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DEGREES OF CATEGORICITY ON A CONE VIA η-SYSTEMS

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Abstract. We investigate the complexity of isomorphisms of computable structures on cones in the Turing degrees. We show that, on a cone, every structure has a strong degree of categoricity, and that degree of categoricity is Δ^0_α-complete for some α. To prove this, we extend Montalbán's η-system framework to deal with limit ordinals in a more general way. We also show that, for any fixed computable structure, there is an ordinal α and a cone in the Turing degrees such that the exact complexity of computing an isomorphism between the given structure and another copy $\mathcal{B}$ in the cone is a c.e. degree in $\Delta^0_\alpha(\mathcal{B})$. In each of our theorems the cone in question is clearly described in the beginning of the proof, so it is easy to see how the theorems can be viewed as general theorems with certain effectiveness conditions.

§1. Introduction. In this paper, we will consider the complexity of computing isomorphisms between computable copies of a structure after relativizing to a cone. By relativizing to a cone, we are able to consider *natural* structures, that is, those structures which one might expect to encounter in normal mathematical practice. The main result of this paper is a complete classification of the natural degrees of categoricity: the degrees of categoricity of natural computable structures. Unless otherwise stated, all notation and conventions will be as in the book by Ash and Knight [6]. We consider countable structures over at most countable languages.

Recall that a computable structure is said to be computably categorical if any two computable copies of the structure are computably isomorphic. As an example, consider the rationals as a linear order; the standard back-and-forth argument shows that the rationals are computably categorical. It is easy to see, however, that not all computable structures are computably categorical. The natural numbers as a linear order is one example.

There has been much work in computable structure theory dedicated to characterizing computable categoricity for various classes of structures (e.g., a linear order is computably categorical if and only if it has at most finitely many successivities [15, 26]). For those structures that are not computably categorical, what can we say about the isomorphisms between computable copies, or more generally, about the complexities of the isomorphisms relative to that of the structure?

We can extend the definition of computable categoricity as follows:

DEFINITION 1.1. A computable structure $\mathcal{A}$ is **d**-*computably categorical* if for all computable $\mathcal{B} \cong \mathcal{A}$ there exists a **d**-computable isomorphism between $\mathcal{A}$ and $\mathcal{B}$.

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It is easy to see, for example, that the natural numbers as a linear order, $\mathcal{N}$, is $\mathbf{0}'$-computably categorical. Indeed, it is also easy to construct a computable copy $\mathcal{A}$ of $\mathcal{N}$ such that every isomorphism between $\mathcal{A}$ and $\mathcal{N}$ computes $\mathbf{0}'$. Thus $\mathbf{0}'$ is the least degree $\mathbf{d}$ such that $\mathcal{N}$ is $\mathbf{d}$-computably categorical. This motivates the following definitions.

Definition 1.2. We say a computable structure $\mathcal{A}$ has *degree of categoricity* $\mathbf{d}$ if

(1) $\mathcal{A}$ is $\mathbf{d}$-computably categorical.
(2) If $\mathcal{A}$ is $\mathbf{c}$-computably categorical, then $\mathbf{c} \geq \mathbf{d}$.

Definition 1.3. We say that a Turing degree $\mathbf{d}$ is a *degree of categoricity* if there exists a computable structure $\mathcal{A}$ with degree of categoricity $\mathbf{d}$.

The notion of a degree of categoricity was first introduced by Fokina, Kalimullin, and R. Miller [12]. They showed that if $\mathbf{d}$ is d.c.e. (difference of c.e.) in and above $\mathbf{0}^{(n)}$, then $\mathbf{d}$ is a degree of categoricity. They also showed that $\mathbf{0}^{(\omega)}$ is a degree of categoricity. For the degrees c.e. in and above $\mathbf{0}^{(n)}$, they exhibited rigid structures capturing the degrees of categoricity. In fact, all their examples had the following, stronger property.

Definition 1.4. A degree of categoricity $\mathbf{d}$ is a *strong* degree of categoricity if there is a structure $\mathcal{A}$ with computable copies $\mathcal{A}_0$ and $\mathcal{A}_1$ such that $\mathbf{d}$ is the degree of categoricity for $\mathcal{A}$, and every isomorphism $f : \mathcal{A}_0 \to \mathcal{A}_1$ satisfies $\deg(f) \geq \mathbf{d}$.

In [8], Csima, Franklin, and Shore showed that for every computable ordinal α, $\mathbf{0}^{(\alpha)}$ is a strong degree of categoricity. They also showed that if α is a computable successor ordinal and $\mathbf{d}$ is d.c.e. in and above $\mathbf{0}^{(\alpha)}$, then $\mathbf{d}$ is a strong degree of categoricity.

In [12] it was shown that all strong degrees of categoricity are hyperarithmetical, and in [8] it was shown that all degrees of categoricity are hyperarithmetical. There are currently no examples of degrees of categoricity that are not strong degrees of categoricity. Indeed, we do not even have an example of a structure that has a degree of categoricity but not strongly.

All known degrees of categoricity satisfy $\mathbf{0}^{(\alpha)} \leq \mathbf{d} \leq \mathbf{0}^{(\alpha+1)}$ for some computable ordinal α. So in particular, all known noncomputable degrees of categoricity are hyperimmune. In [1], Anderson and Csima showed that no noncomputable hyperimmunefree degree is a degree of categoricity. They also showed that there is a Σ^0_2 degree that is not a degree of categoricity, and that if G is 2-generic (relative to a perfect tree), then $\deg(G)$ is not a degree of categoricity. The question of whether there exist Δ^0_2 degrees that are not degrees of categoricity remains open.

Turning to look at the question of degree of categoricity for a given structure, R. Miller showed that there exists a field that does not have a degree of categoricity [23], and Fokina, Frolov, and Kalimullin [11] showed that there exists a rigid structure with no degree of categoricity.

In this paper, we claim that the only *natural* degrees of categoricity are the Δ^0_α-complete degrees for some computable ordinal α. By a natural degree of categoricity, we mean the degree of categoricity of a natural structure.

What do we mean by *natural*? By a natural structure, we mean one which might show up in the normal course of mathematics; we will not include a structure which has been constructed, say via diagonalization, to have some computability-theoretic property as a natural structure. So, for example, we will not consider a structure which is computably categorical but not relatively computably categorical to be a natural structure. On the other hand, the infinite-dimensional vector space is a natural structure. Of course, this is not a rigorous definition. Instead, we note that arguments involving natural structures tend to relativize, and so a natural structure will have property P if and only if it has property P on a cone (i.e., there is a Turing degree $\mathbf{d}$ such that for all $\mathbf{c} \geq \mathbf{d}$, P holds relative to $\mathbf{c}$). Thus by considering arbitrary structures on a cone, we can prove results about natural structures.

The second author previously considered degree spectra of relations on a cone [18]. McCoy [22] has also shown that on a cone, every structure has computable dimension 1 or ω. Here, we give an analysis of degrees of categoricity along similar lines.

Our main theorem is:

THEOREM 1.5. *Let $\mathcal{A}$ be a countable structure. Then, on a cone: $\mathcal{A}$ has a strong degree of categoricity, and this degree of categoricity is Δ^0_α-complete.*

There are three important parts to this theorem: first, that every natural structure has a degree of categoricity; second, that this degree of categoricity is a strong degree of categoricity; and third, that the degree of categoricity is Δ^0_α-complete. The ordinal α is the least ordinal α such that $\mathcal{A}$ is Δ^0_α categorical on a cone. This is related to the Scott rank of $\mathcal{A}$ under an appropriate definition of Scott rank [25]: α is the least ordinal α such that $\mathcal{A}$ has a $\Sigma^{\mathrm{inf}}_{\alpha+2}$ Scott sentence if α. (While α may not be computable, every ordinal is computable on some cone. The reader may be uncomfortable with talking about Δ^0_α-complete degrees on a cone when α is not computable; precisely what we mean will be clarified in Section 2.)

The construction of a structure with degree of categoricity some d.c.e. (but not c.e.) degree uses a computable approximation to the d.c.e. degree; this requires the choice of a particular index for the approximation, and hence the argument that the resulting structure has degree of categoricity d.c.e. but not c.e. does not relativize. By our theorem, there is no possible construction which does relativize. Moreover, our theorem says something about what kinds of constructions would be required to solve the open problems about degrees of categoricity, for example whether there is a 3-c.e. but not d.c.e. degree of categoricity, or whether there is a degree of categoricity which is not a strong degree of categoricity—the proof must be by constructing a structure which is not natural, using a construction which does not relativize.

The proof of Theorem 1.5 also gives an effectiveness condition which, if it holds of some computable structure, means that the conclusion of the theorem is true of that structure without relativizing to a cone. See, for example, the definition of the degree $\mathbf{e}$ in Theorem 6.2. If $\mathcal{A}$ is a computable structure, α is a computable ordinal and is least such that $\mathcal{A}$ is Δ^0_α categorical, and one can take $\mathbf{e} = \mathbf{0}$ (which, in particular, means that it is effectively witnessed that α is the least ordinal such that $\mathcal{A}$ is Δ^0_α categorical), then $\mathcal{A}$ has strong degree of categoricity Δ^0_α.

COROLLARY 1.6. *The degrees of categoricity on a cone are the Δ^0_α-complete degrees for some α.*

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Indeed, each Δ^0_α-complete degree is a degree of categoricity on a cone. To see this, examine the proof of Theorem 3.1 of [8] showing that each Δ^0_α-complete degree is a degree of categoricity, and note that the proof relativizes.

In 2012, Csima, Kach, Kalimullin, and Montalbán worked out a proof of Theorem 1.5 in the case where $\mathcal{A}$ is Δ^0_2 categorical on a cone. That is, they showed that if $\mathcal{A}$ is Δ^0_2 categorical on a cone, but not computably categorical on a cone, then $\mathcal{A}$ has Δ^0_2-complete strong degree of categoricity on a cone. They also conjectured the general result at that time. The work was not written up. The result was later independently suggested by the second author. The proof of the general result requires not only the machinery of α-systems but also some new ideas. The proof of the special case is quite similar to a result of Harizanov [17, Theorem 2.5], who answered an analogous question for degree spectra of relations; the corresponding general case for degree spectra of relations is still open (though some more general results are proved in [18]). On the other hand, our proof of the general result for categoricity uses, in an integral way, certain facts about automorphisms (which were not used in the case of a Δ^0_2 categorical structure), and so our proof does not work for degree spectra. We discuss in Section 6 the new difficulties which arise in the general case.

The second result of this paper concerns the difficulty of computing isomorphisms between two given copies $\mathcal{A}$ and $\mathcal{B}$ of a structure. We show that, on a cone, there is an isomorphism of least degree between $\mathcal{A}$ and $\mathcal{B}$, and that it is of c.e. degree.

THEOREM 1.7. *Let $\mathcal{A}$ be a countable structure. Let α be such that $\mathcal{A}$ is Δ^0_α categorical on a cone. Then, on a cone: for every copy $\mathcal{B}$ of $\mathcal{A}$, there is a degree* $\mathbf{d}$ *that is $\Sigma^0_{\alpha-1}$ in $\mathcal{B}$ if α is a successor, or Δ^0_α in $\mathcal{B}$ if α is a limit, such that* $\mathbf{d}$ *computes an isomorphism between $\mathcal{A}$ and $\mathcal{B}$ and such that all isomorphisms between $\mathcal{A}$ and $\mathcal{B}$ compute* $\mathbf{d}$.

The degree $\mathbf{d}$ is the least degree of an isomorphism between $\mathcal{A}$ and $\mathcal{B}$.

We begin in Section 2 by giving the technical definitions for what we mean by "on a cone." In Section 3 we prove Theorem 1.7. In Section 4 we prove a stronger version of Theorem 1.5 in the restricted case of structures which are Δ^0_2 categorical on a cone; it will follow that the only possible degrees of categoricity on a cone for such structures are Δ^0_1-complete or Δ^0_2-complete. In order to prove the general case of Theorem 1.5, we need to use the method of α-systems. These were introduced by Ash, see [6]. Montalbán [24] introduced η-systems, which are similar to Ash's α-systems but give more control. They also deal with limit ordinals in a different way. We need the extra control of Montalbán's η-systems, but we need to deal with limit ordinals as in Ash's α-systems. So in Section 5 we introduce a modified version of Montalbán's η-systems. We conclude in Section 6 with a complete proof of Theorem 1.5.

§2. Relativizing to a cone.

A *cone of Turing degrees* is a set $C_{\mathbf{d}} = \{\mathbf{c} : \mathbf{c} \geq \mathbf{d}\}$. Martin [20] showed that under set-theoretic assumptions of determinacy, every set of Turing degrees either contains a cone or is disjoint from a cone. Noting that every countable intersection of cones contains a cone, we see that we can form a $\{0, 1\}$-valued measure on sets of degrees by assigning measure one to those sets which contain a cone. In this paper, all of the sets of degrees which we will consider arise from Borel sets, and by Borel determinacy [21], such sets either contain or are disjoint from a cone.

