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Globally subanalytic CMC surfaces in $\mathbb{R}^3$ with singularities

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In this paper we present a classification of a class of globally subanalytic CMC surfaces in $\mathbb{R}^3$ that generalizes the recent classification made by Barbosa and do Carmo in 2016. We show that a globally subanalytic CMC surface in $\mathbb{R}^3$ with isolated singularities and a suitable condition of local connectedness is a plane or a finite union of round spheres and right circular cylinders touching at the singularities. As a consequence, we obtain that a globally subanalytic CMC surface in $\mathbb{R}^3$ that is a topological manifold does not have isolated singularities. It is also proved that a connected closed globally subanalytic CMC surface in $\mathbb{R}^3$ with isolated singularities which is locally Lipschitz normally embedded needs to be a plane or a round sphere or a right circular cylinder. A result in the case of non-isolated singularities is also presented. It also presented some results on regularity of semialgebraic sets and, in particular, it proved a real version of Mumford's Theorem on regularity of normal complex analytic surfaces and a result about C^1 regularity of minimal varieties.

Keywords: Classification of surfaces; Constant mean curvature surfaces; Semialgebraic Sets

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1. Introduction

The question of describing minimal surfaces or, more generally, surfaces of constant mean curvature (CMC surfaces) is known in Analysis and Differential Geometry since the classical papers of Bernstein [**4**], Bombieri, De Giorgi and Giusti [**9**], Hopf [**23**] and Alexandrov [**1**]. Recently, in the paper [**2**], Barbosa and do Carmo showed that the connected algebraic smooth CMC surfaces in $\mathbb{R}^3$ are only the planes, round spheres and right circular cylinders. A generalization of this result was proved in Barbosa et al. [**3**], it obtained the same conclusion in the case of connected globally subanalytic smooth CMC surfaces in $\mathbb{R}^3$. In [**30**], Perdomo showed that there are no algebraic smooth surfaces of degree 3 in $\mathbb{R}^3$, with nonzero constant mean curvature. Recently, this result was generalized by Perdomo and Tkachev in [**31**], they showed that there are no algebraic smooth hypersurfaces of degree 3 in $\mathbb{R}^n$, $n \geqslant 3$, with nonzero constant mean curvature.

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When we know well about smooth sets in a certain category, it is natural to think about objects in this category that have singularities. Thus, it is natural to study and classify algebraic CMC surfaces with singularities. In fact, minimal surfaces with singularities play an important role in the study of minimal submanifolds and there are already many works on minimal surfaces with singularities (see [**5**, **9**, **10**, **17**, **28**, **35**]). Let us remark that these papers are devoted to give conditions on CMC surfaces with singularities such that their singularities are removable or to present examples with non-removable singularities.

Thus, this paper is devoted to study and classify algebraic CMC surfaces in $\mathbb{R}^3$ with possibly non-removable singularities. For instance, it is easy to find examples of algebraic surfaces with non-removable isolated singularities such that the smooth part have non-zero constant mean curvature, namely, finite unions of round spheres and right circular cylinders touching at the singularities.

A first natural question is: Are there further examples?

The main aim of this paper is to show that the answer to the question above is no, when we impose a suitable condition of local connectedness called here by connected links (see definition 2.19).

As it was said before, some of the papers quoted above are devoted to give conditions on CMC surfaces with singularities such that their singularities are removable. Looking for minimal surfaces with removable singularities is also a subject studied in Complex Algebraic Geometry, since any complex analytic set is a minimal variety (possibly with singularities) (see p. 180 in [**11**]). A pioneer result in the topology of singular analytic surfaces is Mumford's Theorem (see [**27**]) that in $\mathbb{C}^3$ can be formulated as follows: *if $X \subset \mathbb{C}^3$ is a complex analytic surface with an isolated singularity p and its link at p has trivial fundamental group, then X is smooth at p.* We can find some results related with Mumford's Theorem in [**7**, **32**, **33**].

Let us describe how this paper is organized. In § 2 are presented some definitions and main properties used in the paper about globally subanalytic sets. Section 3 is devoted to show the main result of this paper. A classification is presented in theorem 3.1 of the globally subanalytic CMC surfaces $X \subset \mathbb{R}^3$ with $\dim \mathrm{Sing}(X) < 1$ and such that each connected component of $X \setminus \mathrm{Sing}(X)$ is a CMC surface and has connected links (see definition 2.19). In § 4 are presented some consequences of theorem 3.1 and its proof. For instance, it is presented in corollary 4.2 a classification of the closed connected globally subanalytic CMC surfaces $X \subset \mathbb{R}^3$ with $\dim \mathrm{Sing}(X) < 1$ which have connected links and in corollary 4.11 a classification of the closed connected globally subanalytic CMC surfaces $X \subset \mathbb{R}^3$ with $\dim \mathrm{Sing}(X) < 1$ which are locally LNE (see definition 4.10). In particular, these results generalize the main results of [**2**, **3**]. Moreover, it is given a classification of globally subanalytic CMC surfaces $X \subset \mathbb{R}^3$ when each connected component of $X \setminus \mathrm{Sing}(X)$ has smooth closure (see proposition 4.12). Some results on regularity of subanalytic sets are also presented and, in particular, it is proved a real version of Mumford's Theorem on regularity of normal complex analytic surfaces (see corollary 4.19) and it is also given a proof that a minimal variety in $\mathbb{R}^n$ which is a subanalytic C^1 submanifold needs to be smooth (see proposition 4.26).

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2. Preliminaries

In this section, we make a brief exposition about globally subanalytic sets. In order to know more about globally subanalytic sets, for example, see [**12**, **14**–**16**].

Definition 2.1 Algebraic sets. A subset $X \in \mathbb{R}^n$ is called **algebraic** if there are polynomials $p_1, \dots, p_k : \mathbb{R}^n \to \mathbb{R}$ such that $X = \{x \in \mathbb{R}^n;\ p_1(x) = \cdots = p_k(x) = 0\}$.

Definition 2.2 Semialgebraic sets. A subset $X \in \mathbb{R}^n$ is called **semialgebraic** if X can be written as a finite union of sets of the form $\{x \in \mathbb{R}^n;\ p(x) = 0, q_1(x) > 0, \dots, q_k(x) > 0\}$, where $p, q_1, \dots, q_k$ are polynomials on $\mathbb{R}^n$. A function $f : X \subset \mathbb{R}^n \to \mathbb{R}^p$ is said to be a **semialgebraic function** if its graph is a semialgebraic set.

Definition 2.3. A subset $X \in \mathbb{R}^n$ is called *semianalytic* at $x \in \mathbb{R}^n$ if there exists an open neighbourhood U of x in $\mathbb{R}^n$ such that $U \cap X$ can be written as a finite union of sets of the form $\{x \in \mathbb{R}^n \,|\, p(x) = 0, q_1(x) > 0, \dots, q_k(x) > 0\}$, where $p, q_1, \dots, q_k$ are analytic functions on U. A subset $X \subset \mathbb{R}^n$ is called *semianalytic* if X is semianalytic at each point $x \in \mathbb{R}^n$.

Definition 2.4. A subset $X \subset \mathbb{R}^n$ is called *subanalytic at* $x \in \mathbb{R}^n$ if there exists an open neighbourhood U of x in $\mathbb{R}^n$ and a relatively compact semianalytic subset $S \subset \mathbb{R}^n \times \mathbb{R}^m$, for some m, such that $U \cap X = \pi(S)$ where $\pi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is the orthogonal projection map. A subset $X \subset \mathbb{R}^n$ is called *subanalytic* if X is subanalytic at each point of $\mathbb{R}^n$.

