^{(2)}(\Gamma)$, then, by Lemma 4.3, there are loxodromic elements in Γ with attracting point in W_+ and repelling point in $\mathcal{U}_-$. By Lemma 3.14, such a loxodromic element γ_1 contracts points that are in general position with $\gamma_1^- \in \mathcal{U}_-$ toward $\gamma_1^+ \in W_+$. Now apply statement (iv) of Lemma 3.14, to loxodromic element γ_1, with W_- in general position with γ_1^+ and U_- containing γ_1^-. Hence for any n large enough, $\gamma_1^n\mathcal{U}^{(2)} \cap W^{(2)} \neq \emptyset$.

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We take an open subset $\mathcal{W}^{(2)}$ of $\gamma_1^n\mathcal{U}^{(2)} \cap W^{(2)}$ of the form $\mathcal{W}^{(2)} = \mathcal{W}_+ \times \mathcal{W}_-$. Then $\mathcal{V}_+ \times \mathcal{W}_- \subset \mathcal{V}_+ \times W_- \subset L^{(2)}(\Gamma)$. Likewise, we choose a loxodromic element $\gamma_2 \in \Gamma$ such that $\gamma_2\mathcal{W}^{(2)} \cap \mathcal{V}^{(2)} \neq \emptyset$. Then

$$(\gamma_2\gamma_1^n\mathcal{U}^{(2)} \cap \gamma_2 W^{(2)}) \cap \mathcal{V}^{(2)} \supset \gamma_2\mathcal{W}^{(2)} \cap \mathcal{V}^{(2)} \neq \emptyset.$$

Finally, the element $g = \gamma_2\gamma_1^n$ satisfies $g\mathcal{U}^{(2)} \cap \mathcal{V}^{(2)} \neq \emptyset$. □

The following theorem describes the set of directions $\theta \in \mathfrak{a}_1^+$ for which we will show that ϕ_t^θ is topologically mixing.

THEOREM 4.6. [**Ben97**] *We define the limit cone of* Γ *by* $\mathcal{C}(\Gamma) := \overline{\bigcup_{\gamma\in\Gamma} \mathbb{R}\lambda(\gamma)}$. *Then*

$$\mathcal{C}(\Gamma) = \bigcap_{n\geq 1} \overline{\bigcup_{\substack{\|\gamma\|\geq n\\ \gamma\in\Gamma}} \mathbb{R}\mu(\gamma)},$$

and the limit cone is closed, convex, of non-empty interior.

The limit cone is also called the *Benoist cone*.

4.2. *Topological transitivity properties.* Recall the definition of the subset of non-wandering Weyl chambers $\widetilde{\Omega}(X) = \mathcal{H}^{-1}(L^{(2)}(\Gamma) \times \mathfrak{a})$.

PROPOSITION 4.7. *Let* $\theta \in \mathfrak{a}^{++}$. *If the flow* $(\Omega(X), \phi_t^\theta)$ *is topologically transitive then* $\theta \in \mathring{\mathcal{C}}(\Gamma)$.

Proof. We assume that the dynamical system $(\Omega(X), \phi_t^\theta)$ is topologically transitive, that is, there exists a dense orbit. Let $x \in \Omega(X)$ be a point of ϕ_t^θ-dense orbit and choose $g_x \in G$ a lift of x.

By density of $(\phi_t^\theta(x))_{x\in\mathbb{R}}$, for any $yM \in \widetilde{\Omega}(X) \subset G/M$, there exists $t_n \to +\infty$, $\delta_n \to id_G$, $m_n \in M$ and $\gamma_n \in \Gamma$ such that

$$\phi_{t_n}^\theta(g_x) = g_x e^{t_n\theta} = \gamma_n y \delta_n m_n.$$

Thanks to Lemma 3.12, we deduce the estimates

$$\mu(g_x e^{t_n\theta}) \in t_n\theta + B_{\mathfrak{a}}(0, C_{g_x}),$$
$$\mu(\gamma_n y \delta_n m_n) \in \mu(\gamma_n) + B_{\mathfrak{a}}(0, C_y + C_{\delta_n}).$$

Therefore, $\mu(\gamma_n)$ and $t_n\theta$ are at bounded distance, and by Theorem 4.6 the direction θ must lie in the (closed) limit cone.

Let us now argue by contradiction that θ cannot be in the boundary of the limit cone. First, we choose a point $v \in \mathfrak{a}$ such that the line $v + \mathbb{R}\theta$ is far from the limit cone (the distance depends on g_x). Since $g_x e^{-v}$ is in $g_x e^{\mathfrak{a}} M$, which is, by A-invariance, a subset of $\widetilde{\Omega}(X)$, we use as above topological transitivity on $g_x e^{-v}$. Then we prove that elements of the form $g_x e^{v+t\theta} m g_x^{-1}\delta$, with $m \in M$, are loxodromic and very contracting when t is large enough and δ sufficiently close to id_G. In the last step, we estimate the Jordan projection of such elements: using the proximal representations of Lemma 3.9, we prove that they are

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in a bounded neighborhood of $v + t\theta$ when t is sufficiently large and δ sufficiently close to id_G. Finally, we find a contradiction with the choice of $v \in \mathfrak{a}$.