If P is a statement which relativizes to any degree, we say that P holds on a cone if there is a degree $\mathbf{d}$ (the base of the cone) such that for all $\mathbf{c} \geq \mathbf{d}$, P holds relative to $\mathbf{c}$. Thus a statement holds on a cone if and only if it holds almost everywhere relative to the Martin measure. In the rest of this section, we will relativize the definitions we are interested in.

Definition 2.1. The structure $\mathcal{A}$ is *computably categorical on the cone above* $\mathbf{d}$ if for all $\mathbf{c} \geq \mathbf{d}$, whenever $\mathcal{B}$ and $\mathcal{C}$ are $\mathbf{c}$-computable copies of $\mathcal{A}$, there exists a $\mathbf{c}$-computable isomorphism between $\mathcal{B}$ and $\mathcal{C}$. More generally, a structure is Δ^0_α *categorical on the cone above* $\mathbf{d}$ if for all $\mathbf{c} \geq \mathbf{d}$ whenever $\mathcal{B}$ and $\mathcal{C}$ are $\mathbf{c}$-computable copies of $\mathcal{A}$, there exists a $\Delta^0_\alpha(\mathbf{c})$-computable isomorphism between $\mathcal{B}$ and $\mathcal{C}$.

Note that even if α is not computable, there is a cone on which α is computable, and for $\mathbf{c}$ on this cone, $\Delta^0_\alpha(\mathbf{c})$ makes sense. In a similar way, we do not have to assume that the structure $\mathcal{A}$ is computable. If $\mathcal{A}$ is Δ^0_α-categorical on a cone, there is a degree $\mathbf{d}$ which computes $\mathcal{A}$ and α, and $\mathcal{A}$ is Δ^0_α-categorical on the cone above $\mathbf{d}$.

Recall that a computable structure $\mathcal{A}$ is relatively Δ^0_α categorical if for all $\mathcal{B} \cong \mathcal{A}$, some isomorphism from $\mathcal{A}$ onto $\mathcal{B}$ is $\Delta^0_\alpha(\mathcal{B})$, and that there exist structures that are Δ^0_α categorical but not relatively so [10, 14, 16]. If we were to modify the definition of relatively Δ^0_α categorical to be on a cone, it would be equivalent to Definition 2.1. That is, there is no difference between relatively Δ^0_α categorical on a cone and Δ^0_α categorical on a cone.

The notion of relatively Δ^0_α categoricity is intimately related to that of a Scott family.

Notation 2.2. *All formulas in this paper will be infinitary formulas, that is, formulas in* $L_{\omega_1\omega}$. *See Chapter* 6 *of* [6] *for background on infinitary formulas and computable infinitary formulas. We will denote by* $\Sigma^{\mathrm{inf}}_\alpha$ *the infinitary* Σ_α *formulas and by* $\Sigma^{\mathrm{c}}_\alpha$ *the computable* Σ_α *formulas.*

Definition 2.3. A *Scott family* for a structure $\mathcal{A}$ is a countable family Φ of formulas over a finite parameter such that

- for each $\bar{a} \in \mathcal{A}$, there exists $\varphi \in \Phi$ such that $\mathcal{A} \models \varphi(\bar{a})$,
- if $\varphi \in \Phi$, $\mathcal{A} \models \varphi(\bar{a})$, and $\mathcal{A} \models \varphi(\bar{b})$, then there is an automorphism of $\mathcal{A}$ taking $\bar{a}$ to $\bar{b}$.

It follows from work of Scott [27] (see [6]) that every countable structure has a Scott family consisting of $\Sigma^{\mathrm{inf}}_\alpha$ formulas for some countable ordinal α.

Theorem 2.4 (Ash-Knight-Manasse-Slaman [7] and Chisholm [9]). *A computable structure* $\mathcal{A}$ *is relatively* Δ^0_α *categorical if and only if it has a Scott family which is a c.e. set of* $\Sigma^{\mathrm{c}}_\alpha$ *formulas.*

Now we can see the power of working on a cone.

Remark 2.5. Let $\mathcal{A}$ be a countable structure. Then $\mathcal{A}$ has a Scott family consisting of $\Sigma^{\mathrm{inf}}_\alpha$ formulas for some countable ordinal α. Let $\mathbf{d}$ be such that $\mathcal{A}$ and α are $\mathbf{d}$-computable, and such that the Scott family for $\mathcal{A}$ is c.e. and consists of $\Sigma^{\mathrm{c}}_\alpha$ formulas relative to $\mathbf{d}$. Then $\mathcal{A}$ is Δ^0_α categorical on the cone above $\mathbf{d}$. That is, every countable structure is Δ^0_α categorical on a cone for some α.

There is also an analogue of Theorem 2.4 for (nonrelative) Δ^0_α categoricity. Historically, this came first; the $\alpha = 1$ case is due to Goncharov [13] and the general case is due to Ash [4].

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We now recall some definitions from [6].

DEFINITION 2.6 (Back-and-forth relations). For a structure $\mathcal{A}$ tuples $\bar{a}, \bar{b} \in \mathcal{A}$ of the same length

- $\bar{a} \leq_0 \bar{b}$ if and only if for every quantifier-free formula $\varphi(\bar{x})$ with Gödel number less than $length(\bar{a})$, if $\mathcal{A} \models \varphi(\bar{a})$ then $\mathcal{B} \models \varphi(\bar{b})$,
- for $\alpha > 0$, $\bar{a} \leq_\alpha \bar{b}$ if and only if, for each $\bar{d}$ in $\mathcal{A}$ and each $0 \leq \beta < \alpha$, there exists $\bar{c}$ in $\mathcal{A}$ such that $\bar{b}, \bar{d} \leq_\beta \bar{a}, \bar{c}$.

DEFINITION 2.7 (p. 269 Ash-Knight [6]). For tuples $\bar{c}$ and $\bar{a}$ in $\mathcal{A}$, we say that $\bar{a}$ is α-free over $\bar{c}$ if for any $\bar{a}_1$ and for any $\beta < \alpha$, there exist $\bar{a}'$ and $\bar{a}_1'$ such that $\bar{c}, \bar{a}, \bar{a}_1 \leq_\beta \bar{c}, \bar{a}', \bar{a}_1'$ and $\bar{c}, \bar{a}' \nleq_\alpha \bar{c}, \bar{a}$.

DEFINITION 2.8 (p. 241 Ash-Knight [6]). A structure $\mathcal{A}$ is *α-friendly* if for $\beta < \alpha$, the standard back-and-forth relations $\leq_\beta$ are c.e. uniformly in β.

There is a version of Theorem 2.4 for the nonrelative notion of categoricity. It comes in two parts:

PROPOSITION 2.9 (Proposition 17.6 from [6]). *Let $\mathcal{A}$ be a computable structure. Suppose $\mathcal{A}$ is α-friendly, with computable existential diagram. Suppose that there is a tuple $\bar{c}$ in $\mathcal{A}$ over which no tuple $\bar{a}$ is α-free. Then $\mathcal{A}$ has a formally Σ^0_α Scott family, with parameters $\bar{c}$.*

THEOREM 2.10 (Theorem 17.7 from [6]). *Let $\mathcal{A}$ be α-friendly. Suppose that for each tuple $\bar{c}$ in $\mathcal{A}$, we can find a tuple $\bar{a}$ that is α-free over $\bar{c}$. Finally, suppose that the relation $\nleq_\alpha$ is c.e. Then there is a computable $\mathcal{B} \cong \mathcal{A}$ with no Δ^0_α isomorphism from $\mathcal{A}$ to $\mathcal{B}$.*

COROLLARY 2.11. *Suppose that $\mathcal{A}$ is not Δ^0_α categorical on any cone. Then for any $\bar{c}$ in $\mathcal{A}$, there is some $\bar{a} \in \mathcal{A}$ that is α-free over $\bar{c}$.*

We now give the definitions needed to discuss degrees of categoricity on a cone.

DEFINITION 2.12. The structure $\mathcal{A}$ has *degree of categoricity* $\mathbf{d}$ *relative to* $\mathbf{c}$ if $\mathbf{d}$ can compute an isomorphism between any two $\mathbf{c}$-computable copies of $\mathcal{A}$, and moreover $\mathbf{d} \geq \mathbf{c}$ is the least degree with this property. If in addition to this there exist two $\mathbf{c}$-computable copies of $\mathcal{A}$ such that for every isomorphism f between them, $f \oplus \mathbf{c} \geq_T \mathbf{d}$, then we say $\mathcal{A}$ has *strong degree of categoricity* $\mathbf{d}$ *relative to* $\mathbf{c}$.

DEFINITION 2.13. We say that a structure $\mathcal{A}$ has a (*strong*) *degree of categoricity on a cone*, if there is some $\mathbf{d}$ such that for every $\mathbf{c} \geq \mathbf{d}$, $\mathcal{A}$ has a (strong) degree of categoricity relative to $\mathbf{c}$.

DEFINITION 2.14. We say that a structure $\mathcal{A}$ has Δ^0_α-complete (*strong*) *degree of categoricity on a cone*, if there is some $\mathbf{d}$ such that for every $\mathbf{c} \geq \mathbf{d}$, $\mathcal{A}$ has Δ^0_α-complete (strong) degree of categoricity relative to $\mathbf{c}$.

§3. Isomorphism of c.e. degree.

Theorem 1.7 follows from the following more technical statement.

THEOREM 3.1. *Let $\mathcal{A}$ be a structure. Suppose that $\mathcal{A}$ is Δ^0_α categorical on a cone. Then there is a degree* $\mathbf{c}$ *such that for every copy $\mathcal{B}$ of $\mathcal{A}$, there is a degree* $\mathbf{d}$ *that is*

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$\Sigma^0_{\alpha-1}$ *in and above* $\mathcal{B} \oplus \mathbf{c}$ *if* α *is a successor ordinal, or* Δ^0_α *in and above* $\mathcal{B} \oplus \mathbf{c}$ *if* α *is a limit ordinal, such that*

(1) $\mathbf{d}$ *computes some isomorphism between* $\mathcal{A}$ *and* $\mathcal{B}$ *and*
(2) *for every isomorphism* f *between* $\mathcal{A}$ *and* $\mathcal{B}$, $f \oplus \mathbf{c} \geq_T \mathbf{d}$.

Before giving the proof, we consider two motivating examples.

EXAMPLE 3.2. Let $\mathcal{N}$ be the standard presentation of $(\omega, <)$. If $\mathcal{A}$ is any other presentation, let $\mathrm{Succ}(\mathcal{A})$ be the successor relation in $\mathcal{A}$. Then the unique isomorphism between $\mathcal{N}$ and $\mathcal{A}$ has the same Turing degree as $\mathrm{Succ}(\mathcal{A})$. Note that $\mathrm{Succ}(\mathcal{A})$ is Π^0_1.

EXAMPLE 3.3. Let $\mathcal{V}$ be an infinite-dimensional $\mathbb{Q}$-vector space with a computable basis. If $\mathcal{W}$ is any other presentation of $\mathcal{V}$, let $\mathrm{Indep}(\mathcal{W})$ be the independence relation in $\mathcal{W}$, as a subset of $\mathcal{W}^{<\omega}$. Then any isomorphism between $\mathcal{V}$ and $\mathcal{W}$ computes $\mathrm{Indep}(\mathcal{W})$, and $\mathrm{Indep}(\mathcal{W})$ computes a basis for $\mathcal{W}$ and hence an isomorphism between $\mathcal{V}$ and $\mathcal{W}$. Note that $\mathrm{Indep}(\mathcal{W})$ is Π^0_1.

Theorem 1.7 says that this is the general situation for natural structures.

PROOF OF THEOREM 3.1. Let $\mathbf{c}$ be a degree such that $\mathcal{A}$ is $\mathbf{c}$-computable and Δ^0_α-categorical on the cone above $\mathbf{c}$. By increasing $\mathbf{c}$ to absorb the effectiveness conditions of Proposition 2.9 and Theorem 2.10, $\mathcal{A}$ has a c.e. Scott family S consisting of $\Sigma^{\mathtt{c}}_\alpha$ formulas relative to $\mathbf{c}$. Increasing $\mathbf{c}$, we may assume that S consists of formulas of the form $(\exists \bar{x})\varphi$ where φ is $\Pi^{\mathtt{c}}_\beta$ relative to $\mathbf{c}$ for some $\beta < \alpha$. Further increasing $\mathbf{c}$, we may assume that $\mathbf{c}$ can decide whether two formulas from S are satisfied by the same elements. Then we can replace S by a Scott family in which every tuple from $\mathcal{A}$ satisfies a unique formula from S. Finally, by replacing $\mathbf{c}$ with a higher degree, we may assume that $\mathbf{c}$ can compute, for an element of $\mathcal{A}$, the unique formula of S which it satisfies, and can decide, for each tuple of the appropriate arity, whether or not it is a witness to the existential quantifier in that formula. This is the degree $\mathbf{c}$ from the statement of the theorem.