Definition 2.5. Let $X \subset \mathbb{R}^n$ be a subanalytic set. A map $f : X \to \mathbb{R}^k$ is called a *subanalytic map* if its graph is subanalytic.

Remark 2.6. The complement, the closure and the interior of a subanalytic set are subanalytic sets. A finite intersection of subanalytic sets is still a subanalytic set.

Definition 2.7. A subset $X \subset \mathbb{R}^n$ is called *globally subanalytic* if its image under the map, from $\mathbb{R}^n$ to $\mathbb{R}^n$,

$$(x_1, \dots, x_n) \mapsto \left(\frac{x_1}{\sqrt{1 + x_1^2}}, \dots, \frac{x_n}{\sqrt{1 + x_n^2}} \right)$$

is subanalytic.

Remark 2.8.

(1) Any semialgebraic set is a globally subanalytic set;

(2) Any bounded subanalytic set is a globally subanalytic set;

(3) Any globally subanalytic set is a subanalytic set;

(4) The collection of all globally subanalytic sets form an O-minimal structure (see the main theorem in [**15**]). In particular, a bounded subanalytic subset of $\mathbb{R}$ is a finite union of intervals.

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Notation. Let p be a point in $\mathbb{R}^n$, $Y \subset \mathbb{R}^n$ and $\varepsilon > 0$. Then we denote the sphere with centre p and radius ε by

$$\mathbb{S}^{n-1}_\varepsilon(p) := \{x \in \mathbb{R}^n;\ \|x - p\| = \varepsilon\},$$

the open ball with centre p and radius ε by

$$B^n_\varepsilon(p) := \{x \in \mathbb{R}^n;\ \|x - p\| < \varepsilon\},$$

and the cone over Y with vertex p by

$$\mathrm{Cone}_p(Y) := \{tx + p \in \mathbb{R}^n;\ x \in Y \text{ and } t \in [0, 1]\}.$$

Here smooth means C^∞ smooth.

Definition 2.9. Let $X \subset \mathbb{R}^n$ be a subset. The **singular set** of X, denoted by $\mathrm{Sing}(X)$, is the set of points $x \in X$ such that $U \cap X$ is not a smooth submanifold of $\mathbb{R}^n$ for any open neighbourhood U of x. A point of $\mathrm{Sing}(X)$ is called a **singular point** (or a **singularity**) of X. If $p \in \mathrm{Reg}(X) := X \setminus \mathrm{Sing}(X)$, we say that X is smooth at p.

Thus, if $p \in \mathrm{Reg}(X) = X \setminus \mathrm{Sing}(X)$, there is open neighbourhood $U \subset \mathbb{R}^n$ of p such that $X \cap U$ is a smooth submanifold of $\mathbb{R}^n$ and, then, we define the **dimension of X at p** by $\dim_p X = \dim X \cap U$. Thus, we define the **dimension of X** by

$$\dim X = \max_{p \in \mathrm{Reg}(X)} \dim_p X.$$

We say that X has **pure dimension**, if $\dim X = \dim_p X$ for all $p \in \mathrm{Reg}(X)$.

In the case that $X \subset \mathbb{R}^n$ is a subanalytic set, we have that $\mathrm{Sing}(X)$ is also subanalytic with $\dim \mathrm{Sing}(X) < \dim X$ (see [**6**, theorem 7.2]).

Here we assume that the sets have pure dimension.

Proposition 2.10. *Let $Y \subset \mathbb{R}^n$ be a subanalytic set. Suppose that there exists an open neighbourhood U of p such that $(Y \setminus \{p\}) \cap U$ is a smooth submanifold of $\mathbb{R}^n$. For any small enough $\varepsilon > 0$, $Y \cap \mathbb{S}^{n-1}_\varepsilon(p)$ is a smooth submanifold of $\mathbb{R}^n$.*

Proof. We can assume that $U = B_\delta(p)$ for some $\delta > 0$. Let $M = (Y \setminus \{p\}) \cap B_\delta(p)$ and let $\rho\colon M \to \mathbb{R}$ be the function given by $\rho(x) = \|x - p\|$. Since M is a bounded subanalytic set, we have that ρ is a globally subanalytic function and, in particular, $\Sigma = \{x \in M; d\rho_x = 0\}$ and $\Delta = \rho(\Sigma)$ are globally subanalytic sets. Since ρ is also smooth, by Sard's Theorem, Δ has zero Lebesgue measure and therefore $\dim \Delta = 0$, which implies that Δ is a finite number of points, since it is a globally subanalytic set. Then, for any $0 < \varepsilon < \min \Delta$, $\rho^{-1}(\varepsilon) = Y \cap \mathbb{S}^{n-1}_\varepsilon(p)$ is a smooth submanifold of $\mathbb{R}^n$. □

Proposition 2.11 [**12**, theorem 4.10]. *Let $Y \subset \mathbb{R}^n$ be a subanalytic set and $p \in \mathbb{R}^n$. For any sufficiently small $\varepsilon > 0$, the pair $(\overline{B^n_\varepsilon(p)}, Y \cap \overline{B^n_\varepsilon(p)})$ is homeomorphic to the pair $(\overline{B^n_\varepsilon(p)}, \mathrm{Cone}_p(Y \cap \mathbb{S}^{n-1}_\varepsilon(p)))$.*

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In this case, we denote the set $Y \cap \mathbb{S}^{n-1}_\varepsilon(p)$ by $\text{link}_p(Y)$ and it is called **the link of Y at p**.

Definition 2.12. Let $X \subset \mathbb{R}^n$ be a set and $x_0 \in \overline{X}$ be a non-isolated point. Suppose that $X \setminus \text{Sing}(X)$ is a smooth manifold with dimension d. We denote by $\mathcal{N}(X, x_0)$ the subset of the Grassmannian $Gr(d, \mathbb{R}^n)$ of all d-dimensional linear subspaces $T \subset \mathbb{R}^n$ such that there is a sequence of points $\{x_i\} \subset X \setminus \text{Sing}(X)$ tending to x_0 and $\lim T_{x_i} X = T$. We denote by $\widetilde{\mathcal{N}}(X, x_0)$ the subset of $\mathbb{R}^n$ given by the union of all $T \in \mathcal{N}(X, x_0)$.

Definition 2.13. Let $X \subset \mathbb{R}^n$ be a set and $x_0 \in \overline{X}$. We say that $v \in \mathbb{R}^n$ is a tangent vector of X at $x_0 \in \mathbb{R}^n$ if there are a sequence of points $\{x_i\} \subset X$ tending to x_0 and sequence of positive real numbers $\{t_i\}$ such that

$$\lim_{i \to \infty} \frac{1}{t_i}(x_i - x_0) = v.$$

Let $C(X, x_0)$ denote the set of all tangent vectors of X at $x_0 \in \mathbb{R}^n$. We call $C(X, x_0)$ the **tangent cone** of X at x_0.

