Bull. Aust. Math. Soc. **106** (2022), 320–332
doi:10.1017/S0004972722000314

COMPLEX STRUCTURES ON STRATIFIED LIE ALGEBRAS

JUNZE ZHANG

(Received 2 January 2022; accepted 2 March 2022; first published online 19 April 2022)

Abstract

We investigate some properties of complex structures on Lie algebras. In particular, we focus on nilpotent complex structures that are characterised by suitable J-invariant ascending or descending central series, $\mathfrak{d}^j$ and $\mathfrak{d}_j$, respectively. We introduce a new descending series $\mathfrak{p}_j$ and use it to prove a new characterisation of nilpotent complex structures. We also examine whether nilpotent complex structures on stratified Lie algebras preserve the strata. We find that there exists a J-invariant stratification on a step 2 nilpotent Lie algebra with a complex structure.

2020 *Mathematics subject classification*: primary 22E25; secondary 20G07.

Keywords and phrases: complex structure, stratified Lie algebra, nilpotent Lie algebra, J-invariant central series.

1. Introduction

In recent years, complex structures on nilpotent Lie algebras have been shown to be very useful for understanding some geometric and algebraic properties of nilmanifolds. In [3, 4], Cordero *et al.* introduced *nilpotent complex structures*, studied 6-dimensional nilpotent Lie algebras with nilpotent complex structures and provided a classification. Since the ascending central series is not necessarily J-invariant, they introduced a J-invariant ascending central series to characterise nilpotent complex structures. More recently, Latorre, Ugarte and Villacampa defined the space of nilpotent complex structures on nilpotent Lie algebras and studied complex structures on nilpotent Lie algebras with one-dimensional centre [10, 11]. They also provided a theorem describing the ascending central series of 8-dimensional nilpotent Lie algebras with complex structures. In [7], Gao *et al.* studied the relation between the step of a nilpotent Lie algebra and the smallest integer j_0 such that the J-invariant ascending central series stops. Furthermore, they introduced a J-invariant descending central series, which is another tool to characterise nilpotent complex structures. These papers use the language of differential forms to characterise nilpotent complex structures. Our proofs are purely Lie algebraic.

Let G be a Lie group and $\mathfrak{g} \cong T_eG$ be its Lie algebra, which we always assume to be real, unless otherwise stated. A linear isomorphism $J : TG \to TG$ is an *almost complex structure* if $J^2 = -I$. By the Newlander–Nirenberg theorem [13], an almost complex structure J corresponds to a left invariant complex structure on G if and only if

$$[J_eX, J_eY] - [X, Y] - J_e([J_eX, Y] + [X, J_eY]) = 0, \tag{1.1}$$

for all $X, Y \in \mathfrak{g}$. Since we are interested only in Lie algebras in this paper, from now on, we will write J for J_e. We will refer to (1.1) as the *Newlander–Nirenberg condition*.

2. Complex structures on nilpotent Lie algebras

In this section, we consider some properties of the central series of a nilpotent Lie algebra with complex structure J and define a J-invariant central series. We define nilpotent complex structures, and relate their properties to the dimension of the centre $\mathfrak{z}$ of a nilpotent Lie algebra.

Definition 2.1 (see, for example, [9]). Let $\mathfrak{g}$ be a Lie algebra. The *descending central series* and *ascending central series* of $\mathfrak{g}$ are denoted by $\mathfrak{c}_j(\mathfrak{g})$ and $\mathfrak{c}^j(\mathfrak{g})$, respectively, for all $j \geq 0$, and defined inductively by

$$\mathfrak{c}_0(\mathfrak{g}) = \mathfrak{g}, \quad \mathfrak{c}_j(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{c}_{j-1}(\mathfrak{g})];$$
$$\mathfrak{c}^0(\mathfrak{g}) = \{0\}, \quad \mathfrak{c}^j(\mathfrak{g}) = \{X \in \mathfrak{g} : [X, \mathfrak{g}] \subseteq \mathfrak{c}^{j-1}(\mathfrak{g})\}.$$

Remark 2.2.

(i) Notice that $\mathfrak{c}^1(\mathfrak{g}) = \mathfrak{Z}(\mathfrak{g})$, $\mathfrak{c}_1(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{c}^j(\mathfrak{g})/\mathfrak{c}^{j-1}(\mathfrak{g}) = \mathfrak{Z}(\mathfrak{g}/\mathfrak{c}^{j-1}(\mathfrak{g}))$ for all $j \geq 1$, where $\mathfrak{Z}(\cdot)$ means the centre of a Lie algebra. Furthermore, $\mathfrak{c}_j(\mathfrak{g})/\mathfrak{c}_{j+1}(\mathfrak{g}) \subseteq \mathfrak{Z}(\mathfrak{n}/\mathfrak{c}_{j+1}(\mathfrak{g}))$ for all $j \geq 0$. It is clear that $\mathfrak{c}^j(\mathfrak{g})$ and $\mathfrak{c}_j(\mathfrak{g})$ are ideals of $\mathfrak{g}$ for all $j \geq 0$.