First, by considering the maps of Corollary 3.10, we set

$$r := \frac{1}{7}\mathrm{d}(y(g_x\eta_0), Y(g_x\check{\eta}_0)).$$

For all $\alpha \in \Pi$, we choose p_{α,g_x} a rank-one projection of image $y_\alpha(g_x\eta_0)$ and kernel $Y_\alpha(g_x\check{\eta}_0)$. Set

$$D_r := \sup_{\xi\in B(g_x\eta_0,r)} \sup_{\alpha\in\Pi} \{|\|p_{\alpha,g_x}(v_{y_\alpha(\xi)})\| - 1| \mid v_{y_\alpha(\xi)} \in \partial B(0, 1) \cap y_\alpha(\xi)\}.$$

Note that D_r does not depend on the choice of the rank one projection and only depends on r and g_x. Assume by contradiction that θ is in the boundary of $\mathcal{C}(\Gamma)$. Let H be a supporting hyperplane of the convex $\mathcal{C}(\Gamma)$, tangent at θ and H^+ the half space not containing $\mathcal{C}(\Gamma)$. Pick $v \in H^+$ such that

$$d(v, H) \geq 4D_r.$$

Then

$$d(v + \mathbb{R}^+\theta, \mathcal{C}(\Gamma)) = d(v + \mathbb{R}^+\theta, H) = d(v, H) \geq 4D_r. \tag{6}$$

Let us now use topological transitivity. Since $g_xe^{-v} \in \widetilde{\Omega}(X)$, the trajectory $(\phi_t^\theta(x))_{x\in\mathbb{R}}$ comes back infinitely often in any small neighborhood of Γg_xe^{-v}. Hence there exist $t_n \to +\infty$, $\delta_n \to id_G$, $m_n \in M$ and $\gamma_n \in \Gamma$ such that

$$\phi_{t_n}^\theta(g_xm_n) = \gamma_ng_xe^{-v}\delta_n. \tag{7}$$

Rewrite the previous equation as

$$g_xe^{v+t_n\theta}m_ng_x{-1}\,(g_xe^{-v}\delta_n^{-1}e^vg_x^{-1}) = \gamma_n.$$

For every $n \geq 1$ we set $\delta_n' := g_xe^{-v}\delta_n^{-1}e^vg_x^{-1}$. The sequence $(\delta_n')_{n\geq1}$ converges toward id_G, and we have

$$g_xe^{v+t_n\theta}m_ng_x^{-1}\delta_n' = \gamma_n. \tag{8}$$

Since by hypothesis $\theta \in \mathfrak{a}^{++}$, we choose a positive number $t_v > 0$ such that $v + t_v\theta \in \mathfrak{a}^{++}$. For every n large enough such that $t_n \geq t_v$, denote by $\lfloor t_n - t_v\rfloor$ the integer part of $t_n - t_v$ and $\{t_n - t_v\}$ its fractionary part. Now set $v_n := v + (t_v + \{t_n - t_v\})\theta \in \mathfrak{a}^{++}$ and $h_\theta := g_xe^\theta g_x^{-1}$. Rewrite equation (8) for large integers:

$$g_xe^{v_n}m_ng_x^{-1}h_\theta^{\lfloor t_n-t_v\rfloor}\delta_n' = \gamma_n. \tag{9}$$

For every $n \geq 1$ we set $g_n := g_xe^{v+t_n\theta}m_ng_x^{-1} = g_xe^{v_n}m_ng_x^{-1}h_\theta^{\lfloor t_n-t_v\rfloor}$.

Let us now prove that for n large, the g_n are very contracting elements. We apply Lemma 3.15 on the loxodromic elements $h_\theta^{\lfloor t_n-t_v\rfloor}$. There is a sequence of $\varepsilon_n \to 0$ so that $h_\theta^{\lfloor t_n-t_v\rfloor}$ is (r, ε_n)-loxodromic. Then for any $n \geq n_0$ large enough, g_n is the product of an (r, ε_n)-loxodromic element and a loxodromic element of the form $g_xe^{v_n}m_ng_x^{-1}$, where $v_n \in \mathfrak{a}^{++}$ is bounded, and with $\lfloor t_n - t_v\rfloor \to +\infty$. Since $g_xe^{v_n}m_ng_x^{-1}$ and $h_\theta^{\lfloor t_n-t_v\rfloor}$ have the same attracting and repelling point in $\mathcal{F}(X)$, we deduce that g_n is (r, ε_n)-loxodromic for $n \geq n_0$. Now take $\varepsilon_n' = 2\max(\varepsilon_n, \|\delta_n' - id_G\|)$. Then there exists n_1 so that for $n \geq$

$\max(n_0, n_1)$, we have that $0 < \varepsilon'_n \leq r$, and g_n is $(r, \varepsilon'_n/2)$-loxodromic. Corollary 3.16 shows that $g_n\delta'_n$ is $(2r, 2\varepsilon'_n)$-loxodromic for n large enough, and $(g_n\delta'_n)^+ \in B(g_x\eta_0, \varepsilon'_n)$.