Let $\mathcal{B}$ be a copy of $\mathcal{A}$. Consider the set

$$S(\mathcal{B}) = \{(\bar{b}, \varphi) : \mathcal{B} \models \varphi(\bar{b}), \varphi \in S\}.$$

Let $\mathbf{d}$ be the degree of $S(\mathcal{B}) \oplus \mathcal{B} \oplus \mathbf{c}$. First, note that the set

$$S(\mathcal{A}) = \{(\bar{a}, \varphi) : \mathcal{A} \models \varphi(\bar{a}), \varphi \in S\}$$

is $\mathbf{c}$-computable. If f is an isomorphism between $\mathcal{A}$ and $\mathcal{B}$, then $f \oplus \mathbf{c}$ computes $S(\mathcal{A})$. Then using f and $S(\mathcal{A})$, we can compute $S(\mathcal{B})$. Thus

$$f \oplus \mathbf{c} \geq_T S(\mathcal{B}) \oplus \mathcal{B} \oplus \mathbf{c} \equiv_T \mathbf{d}$$

for every isomorphism f between $\mathcal{A}$ and $\mathcal{B}$.

On the other hand, $\mathbf{c}$ computes $S(\mathcal{A})$. Using $S(\mathcal{B})$ and $S(\mathcal{A})$ we can compute an isomorphism between $\mathcal{A}$ and $\mathcal{B}$. So there is an isomorphism f between $\mathcal{A}$ and $\mathcal{B}$ such that

$$f \oplus \mathbf{c} \equiv_T S(\mathcal{B}) \oplus \mathcal{B} \oplus \mathbf{c} \equiv_T \mathbf{d}.$$

We now introduce a related set $T(\mathcal{B})$. We will show that $T(\mathcal{B}) \oplus \mathcal{B} \oplus \mathbf{c} \equiv_T \mathbf{d}$. If α is a successor ordinal, then $T(\mathcal{B})$ will be $\Pi^0_{\alpha-1}$ in $\mathcal{B} \oplus \mathbf{c}$, and if α is a limit ordinal then $T(\mathcal{B})$ will be Δ^0_α in $\mathcal{B} \oplus \mathbf{c}$. Thus $\mathbf{d}$ will be a degree of the appropriate type.

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We may consider the elements of $\mathcal{B}$ to be ordered, and hence order tuples from $\mathcal{B}$ via the lexicographic order. Let $T(\mathcal{B})$ be the set of tuples $(\bar{a}, \bar{b}, \varphi)$ where:

(1) $\varphi(\bar{x}, \bar{y})$ is a **c**-computable Π^0_β formula, for some $\beta < \alpha$,
(2) $(\exists \bar{y})\varphi(\bar{x}, \bar{y})$ is in S, and
(3) $\mathcal{B} \models \varphi(\bar{a}, \bar{c})$, for some $\bar{c} \leq \bar{b}$ in the lexicographical ordering of tuples from $\mathcal{B}$.

It is easy to see that if α is a successor ordinal, then $T(\mathcal{B})$ is $\Pi^0_{\alpha-1}$ in $\mathcal{B} \oplus \mathbf{c}$, and if α is a limit ordinal then $T(\mathcal{B})$ is Δ^0_α in $\mathcal{B} \oplus \mathbf{c}$. Now we will argue that $T(\mathcal{B}) \oplus \mathcal{B} \oplus \mathbf{c} \equiv_T S(\mathcal{B}) \oplus \mathcal{B} \oplus \mathbf{c}$.

Suppose we want to check whether $(\bar{a}, \bar{b}, \varphi) \in T(\mathcal{B})$ using $S(\mathcal{B}) \oplus \mathcal{B} \oplus \mathbf{c}$. Using **c**, we first compute whether (1) and (2) hold for φ. Then using $S(\mathcal{B}) \oplus \mathcal{B} \oplus \mathbf{c}$ we can compute an isomorphism $f : \mathcal{B} \to \mathcal{A}$. Now for each $\bar{c} \leq \bar{b}$ in $\mathcal{B}$, $\mathcal{B} \models \varphi(\bar{a}, \bar{c})$ if and only if $\mathcal{A} \models \varphi(f(\bar{a}), f(\bar{c}))$. In $\mathcal{A}$, using **c** we can decide whether $\mathcal{A} \models \varphi(f(\bar{a}), f(\bar{c}))$.

On the other hand, to see whether $(\bar{a}, (\exists \bar{y})\varphi(\bar{x}, \bar{y}))$ is in $S(\mathcal{B})$ using $T(\mathcal{B})$, look for $\bar{b}$ and ψ such that $(\bar{a}, \bar{b}, \psi) \in T(\mathcal{B})$. Some such ψ and witness $\bar{b}$ must exist, since $\bar{a}$ satisfies some formula from S. Then $(\bar{a}, (\exists \bar{y})\varphi(\bar{x}, \bar{y})) \in S(\mathcal{B})$ if and only if $\varphi = \psi$ (recall that we assumed that each element of $\mathcal{A}$ satisfied a unique formula from the Scott family). ⊣

§4. Not computably categorical on any cone.

This section is devoted to the proof of Theorem 1.5 for structures which are Δ^0_2 categorical on a cone. The general case of the theorem will require the η-systems developed in Section 5, and will be significantly more complicated, so the proof of this simpler case should be helpful in following the proof in the general case, and in fact, we have a slightly stronger theorem in this case.

THEOREM 4.1. *Let $\mathcal{A}$ be a countable structure. If $\mathcal{A}$ is not computably categorical on any cone, then there exists an* **e** *such that for all* $\mathbf{d} \geq \mathbf{e}$, *if* **c** *is c.e. in and above* **d**, *then there exists a* **d**-*computable copy $\mathcal{B}$ of $\mathcal{A}$ such that*

(1) *there is a* **c**-*computable isomorphism between $\mathcal{A}$ and $\mathcal{B}$ and*
(2) *for every isomorphism f between $\mathcal{A}$ and $\mathcal{B}$, $f \oplus \mathbf{d}$ computes* **c**.

PROOF. Suppose $\mathcal{A}$ is not computably categorical on any cone. Before we begin, note that since $\mathcal{A}$ is not computably categorical on any cone, for any tuple $\bar{c}$ in $\mathcal{A}$, there exist a tuple $\bar{a}$ in $\mathcal{A}$ that is 1-free over $\bar{c}$. Let **e** be such that:

(1) $\mathcal{A}$ is **e**-computable,
(2) **e** computes a Scott family for $\mathcal{A}$ where each tuple satisfies a unique formula, and **e** can compute which which formula a tuple of $\mathcal{A}$ satisfies,
(3) $\mathcal{A}$ is 1-friendly relative to **e**, and
(4) given $\bar{c}$, **e** can compute the least tuple $\bar{a}$ that is 1-free over $\bar{c}$.

Let $\mathbf{d} \geq \mathbf{e}$, and let **c** be c.e. in and above **d**. Let $C \in \mathbf{c}$ be such that we have a **d**-computable approximation to C where at most one number is enumerated at each stage, and there are infinitely many stages when nothing is enumerated.

We will build $\mathcal{B}$ with domain ω by a **d**-computable construction. We will build a bijection $f : \omega \to \mathcal{A}$ and $\mathcal{B}$ will be the pullback, along f, of $\mathcal{A}$. At each stage s, we will have a finite approximation f_s to f, and $\mathcal{B}[s]$ a finite part of the diagram of $\mathcal{B}$ so that f_s is a partial isomorphism between $\mathcal{B}[s]$ and $\mathcal{A}$. Once we put something into the diagram of $\mathcal{B}$, we will not remove it, and so $\mathcal{B}$ will be $\mathbf{d}$-computable. While the approximation f_s will be $\mathbf{d}$-computable, f will be C-computable.

We will have distinguished tuples $\bar{a}_0 \in \mathcal{A}$ and $\bar{b}_0 \in \mathcal{B}$, such that for any isomorphism $g : \mathcal{B} \to \mathcal{A}$, we will have $0 \notin C$ if and only if $g(\bar{b}_0)$ is automorphic to $\bar{a}_0$ in $\mathcal{A}$. For $n > 0$ the strategy for coding whether $n \in C$ will be the same, but our $\bar{a}_n$ and $\bar{b}_n$ will be re-defined each time some $m < n$ is enumerated into C. When n is enumerated into C, we will be able to redefine f on $\bar{b}_n$ and on all greater values. At each stage s, we have current approximations $\bar{a}_n[s]$ and $\bar{b}_n[s]$ to these values. The tuple $\bar{b}_n[s]$ will be a series of consecutive elements of ω; by $\mathcal{B} \upharpoonright\!\upharpoonright \bar{b}$ we mean the elements of $\mathcal{B}$ up to, and including, those of $\bar{b}$, and by $\mathcal{B} \upharpoonright \bar{b}$ we mean those up to, but not including, $\bar{b}$.

At each stage, if $n \notin C$, for those $\bar{a}_n$ and $\bar{b}_n$ which are defined at that stage we will have $f(\bar{b}_n)$ is 1-free over $f(\mathcal{B} \upharpoonright \bar{b}_n)$; otherwise, we will have $f(\mathcal{B} \upharpoonright \bar{b}_n)f(\bar{b}_n) \not\cong f(\mathcal{B} \upharpoonright \bar{b}_n)\bar{a}_n$.

Construction.

STAGE 0: Let $\bar{a}_0[0]$ be the least tuple of $\mathcal{A}$ that is 1-free, and let $\bar{b}_0[0]$ be the first $|\bar{a}_0|$-many elements of ω. Define f_0 to be the map $\bar{b}_0[0] \mapsto \bar{a}_0[0]$. Let $\mathcal{B}[0]$ be the pullback, along f_0, of $\mathcal{A}$, using only the first $|\bar{a}_0[0]|$-many symbols from the language.

STAGE $s+1$: Suppose n enters C at stage $s+1$. Let $\bar{b} = \mathcal{B}[s] \upharpoonright \bar{b}_n[s]$. Let $\bar{b}'$ be those elements of $\mathcal{B}[s]$ which are not in $\bar{b}$ or $\bar{b}_n[s]$. Then, since $\bar{a}_n[s]$ is 1-free over $f(\bar{b})$, there are $\bar{a}, \bar{a}' \in \mathcal{A}$ such that

$$f(\bar{b}), \bar{a}_n[s], f(\bar{b}') \leq_0 f(\bar{b}), \bar{a}, \bar{a}', \text{ but } f(\bar{b}), \bar{a} \not\cong f(\bar{b}), \bar{a}_n[s].$$

Define f_{s+1} to map $\bar{b}, \bar{b}_n[s], \bar{b}'$ to $f(\bar{b}), \bar{a}, \bar{a}'$. For $m \leq n$, let $\bar{a}_m[s+1] = \bar{a}_m[s]$ and $\bar{b}_m[s+1] = \bar{b}_m[s]$. For $m > n$, $\bar{a}_m[s+1]$ and $\bar{b}_m[s+1]$ are undefined.

If nothing enters C at stage $s+1$, let n be least such that $\bar{a}_n[s]$ is undefined. For $m < n$, let $\bar{a}_m[s+1] = \bar{a}_m[s]$ and $\bar{b}_m[s+1] = \bar{b}_m[s]$. Let $\bar{a}_n[s+1]$ be the least tuple that is 1-free over $\operatorname{ran}(f_s)$. Extend f_s to f_{s+1} with range $\mathcal{A} \upharpoonright\!\upharpoonright \bar{a}_n[s+1]$ by first mapping new elements $\bar{b}_n[s+1]$ of ω to $\bar{a}_n[s+1]$, and then mapping more elements to the rest of $\mathcal{A} \upharpoonright\!\upharpoonright \bar{a}_n[s+1]$. If $n \in C$, we must modify f_{s+1} as described above in the case n entered C.

In all cases, let $\mathcal{B}[s+1]$ be the pullback, along f_{s+1}, of $\mathcal{A}$. We have $\mathcal{B}[s] \subseteq \mathcal{B}[s+1]$.

End of construction.

Since $\bar{a}_n$ and $\bar{b}_n$ are only re-defined when there is an enumeration of some $m \leq n$ into C, it is easy to see that for each n, $\bar{a}_n$, and $\bar{b}_n$ eventually reach a limit. Moreover, since the $\bar{a}_n$ and $\bar{b}_n$ form infinite sequences in $\mathcal{A}$ and $\mathcal{B}$, respectively, and since f is not re-defined on $\mathcal{B} \upharpoonright\!\upharpoonright \bar{b}_n$ unless there is an enumeration of $m \leq n$ into C, we see that f is an isomorphism between $\mathcal{B}$ and $\mathcal{A}$. Moreover, C can compute a stage when $\bar{a}_n$ and $\bar{b}_n$ have reached their limit, and hence f is $\mathbf{c}$-computable.