(ii) A Lie algebra $\mathfrak{g}$ is called *nilpotent of step* k, for some $k \in \mathbb{N}$, if $\mathfrak{c}_k(\mathfrak{g}) = \{0\}$ and $\mathfrak{c}_{k-1}(\mathfrak{g}) \neq \{0\}$. We will denote nilpotent Lie algebras by $\mathfrak{n}$ in this paper. (See, for example, [8, Section 5.2] or [9].)

2.1. *J*-invariant central series and nilpotent complex structures.

Following [3, Definition 1], we define the *J-invariant ascending central series* $\mathfrak{d}^j$ for nilpotent Lie algebras and introduce *nilpotent complex structures* on nilpotent Lie algebras. Furthermore, we recall the definition of the *J-invariant descending central series* $\mathfrak{d}_j$ [7, Definition 2.7].

Definition 2.3. Let $\mathfrak{n}$ be a Lie algebra with a complex structure J. Define a sequence of J-invariant ideals of $\mathfrak{n}$ by $\mathfrak{d}^0 = \{0\}$ and

$$\mathfrak{d}^j = \{X \in \mathfrak{n} : [X, \mathfrak{n}] \subseteq \mathfrak{d}^{j-1}, [JX, \mathfrak{n}] \subseteq \mathfrak{d}^{j-1}\} \tag{2.1}$$

for all $j \geq 1$. We call the sequence $\mathfrak{d}^j$ the *J-invariant ascending central series*. The complex structure J is called *nilpotent of step* j_0 if there exists $j_0 \in \mathbb{N}$ such that $\mathfrak{d}^{j_0} = \mathfrak{n}$ and $\mathfrak{d}^{j_0-1} \subset \mathfrak{n}$.

We define inductively the *J-invariant descending central series* by

$$\mathfrak{d}_0 = \mathfrak{n}, \qquad \mathfrak{d}_j = [\mathfrak{d}_{j-1}, \mathfrak{n}] + J[\mathfrak{d}_{j-1}, \mathfrak{n}] \quad \text{for all } j \geq 1. \tag{2.2}$$

Remark 2.4.

(i) For the ascending J-invariant central series $\mathfrak{d}^j$,

$$\mathfrak{d}^j/\mathfrak{d}^{j-1} = \mathfrak{Z}(\mathfrak{n}/\mathfrak{d}^{j-1}) \cap J\mathfrak{Z}(\mathfrak{n}/\mathfrak{d}^{j-1}) \quad \text{for all } j \geq 1.$$

In particular, $\mathfrak{d}^1 = \mathfrak{z} \cap J\mathfrak{z}$, which is the largest J-invariant subspace of $\mathfrak{z}$ and, if J is nilpotent, then $\mathfrak{d}^1 \neq \{0\}$. The nilpotency of J implies that the ascending J-invariant central series $\mathfrak{d}^j$ of $\mathfrak{n}$ is strictly increasing until $\mathfrak{d}^{j_0} = \mathfrak{n}$. Furthermore, if $\mathfrak{n}$ is a step k nilpotent Lie algebra with a nilpotent complex structure J of step j_0, then $k \leq j_0 \leq \frac{1}{2}\dim \mathfrak{n}$ (see, for example, [3, 7]).

(ii) By definition, if $\mathfrak{n}$ admits a nilpotent complex structure, then $\mathfrak{n}$ is nilpotent.

(iii) For all $j \geq 0$, it is clear that $\mathfrak{c}_j(\mathfrak{n}) + J\mathfrak{c}_j(\mathfrak{n}) \subseteq \mathfrak{d}_j$; furthermore, $\mathfrak{d}_j \trianglelefteq \mathfrak{n}$ and $\mathfrak{d}^j \trianglelefteq \mathfrak{n}$ where $\trianglelefteq$ is the notation for an ideal.

(iv) Let $\mathfrak{n}$ be a Lie algebra with a complex structure J. Then J preserves all terms of $\mathfrak{c}^j(\mathfrak{n})$ if and only if $\mathfrak{d}^j = \mathfrak{c}^j(\mathfrak{n})$ for all $j \geq 0$ [3, Corollary 5]. Similarly, J preserves all terms of $\mathfrak{c}_j(\mathfrak{n})$ if and only if $\mathfrak{d}_j = \mathfrak{c}_j(\mathfrak{n})$ for all j.

The following lemma provides a connection between J-invariant ascending and descending central series.