Using Fact 2.6, we compute $\lambda(g_n\delta'_n)$:

$$\begin{aligned}\lambda(g_n\delta'_n) &= \sigma(g_n\delta'_n, (g_n\delta'_n)^+)\\ &= \sigma(g_n, \delta'_n(g_n\delta'_n)^+) + \sigma(\delta'_n, (g_n\delta'_n)^+)\\ &= \sigma(g_n, g_x\eta_0)\\ &\quad + (\sigma(g_n, \delta'_n(g_n\delta'_n)^+) - \sigma(g_n, g_x\eta_0))\\ &\quad + \sigma(\delta'_n, (g_n\delta'_n)^+).\end{aligned}$$

Remark that $\sigma(g_n, g_x\eta_0) = \lambda(g_n) = v + t_n\theta$. Hence

$$\lambda(g_n\delta'_n) - (v + t_n\theta) = (\sigma(g_n, \delta'_n(g_n\delta'_n)^+) - \sigma(g_n, g_x\eta_0)) + \sigma(\delta'_n, (g_n\delta'_n)^+). \tag{10}$$

We analyze separately the two terms of the right-hand side of the last equality. For the last term, by Lemma 3.12(ii),

$$\|\sigma(\delta'_n, (g_n\delta'_n)^+)\| \leq C_{\delta'_n}.$$

Now we will bound, independently of v, the term $\sigma(g_n, \delta'_n(g_n\delta'_n)^+) - \sigma(g_n, g_x\eta_0)$. Let $\alpha \in \Pi$ be a simple root and consider the proximal representation of G associated to α. By Lemma 3.11(b)(iii), for any $\xi \in \delta'_n B(g_x\eta_0, \varepsilon'_n)$, there exists a non-zero representative $v_\xi \in V_\alpha$ such that

$$\chi_{\rho_\alpha,\max}(\sigma(g_n, \xi)) = \log \frac{\|\rho_\alpha(g_n)v_\xi\|}{\|v_\xi\|}.$$

Let $\xi = \delta'_n(g_n\delta'_n)^+$ and consider a unitary vector $v_\xi \in V_\alpha$. Then

$$\frac{\rho_\alpha(g_n)}{\lambda_1(\rho_\alpha(g_n))}(v_\xi) = p_{\alpha,g_x}(v_\xi) + \frac{\rho_\alpha(g_n)}{\lambda_1(\rho_\alpha(g_n))}(v_\xi - p_{\alpha,g_x}(v_\xi)).$$

By the triangle inequality,

$$\begin{aligned}\|p_{\alpha,g_x}(v_\xi)\| - \left\|\frac{\rho_\alpha(g_n)(v_\xi - p_{\alpha,g_x}(v_\xi))}{\lambda_1(\rho_\alpha(g_n))}\right\| &\leq \frac{\|\rho_\alpha(g_n)v_\xi\|}{\lambda_1(\rho_\alpha(g_n))}\\ &\leq \|p_{\alpha,g_x}(v_\xi)\| + \left\|\frac{\rho_\alpha(g_n)(v_\xi - p_{\alpha,g_x}(v_\xi))}{\lambda_1(\rho_\alpha(g_n))}\right\|.\end{aligned}$$

The eigenvalues of $\rho_\alpha(g_n)/(\lambda_1(\rho_\alpha(g_n)))$ restricted to the repelling hyperplane $X_-(g_n) = Y_\alpha(g_x\check{\eta}_0)$ are $\exp(\chi_\alpha(\lambda(g_n)) - \chi_{\rho_\alpha,\max}(\lambda(g_n)))$, where $\chi_\alpha \neq \chi_{\rho_\alpha,\max}$ is a restricted weight of $\Sigma(\rho_\alpha)$. They converge to zero and these endomorphisms are all diagonalizable. Hence,

$$\left\|\frac{\rho_\alpha(g_n)_{|Y_\alpha(g_x\check{\eta}_0)}}{\lambda_1(\rho_\alpha(g_n))}\right\| \underset{n\to+\infty}{\to} 0.$$

Taking the logarithm and the upper bound of $\|\rho_\alpha(g_n)v_\xi\|/(\lambda_1(\rho_\alpha(g_n)))$ and its inverse, we obtain for n large enough,

$$\|\sigma(g_n, \xi) - \sigma(g_n, g_x\eta_0)\| \leq D_r + \sup_{\alpha\in\Pi}\left\|\frac{\rho_\alpha(g_n)_{|Y_\alpha(g_x\check{\eta}_0)}}{\lambda_1(\rho_\alpha(g_n))}\right\| \|id_{V_\alpha} - p_{\alpha,g_x}\|.$$

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Finally, for any $v \in \mathfrak{a}$, there exist $t_n \to +\infty$, $\delta'_n \to id_G$, such that for any n large enough,

$$\|\lambda(\gamma_n) - (v + t_n\theta)\| \leq D_r + \sup_{\alpha\in\Pi} \left\| \frac{\rho_\alpha(g_n)_{|Y_\alpha(g_x\check{\eta}_0)}}{\lambda_1(\rho_\alpha(g_n))} \right\| \|id_{V_\alpha} - p_{\alpha,g_x}\| + C_{\delta'_n}. \tag{11}$$

The two last terms converge to zero when $n \to +\infty$, hence, for n large enough, the norm of $\lambda(\gamma_n) - (v + t_n\theta)$ is uniformly bounded by $3D_r$.

To conclude, recall that the limit cone is the smallest closed cone containing all the Jordan projections of Γ. Hence, this implies that for n large enough, the distance $d(v + t_n\theta, \mathcal{C}(\Gamma))$ is smaller than $3D_r$. This is contradictory with the choice of v given by equation (6).