Remark 2.14. It follows from Curve Selection Lemma for subanalytic sets that, if $X \subset \mathbb{R}^n$ is a subanalytic set and $x_0 \in \overline{X \setminus \{x_0\}}$, then

$$\begin{aligned} C(X, x_0) = \{v;\ &\text{there exists a subanalytic arc } \alpha : [0, \varepsilon) \to \mathbb{R}^n \text{ s.t. } \alpha((0, \varepsilon)) \subset X \\ &\text{and} \quad \alpha(t) - x_0 = tv + o(t)\}, \end{aligned}$$

where $g(t) = o(t)$ means that $g(0) = 0$ and $\lim_{t \to 0^+} g(t)/t = 0$.

Remark 2.15. If $X \subset \mathbb{R}^n$ is a subanalytic set, it follows from lemma 4 in [29], which works also in the subanalytic setting, that $C(X, p) \subset \widetilde{\mathcal{N}}(X, p)$.

Remind that if $X \subset \mathbb{R}^n$ is a smooth hypersurface and there exists a constant $H \in \mathbb{R}$ such that if X is locally expressed as the graph of a smooth function $u\colon B^{n-1}_\varepsilon(p) \to \mathbb{R}$, then u is a solution of the following PDE

$$\text{div}\left((1 + |\nabla u|^2)^{-1/2} \nabla u\right) = (n - 1)H, \tag{2.1}$$

we say that X is a **smooth CMC hypersurface** (with mean curvature H) and when $n = 3$, we say also that X is a **smooth CMC surface** (with mean curvature H).

In the following definition we are going to generalize the notion of smooth CMC hypersurface.

Definition 2.16. We say that a subset $X \subset \mathbb{R}^n$ is a **CMC hypersurface** (with mean curvature H) if $X \setminus \text{Sing}(X)$ is a $(n-1)$-dimensional smooth CMC hypersurface (with mean curvature H) and $\mathcal{H}^{n-1}(\text{Sing}(X)) = 0$, where $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure. When X is a CMC hypersurface with mean curvature $H = 0$, we say that X is a **minimal hypersurface**.

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When $n = 3$, in the above definition we use the word surface instead of hypersurface.

More generally, we say that a smooth submanifold $X \subset \mathbb{R}^n$ is a **minimal submanifold** if X is locally expressed as the graph of a smooth mapping $u = (u_1, \ldots, u_k)\colon B^m_\varepsilon(p) \to \mathbb{R}^k$, satisfying the following equations:

$$\begin{cases} \displaystyle\sum_{i,j=1}^{m} \frac{\partial}{\partial x_i}\left(\sqrt{g}g^{ij}\frac{\partial u_r}{\partial x_j}\right) = 0, & r = 1, \ldots, k \\ \displaystyle\sum_{i=1}^{m} \frac{\partial}{\partial x_i}(\sqrt{g}g^{ij}) = 0, & j = 1, \ldots, m, \end{cases} \tag{2.2}$$

where (g^{ij}) is the inverse matrix of the matrix (g_{ij}), where $g_{ij} = \delta_{ij} + \langle \partial u/\partial x_i, \partial u/\partial x_j\rangle$, and where $g = \det(g_{ij})$.

Definition 2.17. We say that a subset $X \subset \mathbb{R}^n$ is a **minimal variety** if $X \setminus \operatorname{Sing}(X)$ is a m-dimensional minimal submanifold and $\mathcal{H}^m(\operatorname{Sing}(X)) = 0$, where $\mathcal{H}^m$ denotes the m-dimensional Hausdorff measure.

Remark 2.18. It is well known that a smooth submanifold $M^m \subset \mathbb{R}^n$ that locally minimizes volume is a minimal submanifold. Thus, we have that if $X \subset \mathbb{R}^n$ is a set that locally minimizes volume then X is a minimal variety.

Definition 2.19. We say that a subset $Y \subset \mathbb{R}^n$ has **connected link at** $y \in \overline{Y}$ if there exists $\varepsilon_0 > 0$ such that for each $0 < \varepsilon \leqslant \varepsilon_0$, $(Y \setminus \{y\}) \cap B^n_\varepsilon(y)$ has a unique connected component C with $y \in \overline{C}$. We say that Y has **connected links** if Y has connected link at y for all $y \in \overline{Y}$.

Remark 2.20. The definition that a set has connected links is essentially more general than the definition of topological submanifolds in $\mathbb{R}^n$, since any topological submanifold $Y^m \subset \mathbb{R}^n$ $(m > 1)$ has connected links and there exist sets that have connected links and are not topological submanifolds, for example, the union of two transversal planes in $\mathbb{R}^3$.

Definition 2.21. Let $\pi\colon \mathbb{R}^n \to \mathbb{R}^{n-1}$ be a linear projection. We say that a subset $Y \subset \mathbb{R}^n$ is a **π-graph of a continuous function** $f\colon \Omega \subset \mathbb{R}^{n-1} \to \mathbb{R}$ if there exists a rotation mapping $L\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $L(\pi^{-1}(0)) = \{(0, \ldots, 0, x_n); x_n \in \mathbb{R}\}$ and $L(Y) = \{(x, f(x)); x \in \Omega\}$.

3. CMC surfaces with singularities

Theorem 3.1. *Let $X \subset \mathbb{R}^3$ be a closed and connected globally subanalytic set such that* $\dim \operatorname{Sing}(X) < 1$. *Suppose that each connected component of $X \setminus \operatorname{Sing}(X)$ has connected links. If X is a CMC surface with mean curvature H, then we have the following*:

(1) *if $H = 0$, then* $\operatorname{Sing}(X) = \emptyset$ *and X is a plane*;

(2) *if $H \neq 0$, then*
  (i) *X is a round sphere or a right circular cylinder, when* $\operatorname{Sing}(X) = \emptyset$;

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(ii) *X is a finite union of round spheres and right circular cylinders touching at the points of* $\mathrm{Sing}(X)$, *when* $\mathrm{Sing}(X) \neq \emptyset$.

Proof. Let $X_1, \ldots, X_m$ be the connected components of $X \setminus \mathrm{Sing}(X)$. Thus, $X = \bigcup_{i=1}^m \overline{X_i}$. Fixed $i \in \{1, \ldots, m\}$, it is enough to show that $Z := \overline{X}_i$ is a plane or a round sphere or a right circular cylinder.

Claim 1. *Z is a topological manifold.*

Proof of claim 1. Let $p \in Z$. If $p \notin \mathrm{Sing}(X)$, it is clear that Z is a topological manifold around p and that $\mathcal{N}(Z,p)$ contains a unique plane. Thus, we can assume that $p \in \mathrm{Sing}(X)$. Let $\varepsilon_0 > 0$ be a number that satisfies definition 2.19 and propositions 2.10 and 2.11. Then, by definition 2.19, $(Z \cap B^3_{\varepsilon_0}(p)) \setminus \{p\}$ is connected and by proposition 2.10, $Z \cap \mathbb{S}^2_{\varepsilon_0}(p)$ is smooth (and compact). By proposition 2.11, $Z \cap \mathbb{S}^2_{\varepsilon_0}(p)$ is connected and using the classification of compact smooth curves, we have that $Z \cap \mathbb{S}^2_{\varepsilon_0}(p)$ is homeomorphic to $\mathbb{S}^1$. Therefore, by proposition 2.11, once again, $Y := Z \cap B^3_{\varepsilon_0}(p)$ is a topological manifold. □

Claim 2. *$\mathcal{N}(Z,p)$ contains a unique plane and $C(Z,p)$ has at least 2 different lines passing through the origin, for all $p \in Z$.*