Lemma 2.5. *Let $\mathfrak{n}$ be a Lie algebra with a complex structure J.*

(i) *If J is nilpotent of step j_0, then $\mathfrak{n}/\mathfrak{d}^{j_0-1}$ is Abelian. Conversely, if there exists $j_0 \in \mathbb{N}$ such that $\mathfrak{n}/\mathfrak{d}^{j_0-1}$ is Abelian, then J is nilpotent of step at most j_0.*

(ii) *If J is nilpotent of step j_0, then $\mathfrak{d}_j \subseteq \mathfrak{d}^{j_0-j}$ for all $j \geq 0$. Conversely, if there exists $j_0 \in \mathbb{N}$ such that $\mathfrak{d}_j \subseteq \mathfrak{d}^{j_0-j}$ for all $j \geq 0$, then J is nilpotent of step at most j_0.*

Proof. For part (i), suppose that J is nilpotent of step j_0. By definition, $\mathfrak{d}^{j_0} = \mathfrak{n}$ and $\mathfrak{d}^{j_0-1} \subset \mathfrak{n}$. Then $\mathfrak{Z}(\mathfrak{n}/\mathfrak{d}^{j_0-1}) \cap J\mathfrak{Z}(\mathfrak{n}/\mathfrak{d}^{j_0-1}) = \mathfrak{n}/\mathfrak{d}^{j_0-1}$. It is obvious that $\mathfrak{Z}(\mathfrak{n}/\mathfrak{d}^{j_0-1}) = \mathfrak{n}/\mathfrak{d}^{j_0-1}$. Hence, $\mathfrak{n}/\mathfrak{d}^{j_0-1}$ is Abelian.

Conversely, suppose that there exists $j_0 \in \mathbb{N}$ such that $\mathfrak{n}/\mathfrak{d}^{j_0-1}$ is Abelian. Then $\{0\} \neq \mathfrak{c}_1(\mathfrak{n}) \subseteq \mathfrak{d}^{j_0-1}$. For all $X \in \mathfrak{n}$, we have $[X, \mathfrak{n}] \subseteq \mathfrak{d}^{j_0-1}$ and $[JX, \mathfrak{n}] \subseteq \mathfrak{d}^{j_0-1}$. We deduce that $\mathfrak{n} = \mathfrak{d}^{j_0}$ and therefore J is nilpotent of step at most j_0.

For part (ii), assume that J is nilpotent of step j_0. By definition, $\mathfrak{d}_0 = \mathfrak{n} = \mathfrak{d}^{j_0}$. Next, assume that $\mathfrak{d}_{s-1} \subseteq \mathfrak{d}^{j_0-s+1}$ for some $s \in \mathbb{N}$. Then

$$\mathfrak{d}_s = [\mathfrak{d}_{s-1}, \mathfrak{n}] + J[\mathfrak{d}_{s-1}, \mathfrak{n}] \subseteq [\mathfrak{d}^{j_0-s+1}, \mathfrak{n}] + J[\mathfrak{d}^{j_0-s+1}, \mathfrak{n}] \subseteq \mathfrak{d}^{j_0-s} + J\mathfrak{d}^{j_0-s} = \mathfrak{d}^{j_0-s}.$$

Hence, by induction, $\mathfrak{d}_j \subseteq \mathfrak{d}^{j_0-j}$ for all $j \geq 0$.

Conversely, suppose that there exists $j_0 \in \mathbb{N}$ such that $\mathfrak{d}_j \subseteq \mathfrak{d}^{j_0-j}$ for all $j \geq 0$. In particular, $\mathfrak{d}_1 \subseteq \mathfrak{d}^{j_0-1}$. By definition, $\mathfrak{c}_1(\mathfrak{n}) \subseteq \mathfrak{d}_1$. It follows that

$$[\mathfrak{n}/\mathfrak{d}^{j_0-1}, \mathfrak{n}/\mathfrak{d}^{j_0-1}] \subseteq [\mathfrak{n}, \mathfrak{n}] + \mathfrak{d}^{j_0-1} = \mathfrak{c}_1(\mathfrak{n}) + \mathfrak{d}^{j_0-1} \subseteq \mathfrak{d}_1 + \mathfrak{d}^{j_0-1} \subseteq \mathfrak{d}_{j_0-1},$$

and thus $\mathfrak{n}/\mathfrak{d}^{j_0-1}$ is Abelian. From Lemma 2.5, J is nilpotent of step at most j_0. □

REMARK 2.6. Under the condition of Lemma 2.5, if J is nilpotent of step j_0, then $\mathfrak{d}_{j_0-1} \subseteq \mathfrak{d}^1 \subseteq \mathfrak{z}$ and $\mathfrak{d}_{j_0-1}$ is Abelian. Furthermore, there exists $j_0 \in \mathbb{N}$ such that $\mathfrak{n}/\mathfrak{d}^{j_0-1}$ is Abelian if and only if $\mathfrak{d}_j \subseteq \mathfrak{d}^{j_0-j}$ for all $j \geq 0$. This is proved by induction as in the proof of Lemma 2.5.