Hence, topological transitivity of the dynamical system $(\Omega(X), \phi_t^\theta)$ implies that $\theta \in \overset{\circ}{\mathcal{C}}(\Gamma)$. □

5. *Topological mixing*

Recall the definition of topological mixing.

Definition 5.1. Fix a direction $\theta \in \mathfrak{a}_1^{++}$. The Weyl chamber flow $\phi_{\mathbb{R}}^\theta$ is *topologically mixing* on $\Omega(X)$ if for all open subsets $U, V \subset \Omega(X)$, there exists $T > 0$ such that for all $t \geq T$, we have $U \cap \phi_t^\theta(V) \neq \emptyset$.

It will sometimes be more convenient to set proofs in the cover $\widetilde{\Omega}(X)$, where the topological mixing takes the following form: for all open subsets $\widetilde{U}, \widetilde{V} \subset \widetilde{\Omega}(X)$, there exists $T > 0$ such that for all $t \geq T$ there exists $\gamma_t \in \Gamma$ with $\gamma_t\widetilde{U} \cap \phi_t^\theta(\widetilde{V}) \neq \emptyset$.

5.1. *Non-arithmetic spectrum.*

Denote by Γ^{lox} the set of loxodromic elements of Γ. Dal'bo [**Dal00**] introduced the notion of *non-arithmetic spectrum* for subgroups of $Isom(H^n)$, meaning that the length spectrum of such a group is not contained in a discrete subgroup of $\mathbb{R}$.

We generalize this definition for isometry groups in higher rank.

Definition 5.2. We say that Γ has *non-arithmetic spectrum* if the length spectrum $\lambda(\Gamma^{\mathrm{lox}})$ spans a dense subgroup of $\mathfrak{a}$.

PROPOSITION 5.3. *Every Zariski dense semigroup Γ contains loxodromic elements, strong (r, ε)-Schottky Zariski dense semigroups and has non-arithmetic spectrum.*

Proof. For a general semisimple, connected, real linear Lie group, Benoist proves in [**Ben00**, Proposition] that when Γ is a Zariski dense semigroup of G, the additive group generated by the full length spectrum $\lambda(\Gamma)$ is dense in $\mathfrak{a}$. Thus, this Proposition implies that Zariski dense semigroups containing only loxodromic elements have non-arithmetic length spectrum. In particular, strong (r, ε)-Schottky Zariski dense semigroups have non-arithmetic length spectrum. Finally, the existence of Zariski dense Schottky semigroups in Zariski dense subgroups of G follows from [**Ben97**, Proposition 4.3 for $\theta = \Pi$]. □

5.2. *A key proposition for mixing.* The following proposition is the technical point for proving the topological mixing of the Weyl chamber flow. Roughly, it shows that among elements of Γ which do not move too much a flat, (ie. $(\gamma_t^+, \gamma_t^-) \in \mathcal{U}^{(2)}$) for any given $x \in \mathfrak{a}$, we can find an element which send 0 to $x + \theta t$ for large t (ie. $\lambda(\gamma_t) \in B(x + t\theta, \eta)$)

PROPOSITION 5.4. *Fix $\theta \in \mathfrak{a}_1^{++}$ in the interior of the limit cone $\mathcal{C}(\Gamma)$. Then for every non-empty open subset $\mathcal{U}^{(2)} \subset L^{(2)}(\Gamma)$, for all $x \in \mathfrak{a}$ and $\eta > 0$ there exists $T > 0$ such that for all $t \geq T$ there exists a loxodromic element $\gamma_t \in \Gamma$ with*

$$\begin{cases} (\gamma_t^+, \gamma_t^-) \in \mathcal{U}^{(2)}, \\ \lambda(\gamma_t) \in B(x + t\theta, \eta). \end{cases} \tag{12}$$

We will need the following classical density lemma; see, for example, [**Ben00**, Lemma 6.2].

LEMMA 5.5. *Let V be a real vector space of finite dimension. Let $l_0, l_1, \ldots, l_t$ be vectors of V and $\eta > 0$. Set*

$$L := \sum_{0 \leq i \leq t} \mathbb{R}_+ l_i, \quad M := \sum_{0 \leq i \leq t} \mathbb{Z} l_i, \quad \textit{and} \quad M_+ := \sum_{0 \leq i \leq t} \mathbb{N} l_i.$$

Assume that M is η-dense in V. Then there exists $v_0 \in V$ such that M_+ is η-dense in $v_0 + L$.

Remark that if M_+ is η-dense in $v_0 + L$ then it is η-dense in $v + L$ for every $v \in v_0 + L$. The following lemma is a consequence of [**Ben97**, Proposition 4.3].

LEMMA 5.6. *For all θ in the interior of the limit cone $\mathcal{C}(\Gamma)$, there exist a finite set $S \subset \Gamma$, a positive number $\rho > 0$ and $\varepsilon_n \underset{+\infty}{\to} 0$ such that:*

(i) *θ is in the interior of the convex cone $L(S) := \sum_{\gamma \in S} \mathbb{R}_+ \lambda(\gamma)$;*

(ii) *the elements of $\lambda(S)$ form a basis of $\mathfrak{a}$;*

(iii) *for all $n \geq 1$, the family $S_n := (\gamma^n)_{\gamma \in S}$ spans a Zariski-dense strong (ρ, ε_n)-Schottky semigroup of Γ.*

Proof. Fix θ in the interior of $\mathcal{C}(\Gamma)$.