Now suppose $g : \mathcal{B} \to \mathcal{A}$ is an isomorphism. To compute C from $g \oplus \mathbf{d}$, proceed as follows. Compute $g(\bar{b}_0)$. Ask $\mathbf{d}$ whether $(\mathcal{A}, g(\bar{b}_0)) \cong (\mathcal{A}, \bar{a}_0)$. If yes, then $0 \notin C$. We also know that $\bar{b}_1 = \bar{b}_1[0]$ and that $\bar{a}_1 = \bar{a}_1[0]$. If $(\mathcal{A}, g(\bar{b}_0)) \not\cong (\mathcal{A}, \bar{a}_0)$, then $0 \in C$. Compute s such that $0 \in C[s]$. Then $\bar{b}_1 = \bar{b}_1[s]$ and $\bar{a}_1 = \bar{a}_1[s]$. Continuing in this way, given $\bar{b}_n$ and $\bar{a}_n$, we ask $\mathbf{d}$ whether $(\mathcal{A}, g(\bar{b}_n)) \cong (\mathcal{A}, \bar{a}_n)$, using the answer to decide whether $n \in C$ and to compute $\bar{b}_{n+1}$ and $\bar{a}_{n+1}$. ⊣

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Using Knight's theorem on the upwards closure of degree spectra [19], we get a slight strengthening of the above theorem.

Corollary 4.2. *Let $\mathcal{A}$ be a countable structure. If $\mathcal{A}$ is not computably categorical on any cone, then there exists an* $\mathbf{e}$ *such that for all* $\mathbf{d} \geq \mathbf{e}$, *if* $\mathbf{c}$ *is c.e. in and above* $\mathbf{d}$, *then there exists a* $\mathbf{d}$*-computable copy $\mathcal{B}$ of $\mathcal{A}$ such that every isomorphism between $\mathcal{A}$ and $\mathcal{B}$ computes* $\mathbf{c}$, *and such that there exists a* $\mathbf{c}$*-computable isomorphism between $\mathcal{A}$ and $\mathcal{B}$.*

Proof. Take $\mathbf{e}$ as guaranteed by the theorem, with $\mathbf{e}$ computing $\mathcal{A}$, and fix $\mathbf{d} \geq \mathbf{e}$, and let $\mathbf{c}$ be c.e. in $\mathbf{d}$. Let $\mathcal{C}$ be as guaranteed by the theorem. Since $\mathcal{C}$ is $\mathbf{d}$-computable, by the proof of Knight's upward closure theorem [19] (and noting that a "trivial" structure is computably categorical on a cone), there exists $\mathcal{B}$ such that $\deg(\mathcal{B}) = \mathbf{d}$ and such that there exists a $\mathbf{d}$-computable isomorphism $h : \mathcal{B} \cong \mathcal{C}$. Now since $\mathcal{A}$ is $\mathbf{e}$-computable and $\deg(\mathcal{B}) = \mathbf{d}$, any isomorphism $g : \mathcal{A} \cong \mathcal{B}$ computes $\mathbf{d}$. Since $\mathbf{d}$ computes h, g computes the isomorphism $g \circ h : \mathcal{C} \cong \mathcal{A}$ and hence it computes $\mathbf{c}$. On the other hand, since $\mathbf{c}$ computes $\mathbf{d}$ and hence h, and since $\mathbf{c}$ computes an isomorphism between $\mathcal{A}$ and $\mathcal{C}$, we have that $\mathbf{c}$ computes an isomorphism between $\mathcal{A}$ and $\mathcal{B}$. ⊣

Corollary 4.3. *On a cone, a structure cannot have degree of categoricity which is Δ^0_2 but not Δ^0_1 or Δ^0_2-complete. That is, if $\mathcal{A}$ is not computably categorical on any cone, and if $\mathcal{A}$ has a degree of categoricity on a cone, then there is some* $\mathbf{e}$ *such that for all* $\mathbf{d} \geq \mathbf{e}$, *the degree of categoricity of $\mathcal{A}$ relative to* $\mathbf{d}$ *is at least* $\mathbf{d}'$.

Corollary 4.4. *If $\mathcal{A}$ is Δ^0_2 categorical on a cone then $\mathcal{A}$ has Δ^0_1-complete or Δ^0_2-complete degree of categoricity on a cone.*

§5. A version of Ash's metatheorem. The goal of the remainder of the paper is to prove Theorem 1.5. Our main tool will be a version of Ash's metatheorem for priority constructions which was first introduced in [2, 3, 5]. Ash and Knight's book [6] is a good reference. Montalbán [24] has recently developed a variant of Ash's metatheorem using computable approximations. Montalbán's formulation of the metatheorem also provides more control over the construction; for the proof of Theorem 1.5, we will require this extra control. However, Montalbán's version of the metatheorem, as written, only covers $\mathbf{0}^{(\eta)}$-priority constructions for η a successor ordinal. In this section, we will introduce the metatheorem and expand it to include the case of limit ordinals.

Fix a computable ordinal η for which we will define η-systems and the metatheorem for constructions guessing at a Δ^0_η-complete function. Here our notation differs from Montalbán's but corresponds to Ash's original notation. What we call an η-system corresponds to what Ash would have called an η-system, but what Montalbán calls an η-system we will call an $\eta + 1$-system. This will allow us to consider, for limit ordinals η, what Montalbán might have called a $< \eta$-system.

5.1. Some Δ^0_ξ-complete functions, their approximations, and true stages. Before defining an η-system and stating the metatheorem, we discuss some Δ^0_ξ-complete functions and their approximations as introduced by Montalbán [24]. We will introduce orderings on ω to keep track of our beliefs on the correctness of the approximations.

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For each computable ordinal $\xi \leq \eta$, Montalbán defines a Δ^0_ξ-complete function $\nabla^\xi \in \omega^\omega$, and for each stage $s \in \omega$ a computable approximation ∇^ξ_s to ∇^ξ. ∇^ξ_s is a finite string which guesses at an initial segment of ∇^ξ. The approximations are all uniformly computable in both s and ξ. Montalbán shows that the approximation has the following properties (see Lemmas 7.3, 7.4, and 7.5 of [24]):

(N1) For every ξ, the sequence of stages $t_0 < t_1 < t_2 < \cdots$ for which ∇^ξ_t is correct is an infinite sequence with $\nabla^\xi_{t_0} \subseteq \nabla^\xi_{t_1} \subseteq \cdots$ and $\bigcup_{i \in \omega} \nabla^\xi_{t_i} = \nabla^\xi$.

(N2) For each stage s, there are only finitely many ξ with $\nabla^\xi_s \neq \langle\rangle$, and these ξs can be computed uniformly in s.

(N3) If $\gamma \leq \xi$, $s \leq t$, and $\langle\rangle \neq \nabla^\xi_s \subseteq \nabla^\xi_t$, then $\nabla^\gamma_s \subseteq \nabla^\gamma_t$.

We say that s is a *true stage* or *η-true stage* if $\nabla^\eta_s \subseteq \nabla^\eta$.

Montalbán defines relations $(\leq_\xi)_{\xi<\eta}$ on ω, to be thought of as a relation on stages in an approximation. We will define relations $(\leq_\xi)_{\xi<\eta}$ which are almost, but not exactly, the same as Montalbán's (we leave the definition of these relations, and the proofs of their properties, to Lemma 5.3). An instance $s \leq_\xi t$ of the relation should be interpreted as saying that, from the point of view of t, s is a ξ-true stage. A relation $s \leq_\xi t$ is almost, but not exactly, equivalent to saying that for all $\gamma \leq \xi + 1$, $\nabla^\gamma_s \subseteq \nabla^\gamma_t$. The problem is that we require the property (B4) below.

Definition 5.1. Let $s \trianglelefteq t$ if and only if, for all $\xi < \eta$, $\nabla^{\xi+1}_s \subseteq \nabla^{\xi+1}_t$.

We can interpret $s \trianglelefteq t$ as saying that s appears to be a true stage (or η-true stage) from stage t. This relation is computable by (N2) above.

We will see that the relations $\leq_\xi$ satisfy the following properties:

(B0) $\leq_0$ is the standard ordering on ω.

(B1) The relations $\leq_\xi$ are uniformly computable.

(B2) Each $\leq_\xi$ is a *preordering* (i.e., reflexive and transitive).

(B3) The sequence of relations is *nested* (i.e., if $\gamma \leq \xi$ and $s \leq_\xi t$, then $s \leq_\gamma t$).

(B4) The sequence of relations is *continuous* (i.e., if λ is a limit ordinal, then $\leq_\lambda = \bigcap_{\xi<\lambda} \leq_\xi$).

(B5) For every $s < t$ in ω, if $s \leq_\xi t$ then $\nabla^{\xi+1}_s \subseteq \nabla^{\xi+1}_t$.

(B6) The sequence $t_0 < t_1 < \cdots$ of true stages satisfies $t_0 \trianglelefteq t_1 \trianglelefteq \cdots$ and $\bigcup_{i\in\omega} \nabla^\eta_{t_i} = \nabla^\eta$. We call the sequence of true stages the *true path*.

(B7) For $s \in \omega$, we can compute $H(s) = \max\{\xi < \eta \mid \nabla^\xi_s \neq \langle\rangle\}$. $H(s)$ has the property that if $t > s$ and $s \ntrianglelefteq t$, then $s \nleq_{H(s)} t$. We call $H(s)$ the *height* of s.

(B8) For every ξ with $\xi < \eta$, and $r < s < t$, if $r \leq_\xi t$ and $s \leq_\xi t$, then $r \leq_\xi s$. Moreover, if ξ is a successor ordinal, then it suffices to assume that $s \leq_{\xi-1} t$.

(B9) $s \trianglelefteq t$ if and only if for all $\xi < \eta$, $s \leq_\xi t$.

(B10) If t is a true stage and $s \trianglelefteq t$, then s is also a true stage.

Properties (B0)–(B5) are as in Montalbán [24]. Our (B6) is a modification of Montalbán's (B6). (B7), (B9) and (B10) are new properties. (B8) is Montalbán's (♣) together with his Observation 2.1.

We will define, for convenience, the relations $\trianglelefteq_\xi$ for $\xi < \eta$.

Definition 5.2. Let $s \trianglelefteq_\xi t$ if for all $\gamma \leq \xi + 1$, $\nabla^\gamma_s \subseteq \nabla^\gamma_t$.

These relations are uniformly computable because by (N2), we only need to check whether $\nabla^\gamma_s \subseteq \nabla^\gamma_t$ for finitely many γ.

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Following Montalbán, we will construct the desired relations $(\leq_\xi)_{\xi<\eta}$.

Proposition 5.3. *There is a sequence* $(\leq_\xi)_{\xi<\eta}$ *satisfying* $(B0)$–$(B10)$.

In order to prove this proposition, we will use a number of lemmas from [24], as well as properties (N1)–(N3).

Lemma 5.4 (Lemma 7.3 of [24]). *For each* ξ, *there is a subsequence* $\{t_i : i \in \omega\}$ *such that* $\bigcup_{i\in\omega} \nabla^\xi_{t_i} = \nabla^\xi$.

Lemma 5.5 (Lemma 7.6 of [24]). *Let* $\lambda \leq \eta$ *be a limit ordinal, and* $s < t \in \omega$. *Suppose that* $\nabla^\lambda_s \neq \langle\rangle$. *Then* $\nabla^\lambda_s \subseteq \nabla^\lambda_t$ *if and only if* $(\forall \xi < \lambda)\nabla^\xi_s \subseteq \nabla^\xi_t$.

Lemma 5.6 (Lemma 7.7 of [24]). $(\trianglelefteq_\xi)_{\xi\leq\eta}$ *is a nested computable sequence of pre-orderings satisfying*:

(♣) *For every* $\xi < \eta$, *and every* $r < s < t$, *if* $r \trianglelefteq_{\xi+1} t$ *and* $s \trianglelefteq_\xi t$, *then* $r \trianglelefteq_{\xi+1} s$.

Lemma 5.7. *We have*:

(♡) *For every limit ordinal* $\xi \leq \eta$, *and every* $r < s < t$, *if* $\nabla^\xi_r \subseteq \nabla^\xi_t$ *and* $\nabla^\xi_s \subseteq \nabla^\xi_t$, *then* $\nabla^\xi_r \subseteq \nabla^\xi_s$.