Proof of claim 2. Claim 1 gives that $\Sigma := X_i \cap B^3_{\varepsilon_0}(p) = (Z \setminus \{p\}) \cap B^3_{\varepsilon_0}(p)$ is smooth and homeomorphic to a punctured disc. By hypothesis, we have that Σ has constant mean curvature and as it is smooth, Σ is $C^{2,1}$. Then by the main theorem in [**20**], there exists a mapping $\mathbf{x}\colon B^2_r(0) \to \Sigma \cup \{p\}$ for some $r > 0$ such that $\mathbf{x}(0) = p$, $\mathbf{x}|_{B^2_r(0)\setminus\{0\}}$ is a smooth parametrization of Σ, $\mathbf{x}$ is $C^{1,\alpha}$ for all $\alpha \in (0,1)$ and satisfies the following isothermal conditions

$$\|\mathbf{x}_u\|^2 - \|\mathbf{x}_v\|^2 = \langle \mathbf{x}_u, \mathbf{x}_v \rangle = 0. \tag{3.1}$$

Moreover, by remark 3 in [**20**], $\mathbf{x}$ satisfies, for some non-negative integer n and $a \in \mathbb{C}^3 \setminus \{0\}$, the following asymptotic condition:

$$\mathbf{x}_u(w) + i\mathbf{x}_v(w) = aw^n + o(w^n) \tag{3.2}$$

when $w := u + iv \to (0,0)$. Here, we are doing the canonical identification $\mathbb{C} \cong \mathbb{R}^2$. Now, since $\mathbf{x}$ is C^1 and satisfies (3.1) and (3.2), it follows that $\mathcal{N}(Z,p)$ contains a unique plane P generated by the vectors $Re(a)$ and $Im(a)$ (see also lemma 3.1 in [**21**]). Moreover, after we integrate equation (3.2) (like as in lemma 3.3 in [**19**]), it is easy to verify that $C(Z,p)$ has 2 different lines passing through the origin, which finishes the proof of claim 2. □

Let $p \in Z$ and let $P \subset \mathbb{R}^3$ be the plane as in the proof of claim 2, i.e., $P = \widetilde{\mathcal{N}}(Z,p)$. Let $\pi\colon \mathbb{R}^3 \to P \cong \mathbb{R}^2$ be the orthogonal projection.

Claim 3. *There exists an open neighbourhood A of p in Z such that $\pi|_A$ is an open mapping.*

Proof of claim 3. By remark 2.15, we have $C(Z,p) \subset P = \widetilde{\mathcal{N}}(Z,p)$ and, in particular, $C(Z,p) \cap P^\perp = \{0\}$. Since $\dim \mathrm{Sing}(X) < 1$, we can choose an open neighbourhood U of p so small such that $\widetilde{\mathcal{N}}(Z,q) \cap P^\perp = \{0\}$ for all $q \in U \cap Z$, $Z \cap U \setminus \{p\}$

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is smooth and $(Z \cap U) \cap \pi^{-1}(\pi(p)) = \{p\}$. In particular, $C(Z,q) = T_qZ$ for all $q \in Z \cap U \setminus \{p\}$ and $C(Z,q) \cap P^{\perp} = \{0\}$ for all $q \in Z \cap U$. We claim that π is then an open mapping on $A := Z \cap U$. To see this, fix $q \in A$. Since Z is closed, there is a $r > 0$ small enough such that $V := \overline{B^3_r(q)} \cap Z \subset A$ is compact, and the topological boundary ∂V of V in Z lies on $\partial B^3_r(q)$, which implies that $q \notin \partial V$. Moreover, for a small enough $r > 0$, we can assume that

$$\pi(q) \notin \pi(\partial V)$$

otherwise, there exists a sequence of positive numbers $\{r_k\}$ tending to 0 such that for each k, there is a point $q_k \in Z \cap \partial \overline{B^3_{r_k}(q)}$ with $\pi(q_k) = \pi(q)$. So, extracting a subsequence if necessary, we can assume that $\lim_{k\to\infty}((q_k - q)/\|q_k - q\|) = v \neq 0$. This implies that $v \in P^{\perp} \cap C(Z,q)$, which is a contradiction, since $C(Z,q) \cap P^{\perp} = \{0\}$. Since $\pi(\partial V)$ is a compact set, there is $s > 0$ such that

$$\overline{B^2_s(\pi(q))} \cap \pi(\partial V) = \emptyset. \tag{3.3}$$

It is enough to show that $\pi(q)$ is an interior point of $\pi(V)$ in P.

Thus, suppose by contradiction that $\pi(q)$ is not an interior point of $\pi(V)$. Then $B^2_\delta(\pi(q)) \not\subset \pi(V)$, for any $\delta > 0$. In particular, there is a point

$$x \in B^2_{s/2}(\pi(q)) \setminus \pi(V).$$

Since $x \notin \pi(V)$ and $\pi(V)$ is compact, for $t = dist(x, \pi(V))$, we have that $\overline{B^2_t(x)} \subset P$ intersects $\pi(V)$ while $B^2_t(x) \cap \pi(V) = \emptyset$. Moreover, since $q \in V$, we have $t \leqslant \|x - \pi(q)\| < s/2$, and if $y \in \overline{B^2_t(x)}$ then

$$\|y - \pi(q)\| \leqslant \|y - x\| + \|x - \pi(q)\| \leqslant t + s/2 < s,$$

which yields

$$\overline{B^2_t(x)} \subset B^2_s(\pi(q)).$$

Thus $\overline{B^2_t(x)} \cap \pi(\partial V) = \emptyset$ by (3.3). Take $y' \in \overline{B^2_t(x)} \cap \pi(V)$ and $y \in \pi^{-1}(y') \cap V$. Note that $y \notin \partial V$, so y is an interior point of V, and hence $C(V,y) = C(Z,y)$.

Since $B^2_t(x) \cap \pi(V) = \emptyset$, no point of V is contained in the cylinder $C := \pi^{-1}(B^2_t(x))$. This implies that $\ell \subset T_y\partial C$ for each line $\ell \subset C(Z,y)$ passing through the origin. In fact, let $\ell \subset C(Z,y)$ be a line passing through the origin, then we have two arcs $\gamma_1, \gamma_2 \colon [0,\varepsilon) \to V$ such that $\gamma_i(\tau) - y = (-1)^i \tau u + o(\tau)$ for $i = 1, 2$, where $u \in \mathbb{R}^3 \setminus \{0\}$ satisfies $\ell = \{\tau u; \tau \in \mathbb{R}\}$. Thus, suppose that $\ell \not\subset T_y\partial C$, then the line $y + \ell = \{y + \tau u; \tau \in \mathbb{R}\}$ and C intersect transversally at y, this forces the image of γ_1 or γ_2 to intersect C, which is a contradiction, since $V \cap C = \emptyset$. Therefore, $\ell \subset T_y\partial C$. Moreover, by our choice of the open neighbourhood U of p, $C(Z,y) = T_yZ$ or $C(Z,y) = C(Z,p) \subset P$ and, in any case, $C(Z,y)$ is contained in some plane $\tilde{P}$ such that $\tilde{P} \cap P^{\perp} = \{0\}$. Since $C(Z,y)$ has at least two different lines passing through the origin and $P^{\perp} \subset T_y\partial C$, we obtain that $T_y\partial C$ has three linearly independent vectors, which is a contradiction, since $T_y\partial C$ is a 2-dimensional linear subspace.