Let us now construct a family of r_G open cones in the limit cone $\mathcal{C}(\Gamma)$. We consider an affine chart of $\mathbb{P}(\mathfrak{a})$ centered in $\mathbb{R}\theta$. Since $\mathbb{R}\theta$ is in the open set $\mathbb{P}(\mathring{\mathcal{C}}(\Gamma))$, it admits an open, polygonal, convex neighborhood with r_G distinct vertices centered in $\mathbb{R}\theta$ and included in $\mathbb{P}(\mathring{\mathcal{C}}(\Gamma))$. We denote by $p := (\mathbb{R}p_i)_{1 \leq i \leq r_G}$ the family of vertices of that convex neighborhood, and $\mathscr{H}_p$ its convex hull. Without loss of generality we can assume that there exists $\delta_0 > 0$ such that the δ_0-neighborhood of $\mathscr{H}_p$, $\mathcal{V}_{\delta_0}(\mathscr{H}_p)$ is included in $\mathbb{P}(\mathring{\mathcal{C}}(\Gamma))$.

For any $\delta > 0$, we denote by $\mathcal{V}_\delta(\partial \mathscr{H}_p)$ the δ-neighborhood of the boundary $\partial \mathscr{H}_p$. Choose $0 < \delta \leq \inf(\delta_0, \frac{1}{3} d(\mathbb{R}\theta, \partial \mathscr{H}_p))$ so that $\mathbb{R}\theta \in \mathscr{H}_p \setminus \mathcal{V}_\delta(\mathscr{H}_p)$.

Denote by $L_p \subset \mathring{\mathcal{C}}(\Gamma)$ (respectively, $\mathcal{V}_\delta(\partial L_p)$) the closed (respectively, open) cone whose projective image is $\mathscr{H}_p$ (respectively, $\mathcal{V}_\delta(\partial \mathscr{H}_p)$). For all $1 \leq i \leq r_G$, denote by $(\Omega_i)_{1 \leq i \leq r_G}$ the family of open cones such that $\mathbb{P}(\Omega_i) := B_{\mathbb{P}(\mathfrak{a})}(p_i, \delta)$.

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By [**Ben97**, Proposition 4.3] applied to the finite family of disjoint open cones $(\Omega_i)_{1\leq i\leq r_G}$ there exist $0<\varepsilon_0\leq\rho$ and a generating set $S:=\{\gamma_i\}_{1\leq i\leq r_G}\subset\Gamma$ of a Zariski dense (ρ,ε)-Schottky semigroup such that for all $1\leq i\leq r_G$ the Jordan projection $\lambda(\gamma_i)$ is in Ω_i. By Lemma 3.3, for any $n\geq 1$, the elements of S_n are (ρ,ε_n)-loxodromic. Thus, for n large, condition (iii) holds. By construction, $\lambda(S)$ form a family of r_G linearly independent vectors of $\mathfrak{a}$, hence (ii) holds. Set $L(S):=\sum_{\gamma\in S}\mathbb{R}_+\lambda(\gamma)$. The construction of L_p and $\mathcal{V}_\delta(\partial L_p)$ implies that $\theta\in L_p\setminus\mathcal{V}_\delta(\partial L_p)$. Since $\lambda(\gamma_i)\in\Omega_i\subset\mathcal{V}_\delta(\partial L_p)$ for all $1\leq i\leq r_G$, the boundary of the cone $\partial L(S)\subset\mathcal{V}_\delta(\partial L_p)$. Hence $L_p\setminus\mathcal{V}_\delta(\partial L_p)\subset\mathring{L}(S)$ and, finally, condition (i) holds: θ is in the interior of the cone $L(S)$. □

Let us give a proof of the key proposition.

Proof of Proposition 5.4. We fix a point θ in the interior of $\mathcal{C}(\Gamma)$, an open, non-empty set $\mathcal{U}=\mathcal{U}^+\times\mathcal{U}^-\subset L^{(2)}(\Gamma)$, a point $x\in\mathfrak{a}$ and $\eta>0$.

Consider S and $\rho>0$ as in Lemma 5.6. Denote by Γ_n the semigroup spanned by S^n.

By Lemma 3.7, one can pick $h\in\Gamma^{\mathrm{lox}}$ such that $(h^+,h^-)\in\mathcal{U}^{(2)}\setminus(\gamma_1^-,\gamma_{r_G}^+)$. Choose $r>0$ so that

$$r\leq\inf(\rho,\tfrac{1}{6}d(h^+,h^-),\tfrac{1}{6}d(\gamma_{r_G}^+,h^-),\tfrac{1}{6}d(h^+,\gamma_1^-)).$$

In particular, Proposition 3.17 holds for elements of the form $h\gamma_{r_G}^{n_{r_G}}g\gamma_1^{n_1}h$, where $g\in\Gamma_n$.