Proof. Fix $\xi \leq \eta$ and $r < s < t$ such that $\nabla^\xi_r \subseteq \nabla^\xi_t$ and $\nabla^\xi_s \subseteq \nabla^\xi_t$. For each $\gamma < \xi$, $r \trianglelefteq_{\gamma+1} t$ and $s \trianglelefteq_\gamma t$, so that by (♣) we have $r \trianglelefteq_{\gamma+1} s$. Then, by Lemma 5.5, $\nabla^\xi_r \trianglelefteq_\xi \nabla^\xi_s$. ⊣

In verifying that the relations $(\leq_\xi)_{\xi<\eta}$ have the desired properties, we will also need to use several facts which Montalbán uses without proof (and without explicitly isolating them as, say, a lemma). We will isolate these in the following lemma, and prove them. Unfortunately, the proofs require notation that is introduced in [24] which we have not introduced here (and which would require repeating most of [24] in order to introduce). We suggest that the reader either take these statements for granted, or if the reader is interested in the proofs of these statements, we suggest that they consult [24] for the required background and definitions.

Lemma 5.8.

(i) ∇^1_s *is the string of* s 0*'s.*

(ii) *Fix* $s < t$ *and* $\xi < \eta$. *If* $\nabla^\xi_s = \nabla^\xi_t$, *then* $\nabla^{\xi+1}_s = \nabla^{\xi+1}_t$.

(iii) *Fix* $s < t, r$ *and* $\xi < \eta$. *If* $\nabla^\xi_s \subseteq \nabla^\xi_t \subseteq \nabla^\xi_r$, *and* $\nabla^{\xi+1}_s \subseteq \nabla^{\xi+1}_r$, *then* $\nabla^{\xi+1}_s \subseteq \nabla^{\xi+1}_t$.

(iv) *Let* ξ *be a limit ordinal. There is an increasing sequence* $\gamma_1, \gamma_2, \gamma_3, \ldots$ *with limit* ξ *such that for all* s,

$$\nabla^\xi_s = \langle \nabla^{\gamma_1}_s(0), \nabla^{\gamma_2}_s(0), \ldots, \nabla^{\gamma_{n_s}}_s(0)\rangle$$

where n_s *is the greatest such that* $\nabla^{\gamma_n}_s(0) \neq \varnothing$.

(v) *Given* $s < t$, *if* $\nabla^{\gamma_{n_s}}_s(0) = \nabla^{\gamma_{n_s}}_t(0)$, *then* $\nabla^\xi_s \subseteq \nabla^\xi_t$.

Proof. All of the notation in this proof is as in [24].

(i) This is just Definition 7.2 of [24].

(ii) Since $\xi + 1$ is a successor ordinal, if we unwrap Definitions 6.15 and 7.2 of [24], we find that $\nabla^{\xi+1}_s = J(\nabla^\xi_s)$ and $\nabla^{\xi+1}_t = J(\nabla^\xi_t)$.[1] If $\nabla^\xi_s = \nabla^\xi_t$, then $\nabla^{\xi+1}_s = \nabla^{\xi+1}_t$.

[1]This requires some effort to check.

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(iii) Once again we have that $\nabla_s^{\xi+1} = J(\nabla_s^\xi)$, $\nabla_t^{\xi+1} = J(\nabla_t^\xi)$, and $\nabla_r^{\xi+1} = J(\nabla_r^\xi)$. Then Lemma 6.4 of [24] gives the desired conclusion.

(iv) Let $\xi = 1 + \eta\langle n_0, \ldots, n_k\rangle$. Then, if $n_k > 0$,

$$\nabla_s^\xi = J^{\omega^\eta}_{\langle n_0,\ldots,n_k\rangle}(\nabla_s^1) = J^{\omega^{\eta[n_0]\ldots[n_k]}} \circ J^{\omega^\eta}_{\langle n_0,\ldots,n_k-1\rangle}(\nabla_s^1).$$

Now

$$J^{\omega^{\eta[n_0]\ldots[n_k]}}(\sigma) = \langle J_1^{\omega^{\eta[n_0]\ldots[n_k]}}(\sigma)(0), J_2^{\omega^{\eta[n_0]\ldots[n_k]}}(\sigma)(0), \ldots, J_j^{\omega^{\eta[n_0]\ldots[n_k]}}(\sigma)(0)\rangle,$$

where j is greatest such that $J_j^{\omega^{\eta[n_0]\ldots[n_k]}}(\sigma) \neq \langle\rangle$. Recall that

$$J_n^{\omega^{\eta[n_0]\ldots[n_k]}} = J^{\omega^{\eta[n_0]\ldots[n_k][n-1]}} \circ J^{\omega^{\eta[n_0]\ldots[n_k][n-2]}} \circ \cdots \circ J^{\omega^{\eta[n_0]\ldots[n_k][0]}}.$$

Thus $\nabla_s^\xi(n)$ is

$$J^{\omega^{\eta[n_0]\ldots[n_k][n-1]}} \circ J^{\omega^{\eta[n_0]\ldots[n_k][n-2]}} \circ \cdots \circ J^{\omega^{\eta[n_0]\ldots[n_k][0]}} \circ J^{\omega^\eta}_{\langle n_0,\ldots,n_k-1\rangle}(\nabla_s^1)(0),$$

which is just

$$J^{\omega^\eta}_{\langle n_0,\ldots,n_k,n-1\rangle}(\nabla_s^1)(0) = \nabla_s^{1+\eta\langle n_0,\ldots,n_k,n-1\rangle}(0).$$

If $n_k = 0$, then

$$\nabla_s^\xi = J^{\omega^\eta}_{\langle n_0,\ldots,n_k\rangle}(\nabla_s^1) = J^{\omega^{\eta[n_0]\ldots[n_k]}} \circ J^{\omega^\eta}_{\langle n_0,\ldots,n_{k-1}\rangle}(\nabla_s^1).$$

In this case, we get that $\nabla_s^\xi(n)$ is

$$J^{\omega^{\eta[n_0]\ldots[n_k][n-1]}} \circ J^{\omega^{\eta[n_0]\ldots[n_k][n-2]}} \circ \cdots \circ J^{\omega^{\eta[n_0]\ldots[n_k][0]}} \circ J^{\omega^\eta}_{\langle n_0,\ldots,n_{k-1}\rangle}(\nabla_s^1)(0),$$

which is again just

$$J^{\omega^\eta}_{\langle n_0,\ldots,n_k,n-1\rangle}(\nabla_s^1)(0) = \nabla_s^{1+\eta\langle n_0,\ldots,n_k,n-1\rangle}(0).$$

(v) Let $\xi = 1 + \eta\langle n_0, \ldots, n_k\rangle$. In (iv), we showed that for each s,

$$\nabla_s^\xi = \langle \nabla_s^{1+\eta\langle n_0,\ldots,n_k,0\rangle}(0), \ldots, \nabla_s^{1+\eta\langle n_0,\ldots,n_k,j-1\rangle}(0)\rangle,$$

where j is the greatest such that $\nabla_s^{1+\eta\langle n_0,\ldots,n_k,j-1\rangle}(0) \neq \langle\rangle$. So given $s < t$, we have

$$\nabla_t^\xi = \langle \nabla_t^{1+\eta\langle n_0,\ldots,n_k,0\rangle}(0), \ldots, \nabla_t^{1+\eta\langle n_0,\ldots,n_k,\ell-1\rangle}(0)\rangle.$$

If $s \trianglelefteq_{1+\eta\langle n_0,\ldots,n_k,j-1\rangle} t$, then

$$\nabla_s^{1+\eta\langle n_0,\ldots,n_k,i\rangle}(0) = \nabla_t^{1+\eta\langle n_0,\ldots,n_k,i\rangle}(0)$$

for $0 \leq i < j$. So $\nabla_s^\xi \subseteq \nabla_t^\xi$. ⊣

Now we will show how to construct the order $(\leq_\xi)_{\xi<\eta}$ and prove Proposition 5.3.

Proof of Proposition 5.3. The proof of this proposition is very similar to the proof of Lemma 7.8 of [24]. The definition of our relations $\leq_\xi$ is the same as Montalbán's, except for one small change. Let C be the set of tuples (λ, u, v) where $\lambda < \eta$ is a limit ordinal, $\nabla_u^\lambda \subsetneq \nabla_v^\lambda$, $\nabla_u^{\lambda+1} \nsubseteq \nabla_v^{\lambda+1}$, and if there is r with $\nabla_u^\lambda \subsetneq \nabla_r^\lambda \subseteq \nabla_v^\lambda$ then $\nabla_u^{\lambda+1} \subseteq \nabla_r^{\lambda+1}$. (The only change here is that we require that $\lambda < \eta$.) Let $\gamma_{\lambda,v}$ be such that the last entry of ∇_v^λ is $\nabla_v^{\gamma_{\lambda,v}}(0)$ (Some such $\gamma_{\lambda,v}$ exists by Lemma 5.8(iv)).

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For $\xi < \eta$, define

$$s \leq_\xi t \Leftrightarrow s \trianglelefteq_\xi t \text{ and } \neg\exists(\lambda, u, v) \in C\,(\gamma_{\lambda,v} < \xi \text{ and } u \leq s < v \trianglelefteq_{\gamma_{\lambda,v}} t).$$

Except for the difference in the definition of C, this is the same as Montalbán's definition.

We must now verify that $\leq_\xi$ satisfies (B0)–(B10). For many of the properties the verification is very similar to, or exactly the same as, Montalbán's, but we will reproduce them here for completeness.

(B0) We can see that $\trianglelefteq_0 = \leq$ as ∇^1_s is the sequence of s zeros (Lemma 5.8(i)).

(B1) The relations $\leq_\xi$ are uniformly computable as the relations $\trianglelefteq_\xi$ are, $\gamma_{\lambda,v}$ is computable in λ and v by (N2), and the existential quantifier is bounded, as $u, v \leq t$ and by (N2), there are only finitely many λ's with $\nabla^{\lambda+1}_v \neq \langle\rangle$.

(B2) Fix ξ and s. Then note that $s \trianglelefteq_\xi s$, and there is no v with $s < v \trianglelefteq_{\gamma_{\lambda,v}} s$. Hence $s \leq_\xi s$.

Now for transitivity, suppose that $s \leq_\xi t \leq_\xi r$, but that $s \not\leq_\xi r$. Since $\trianglelefteq_\xi$ is transitive, $s \trianglelefteq_\xi r$, and so it must be that there is $(\lambda, u, v) \in C$ such that $\gamma_{\lambda,v} < \xi$ and $u \leq s < v \trianglelefteq_{\gamma_{\lambda,v}} r$. If $t < v$, then $u \leq t < v \trianglelefteq_{\gamma_{\lambda,v}} r$ and so (λ, u, v) witnesses that $t \not\leq_\xi r$, a contradiction. So it must be that $v \leq t$. Now $v \trianglelefteq_{\gamma_{\lambda,v}} r$, so by Lemma 5.8 (v), $\nabla^\lambda_v \subseteq \nabla^\lambda_r$. By (N3), $v \trianglelefteq_{\gamma_{\lambda,v}+1} r$. Also, $t \trianglelefteq_\xi r$. Since ξ is greater than $\gamma_{\lambda,v}$, by (♣), $v \trianglelefteq_{\gamma_{\lambda,v}} t$. Then $u \leq s < v \trianglelefteq_{\gamma_{\lambda,v}} t$ and so (λ, u, v) witnesses that $s \not\leq_\xi t$. This is again a contradiction. So $\leq_\xi$ is transitive.

(B3) Suppose that $\gamma \leq \xi$ and $s \leq_\xi t$. We claim that $s \leq_\gamma t$. Since $s \leq_\xi t$, $s \trianglelefteq_\xi t$, and so $s \trianglelefteq_\gamma t$ as $\trianglelefteq$ is nested. We must show that there is no $(\lambda, u, v) \in C$ with $\gamma_{\lambda,v} < \gamma$ and $u \leq s < v \trianglelefteq_{\gamma_{\lambda,v}} t$. If there was, then since $\gamma < \xi$, (λ, u, v) witnesses that $s \not\leq_\xi t$. Since in fact $s \leq_\xi t$, $s \leq_\gamma t$.

(B4) Suppose to the contrary that for some limit ordinal $\alpha < \eta$, $s \not\leq_\alpha t$, but that for all $\xi < \alpha$, $s \leq_\xi t$. If $s \not\leq_\alpha t$ due to the existence of some $(\lambda, u, v) \in C$ with $\gamma_{\lambda,v} < \alpha$ and $u \leq s < v \trianglelefteq_{\gamma_{\lambda,v}} t$, then (λ, u, v) also witnesses that $s \not\leq_{\gamma_{\lambda,v}+1} t$, and $\gamma_{\lambda,v} + 1 < \alpha$, contrary to our initial assumption. So it must be that $s \not\leq_\alpha t$ because $s \ntrianglelefteq_\alpha t$. Now, for all $\xi < \alpha$, $s \trianglelefteq_\xi t$, and so by Lemma 5.5 it must be that $\nabla^\alpha_s \subseteq \nabla^\alpha_t$, but $\nabla^{\alpha+1}_s \not\subseteq \nabla^{\alpha+1}_t$. Let v be the least such that $\nabla^\alpha_s \subsetneq \nabla^\alpha_v \subseteq \nabla^\alpha_t$ and $\nabla^{\alpha+1}_s \not\subseteq \nabla^{\alpha+1}_v$. Some such v exists because, by Lemma 5.8 (ii), if $\nabla^\alpha_s = \nabla^\alpha_t$, then $\nabla^{\alpha+1}_s = \nabla^{\alpha+1}_t$. Then $(\alpha, s, v) \in C$. And $v \trianglelefteq_{\gamma_{\alpha,v}} t$ by Lemma 5.5 because $\langle\rangle \neq \nabla^\alpha_v \subseteq \nabla^\alpha_t$. So $s \not\leq_{\gamma_{\alpha,v}+1} t$ contradicting our assumptions.