Therefore, $\pi(q)$ is an interior point of $\pi(V)$ and this finishes the proof of claim 3. □

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CLAIM 4. *There exists an open neighbourhood $W \subset \mathbb{R}^3$ of p such that $Z \cap W$ is a π-graph of a continuous function $f\colon B^2_r(0) \to \mathbb{R}$, for some $r > 0$.*

Proof of claim 4. After a translation, if necessary, we can assume that $p = 0$. By claim 3, there exists a bounded neighbourhood A of the origin in Z such that $\pi = \pi|_A : A \to \pi(A)$ is an open mapping and $\pi(A)$ is an open set in $\mathbb{R}^2$ and, moreover, $\widetilde{\mathcal{N}}(Z, q) \cap P^{\perp} = \{0\}$ for all $q \in A$, $A \setminus \{0\}$ is smooth and $A \cap \pi^{-1}(0) = \{0\}$. After a rotation, if necessary, we can suppose that π is the projection on the two first coordinates. Let $B \subset A$ be a bounded open subset of Z such that $K = \overline{B} \subset A$ and $0 \in B$. Let $f : \Omega := \pi(K) \to \mathbb{R}$ given by $f(x) = \max\{y_3 \in \mathbb{R};\ (y_1, y_2, y_3) \in \pi^{-1}(x) \cap K\}$. Then, $f(0) = 0$ and $0 \in \mathrm{Graph}(f)$.

We are going to show that f is a continuous function. Since Z is globally subanalytic, we have that if $x \in \Omega$, then $\pi^{-1}(x) \cap K$ is finite and, in particular, $\pi^{-1}(x) \cap K$ is discrete. Let $\{x_k\}_{k\in\mathbb{N}} \subset \Omega$ with $x_k \to x \in \Omega$ and for each $k \in \mathbb{N}$ we define $\bar{x}_k = (x_k, f(x_k))$. Then, taking a subsequence, if necessary, we have that $\bar{x}_k \to \bar{x}' \in K$, since K is compact. But $\bar{x}' \in \pi^{-1}(x) \cap K$, then $\pi_3(\bar{x}') \leqslant \pi_3(\bar{x}) = f(x)$, where $\pi_3 : \mathbb{R}^3 \to \mathbb{R}$ is the projection on the third coordinate. Suppose that $\pi_3(\bar{x}') < \pi_3(\bar{x})$, then there exist B' and B'' disjoint open balls with centre $\bar{x}'$ and $\bar{x}$, resp., such that $\pi(B' \cap Z)$ and $\pi(B'' \cap Z)$ are open neighbourhoods of x in $\mathbb{R}^2$. Hence, $V' := B' \cap Z$ is below $V'' := B'' \cap Z$. Then, for k sufficiently large, $\bar{x}_k \in V'$ and $x_k \in \pi(V') \cap \pi(V'')$. Therefore, $\pi^{-1}(x_k) \cap V''$ is above $\pi^{-1}(x_k) \cap V'$, but this is a contradiction. Thus f is continuous at x.

There exist an open neighbourhood W of the origin in $\mathbb{R}^3$ and an open ball $B^2_r(0)$ such that $\mathrm{Graph}(f|_{B^2_r(0)}) = Z \cap W$. In fact, since Z is a topological manifold by claim 1, there exists a homeomorphism $\varphi : W_1 \cap Z \to B^2_s(0)$, where W_1 is an open neighbourhood of the origin in $\mathbb{R}^3$ and using that f is a continuous function, we can take $r > 0$ such that $\mathrm{Graph}(f|_{B^2_r(0)}) \subset W_1 \cap Z$. Moreover, let $\psi : B^2_r(0) \to W_1 \cap Z$ be the mapping given by $\psi(y) = (y, f(y))$, we have that $\varphi \circ \psi : B^2_r(0) \to B^2_s(0)$ is a continuous and injective map. Then, by the Invariance of Domain Theorem, $\varphi \circ \psi$ is an open mapping. In particular, $\psi(B^2_r(0))$ is an open set of Z and, therefore, there exists an open neighbourhood W of the origin in $\mathbb{R}^3$ such that $\psi(B^2_r(0)) = \mathrm{Graph}(f|_{B^2_r(0)}) = Z \cap W$, which finishes the proof of claim 4. □

CLAIM 5. *Z is a smooth CMC surface.*

Proof of claim 5. Let $p \in Z$. It is enough to show that Z is smooth at p, i.e., $p \notin \mathrm{Sing}(Z)$. We can suppose that $p = 0$ and by claim 4, we can assume that there are $r > 0$ small enough, a continuous function $f : B^2_r(0) \to \mathbb{R}$ and an open subset $W \subset \mathbb{R}^3$ such that $\mathrm{Graph}(f) = Z \cap W$ and $\widetilde{\mathcal{N}}(Z, q) \cap (\{0\} \times \mathbb{R}) = \{0\}$ for all $q \in Z \cap W$. Therefore, by Implicit Function Theorem, f is smooth on $B^2_\delta(0) \setminus \{0\}$ for some $0 < \delta \leqslant r$. Thus, by theorem 3 in [**34**, p. 168] (or main theorem in [**18**, p. 170]), f is smooth on $B^2_\delta(0)$. Therefore, Z is smooth at p. □

Now, we are ready to complete the proof of the theorem. By theorem 3.2 in [**3**], Z is a plane, a round sphere or a right circular cylinder, since Z is a smooth CMC surface and also a closed globally subanalytic set. Thus, it is clear that X is a finite union of planes or a finite union of round spheres and right circular cylinders

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touching at the singularities. However, since X is connected and $\dim \operatorname{Sing}(X) < 1$, if X is a finite union of planes (in the case $H = 0$), then $m = 1$ and X is plane and, in particular, X is smooth. □

By remark 2.18 and since a complex analytic set locally minimizes volume (see page 180 in [11]), we get that theorem 3.1 does not hold true if we consider 2-dimensional minimal varieties in $\mathbb{R}^4 \cong \mathbb{C}^2$. For example, $X = \{(x, y) \in \mathbb{C}^2; y = x^2\}$ is a 2-dimensional minimal submanifold which is not a plane.

The hypothesis $\dim \operatorname{Sing}(X) < 1$ cannot be removed, since there exist CMC surfaces in $\mathbb{R}^3$ with non-isolated singularities that do not satisfy the conclusion of theorem 3.1, as we can see in the next example.

Example 3.2 Enneper's minimal surface. Let X be the self-intersecting minimal surface generated using the Enneper–Weierstrass parameterization with $f = 1$ and $g = \mathrm{id}$ [13, p. 93, proposition 4]. With some computations, we can see that X is algebraic and it is given by the following equation (see [36])

$$\left(\frac{y^2 - x^2}{2} + \frac{2z^3}{9} + \frac{2z}{3}\right)^3 - 6z\left(\frac{y^2 - x^2}{4} - \frac{z}{4}\left(x^2 + y^2 + \frac{8}{9}z^2\right) + \frac{2z}{9}\right)^2 = 0.$$

Next example shows a C^1 submanifold X of $\mathbb{R}^3$ with $\dim \operatorname{Sing}(X) = 1$ and the connected components of $X \setminus \operatorname{Sing}(X)$ are smooth CMC surfaces with different constants.