Choose $0<\varepsilon\leq r$ small enough so that

$$\begin{cases}(3r_G+2)C_{r,\varepsilon}\leq\eta/2,\\ B(h^+,\varepsilon)\times B(h^-,\varepsilon)\subset\mathcal{U}^{(2)},\end{cases}\tag{13}$$

where $(C_{r,\varepsilon})_{\varepsilon\geq 0}$ are constants given by the proposition.

We use Lemma 3.3 and choose n large so that h^n, S^n are (r,ε_n)-loxodromic elements with $\varepsilon_n\leq\varepsilon$.

By Proposition 5.3, the subgroup generated by $\lambda(\Gamma_n)$ is dense in $\mathfrak{a}$. By Lemma A.1 applied to $\lambda(\Gamma_n)$, there exists a finite subset $F\subset\Gamma_n$ containing at most $2r_G$ elements so that $\lambda(S^n)\cup\lambda(F)$ spans an $\eta/2$-dense subgroup of $\mathfrak{a}$. We denote by l the number of elements in $S':=S^n\cup F$ and we enumerate the elements of $S^n\cup F$ by $(g_1,\ldots,g_l)$, where $g_1:=\gamma_1^n$ and $g_l:=\gamma_{r_G}^n$. A crucial fact is that $l\leq 3r_G$ is bounded independently of $\lambda(\Gamma_n)$.

The additive subgroup generated by $\lambda(S')$ is $\eta/2$-dense in $\mathfrak{a}$. Furthermore, θ is still in the interior of the convex cone $L(S'):=\sum_{g\in S'}\mathbb{R}_+\lambda(g)$ by (i). Lemma 5.5 gives the existence of $v_0\in\mathfrak{a}$ such that $M_+(S'):=\sum_{g\in S'}\mathbb{N}\lambda(g)$ is $\eta/2$- dense in $v_0+L(S')$.

The interior of $L(S')$ contains θ. Hence for any $v\in\mathfrak{a}$, the intersection $(v+\mathbb{R}_+\theta)\cap(v_0+L(S'))$ is a half line.

Consider such a half line $x-\nu(h^n,g_l,\ldots,g_1,h^n)-2\lambda(h^n)+\theta[T,+\infty)$ contained in $v_0+L(S')$, for some $T\in\mathbb{R}$. For all $t\geq T$, there exists $n_t:=(n_t(1),\ldots,n_t(l))\in\mathbb{N}^l$ such that

$$\left\|\sum_{i=1}^{l}n_t(i)\lambda(g_i)-x+\nu(h^n,g_l,\ldots,g_1,h^n)+2\lambda(h^n)-\theta t\right\|\leq\eta/2.\tag{14}$$

Furthermore, Proposition 3.17 applied to $\gamma_t := h^n g_l^{n_t(l)} \ldots g_1^{n_t(l)} h^n$ gives

$$\left\| \lambda(\gamma_t) - \sum_{i=1}^{l} n_t(i)\lambda(g_i) - 2\lambda(h^n) - \nu(h^n, g_l, \ldots, g_1, h^n) \right\| \leq (l+2)C_{r,\varepsilon} \tag{15}$$

and $(\gamma_t^+, \gamma_t^-) \in B(h^+, \varepsilon) \times B(h^-, \varepsilon) \subset \mathcal{U}^{(2)}$ by (13).

Finally, we have $(3r_G + 2)C_{r,\varepsilon} \leq \eta/2$ by the choice of n, S^n, h^n. Once again, we remark that it is necessary for l to be bounded independently of Γ and n. We get the following bound using the triangle inequality:

$$\|\lambda(\gamma_t) - x - \theta t\| \leq \eta. \tag{16}$$

This concludes the proof. □

Prasad and Rapinchuk [**PR05**, Theorem 3] prove that Schanuel's conjecture in transcendental number theory implies that every Zariski dense semigroup of G contains a finite subset F such that $\lambda(F)$ generates a dense subgroup of $\mathfrak{a}$. Assuming that conjecture, we can remove our Lemma A.1 and simplify our proof as follows. Start by following our proof, choosing $S \subset \Gamma$ and $\rho > 0$ as in Lemma 5.6. Now use Prasad and Rapinchuk's density theorem: there is a finite subset F of the semigroup generated by S such that $\langle \lambda(F) \rangle$ is dense in $\mathfrak{a}$. Remark that for any $n \in \mathbb{N}$, the subset $S_n'' := F^n \cup S^n$ is finite, has at most $|F| + r_G$ elements and the subgroup generated by $\lambda(S_n'')$ is also dense in $\mathfrak{a}$. It suffices then to follow the end of the proof by taking $S' = S_n''$ for n large enough so that S' is a (r, ε_n)-Schottky semigroup with $(|F| + r_G + 2)C_{r,\varepsilon_n} \leq \eta/2$.

5.3. *Proof of Theorem 1.2.* We end the proof of the main theorem with Proposition 5.4 and Theorem 4.5.

If $(\Omega(X), \phi_t^\theta)$ is topologically mixing, it is in particular topologically transitive. Therefore, by Proposition 4.7, if $(\Omega(X), \phi_t^\theta)$ is topologically mixing, θ is in the interior of the limit cone.

Let us prove that if $\theta \in \overset{\circ}{\mathcal{C}}(\Gamma) \cap \mathfrak{a}_1^{++}$ then $(\Omega(X), \phi_t^\theta)$ is topologically mixing.