(B5) Fix $s, t \in \omega$ with $s \leq_\xi t$. Then $s \trianglelefteq_\xi t$, and so $\nabla^{\xi+1}_s \subseteq \nabla^{\xi+1}_t$ by definition.

(B6) Let $t_0 < t_1 < \cdots$ be the true stages. Then, for each ξ, this is a subsequence of the sequence from (N1), and so $\nabla^\xi_{t_0} \subseteq \nabla^\xi_{t_1} \subseteq \cdots$ and $\bigcup_{i\in\omega} \nabla^\xi_{t_i} = \nabla^\xi$. Thus $t_0 \trianglelefteq t_1 \trianglelefteq t_2 \trianglelefteq \cdots$. Also, by Lemma 5.4, we get that $\nabla^\xi_{t_0} \subseteq \nabla^\xi_{t_1} \subseteq \cdots$. So $t_0, t_1, \ldots$ is a subsequence of the sequence from (N1) for $\xi = \eta$, and so $\bigcup_{i\in\omega} \nabla^\eta_{t_i} = \nabla^\eta$.

(B7) Fix s. By (N2) there are only finitely many ξ with $\nabla^\xi_s \neq \langle\rangle$, and we can compute $H(s) = \{\xi < \eta \mid \nabla^{\xi+1}_s \neq \langle\rangle\}$. Suppose that $t > s$ and $s \ntrianglelefteq_\xi t$. Then, for some $\xi < \eta$, $\nabla^{\xi+1}_s \not\subseteq \nabla^{\xi+1}_t$. Since we must have $\nabla^{\xi+1}_s \neq \langle\rangle$, $\xi \leq H(s)$. Thus $s \not\leq_{H(s)} t$.

(B8) First, we will prove the successor case. Suppose that $r < s < t$, $r \leq_{\xi+1} t$, and $s \leq_\xi t$. Suppose towards a contradiction that $r \not\leq_{\xi+1} s$. By (♣), we get that $r \trianglelefteq_{\xi+1} s$. So it must be that there is some $(\lambda, u, v) \in C$ which witnesses that $r \not\leq_{\xi+1} s$.

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So $v \trianglelefteq_{\gamma_{\lambda,v}} s \trianglelefteq_\xi t$, and so since $\xi + 1 > \gamma_{\lambda,v}$, $v \trianglelefteq_{\gamma_{\lambda,v}} t$. Thus (λ, u, v) witnesses that $r \nleq_{\xi+1} t$.

Now we will show the limit case. This is the content of Observation 2.1 of [24]. Suppose that $r < s < t$, $r \leq_\xi t$, and $s \leq_\xi t$. If ξ is a successor, then we just use the previous case and the fact that $s \leq_{\xi-1} t$. For the limit case, for every $\gamma < \xi$, we have $r \leq_{\gamma+1} t$ and $s \leq_\gamma t$ and so by the previous case we have $r \leq_{\gamma+1} s$. But then, by (B4), we get $r \leq_\xi s$.

(B9) If, for all $\xi < \eta$, $s \leq_\xi t$, then for all $\xi < \eta$, $\nabla_s^{\xi+1} \subseteq \nabla_t^{\xi+1}$, and so $s \trianglelefteq t$. On the other hand, suppose that $s \trianglelefteq t$. Fix $\xi < \eta$. Then $s \trianglelefteq_\xi t$, so to show that $s \leq_\xi t$, it suffices to show that there is no $(\lambda, u, v) \in C$ with $\gamma_{\lambda,v} < \xi$ and $u \leq s < v \trianglelefteq_{\gamma_{\lambda,v}} t$. Suppose to the contrary that there was such a (λ, u, v). Since $v \trianglelefteq_{\gamma_{\lambda,v}} t$, $\nabla_v^{\gamma_{\lambda,v}}(0) = \nabla_t^{\gamma_{\lambda,v}}(0)$, and since $\nabla_v^{\gamma_{\lambda,v}}(0)$ is the last entry of ∇_v^λ, by Lemma 5.8 (v) we have $\nabla_v^\lambda \subseteq \nabla_t^\lambda$. Since $s \trianglelefteq t$, $\nabla_s^\lambda \subseteq \nabla_t^\lambda$. Since $(\lambda, u, v) \in C$, $\nabla_u^\lambda \subseteq \nabla_v^\lambda$. Since λ is a limit ordinal, applying Lemma 5.5 and using ($\heartsuit$) we get that $\nabla_u^\lambda \subseteq \nabla_s^\lambda$ and $\nabla_s^\lambda \subseteq \nabla_v^\lambda$. So $\nabla_u^\lambda \subseteq \nabla_s^\lambda \subseteq \nabla_v^\lambda \subseteq \nabla_t^\lambda$. By the minimality of v, we get $\nabla_u^{\lambda+1} \subseteq \nabla_s^{\lambda+1}$, and so since $\nabla_u^{\lambda+1} \nsubseteq \nabla_v^{\lambda+1}$, $\nabla_s^{\lambda+1} \nsubseteq \nabla_v^{\lambda+1}$. Since $\nabla_s^\lambda \subseteq \nabla_v^\lambda \subseteq \nabla_t^\lambda$ and $\nabla_s^{\lambda+1} \nsubseteq \nabla_v^{\lambda+1}$, by Lemma 5.8 (ii) $\nabla_s^{\lambda+1} \nsubseteq \nabla_t^{\lambda+1}$. This is a contradiction (as $s \trianglelefteq t$), and so $s \leq_\xi t$.

(B10) Suppose that t is a true stage, and $s \trianglelefteq t$. If η is a successor ordinal, say $\eta = \xi + 1$, then $\nabla_s^\eta \subseteq \nabla_t^\eta$. If η is a limit ordinal, then by Lemma 5.5, $\nabla_s^\eta \subseteq \nabla_t^\eta$. ⊣

5.2. η-systems and the metatheorem. We are now ready to define an η-system. The definition is essentially the same as for Montalbán, except that what Montalbán would have called an η-system, we call an $\eta + 1$-system.

Definition 5.9. An η-system is a tuple $(L, P, (\leq_\xi^L)_{\xi<\eta}, E)$ where:

(1) L is a c.e. subset of ω called the set of *states*.
(2) P is a c.e. subset of $L^{<\omega}$ called the *action tree*.
(3) $(\leq_\xi^L)_{\xi<\eta}$ is a nested sequence of c.e. pre-orders on L called the *restraint relations*.
(4) $\ell \trianglelefteq^L \ell'$ is c.e., where we define $\ell \trianglelefteq^L \ell'$ if and only if $\ell \leq_\xi^L \ell'$ for all $\xi < \eta$.
(5) $E \subseteq L \times \omega$ is a c.e. set called the *enumeration function*, and is interpreted as $E(l) = \{k \in \omega : (l, k) \in E\}$. We require that for $\ell_0, \ell_1 \in L$ with $\ell_0 \leq_0^L \ell_1$, $E(\ell_0) \subseteq E(\ell_1)$.

Definition 5.10. A *0-run* for $(L, P, (\leq_\xi^L)_{\xi<\eta}, E)$ is a finite or infinite sequence $\pi = (\ell_0, \ell_1, \dots)$ which is in P if it is a finite sequence, or is a path through P if it is an infinite sequence, such that for all $s, t < |\pi|$ and $\xi < \eta$,

$$s \leq_\xi t \Rightarrow \ell_s \leq_\xi^L \ell_t.$$

If π is a 0-run, let $E(\pi) = \bigcup_{s<|\pi|} E(\ell_i)$.

Given an infinite 0-run $\ell_0, \ell_1, \dots$ of an η-system $(L, P, (\leq_\xi^L)_{\xi<\eta}, E)$, let $t_0 \trianglelefteq t_1 \trianglelefteq t_2 \trianglelefteq \cdots$ be the true stages. Then by the properties of E above, $E(\pi) = \bigcup_{i\in\omega} E(\ell_{t_i})$. So $E(\pi)$ is c.e., but it is determined by the true stages.

Montalbán defines an extendability condition and a weak extendability condition. For our extendability condition, we weaken Montalbán's extendability condition even further (as well as modifying it slightly to allow limit ordinals). In order to define our extendability condition, we need the following definition.

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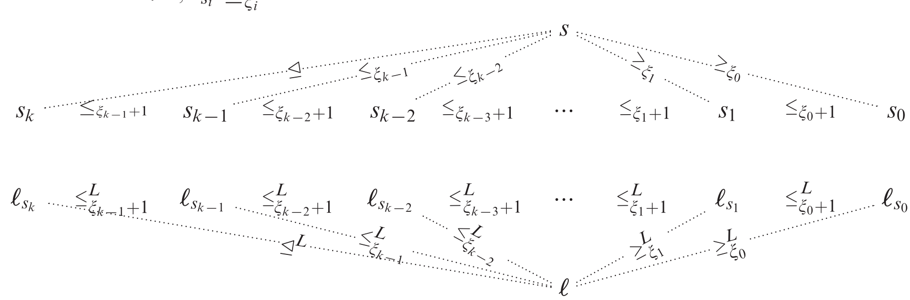
Definition 5.11. To any stage $s > 0$, we effectively associate a sequence of stages and ordinals as follows.

Choose $t^* < s$ greatest such that $t^* \trianglelefteq s$. Some such t^* exists as $0 \trianglelefteq s$. Now for each $\xi < \eta_0$, let $t_\xi < s$ be the largest such that $t_\xi \leq_\xi s$. Note that $t^* \leq t_\xi$ for each ξ as by (B9) $t^* \leq_\xi s$.

There may be infinitely many $\xi < \eta$, but there are only finitely many possible values of t_ξ since they are bounded by s. Since the $\leq_\xi$ are nested (B3), if $\gamma \leq \xi < \eta$, then $t_\xi \leq t_\gamma$. Now we will effectively define stages $t^* = s_k < \cdots < s_0 = s-1$ so that $\{s_0, \dots, s_k\} = \{t_\xi : \xi < \eta\}$ as sets. Let $s_0 = t_0 = s-1$. Suppose that we have defined s_i. If $s_i \trianglelefteq s$, then $k = i$ and we are done. Otherwise, let $\xi_i < \eta$ be the greatest such that $s_i = t_{\xi_i}$. By definition of s_i, it is of the form t_ξ for some ξ. We can find the greatest such by computably searching for ξ_i such that $s_i \leq_{\xi_i} s$ but $s_i \not\leq_{\xi_i+1} s$; some such ξ_i exists since the relations are continuous and nested. Let $s_{i+1} = t_{\xi_i+1}$. Since $s_i \not\leq_{\xi_i+1} s$, $s_{i+1} < s_i$. This completes the definition of $s_k < \cdots < s_0 = s-1$ and $\xi_0 < \cdots < \xi_{k-1} < \eta$.

By (B8), for $i < k$, since $s_{i+1} \leq_{\xi_i+1} s$ and $s_i \leq_{\xi_i} s$, $s_{i+1} \leq_{\xi_i+1} s_i$.

Definition 5.12. We say that an η-system $(L, P, (\leq_\xi^L)_{\xi\leq\eta}, E)$ satisfies the *extendability condition* if: whenever we have a finite 0-run $\pi = \langle \ell_0, \dots, \ell_{s-1}\rangle$ such that for all $i < k$, $\ell_{s_{i+1}} \leq_{\xi_i+1}^L \ell_{s_i}$, where $s_k < s_{k-1} < \cdots < s_0 = s-1$ and $\xi_0 < \xi_1 < \cdots < \xi_{k-1} < \eta$ are the associated sequences of stages and ordinals to s as in Definition 5.11, then there exists an $\ell \in L$ such that $\pi^\frown\ell \in P$, $\ell_{s_k} \trianglelefteq^L \ell$, and for all $i < k$, $\ell_{s_i} \leq_{\xi_i}^L \ell$.



Now we are ready for the metatheorem.