Example 3.3. Let $X = \{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 + z^2 = 1 \text{ and } y \leqslant 0\} \cup \{(x, y, z) \in \mathbb{R}^3; x^2 + z^2 = 1 \text{ and } y \geqslant 0\}$. We have that each connected component of $X \setminus \operatorname{Sing}(X)$ is a semialgebraic smooth CMC hypersurface and X is a C^1 submanifold of $\mathbb{R}^3$, but it is not a smooth submanifold of $\mathbb{R}^3$.

4. Some applications of theorem 3.1 and its proof

4.1. CMC surfaces with isolated singularities

Corollary 4.1. *Let $X \subset \mathbb{R}^3$ be a locally closed subanalytic set. Suppose that $\dim \operatorname{Sing}(X) < 1$. If X is a CMC surface, then for each $p \in \operatorname{Sing}(X)$ there exists $\delta > 0$ such that $X \cap B^3_\delta(p) = Z_1 \cap Z_2$, where Z_1 and Z_2 are smooth CMC surfaces satisfying $Z_1 \cap Z_2 = \{p\}$ and $T_pZ_1 = T_pZ_2$.*

Proof. Since X is subanalytic and has an isolated singularity at p, there exists $\delta > 0$ such that $X \cap B^3_\delta(p) \setminus \{p\}$ is smooth and $X \cap \overline{B^3_\delta(p)}$ is homeomorphic to $\operatorname{Cone}(X \cap \mathbb{S}^2_\delta(p))$. Let $X_1, \ldots, X_m$ be the connected components of $X \cap B^3_\delta(p) \setminus \{p\}$. Thus, $X = \bigcup_{i=1}^m Z_i$, $Z_i = X_i \cup \{p\}$ for $i = 1, \ldots, m$. In particular, $Z_i \cap Z_j = \{p\}$ for any $i \neq j$. Fixed $i \in \{1, \ldots, m\}$, it is enough to show that $\tilde{Z} := Z_i$ is smooth. However, by reading the proof of theorem 3.1, everything works to $\tilde{Z}$ up to the end of the proof of claim 5. Then, $\tilde{Z}$ is smooth, which forces $T_pZ_i = T_pZ_j$ for $i, j \in \{1, \ldots, m\}$ and by Maximum Principle for embedded CMC hypersurfaces (see lemma 2.7 in [38]), we must have $m = 2$. □

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Corollary 4.2. *Let $X \subset \mathbb{R}^3$ be a closed connected globally subanalytic set. Suppose that X has connected links and* $\dim \operatorname{Sing}(X) < 1$. *If X is a CMC surface, then* $\operatorname{Sing}(X) = \emptyset$ *and X is a plane or a round sphere or a right circular cylinder.*

Proof. Since X has connected links, we obtain that $X \setminus \operatorname{Sing}(X)$ has only one connected component. Thus, in the proof of theorem 3.1, we can take Z to be X, which implies $\operatorname{Sing}(X) = \emptyset$ and X is a plane or a round sphere or a right circular cylinder. □

The hypothesis that X has connected links in corollary 4.2 cannot be removed neither, as we can see in the next example.

Example 4.3. Let $X = \{(x, y, z) \in \mathbb{R}^3;\ ((x-1)^2 + y^2 + z^2 - 1)((x+1)^2 + y^2 + z^2 - 1) = 0\}$. It is clear that X is an algebraic set and a non-smooth CMC surface. Moreover, X does not have connected links, since for all $0 < \varepsilon < 1$, $X \setminus \{0\} \cap B^3_\varepsilon(0)$ has two connected components such that the closure of each one of them contains the origin.

The hypothesis $\dim \operatorname{Sing}(X) < 1$ in corollary 4.2 cannot be removed, since there exist CMC surfaces in $\mathbb{R}^3$ with non-isolated singularities that have connected links, as we can see in example 3.2. Another example is the following.

Example 4.4. Let $X = \{(x, y, z) \in \mathbb{R}^3;\ xy = 0\}$. Then X is an algebraic set, a CMC surface and has connected links, but it is not smooth.

In fact, the hypothesis $\dim \operatorname{Sing}(X) < 1$ in corollary 4.2 cannot be removed, even if we impose that the CMC surface $X \in \mathbb{R}^3$ is a graph of a global Lipschitz function and, in particular, having connected links.

Example 4.5. Consider the Lipschitz function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = |x|$. Then $X = \{(x, y, z) \in \mathbb{R}^3;\ z = f(x, y)\}$ is a closed semialgebraic set and a CMC surface, but it is not smooth.

Corollary 4.6. *Let $X \subset \mathbb{R}^3$ be a closed and connected globally subanalytic set. Suppose that X is a topological manifold and* $\dim \operatorname{Sing}(X) < 1$. *If X is a CMC surface, then* $\operatorname{Sing}(X) = \emptyset$ *and X is a plane or a round sphere or a right circular cylinder.*

Proof. The proof follows from corollary 4.2, since X is a 2-dimensional topological manifold and this implies that X has connected links. □

Definition 4.7 [**2**]. Let $p\colon \mathbb{R}^3 \to \mathbb{R}$ be a polynomial and $M = p^{-1}(0)$. We say that M is regular if the gradient of p vanishes nowhere in M.

Thus, by corollary 4.2, we obtain also the main results in [**2**, **3**].

Corollary 4.8 [**2**, theorem 1.1 and proposition 4.1]. *Let $p\colon \mathbb{R}^3 \to \mathbb{R}$ be a polynomial and $M = p^{-1}(0)$. Assume that M is regular. If M is a smooth CMC surface then it is a plane or a round sphere or a right circular cylinder.*

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Corollary 4.9 theorem 3.2 in [**3**]. *Let $X \subset \mathbb{R}^3$ be a closed and connected globally subanalytic set. If X is a smooth CMC surface, then X is a plane or a round sphere or a right circular cylinder.*

4.2. CMC surfaces which are locally LNE

Let $Z \subset \mathbb{R}^n$ be a path connected subset. Given two points $q, \tilde{q} \in Z$, we define the *inner distance* in Z between q and $\tilde{q}$ by the number $d_Z(q, \tilde{q})$ below:

$$d_Z(q, \tilde{q}) := \inf\{\text{length}(\gamma) \mid \gamma \text{ is an arc on } Z \text{ connecting } q \text{ to } \tilde{q}\}.$$

Definition 4.10. We say that a set $Z \subset \mathbb{R}^n$ is **Lipschitz normally embedded** (shortly **LNE**), if there is a constant $C > 0$ such that $d_Z(q, \tilde{q}) \leqslant C\|q - \tilde{q}\|$, for all $q, \tilde{q} \in Z$. We say that Z is **locally LNE**, if for each $p \in Z$ there is an open neighbourhood U of p such that $Z \cap U$ is LNE.

Since any smooth submanifold of $\mathbb{R}^n$ is locally LNE, we have that the next result is another generalization of the main results in [**2**, **3**] (see corollaries 4.8 and 4.9).