Let $\widetilde{U}$, $\widetilde{V}$ be two open subsets of $\widetilde{\Omega}(X)$. Without loss of generality, we can assume that $\widetilde{U} = \mathcal{H}^{-1}(\mathcal{U}^{(2)} \times B(u, r))$ (respectively, $\widetilde{V} = \mathcal{H}^{-1}(\mathcal{V}^{(2)} \times B(v, r))$), where $\mathcal{U}^{(2)}$ and $\mathcal{V}^{(2)}$ are open subsets of $L^{(2)}(\Gamma)$, and $B(u, r)$, $B(v, r)$ open balls of $\mathfrak{a}$.

Recall that for all $g \in \Gamma$, using Hopf coordinates

$$\begin{cases} \mathcal{H}^{(2)}(g(\mathcal{U}^{(2)}) \times B(u, r)) = g\mathcal{U}^{(2)}, \\ \mathcal{H}(\phi_t^\theta(\mathcal{V}^{(2)}) \times B(v, r))) = \mathcal{V}^{(2)} \times B(v + \theta t, r). \end{cases} \tag{17}$$

We begin by transforming the coordinates in $L^{(2)}(\Gamma)$ to recover the setting of Proposition 5.4. By Theorem 4.5, there exists $g \in \Gamma$ such that $g\mathcal{U}^{(2)} \cap \mathcal{V}^{(2)} \neq \emptyset$. For such an element $g \in \Gamma$, the subset $g\mathcal{U}^{(2)} \cap \mathcal{V}^{(2)}$ is a non-empty open subset of $L^{(2)}(\Gamma)$. Let $\mathcal{O}^{(2)} := \mathcal{O}_+ \times \mathcal{O}_- \subset g\mathcal{U}^{(2)} \cap \mathcal{V}^{(2)}$ be a non-empty open subset, such that $r := d(\overline{\mathcal{O}_+}, \overline{\mathcal{O}_-}) > 0$.

Remark that $g\widetilde{U} \cap (\mathcal{H}^{(2)})^{-1}(\mathcal{O}^{(2)})$ is open and non-empty. Thus it contains an open box $\mathcal{H}^{-1}(\mathcal{O}^{(2)} \times B(u', r'))$ with $u' \in \mathfrak{a}$ and $r' > 0$. Set $\eta := \min(r, r')$.

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By Proposition 5.4 applied to $\mathcal{O}^{(2)}$, $x = v - u' \in \mathfrak{a}$ and $\eta > 0$, there exists $T > 0$ such that for all $t \geq T$ there exists $\gamma_t \in \Gamma$ with

$$\begin{cases} (\gamma_t^+, \gamma_t^-) \in \mathcal{O}^{(2)}, \\ \lambda(\gamma_t) \in B(v - u' + t\theta, \eta). \end{cases} \tag{18}$$

Remark that every loxodromic element $\gamma \in \Gamma$ fixes its limit points in $L^{(2)}(\Gamma)$. Thus for all such $\gamma \in \Gamma$ with $(\gamma^+, \gamma^-) \in \mathcal{O}^{(2)}$, the subset $\gamma\mathcal{O}^{(2)} \cap \mathcal{O}^{(2)}$ is open and non-empty (it contains (γ^+, γ^-)). Furthermore, $\lambda(\gamma) = \sigma(\gamma, \gamma^+)$ by Fact 2.6. Hence

$$\begin{cases} \gamma_t\mathcal{O}^{(2)} \cap \mathcal{O}^{(2)} \neq \emptyset, \\ u' + \sigma(\gamma_t, \gamma_t^+) \in B(v + t\theta, \eta). \end{cases} \tag{19}$$

The subset $\gamma_t g\tilde{U} \cap (\mathcal{H}^{(2)})^{-1}(\gamma_t\mathcal{O}^{(2)} \cap \mathcal{O}^{(2)})$ is open, non-empty and contains the point of coordinates $(\gamma_t^+, \gamma_t^-, u' + \sigma(\gamma_t, \gamma_t^+)) \in \mathcal{H}^{-1}(\phi_t^\theta(\tilde{V}))$. Finally, $\gamma_t g\tilde{U} \cap \phi_t^\theta(\tilde{V}) \neq \emptyset$; as $\tilde{U}$, $\tilde{V}$ are arbitrary, it proves that ϕ_t^θ is topological mixing.

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A. *Appendix. A density lemma*

The following density lemma is crucial for the proof of Proposition 5.4.

LEMMA A.1. *Let $d \in \mathbb{N}$, and let V be a real vector space of dimension d. For all $E \subset V$ spanning a dense additive subgroup of V, for all $\epsilon > 0$, for any basis $B \subset E$ of V, there exists a finite subset $F \subset E$ of at most $2d$ elements such that $B \cup F$ spans a ϵ-dense additive subgroup of V.*

Proof. We show the lemma by induction.