COROLLARY 2.7. *Let $\mathfrak{n}$ be a step k nilpotent Lie algebra with a complex structure J. Then J is nilpotent of step k if and only if $\mathfrak{d}_j \subseteq \mathfrak{d}^{k-j}$ for all $j \geq 0$.*

PROOF. Suppose that J is nilpotent of step k. By Lemma 2.5, $\mathfrak{d}_j \subseteq \mathfrak{d}^{k-j}$ for all $j \geq 0$. Conversely, assume that $\mathfrak{d}_j \subseteq \mathfrak{d}^{k-j}$ for all j. Again by Lemma 2.5, J is nilpotent of step at most k. Furthermore, it follows that $\{0\} \neq \mathfrak{c}_{k-1}(\mathfrak{n}) \subseteq \mathfrak{d}_{k-1}$. Therefore, $\mathfrak{d}_{k-1} \neq \{0\}$ and J is nilpotent of step k. □

REMARK 2.8. From Remark 2.6, J is nilpotent of step k if and only if $\mathfrak{n}/\mathfrak{d}^{k-1}$ is Abelian.

We introduce a new descending central series whose descending 'rate' is slower than that of $\mathfrak{c}_j(\mathfrak{n})$ but faster than that of $\mathfrak{d}_j$.

DEFINITION 2.9. Let J be a complex structure on a Lie algebra $\mathfrak{n}$. We define the sequence $\mathfrak{p}_j$ inductively by

$$\mathfrak{p}_0 = \mathfrak{n} \quad \text{and} \quad \mathfrak{p}_j = [\mathfrak{p}_{j-1}, \mathfrak{n}] + [J\mathfrak{p}_{j-1}, \mathfrak{n}] \quad \text{for all } j \geq 1. \tag{2.3}$$

REMARK 2.10. It is clear that $\mathfrak{p}_{j+1} \subseteq \mathfrak{p}_j$ for all $j \geq 0$. Furthermore, $\mathfrak{p}_j \trianglelefteq \mathfrak{n}$ since $[\mathfrak{p}_j, \mathfrak{n}] \subseteq \mathfrak{p}_{j+1} \subseteq \mathfrak{p}_j$ for all $j \geq 0$.

LEMMA 2.11. *Let $\mathfrak{n}$ be a Lie algebra with a complex structure J. Then $\mathfrak{c}_j(\mathfrak{n}) \subseteq \mathfrak{p}_j$ for all $j \geq 0$. Furthermore, $\mathfrak{p}_j \subseteq \mathfrak{d}_j$ and $J\mathfrak{p}_j \subseteq \mathfrak{d}_j$ for all $j \geq 0$.*

PROOF. By definition, $\mathfrak{c}_0(\mathfrak{n}) = \mathfrak{n} = \mathfrak{p}_0$. It follows, by induction, that $\mathfrak{c}_j(\mathfrak{n}) \subseteq \mathfrak{p}_j$ for all $j \geq 0$. Using (2.2), $[\mathfrak{d}_{j-1}, \mathfrak{n}] \subseteq \mathfrak{d}_j$. By definition, $\mathfrak{p}_0 = \mathfrak{n} = \mathfrak{d}_0$ and $J\mathfrak{p}_0 = J\mathfrak{n} = \mathfrak{n} = \mathfrak{d}_0$. Next, suppose that $\mathfrak{p}_s \subseteq \mathfrak{d}_s$ and $J\mathfrak{p}_s \subseteq \mathfrak{d}_s$ for some $s \in \mathbb{N}$. Then by (2.3),

$$\mathfrak{p}_{s+1} = [\mathfrak{p}_s, \mathfrak{n}] + [J\mathfrak{p}_s, \mathfrak{n}] \subseteq [\mathfrak{d}_s, \mathfrak{n}] \subseteq \mathfrak{d}_{s+1} \quad \text{and} \quad J\mathfrak{p}_{s+1} \subseteq J[\mathfrak{d}_s, \mathfrak{n}] \subseteq \mathfrak{d}_{s+1}.$$

By induction, $\mathfrak{p}_j \subseteq \mathfrak{d}_j$ and $J\mathfrak{p}_j \subseteq \mathfrak{d}_j$ for all $j \geq 0$. □

REMARK 2.12.

(i) Notice that $\mathfrak{p}_j/\mathfrak{p}_{j+1} \subseteq \mathfrak{Z}(\mathfrak{n}/\mathfrak{p}_{j+1})$ for all $j \geq 0$. Indeed, for all $P \in \mathfrak{p}_j$ and $Y \in \mathfrak{n}$, since $[P, Y] \subseteq \mathfrak{p}_{j+1}$, it is enough to deduce

$$[P + \mathfrak{p}_{j+1}, Y + \mathfrak{p}_{j+1}] = [P, Y] + \mathfrak{p}_{j+1} \subseteq \mathfrak{p}_{j+1}.$$

Hence, $\mathfrak{p}_j/\mathfrak{p}_{j+1} \subseteq \mathfrak{Z}(\mathfrak{n}/\mathfrak{p}_{j+1})$.