Corollary 4.11. *Let $X \subset \mathbb{R}^3$ be a closed connected subanalytic set. Suppose that X is a CMC surface and $\dim \mathrm{Sing}(X) < 1$. If X is locally LNE, then X is smooth. In particular, if X is globally subanalytic and locally LNE then X is a plane or a round sphere or a right circular cylinder.*

Proof. Suppose that there exists $p \in \mathrm{Sing}(X)$. It follows from corollary 4.1 that there exists $0 < r \ll 1$ such that $Y_r = X \cap B_r^3(p)$ is the union of 2 smooth CMC surfaces with the same tangent space P at p and, moreover, they intersect only at p. Let us denote these surfaces by Z_1 and Z_2. Let $v \in P$ be a unitary vector and for each $i = 1, 2$ let $\gamma_i \colon [0, \delta_r) \to Z_i$ be a C^1 arc such that $\gamma'(0) = v$. We can assume that $\|\gamma_i(t) - p\| = t$ for all $t \in [0, \delta_r)$ and $i = 1, 2$. Thus, $d_{Y_r}(\gamma_1(t), \gamma_2(t)) \geqslant 2t$ and since γ_1 and γ_2 have the same tangent vector at 0, we have that

$$\lim_{t \to 0^+} \frac{\|\gamma_1(t) - \gamma_2(t)\|}{t} = 0.$$

Therefore there is no constant $C > 0$ such that $d_{Y_r} \leqslant C\| \cdot \|$ for all small enough $r > 0$, which is a contradiction with the fact that X is locally LNE. Then X is smooth and, thus, the result follows from theorem 3.2 in [**3**]. □

The hypothesis $\dim \mathrm{Sing}(X) < 1$ in corollary 4.11 cannot be removed, since the set X given in example 4.5 is LNE.

4.3. CMC surfaces with non-isolated singularities

Proposition 4.12. *Let $X \subset \mathbb{R}^3$ be a closed connected globally subanalytic set. Suppose that we can write $X \setminus \mathrm{Sing}(X) = \bigcup_{k=1}^r X_k$ such that for each $k \in \{1, \ldots, r\}$, the closure of X_k is smooth and connected and X_k is a union of connected components of $X \setminus \mathrm{Sing}(X)$. If X is a CMC surface, then each $\overline{X_k}$ is a plane or a round sphere or a right circular cylinder. In particular, X is a finite union of*

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planes or a finite union of round spheres and right circular cylinders touching at the points of Sing(X).

Proof. We have that $\overline{X_k}$ is a closed connected globally subanalytic smooth CMC surface. It follows from theorem 3.2 in [**3**] that $\overline{X_k}$ is a plane or a round sphere or a right circular cylinder. Since $X = \bigcup_{k=1}^{r} \overline{X_k}$, we obtain that X is a finite union of planes or a finite union of round spheres and right circular cylinders touching at the points of Sing(X). □

Corollary 4.13. *Let $X \subset \mathbb{R}^3$ be a closed connected globally subanalytic set. Suppose that the closure of each connected component of $X \setminus \mathrm{Sing}(X)$ is smooth,* $\dim \mathrm{Sing}(X) = 1$ *and* Sing(X) *does not have isolated points. If X is a CMC surface, then* Sing(X) *is a union of lines and X is a finite union of right circular cylinders touching at* Sing(X).

Proof. By proposition 4.12, X is a finite union of planes or a finite union of round spheres and right circular cylinders touching at the points of Sing(X). However, if X is a finite union of planes, the closure of each connected component of $X \setminus \mathrm{Sing}(X)$ is a closed half plane, which is not smooth. Therefore, X is a finite union of right circular cylinders touching at the points of Sing(X), since the existence of spheres forces Sing(X) to have isolated points or forces $X \setminus \mathrm{Sing}(X)$ to have a connected component with non-smooth closure. By the same reason X cannot contain two right circular cylinders with intersection being a non-empty compact set. Thus, we obtain that Sing(X) is a union of lines and X is a finite union of right circular cylinders touching at Sing(X). □

4.4. A topological implicit function theorem

Definition 4.14. Let $X \subset \mathbb{R}^n$ be a subset such that $X \setminus \mathrm{Sing}(X)$ is a d-dimensional smooth submanifold. We say that X **has a non-degenerate tangent cone at** $p \in X$ if

(i) $C(X,p)$ is a d-dimensional linear subspace of $\mathbb{R}^n$;

(ii) $\widetilde{\mathcal{N}}(X,p) \subsetneq \mathbb{R}^n$.

When X has a non-degenerate tangent cone at p for all $p \in X$, we say that X **has non-degenerate tangent cones**.

Remark 4.15. Any C^1 submanifold $M^m \subset \mathbb{R}^n$ has non-degenerate tangent cones.

Corollary 4.16. *Let $X \subset \mathbb{R}^n$ be a subanalytic hypersurface with non-degenerate tangent cones and isolated singularities. If X is a topological manifold, then X is locally a graph of continuous functions.*

Proof. The conditions that X has non-degenerate tangent cones and isolated singularities imply that for each p there exists a open neighbourhood A of p in X such that $\pi|_A$ is an open mapping, where $\pi \colon \mathbb{R}^n \to \ell^\perp$ and ℓ is a line such that $\ell \cap \widetilde{\mathcal{N}}(X,p) = \{0\}$, just like in claim 3. By taking ℓ such that $P = \ell^\perp = C(X,p)$, the proof is an easy adaptation of the proof of claim 4. □

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We would like to remark that the 'hypersurface' hypothesis in corollary 4.16 cannot be removed.

Example 4.17. We consider the cusp $C = \{(x, y) \in \mathbb{C}^2;\ \ y^2 = x^3\} \subset \mathbb{C}^2 \cong \mathbb{R}^4$. Thus, if there exists a continuous function $f\colon (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ such that $(C, 0) = (\mathrm{Graph}(f), 0)$, then the pairs $(\mathbb{C}^2, C)$ and $(\mathbb{C}^2, \mathbb{C} \times \{0\})$ are homeomorphic. However, by the main result in [**37**, p. 454], there is no homeomorphism $\phi\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ such that $\phi(C) = \mathbb{C} \times \{0\}$. Therefore, C cannot be a graph of such a function f.

4.5. A real version of Mumford's Theorem

In 1961, D. Mumford proved in [**27**] the following result.

Theorem 4.18 Mumford's Theorem, see [**27**]. *Let $V \subset \mathbb{C}^3$ be a complex analytic surface with an isolated singularity at p. If V is a topological manifold around p, then V is smooth at p.*

In fact, D. Mumford proved that it is enough to impose $\pi_1(\mathrm{link}_p(X)) = \{0\}$ instead of the condition 'V is a topological manifold around p'. However, it is easy to find a non-smooth real analytic surface which is a topological manifold with isolated singularities, for instance, $X = \{(x, y, z) \in \mathbb{R}^3;\ z^3 = x^3y + xy^3\}$. Since any complex analytic set is a minimal variety, then the correct assumptions that we need to impose in a real version of Mumford's Theorem should be an analytic set $X \subset \mathbb{R}^3$ that is a minimal surface (or, more generally, a CMC surface) with isolated singularities and $\pi_1(\mathrm{link}_p(X)) \cong \pi_1(\mathbb{S}^1) \cong \mathbb{Z}$. Thus, we obtain the following real version of Mumford's Theorem.