Theorem 5.13. *For every η-system $(L, P, (\leq_\xi^L)_{\xi<\eta}, E)$ with the extendability condition, there is a computable infinite 0-run π. A 0-run can be built uniformly in the η-system.*

Proof of Theorem 5.13. The proof is essentially the same as the proof of Theorem 3.2 in [24]. By the trivial case of the extendability condition, there is $\ell_0 \in L$ with $\langle \ell_0\rangle \in P$. Now suppose that we have a 0-run $\pi = \langle \ell_0, \dots, \ell_{s-1}\rangle$. We want to define $\ell_s \in L$ such that $\pi^\frown\ell_s \in P$, and such that for every $\xi < \eta$, if $t \leq_\xi s$, then $\ell_t \leq_\xi^L \ell_s$.

Let $\{t_\xi \mid \xi < \eta\}$, $s_k < \cdots < s_0 = s-1$, and $\xi_1 < \cdots < \xi_k$ be as in Definition 5.11. If $t \leq_\xi s$, then $t \leq t_\xi$, and by (B8), $t \leq_\xi t_\xi$, so since π is a 0-run $\ell_t \leq_\xi^L \ell_{t_\xi}$. So it is sufficient to find ℓ with $\pi^\frown\ell \in P$ such that, for $\xi < \eta$, $\ell_{t_\xi} \leq_\xi^L \ell$. That is, we must find an ℓ with $\pi^\frown\ell \in P$, $\ell_{s_k} \trianglelefteq^L \ell$ and $\ell_{s_i} \leq_{\xi_i}^L \ell$ for $0 \leq i < k$.

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By (B8), for $i \le k$, since $s_{i+1} \le_{\xi_i+1} s$ and $s_i \le_{\xi_i} s$, $s_{i+1} \le_{\xi_i+1} s_i$. Since p is a 0-run, $\ell_{s_{i+1}} \le^L_{\xi_i+1} \ell_{s_i}$. By the extendability condition, there is $\ell \in L$ with $p\widehat{\ }\ell \in P$, $\ell_{s_k} \trianglelefteq^L \ell$, and $\ell_{s_i} \le^L_{\xi_i} \ell$ for $i < k$. We can find such an l effectively, since we have described how to compute the s_i and since the relations $\le^L_\xi$ and $\trianglelefteq^L$ are computable. ⊣

§6. **Proof of Theorem 1.5.** In this section, we will give the proof of Theorem 1.5. The proof will use the η-systems as developed in Section 5, together with a strategy expanding on that in the proof of Theorem 4.1. It is not sufficient to simply combine the techniques of Theorem 4.1 with the α-system construction. Consider a Σ^0_2 set C. The difficulty is that in the approximation of C, an element x may enter C, exit C, and then later exit C again (and may continue to enter and exit C infinitely many times). Each time x enters C, we will have to code this in a way that can be distinguished from each other time that x entered C. To do this, we will use that fact that given a tuple $\bar{a}$ in a structure of sufficient length, we can pick a tuple $\bar{b}$ which is automorphic to $\bar{a}$ (coding that x is not in C), or we can pick a tuple $\bar{b}$ which is not isomorphic to $\bar{a}$ (coding that x is in C). In the latter case, we will distinguish between how many times x has entered C by choosing $\bar{b}$ to be in a different automorphism orbit each time. Of course, we must also code whether or not $x+1$ is in C. But the actions that we take towards coding x can interfere with those that we take to code $x+1$, and because x can both enter and exit C, the interactions between the two become much more complicated than they were in the case of Theorem 4.1; in that case, if x entered C, we simply started coding $x+1$ in a new place. Now, if x later exits C, we must return to where we were coding $x+1$ beforehand, and if x enters C again, then we must code $x+1$ in another new place because we may have interfered with the previous coding locations of $x+1$ (and we must have the coding of x tell us where to look for the coding of $x+1$).

To begin, we prove the following lemma which we will use for coding.

LEMMA 6.1. *Let $\mathcal{A}$ be a countable structure. Let $\bar{x}$ be a tuple from $\mathcal{A}$. Let $\alpha_1 > \beta_1, \dots, \alpha_n > \beta_n$ be computable ordinals with $\beta_1 \ge \beta_2 \ge \dots \ge \beta_n$. Let $\bar{u}_1, \dots, \bar{u}_n$ and $\bar{v}_1, \dots, \bar{v}_n$ be tuples from $\mathcal{A}$ such that $|\bar{u}_{i+1}| = |\bar{u}_i| + |\bar{v}_i|$ and such that $\bar{v}_i$ is α_i-free over $\bar{u}_i$. Then there is a tuple $\bar{y}$ from $\mathcal{A}$ such that, for each $i = 1, \dots, n$,*

(1) $\bar{x} \restriction_{|\bar{u}_1|} = \bar{y} \restriction_{|\bar{u}_1|}$,
(2) $\bar{x} \restriction_{|\bar{u}_i|+|\bar{v}_i|} \le_{\beta_i} \bar{y} \restriction_{|\bar{u}_i|+|\bar{v}_i|}$,
(3) $\bar{y} \restriction_{|\bar{u}_i|+|\bar{v}_i|} \ncong \bar{u}_i\bar{v}_i$.

PROOF. We will inductively define tuples $\bar{x}_0, \dots, \bar{x}_n$, so that taking $\bar{y} = \bar{x}_n$ satisfies the lemma.

Begin with $\bar{x}_0 = \bar{x}$, so $\bar{x}_0$ satisfies (1) and (2).

Given $\bar{x}_m$ satisfying (1) and (2) for all i, and (3) for $i = 1, \dots, m$, define $\bar{x}_{m+1}$ as follows. If $\bar{x}_m$ already satisfies (3) for $i = m+1$, set $\bar{x}_{m+1} = \bar{x}_m$. Otherwise, $\bar{x}_m \restriction_{|\bar{u}_{m+1}|+|\bar{v}_{m+1}|} \cong \bar{u}_{m+1}\bar{v}_{m+1}$. Since $\bar{v}_{m+1}$ is α_{m+1}-free over $\bar{u}_{m+1}$, there is $\bar{x}_{m+1}$ with $\bar{x}_m \le_{\beta_{m+1}} \bar{x}_{m+1}$, $\bar{x}_m \restriction_{|\bar{u}_{m+1}|} = \bar{x}_{m+1} \restriction_{|\bar{u}_{m+1}|}$, and $\bar{x}_{m+1} \restriction_{|\bar{u}_{m+1}|+|\bar{v}_{m+1}|} \ncong \bar{u}_{m+1}\bar{v}_{m+1}$. So $\bar{x}_{m+1}$ satisfies (3) for $i = m+1$. Note that since $\bar{x}_m \restriction_{|\bar{u}_{m+1}|} = \bar{x}_{m+1} \restriction_{|\bar{u}_{m+1}|}$, we have $\bar{x}_{m+1} \restriction_{|\bar{u}_i|+|\bar{v}_i|} = \bar{x}_m \restriction_{|\bar{u}_i|+|\bar{v}_i|}$ for $i \le m$, so that $\bar{x}_{m+1}$ satisfies (1) and satisfies (2) and (3) for $1 \le i \le m$. Since $\bar{x}_m \le_{\beta_{m+1}} \bar{x}_{m+1}$, and for $i \ge m+1$, $\beta_i \le \beta_{m+1}$, we have $\bar{x} \restriction_{|\bar{u}_i|+|\bar{v}_i|} \le_{\beta_i} \bar{x}_m \restriction_{|\bar{u}_i|+|\bar{v}_i|} \le_{\beta_i} \bar{x}_{m+1} \restriction_{|\bar{u}_i|+|\bar{v}_i|}$ for such i. So (2) holds for $\bar{x}_{m+1}$. ⊣

Theorem 1.5 will follow easily from the following technical result.

THEOREM 6.2. *Let $\mathcal{A}$ be a countable structure. If η is an ordinal and $\mathcal{A}$ is not Δ^0_β categorical on any cone for any $\beta < \eta$, then there exists an* **e** *such that for all* $\mathbf{d} \geq \mathbf{e}$, *there exists a* **d***-computable copy $\mathcal{B}$ of $\mathcal{A}$ such that*

(1) *there is a $\Delta^0_\eta(\mathbf{d})$-computable isomorphism between $\mathcal{A}$ and $\mathcal{B}$ and*
(2) *for every isomorphism f between $\mathcal{A}$ and $\mathcal{B}$, $f \oplus \mathbf{d}$ computes $\Delta^0_\eta(\mathbf{d})$.*

PROOF. Suppose $\mathcal{A}$ is not Δ^0_β categorical on any cone for any $\beta < \eta$. Let **e** be such that:

(i) $\mathcal{A}$ and η are **e**-computable, and **e** computes a Scott family for $\mathcal{A}$ in which each tuple satisfies a unique formula and also computes, for tuples in $\mathcal{A}$, which formula in the Scott family they satisfy,
(ii) $\mathcal{A}$ is $\eta + 1$-friendly relative to **e**,
(iii) given a tuple $\bar{a}$ and $\beta < \eta$, **e** can decide whether a tuple $\bar{b}$ is β-free over $\bar{a}$. (Such a tuple is guaranteed to exist by Corollary 2.11 since $\mathcal{A}$ is not Δ^0_β-categorical on any cone.)

Fix $\mathbf{d} \geq \mathbf{e}$ and $D \in \mathbf{d}$. Our argument involves a D-computable η-system. To ease notation, we make no further mention of D (e.g., whenever we write ∇^β we really mean $\nabla^\beta(D)$, we will say computable when we mean **d**-computable, etc.).

We will define our η-system. Let B be a computable set of constant symbols not occurring in A. Let L be the set of sequences

$$\langle p; (\bar{a}_0, \bar{b}_0), (\bar{a}_1, \bar{b}_1), \dots, (\bar{a}_r, \bar{b}_r)\rangle,$$

where:

(L1) p is a finite partial bijection $B \to A$,
(L2) $\bar{a}_n, \bar{b}_n \in A$ are tuples with $|\bar{a}_{n+1}| = |\bar{a}_n| + |\bar{b}_n|$,
(L3) $|\operatorname{ran}(p)| = |\bar{a}_r| + |\bar{b}_r|$,
(L4) $\operatorname{dom}(p)$ and $\operatorname{ran}(p)$ include the first r elements of B and A, respectively,
(L5) $\bar{b}_n$ is α-free over $\bar{a}_n$, where $\alpha = \max_{m \trianglelefteq n} H(m)$ (see (B7)).

Note that (L1)–(L4) are clearly computable, and that (L5) is computable by (iii).

If ℓ has first coordinate p, and ℓ' has first coordinate p', then for $\xi < \eta$, we set $\ell \leq^L_\xi \ell'$ if and only if $p \leq_\xi p'$, that is, if and only if $\operatorname{ran}(p) \leq_\xi \operatorname{ran}(p')$ as substructures of $\mathcal{A}$ under the usual back-and-forth relations.

Then $(\leq^L_\xi)_{\xi<\eta}$ is nested since the usual back-and-forth relations are, and $(\leq^L_\xi)_{\xi<\eta}$ and $\trianglelefteq^L$ are computable by (ii).

Let P consist of the sequences $\ell_0, \dots, \ell_r$ such that

(P1) if
$$\ell_n = \langle p; (\bar{a}_0, \bar{b}_0), (\bar{a}_1, \bar{b}_1), \dots, (\bar{a}_n, \bar{b}_n)\rangle$$
then
$$\ell_{n+1} = \langle p^*; (\bar{a}_0, \bar{b}_0), (\bar{a}_1, \bar{b}_1), \dots, (\bar{a}_n, \bar{b}_n), (\bar{a}_{n+1}, \bar{b}_{n+1})\rangle$$
with $\operatorname{dom}(p) \subseteq \operatorname{dom}(p^*)$,
(P2) for each n, if
$$\ell_n = \langle p; (\bar{a}_0, \bar{b}_0), (\bar{a}_1, \bar{b}_1), \dots, (\bar{a}_n, \bar{b}_n)\rangle$$
then for each i, $\operatorname{ran}(p \restriction_{|\bar{a}_i|+|\bar{b}_i|}) \cong \bar{a}_i\bar{b}_i$ if and only if $i \trianglelefteq n$,
(P3) if $m \trianglelefteq n$, ℓ_m has first coordinate p_m, and ℓ_n has first coordinate p_n, then $p_m \subseteq p_n$.

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Note that (P1) and (P3) are computable, and that (P2) is computable by (i).

Given

$$\ell_n = \langle p; (\bar{a}_0, \bar{b}_0), (\bar{a}_1, \bar{b}_1), \dots, (\bar{a}_n, \bar{b}_n)\rangle,$$

let $E(\ell)$ be the partial atomic diagram on $\mathcal{B}$ obtained by the pullback along p (using only the first $|p|$ logical symbols).

Note that $E(\ell)$ is computable, and if $\ell_0 \leq_0^L \ell_1$ with first coordinates p_0 and p_1, respectively, then $p_0 \leq_0 p_1$, so that $E(\ell_0) \subseteq E(\ell_1)$.