(ii) By Lemma 2.11, $\mathfrak{p}_j + J\mathfrak{p}_j \subseteq \mathfrak{d}_j$ for all $j \geq 0$. We show that $\mathfrak{p}_j + J\mathfrak{p}_j \trianglelefteq \mathfrak{n}$ for all $j \geq 0$. Indeed, for all $P, P' \in \mathfrak{p}_j$,

$$\underbrace{[P + JP', \mathfrak{n}]}_{\subseteq [\mathfrak{p}_j + J\mathfrak{p}_j, \mathfrak{n}]} \subseteq \underbrace{[P, \mathfrak{n}]}_{\subseteq \mathfrak{p}_{j+1}} + \underbrace{[JP', \mathfrak{n}]}_{\subseteq \mathfrak{p}_{j+1}} \subseteq \mathfrak{p}_{j+1} \subseteq \mathfrak{p}_{j+1} + J\mathfrak{p}_{j+1} \subseteq \mathfrak{p}_j + J\mathfrak{p}_j.$$

Hence, $\mathfrak{p}_j + J\mathfrak{p}_j \trianglelefteq \mathfrak{n}$. From part (ii), we can show that $\mathfrak{p}_j + J\mathfrak{p}_j$ is a J-invariant descending central series. Indeed, for all $T = P + JP' \in \mathfrak{p}_j + J\mathfrak{p}_j$ and $Y \in \mathfrak{n}$,

$$[T + \mathfrak{p}_{j+1} + J\mathfrak{p}_{j+1}, Y + \mathfrak{p}_{j+1} + J\mathfrak{p}_{j+1}] \subseteq [T, Y] + \mathfrak{p}_{j+1} + J\mathfrak{p}_{j+1} \subseteq \mathfrak{p}_{j+1} + J\mathfrak{p}_{j+1}.$$

THEOREM 2.13. *Let $\mathfrak{n}$ be a Lie algebra with a complex structure J. The following are equivalent:*

(i) *J is nilpotent of step j_0;*
(ii) $\mathfrak{p}_{j_0} = \{0\}$ *and* $\mathfrak{p}_{j_0-1} \neq \{0\}$;
(iii) $\mathfrak{d}_{j_0} = \{0\}$ *and* $\mathfrak{d}_{j_0-1} \neq \{0\}$.

PROOF. We first show that (i) and (ii) are equivalent. Assume that J is nilpotent of step j_0. From Lemma 2.5(ii), $\mathfrak{d}_{j_0-1} \subseteq \mathfrak{d}^1$. Hence, by Lemma 2.11,

$$\mathfrak{p}_{j_0} \subseteq [\mathfrak{d}_{j_0-1}, \mathfrak{n}] \subseteq [\mathfrak{d}^1, \mathfrak{n}] = \{0\}.$$

Thus, $\mathfrak{p}_{j_0} = \{0\}$. Assume, by contradiction, that $\mathfrak{p}_{j_0-1} = \{0\}$. We show by induction that $\mathfrak{p}_{j_0-j-1} + J\mathfrak{p}_{j_0-j-1} \subseteq \mathfrak{d}^j$ for all $j \geq 0$. By definition, $\mathfrak{p}_{j_0-1} + J\mathfrak{p}_{j_0-1} = \{0\} = \mathfrak{d}^0$. Next, suppose that $\mathfrak{p}_{j_0-s-1} + J\mathfrak{p}_{j_0-s-1} \subseteq \mathfrak{d}^s$ for some $s \in \mathbb{N}$. Then from Remark 2.12(ii),

$$[\mathfrak{p}_{j_0-s-2} + J\mathfrak{p}_{j_0-s-2}, \mathfrak{n}] \subseteq \mathfrak{p}_{j_0-s-1} + J\mathfrak{p}_{j_0-s-1} \subseteq \mathfrak{d}^s.$$

This implies, using (2.1), $\mathfrak{p}_{j_0-s-2} + J\mathfrak{p}_{j_0-s-2} \subseteq \mathfrak{d}^{s+1}$. By induction, $\mathfrak{p}_{j_0-j-1} + J\mathfrak{p}_{j_0-j-1} \subseteq \mathfrak{d}^j$ for all $j \geq 0$. In particular, let $j = j_0 - 1$. Then $\mathfrak{n} \subseteq \mathfrak{d}^{j_0-1}$, which implies that J is nilpotent of step $j_0 - 1$ by definition. This is a contradiction. Therefore, $\mathfrak{p}_{j_0-1} \neq \{0\}$.