Corollary 4.19. *Let $X \subset \mathbb{R}^3$ be a closed subanalytic CMC surface. Suppose that $\dim \mathrm{Sing}(X) < 1$. If $H_1(\mathrm{link}_p(X)) \cong \mathbb{Z}$ then X is smooth at p.*

Proof. Let $\varepsilon > 0$ be a number that satisfies propositions 2.10 and 2.11. Thus, $X \cap \mathbb{S}^2_\varepsilon(p)$ is a closed and smooth set and, in particular, the connected components $Y_1, \ldots, Y_r$ of $X \cap \mathbb{S}^2_\varepsilon(p)$ are closed manifolds. By classification of compact smooth curves, for each $i \in \{1, \ldots, r\}$, we have that Y_i is homeomorphic to $\mathbb{S}^1$. Then,

$$H_1(\mathrm{link}_p(X)) \cong \mathbb{Z}^r.$$

Then, $r = 1$ and by corollary 4.1, we obtain that $X \cap B^3_\varepsilon(p)$ is a smooth CMC surface. □

Let us remark that we cannot remove the hypothesis 'CMC surface' in corollary 4.19, even for algebraic sets, as we can see in the following example.

Example 4.20. $X = \{(x, y, z) \in \mathbb{R}^3;\ \ x^3 + y^3 = z^3\}$ is an algebraic set and $H_1(\mathrm{link}_p(X)) \cong \mathbb{Z}$ for any $p \in X$, but it is not smooth at 0.

4.6. $C^{k,\alpha}$ regularity

Corollary 4.21. *Let X be a subanalytic C^k submanifold of $\mathbb{R}^n$. Suppose that X is $(n-1)$-dimensional, has non-degenerate tangent cones and isolated singularities,*

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when $k = 0$. Then, for any compact subset $K \subset X$, there are $\alpha = \alpha(K) \in (0, 1]$ and an open subset, U_K, of X such that $K \subset U_K$ and U_K is a $C^{k,\alpha}$ submanifold of $\mathbb{R}^n$.

Before we prove corollary 4.21, we need of the following results.

PROPOSITION 4.22 [6, theorem 6.4]. *Let $K \subset \mathbb{R}^n$ be a compact subset, and let $f, g\colon K \to \mathbb{R}$ be continuous subanalytic functions such that $f^{-1}(0) \subset g^{-1}(0)$. Then there exist $C > 0$ and $r > 0$ such that*

$$|f(x)| \geqslant C|g(x)|^r, \quad \forall\, x \in K.$$

REMARK 4.23. Let $U \subset \mathbb{R}^n$ be an open subset and $h\colon U \to \mathbb{R}$ be a continuous subanalytic function. Then h is locally a Hölder function. In fact, for each compact subanalytic subset $K \subset U$, we define $f, g\colon K \times K \to \mathbb{R}$ by $f(x, y) = |x - y|$ and $g(x, y) = |h(x) - h(y)|$. Then f and g are subanalytic functions and $f^{-1}(0) \subset g^{-1}(0)$. Therefore, by proposition 4.22, there exist $C > 0$ and $r > 0$ such that

$$|f(x, y)| \geqslant C|g(x, y)|^r, \quad \forall\, (x, y) \in K \times K.$$

Thus

$$|h(x) - h(y)| \leqslant M|x - y|^\alpha, \quad \forall\, x, y \in K,$$

where $\alpha = 1/r$ and $M = 1/C^\alpha$.

PROPOSITION 4.24 [8, proposition 2.9.1]. *Let $U \subset \mathbb{R}^n$ be a open subset and $f : U \to \mathbb{R}$ be a subanalytic C^1 function. Then, the partial derivatives $\partial f/\partial x_i : U \to \mathbb{R}$, $i = 1, \ldots, n$, are subanalytic functions.*

Proof of corollary 4.21*.* Let $d = \dim X$. For each $x \in X$, by Implicit Function Theorem, if $k > 0$ or by corollary 4.16, if $k = 0$, there are open neighbourhoods $U \subset \mathbb{R}^d$ and $W \subset \mathbb{R}^n$ with $x \in W$, a rotation $\phi\colon \mathbb{R}^n \to \mathbb{R}^n$ and a C^k function $g\colon U \to \mathbb{R}$ such that $\phi(X \cap W) = \mathrm{Graph}(g)$. Now, let $B \subset U$ be an open ball and let I be an open interval such that $x \in \phi^{-1}(B \times I^{n-d}) \subset W$. Then, $\mathrm{Graph}(g|_B) = \phi(X \cap \phi^{-1}(B \times I^{n-d})) = \phi(X) \cap (B \times I^{n-d})$. Therefore, $f = g|_B$ is a subanalytic function, since $B \times I^{n-d}$ and $\phi(X)$ are subanalytic sets. By remark 4.23 and proposition 4.24, f is $C^{k,\alpha}$ for some $\alpha \in (0, 1]$. Therefore, by shrinking B, if necessary, we have that $\phi^{-1} \circ \psi\colon B \to X \cap \phi^{-1}(B \times I^{n-d})$ is a $C^{k,\alpha}$ parametrization of X, where $\psi\colon B \to \mathrm{Graph}(f)$ is given by $\psi(u) = (u, f(u))$.

Thus, if K is compact subset of X, we can find a finite open cover of K, $K \subset \bigcup_{i=1}^r \psi_i(B_1^d(0))$, where each ψ_i is a C^{k,α_i} parametrization of X, for some $\alpha_i \in (0, 1]$. Therefore, by taking $U_K = \bigcup_{i=1}^r \psi_i(B_1^d(0))$ and $\alpha = \min\{\alpha_i;\ i = 1, \ldots, r\}$, we obtain that U_K is a $C^{k,\alpha}$ submanifold of $\mathbb{R}^n$, which finishes the proof. □

We would like to remark that theorem 4.21 does not hold true if we remove the hypothesis 'subanalytic'.

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Example 4.25. Let $f\colon (-\frac{1}{2}, \frac{1}{2}) \to \mathbb{R}$ be the function given by

$$f(t) = \int_0^t \frac{1}{\ln |x|}\, \mathrm{d}x.$$

Then, $X = \mathrm{Graph}(f) \subset \mathbb{R}^2$ is a C^1 submanifold, however it is not a $C^{1,\alpha}$ submanifold for any $\alpha > 0$, since f is C^1 function, however f' is not α-Hölder for any $\alpha > 0$.

4.7. C^1 singularities

We can see in example 4.5 that there is a subanalytic minimal hypersurface with non-isolated singularities which is a Lipschitz submanifold of $\mathbb{R}^3$. However, in contrast with example 4.5, we have the following result.

Proposition 4.26. *Let $X \subset \mathbb{R}^n$ be a minimal variety. If X is a subanalytic C^1 submanifold of $\mathbb{R}^n$ then X is a smooth submanifold of $\mathbb{R}^n$.*

Proof. Since the problem is local and X is a C^1 submanifold of $\mathbb{R}^n$, we can assume that X is a graph of a C^1 function $u\colon B_r^m(p) \to \mathbb{R}^k$. Moreover, $\mathrm{Sing}(X)$ is a bounded subanalytic set and $\dim \mathrm{Sing}(X) \leqslant m - 1$. In particular, we have that $\mathcal{H}^{m-1}(\mathrm{Sing}(X)) < +\infty$. Therefore, u is a weak solution of (2.2) (see theorem 1.2 in [**22**]), which implies that u is real analytic (cf. theorem 2.2 in [**24**]; see also [**25, 26**]) and, in particular, X is a smooth submanifold of $\mathbb{R}^n$. □

Finally, we would like to remark that the above results hold true if we assume that the set X is definable in a polynomially bounded O-minimal structure on $\mathbb{R}$, instead of the assumption that X is a globally subanalytic or subanalytic set. In order to know more about O-minimal structures, for example, see [**12**].

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