Let $\mathcal{E} \subset \mathbb{R}^1 = V$ be a subset that generates a dense additive subgroup of $\mathbb{R}$. Let $x \in \mathbb{R}$ be a basis, that is, a non-zero element. Any element y in $\mathcal{E}$ such that $\langle y, x\rangle$ is dense is a solution. We assume that $\mathcal{E}$ contains no such element. Consider the quotient $\mathbb{R}/x\mathbb{Z}$ and the projection $p : \mathbb{R} \to \mathbb{R}/x\mathbb{Z}$. The set $\mathcal{E}$ projects to an infinite subset of $\mathbb{R}/x\mathbb{Z}$, therefore it has an accumulation point. Let $f_1 \neq f_2 \in \mathcal{E}$ be two elements such that $|p(f_1) - p(f_2)| < \epsilon$. Then $\langle x, f_1, f_2\rangle$ generates an ϵ-dense additive subgroup of $\mathbb{R}$, and the lemma is proved for $dim(V) = 1$, where $F = \{f_1, f_2\}$.

Now consider a vector space V of dimension d. Let $\mathcal{E}$ be a subset of V such that $\overline{\langle\mathcal{E}\rangle} = V$ and $\mathcal{B} = (b_1, \ldots, b_d) \subset E$, a basis of V. Without loss of generality we suppose that the basis is the standard basis and the norm is the sup norm: these only affect computations up to a multiplicative constant.

Suppose that we have $f_1, f_2 \in \mathcal{E}$ such that the additive group $\langle f_1, f_2, \mathcal{B}\rangle$ contains a non-zero vector u of norm $\|u\| \leq \epsilon/2$. We will show that it is enough to conclude and then prove the existence of such elements.

Consider $V' = u^\perp$, the decomposition $V = u \oplus V'$ and p' the projection on V'. Let $\mathcal{E}' = p'(\mathcal{E})$ and $\mathcal{B}'$ a basis of V' included in $p'(\mathcal{B})$. By induction, there is a finite subset

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$\mathcal{F}' \subset \mathcal{E}'$ of at most $2(d-1)$ elements such that $\langle \mathcal{F}', \mathcal{B}' \rangle$ generates an $\epsilon/2$-dense additive subgroup of V'. For all $f' \in \mathcal{F}'$ there exist $f \in \mathcal{E}$ and $\lambda_f \in \mathbb{R}$ such that $f' = f + \lambda_f u$. A similar result holds for elements of $\mathcal{B}'$. We denote by $\mathcal{F} \subset \mathcal{E}$ a choice of lifts for elements of $\mathcal{F}'$. We claim that the set $F = \mathcal{F} \cup \{f_1, f_2\} \cup \mathcal{B}$ generates an ϵ-dense additive subgroup of V.

Let $x \in V$, $x = x' + \lambda_x u$. By hypothesis, there exist $(n_{f'})_{f' \in \mathcal{F}'} \in \mathbb{Z}^{|\mathcal{F}'|}$, $(n_{b'})_{b' \mathcal{B}'} \in \mathbb{Z}^{d-1}$ and $\alpha \in V'$ satisfying $\|\alpha'\| < \epsilon/2$ such that

$$x' = \sum_{f' \in \mathcal{F}'} n_{f'} f' + \sum_{b' \in \mathcal{B}'} n_{b'} b' + \alpha'.$$

Therefore,

$$x' = \sum_{f \in \mathcal{F}} n_{f'} f + \sum_{b \in \mathcal{B}} n_b b + \bigg(\sum_{f \in \mathcal{F}} n_{f'} \lambda_f + \sum_{b \in \mathcal{B}} n_b \lambda_b \bigg) u + \alpha'.$$

Finally, we get

$$x = \sum_{f \in \mathcal{F}} n_{f'} f + \sum_{b \in \mathcal{B}} n_b b + [k]u + (k - [k])u + \alpha',$$

where $k = \sum_{f \in \mathcal{F}} n_{f'} \lambda_f + \sum_{b \in \mathcal{B}} n_b + \lambda_x$ and $[k] \in \mathbb{Z}$ denotes the integer part of k. The vector $\sum_{f \in \mathcal{F}} n_{f'} f + \sum_{b \in \mathcal{B}} n_b b + [k]u$ is in the additive group generated by F and $|(k - [k])u + \alpha| \leq \epsilon$. This proves the claim.

To finish the proof we need to show that for any $\epsilon > 0$, there are elements $f_1, f_2 \in \mathcal{E}$ such that $\langle f_1, f_2, \mathcal{B} \rangle$ contains a non-zero vector of norm less than ϵ.

Consider the natural projection $p : \mathbb{R}^d \to \mathbb{R}^d / \bigoplus_{k=1}^d \mathbb{Z} b_k$ into the torus $\mathbb{R}^d / \bigoplus_{k=1}^d \mathbb{Z} b_k$. If there is an element $f \in \mathcal{E}$ such that $p(\mathbb{Z} f)$ contains accumulation points, we choose u, non-zero and small in $\langle \mathcal{B}, f \rangle$. We assume now that there is no such element in E. Choose an integer N such that $N > 2\sqrt{d}/\varepsilon$. By the pigeonhole principle on $N^d + 1$ distinct elements of $\mathcal{E}$, we deduce the existence of $f_1, f_2 \in \mathcal{E}$ with $0 < |p(f_1 - f_2)| < \epsilon/2$. The unique representative of the projection $p(f_1 - f_2)$ in the fundamental domain $\sum_{i=1}^d (0, 1] b_i$ is a suitable choice for u. Indeed, it is an element of the subgroup $\langle f_1, f_2, \mathcal{B} \rangle$ and it is of norm at most $\epsilon/2$. $\square$

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