Conversely, suppose that $\mathfrak{p}_{j_0} = \{0\}$ and $\mathfrak{p}_{j_0-1} \neq \{0\}$. We show that J is nilpotent of step j_0. By definition, $\mathfrak{p}_{j_0} + J\mathfrak{p}_{j_0} = \{0\} = \mathfrak{d}^0$. It follows, by induction, that $\mathfrak{p}_{j_0-j} + J\mathfrak{p}_{j_0-j} \subseteq \mathfrak{d}^j$ for all $j \geq 0$. Hence, $\mathfrak{p}_{j_0-j} \subseteq \mathfrak{d}^j$. In particular, let $j = j_0 - 1$. Then $\mathfrak{p}_1 = [\mathfrak{n}, \mathfrak{n}] \subseteq \mathfrak{d}^{j_0-1}$, which implies that $\mathfrak{n}/\mathfrak{d}^{j_0-1}$ is Abelian. By Lemma 2.5, J is nilpotent of step at most j_0.

We next show that $\mathfrak{d}^{j_0-1} \neq \mathfrak{n}$. Assume, by contradiction, that $\mathfrak{n} = \mathfrak{d}^{j_0-1}$. We show by induction that $\mathfrak{p}_{j-1} \subseteq \mathfrak{d}^{j_0-j}$ for all $j \geq 1$. By definition, $\mathfrak{p}_0 = \mathfrak{n} = \mathfrak{d}^{j_0-1}$. Next, suppose that $\mathfrak{p}_{s-1} \subseteq \mathfrak{d}^{j_0-s}$ for some $s \in \mathbb{N}$. Then

$$\mathfrak{p}_s = [\mathfrak{p}_{s-1}, \mathfrak{n}] + [J\mathfrak{p}_{s-1}, \mathfrak{n}] \subseteq [\mathfrak{d}^{j_0-s}, \mathfrak{n}] + [J\mathfrak{d}^{j_0-s}, \mathfrak{n}] \subseteq \mathfrak{d}^{j_0-s-1}.$$

By induction, $\mathfrak{p}_{j-1} \subseteq \mathfrak{d}^{j_0-j}$ for all $j \geq 1$. In particular, let $j = j_0$. We deduce that $\mathfrak{p}_{j_0-1} \subseteq \mathfrak{d}^0 = \{0\}$. This implies that $\mathfrak{p}_{j_0-1} = \{0\}$ which is a contradiction. Hence, $\mathfrak{d}^{j_0-1} \neq \mathfrak{n}$. By definition, J is nilpotent of step j_0.

We now show (i) and (iii) are equivalent. Since J is nilpotent of step j_0, it follows from Lemma 2.5(ii) that $\mathfrak{d}_j \subseteq \mathfrak{d}^{j_0-j}$ for all $j \geq 0$. In particular, let $j = j_0$. By definition, $\mathfrak{d}_{j_0} = \mathfrak{d}^0 = \{0\}$. We show that $\mathfrak{d}_{j_0-1} \neq \{0\}$. By Lemma 2.11, $\{0\} \neq \mathfrak{p}_{j_0-1} + J\mathfrak{p}_{j_0-1} \subseteq \mathfrak{d}_{j_0-1}$. Hence, $\mathfrak{d}_{j_0-1} \neq \{0\}$.

Conversely, assume that $\mathfrak{d}_{j_0} = \{0\}$ and $\mathfrak{d}_{j_0-1} \neq \{0\}$. From the definition, $[\mathfrak{d}_{j_0-1}, \mathfrak{n}] \subseteq \mathfrak{d}_{j_0} = \{0\}$. Hence, $\{0\} \neq \mathfrak{d}_{j_0-1} \subseteq \mathfrak{d}^1$. Next, assume that $\mathfrak{d}_{j_0-s} \subseteq \mathfrak{d}^s$ for some $s \in \mathbb{N}$. Then by definition,

$$[\mathfrak{d}_{j_0-s-1}, \mathfrak{n}] \subseteq \mathfrak{d}_{j_0-s} \subseteq \mathfrak{d}^s.$$

By (2.1), $\mathfrak{d}_{j_0-s-1} \subseteq \mathfrak{d}^{s+1}$. By induction, $\mathfrak{d}_{j_0-j} \subseteq \mathfrak{d}^j$ for all $j \geq 0$. Let $j = j_0$. We find that $\mathfrak{d}_0 = \mathfrak{n} \subseteq \mathfrak{d}^{j_0}$. Therefore, $\mathfrak{d}^{j_0} = \mathfrak{n}$ and J is nilpotent of step at most j_0.