Thus we have an η-system $(L, P, (\leq_\xi^L)_{\xi<\eta}, E)$.

Lemma 6.3. *The η-system $(L, P, (\leq_\xi^L)_{\xi<\eta}, E)$ has the extendability condition.*

Proof. Suppose we have a finite 0-run $\pi = \langle \ell_0, \dots, \ell_{s-1}\rangle$, and let $s_k < s_{k-1} < \cdots < s_0 = s-1$, and $\xi_0 < \xi_1 < \cdots < \xi_{k-1} < \eta$ be the associated sequences of stages and ordinals to s, as in Definition 5.11. Suppose that for each i, the first coordinate of ℓ_{s_i} is q_{s_i}.

Claim. *There exists $p \supset q_{s_k}$ such that $q_{s_i} \leq_{\xi_i} p$ for $0 \leq i \leq k$.*

Proof. We construct p inductively as follows. We let $q_{s_0}^* = q_{s_0}$, and for $0 \leq i < k$, let $q_{s_{i+1}}^* \supseteq q_{s_{i+1}}$ be such that $q_{s_i}^* \leq_{\xi_i} q_{s_{i+1}}^*$. This is possible since $q_{s_{i+1}} \leq_{\xi_i+1} q_{s_i}$ and since $q_{s_i}^* \supseteq q_{s_i}$. Let $p = q_{s_k}^*$. Then certainly $q_{s_k}^* \leq_{\xi_k}^L p$. As $q_{s_i}^* \leq_{\xi_i} q_{s_{i+1}}^*$ and $\xi_i < \xi_{i+1}$, it follows inductively that each $q_{s_i}^* \leq_{\xi_i} p$. Since $q_{s_i}^* \supseteq q_{s_i}$, we have $q_{s_i} \leq_{\xi_i} p$ as desired. ⊣

Let

$$\ell_{s_0} = \ell_{s-1} = \langle q_{s-1}; (\bar{a}_0, \bar{b}_0), (\bar{a}_1, \bar{b}_1), \dots, (\bar{a}_{s-1}, \bar{b}_{s-1})\rangle.$$

Claim. *There exists $p^* \supset q_{s_k}$ such that $q_{s_i} \leq_{\xi_i} p^*$ for $0 \leq i < k$ and such that* $\operatorname{ran}(p^* \restriction_{|\bar{a}_n|+|\bar{b}_n|}) \not\cong \bar{a}_n\bar{b}_n$ *for $s_k < n \leq s_0 = s-1$.*

Proof. Let $p \supset q_{s_k}$ be as in the previous claim. We will use Lemma 6.1. Let $\bar{x} = \operatorname{ran}(p)$ and $n = s_0 - s_k$. For $i = 1, \dots, n$, let $\bar{u}_i = \bar{a}_{s_k+i}$ and $\bar{v}_i = \bar{b}_{s_k+i}$. For $i = 1, \dots, n$, let $\alpha_i = \max_{1 \leq j \leq s_k+i} H(j)$ and let $\beta_i = \xi_j$ where j is such that $s_{j+1} < i \leq s_j$. Note that by (L5), $\bar{v}_i$ is α_i-free over $\bar{u}_i$ and that $\beta_1 \geq \beta_2 \geq \cdots$. Also, if $s_{j+1} < i \leq s_j$, then since $s_{j+1} = t_{\xi_j+1}$, $i \not\leq_{\xi_j+1} s$. So $\alpha_i \geq H(i) \geq \xi_j + 1 > \xi_j = \beta_i$. Let $\bar{y}$ be the tuple we get by applying Lemma 6.1 and let p^* map the domain of p to $\bar{y}$. Then

$$p^* \restriction_{|\bar{a}_{s_k}|+|\bar{b}_{s_k}|} = p \restriction_{|\bar{a}_{s_k}|+|\bar{b}_{s_k}|} \supset q_{s_k},$$

and so $p^* \supseteq q_{s_k}$. Also,

$$q_{s_i} \leq_{\xi_i} p \restriction_{|\bar{a}_{s_i}|+|\bar{b}_{s_i}|} \leq_{\xi_i} p^* \restriction_{|\bar{a}_{s_i}|+|\bar{b}_{s_i}|},$$

and so $q_{s_i} \leq_{\xi_i} p^*$. Finally, for $i = s_k+1, \dots, s_0$, $p^* \restriction_{|\bar{a}_i|+|\bar{b}_i|} \not\cong \bar{a}_i\bar{b}_i$. ⊣

Let $\bar{a}_s = \operatorname{ran}(p^*)$, and let $\bar{b}_s$ be α-free over $\bar{a}_s$ where $\alpha = \max_{t \leq s} H(t)$, and such that $\bar{a}_s\bar{b}_s$ contains the first s-many elements of $\mathcal{A}$. Let $\bar{c}$ be a new set of constants in B and let $p^{**} = p^* \cup \{\bar{c} \mapsto \bar{b}_s\}$. Let

$$\ell_s = \langle p^{**}; (\bar{a}_0, \bar{b}_0), (\bar{a}_1, \bar{b}_1), \dots, (\bar{a}_{s-1}, \bar{b}_{s-1}), (\bar{a}_s, \bar{b}_s)\rangle.$$

We claim that $\ell_0, \dots, \ell_s$ is in P. That (L1), (L2), and (L3) hold is clear. (L4) and (L5) follow from the choice of $\bar{b}_s$. (P1) is also clear. (P3) follows from the fact that $p^{**} \supseteq q_{s_k}$ and s_k was maximal with $s_k \trianglelefteq s$.

For (P2), if $i \leq s_k$, then since $p^{**} \supseteq q_{s_k}$ and (P2) held at stage s_k, $\operatorname{ran}(p^{**} \restriction_{|\bar{a}_i|+|\bar{b}_i|}) \cong \bar{a}_i\bar{b}_i$ if and only if $i \trianglelefteq s_k$, and since $s_k \trianglelefteq s$, $i \trianglelefteq s_k$ if and only if $i \trianglelefteq s$ by (B8) and (B9). If $s_k < i < s$, then since s_k is maximal with $s_k \trianglelefteq s$, $i \ntrianglelefteq s$ and by choice of p^* in the second claim above, $\operatorname{ran}(p^{**} \restriction_{|\bar{a}_i|+|\bar{b}_i|}) \ncong \bar{a}_i\bar{b}_i$. The case $i = s$ is clear. Hence $\pi\hat{\ }\ell_s \in P$.

Since $p^{**} \supseteq q_{s_k}$, $q_{s_k} \leq_\xi p^{**}$ for all $\xi < \eta$. Given $i < k$, $q_{s_i} \leq_{\xi_i} p^* \subseteq p^{**}$. This completes the proof of the extendability condition. ⊣

By the metatheorem, there is a computable 0-run $\pi = \ell_0\ell_1 \cdots$ for $(L, P, (\leq_i^L)_{i \leq \eta}, E)$. $E(\pi)$ is the diagram of a structure on $\mathcal{B}$. For each j, let

$$\ell_j = \langle p_j; (\bar{a}_0, \bar{b}_0), (\bar{a}_1, \bar{b}_1), \ldots, (\bar{a}_j, \bar{b}_j)\rangle.$$

Then, along the true stages, by (P3) the p_i are nested, and by (L4) they form a bijection $B \to A$. By definition of E, they are an isomorphism $\mathcal{B} \to \mathcal{A}$.

Lemma 6.4. *Let $f : \mathcal{B} \to \mathcal{A}$ be an isomorphism. Then $f \geq_T \Delta_\eta^0$.*

Proof. Using f we will compute the true path $i_1 \trianglelefteq i_2 \trianglelefteq \cdots$. Then we can compute $\nabla^\eta = \bigcup_{n \in \omega} \nabla_{i_n}^\eta$. We claim that ℓ_j is a true stage if and only if

$(*)$ $\operatorname{ran}(f \restriction_{|\bar{a}_j|+|\bar{b}_j|}) \cong \bar{a}_j\bar{b}_j$.

Note that $(*)$ is computable in f, and so this will complete the proof.

If j is a true stage, then p_j extends to an isomorphism $\mathcal{B} \to \mathcal{A}$. Since f is also an isomorphism, there is an automorphism of $\mathcal{A}$ taking $\operatorname{ran}(f \restriction_{\operatorname{dom}(p_j)})$, as an ordered tuple, to $\operatorname{ran}(p_j)$. By (P2), we have $\operatorname{ran}(p_j \restriction_{|\bar{a}_j|+|\bar{b}_j|}) \cong \bar{a}_j\bar{b}_j$ and so we have $(*)$.

If j satisfies $(*)$, then we claim that j is a true stage. Suppose not, and let $p = \bigcup_{n \in \omega} p_{i_n}$ be the isomorphism $\mathcal{B} \to \mathcal{A}$ along the true path. Let i_n be such that $j < i_n$. Then by (B10), $j \ntrianglelefteq i_n$, and so $\operatorname{ran}(p_{i_n} \restriction_{|\bar{a}_j|+|\bar{b}_j|}) \ncong \bar{a}_j\bar{b}_j$. Since $p_{i_n} \subseteq p$ and f is also an isomorphism $\mathcal{B} \to \mathcal{A}$, we have

$$\operatorname{ran}(f \restriction_{|\bar{a}_j|+|\bar{b}_j|}) \cong \operatorname{ran}(p_{i_n} \restriction_{|\bar{a}_j|+|\bar{b}_j|}) \ncong \bar{a}_j\bar{b}_j.$$

This contradicts $(*)$. So j is a true stage. ⊣

Lemma 6.5. *There is an isomorphism $f : \mathcal{B} \to \mathcal{A}$ with $\Delta_\eta^0 \geq_T f$.*

Proof. Using Δ_η^0 we can compute the true path $i_1 \trianglelefteq i_2 \trianglelefteq \cdots$. Then along this path we compute an isomorphism $f = \bigcup_n p_{i_n}$ from $\mathcal{B} \to \mathcal{A}$. ⊣

This completes the proof. ⊣

As before, we can improve the statement of the theorem slightly as follows using Knight's theorem on the upwards closure of degree spectra.

Corollary 6.6. *Let $\mathcal{A}$ be a countable structure. If η is an ordinal and $\mathcal{A}$ is not Δ_β^0 categorical on any cone for any $\beta < \eta$, then there exists an* $\mathbf{e}$ *such that for all* $\mathbf{d} \geq \mathbf{e}$, *there exists a* $\mathbf{d}$*-computable copy $\mathcal{B}$ of $\mathcal{A}$ such that $\Delta_\eta^0(\mathbf{d})$ computes an isomorphism between $\mathcal{A}$ and $\mathcal{B}$, and every such isomorphism computes $\Delta_\eta^0(\mathbf{d})$.*

Proof. Take $\mathbf{e}$ as guaranteed by the theorem, with $\mathbf{e}$ computing $\mathcal{A}$ and η, and fix $\mathbf{d} \geq \mathbf{e}$. Let $\mathcal{B}$ be as guaranteed by Theorem 1.5. Since $\mathcal{B}$ is $\mathbf{d}$-computable, by the proof of Knight's upward closure theorem [19], there exists $\mathcal{C}$ such that $\deg(\mathcal{C}) = \mathbf{d}$ and such that there exists a $\mathbf{d}$-computable isomorphism $h : \mathcal{C} \cong \mathcal{B}$. Now since $\mathcal{A}$

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is $\mathbf{e}$-computable and $\deg(\mathcal{C}) = \mathbf{d}$, any isomorphism $g : \mathcal{A} \cong \mathcal{C}$ computes $\mathbf{d}$. Since $\mathbf{d}$ computes h, g computes the isomorphism $g \circ h : \mathcal{B} \cong \mathcal{A}$ and hence $\Delta^0_\eta(\mathbf{d})$. Moreover, $\mathbf{d}$ computes an isomorphism between $\mathcal{A}$ and $\mathcal{B}$, and hence between $\mathcal{A}$ and $\mathcal{C}$. ⊣

It is now simple to extract Theorem 1.5 from the above result.

Proof of Theorem 1.5. Let $\mathcal{A}$ be a countable structure. By Remark 2.5, there is an ordinal α such that $\mathcal{A}$ is Δ^0_α categorical on a cone. Let $\alpha \geq 1$ be the least such. By Corollary 6.6, there is a cone on which $\mathcal{A}$ and α are computable such that for every $\mathbf{d}$ in the cone, there exists a $\mathbf{d}$-computable copy $\mathcal{B}$ of $\mathcal{A}$ such that every isomorphism between $\mathcal{A}$ and $\mathcal{B}$ computes $\Delta^0_\alpha(\mathbf{d})$. Thus $\mathcal{A}$ has Δ^0_α-complete strong degree of categoricity on this cone. ⊣

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