We next show that $\mathfrak{d}^{j_0-1} \neq \mathfrak{n}$. Suppose not, that is, $\mathfrak{n} = \mathfrak{d}^{j_0-1}$. By definition, $\mathfrak{d}_0 = \mathfrak{n} = \mathfrak{d}^{j_0-1}$. It follows, by induction, that $\mathfrak{d}_{j-1} \subseteq \mathfrak{d}^{j_0-j}$ for all $j \geq 1$. Let $j = j_0$. We find that $\mathfrak{d}_{j_0-1} \subseteq \{0\}$, a contradiction. Hence, $\mathfrak{d}^{j_0-1} \neq \mathfrak{n}$ and J is nilpotent of step j_0.

Finally, since (i) is equivalent to both (ii) and (iii), we conclude that (ii) and (iii) are equivalent. □

REMARK 2.14. Suppose that a Lie algebra $\mathfrak{n}$ admits a nilpotent complex structure J of step j_0. Then

$$\mathfrak{c}_j(\mathfrak{n}) + J\mathfrak{c}_j(\mathfrak{n}) \subseteq \mathfrak{p}_j + J\mathfrak{p}_j \subseteq \mathfrak{d}_j \subseteq \mathfrak{d}^{j_0-j} \tag{2.4}$$

for all $j \geq 0$.

It is shown, in [3, Corollary 7], that if $\mathfrak{c}_j(\mathfrak{n})$ is J-invariant for all $j \geq 0$, then J is nilpotent. We will provide a different approach to this.

COROLLARY 2.15. *Let* $\mathfrak{n}$ *be a step* k *nilpotent Lie algebra with a complex structure* J*. Suppose that all* $\mathfrak{c}_j(\mathfrak{n})$ *are* J*-invariant. Then* $\mathfrak{p}_j = \mathfrak{c}_j(\mathfrak{n})$ *for all* $j \geq 0$*. Furthermore,* J *is nilpotent of step* k*.*

PROOF. Since all $\mathfrak{c}_j(\mathfrak{n})$ are J-invariant, by definition, $\mathfrak{p}_0 = \mathfrak{n} = \mathfrak{c}_0(\mathfrak{n})$. By induction, $\mathfrak{p}_j = \mathfrak{c}_j(\mathfrak{n})$ for all $j \geq 0$. Therefore, $\mathfrak{p}_k = \mathfrak{c}_k(\mathfrak{n}) = \{0\}$ and $\mathfrak{p}_{k-1} = \mathfrak{c}_{k-1}(\mathfrak{n}) \neq \{0\}$. By Theorem 2.13, J is nilpotent of step k. □

COROLLARY 2.16. *Let* $\mathfrak{n}$ *be a step* k *nilpotent Lie algebra with a nilpotent complex structure* J *of step* k*. Suppose that* $\mathfrak{c}_{k-1}(\mathfrak{n}) = \mathfrak{z}$*. Then* $\mathfrak{z}$ *is* J*-invariant.*

PROOF. Since J is nilpotent of step k, by (2.4),

$$\mathfrak{z} + J\mathfrak{z} \subseteq \mathfrak{d}_{k-1} \subseteq \mathfrak{d}^1 \subseteq \mathfrak{z} \Rightarrow [\mathfrak{z} + J\mathfrak{z}, \mathfrak{n}] = \{0\}.$$

Hence, $J\mathfrak{z} = \mathfrak{z}$. □

COROLLARY 2.17. *Let* $\mathfrak{n}$ *be a Lie algebra with a nilpotent complex structure* J *of step* j_0*. Then for all* $j \geq 1$, $\mathfrak{d}_{j_0-j}$ *is not contained in* $\mathfrak{d}^{j-1}$*.*

PROOF. Since J is nilpotent of step j_0, by Theorem 2.13, $\mathfrak{d}_{j_0-1} \neq \{0\} = \mathfrak{d}^0$. Hence, $\mathfrak{d}_{j_0-1}$ is not contained in $\mathfrak{d}^0$. Next, suppose that $\mathfrak{d}_{j_0-s+1}$ is not contained in $\mathfrak{d}^{s-2}$ for some integer $s \geq 2$. We show that $\mathfrak{d}_{j_0-s}$ is not contained in $\mathfrak{d}^{s-1}$. Suppose not, that is, $\mathfrak{d}_{j_0-s} \subseteq \mathfrak{d}^{s-1}$. Then

$$\mathfrak{d}_{j_0-s+1} = [\mathfrak{d}_{j_0-s}, \mathfrak{n}] + J[\mathfrak{d}_{j_0-s}, \mathfrak{n}] \subseteq [\mathfrak{d}^{s-1}, \mathfrak{n}] + J[\mathfrak{d}^{s-1}, \mathfrak{n}] \subseteq \mathfrak{d}^{s-2}.$$

It follows that $\mathfrak{d}_{j_0-s+1} \subseteq \mathfrak{d}^{s-2}$. This is a contradiction. Hence, $\mathfrak{d}_{j_0-s}$ is not contained in $\mathfrak{d}^{s-1}$. By induction, for all $j \geq 1$, $\mathfrak{d}_{j_0-j}$ is not contained in $\mathfrak{d}^{j-1}$. □

We investigate the possible range of $\dim \mathfrak{z}$ for a Lie algebra $\mathfrak{n}$ with a nilpotent complex structure J.

PROPOSITION 2.18. *Let* $\mathfrak{n}$ *be a non-Abelian Lie algebra of dimension* $2n$ *with a nilpotent complex structure* J. *Then* $2 \leq \dim \mathfrak{z} \leq 2n - 2$.

PROOF. Recall that $\mathfrak{d}^1 = \mathfrak{z} \cap J\mathfrak{z}$, which is the largest J-invariant subspace of $\mathfrak{z}$. Since J is nilpotent, it is clear that $\mathfrak{d}^1 \neq \{0\}$. Furthermore, since $\mathfrak{d}^1$ is J-invariant, it follows that $2 \leq \dim \mathfrak{d}^1 \leq \dim \mathfrak{z}$. Then the lower bound of $\dim \mathfrak{z}$ is 2.

Next, we show that the upper bound of $\dim \mathfrak{z}$ is $2n - 2$. Since $\mathfrak{n}$ is non-Abelian, it is possible to find $X, Y \in \mathfrak{n}$ such that $0 \neq [X, Y] \in \mathfrak{c}_1(\mathfrak{n})$. Then $\mathrm{span}\{X, Y\}$ is 2-dimensional and $\mathrm{span}\{X, Y\} \cap \mathfrak{z} = \{0\}$. Hence, $\dim \mathfrak{z} \leq 2n - 2$.

In conclusion, $2 \leq \dim \mathfrak{z} \leq 2n - 2$. □

REMARK 2.19. From Proposition 2.18, we can further conclude that if $\dim \mathfrak{z} = 1$, then the complex structure J on $\mathfrak{n}$ is nonnilpotent. In particular, the Lie algebra of $n \times n$ upper triangular matrices does not admit a nilpotent complex structure.

3. Stratified Lie algebras with complex structures

In this section, we consider a special type of nilpotent Lie algebras: *stratified Lie algebras*. Recent results on nilpotent Lie algebras with a stratification can be found in [5, 6, 12]. We start with the definition of stratified Lie algebras.

DEFINITION 3.1. A nilpotent Lie algebra $\mathfrak{n}$ is said to admit a *step k stratification* if it has a vector space decomposition of the form $\mathfrak{n}_1 \oplus \mathfrak{n}_2 \oplus \cdots \oplus \mathfrak{n}_k$, where $\mathfrak{n}_k \neq \{0\}$, satisfying the bracket generating property $[\mathfrak{n}_1, \mathfrak{n}_k] = \{0\}$ and

$$[\mathfrak{n}_1, \mathfrak{n}_{j-1}] = \mathfrak{n}_j \quad \text{for all } j \in \{2, \ldots, k\}.$$

A Lie algebra $\mathfrak{n}$ that admits a stratification is called a *stratified Lie algebra*. A complex structure J on a stratified Lie algebra $\mathfrak{n}$ is said to be *strata-preserving* if it preserves each layer of the stratification.

REMARK 3.2. Let $\mathfrak{n}$ be a step k stratified Lie algebra. By induction,

$$\mathfrak{c}_j(\mathfrak{n}) = \bigoplus_{j+1 \leq l \leq k} \mathfrak{n}_l \quad \text{for all } j \geq 0. \tag{3.1}$$

PROPOSITION 3.3. *Let* $\mathfrak{n}$ *be a* $2n$*-dimensional step* n *nilpotent Lie algebra for some* $n \in \mathbb{N}$. *Suppose that* $\dim \mathfrak{c}_j(\mathfrak{n}) = 2n - 2j$ *for* $1 \leq j \leq n$. *Then* $\mathfrak{n}$ *does not admit a stratification.*

PROOF. Assume, by contradiction, that $\mathfrak{n}$ admits a stratification. By (3.1), $\mathfrak{c}_j(\mathfrak{n}) = \bigoplus_{j+1 \leq l \leq n} \mathfrak{n}_l$ and $\dim \mathfrak{c}_1(\mathfrak{n}) = 2n - 2$, so $\dim \mathfrak{n}_1 = 2$. Since $\mathfrak{n}$ is a stratified Lie algebra, $\mathfrak{n}_2 = [\mathfrak{n}_1, \mathfrak{n}_1]$. Thus, $\dim \mathfrak{n}_2 = 1$ and $\dim \mathfrak{c}_2(\mathfrak{n}) = 2n - 3 > 2n - 4$. This is a contradiction. □

PROPOSITION 3.4. *Let* $\mathfrak{n}$ *be a step* k *stratified Lie algebra with a complex structure* J *and* $k \geq 2$. *Suppose that* $\dim \mathfrak{n}_1 = 2$. *Then* J *is not strata-preserving.*

PROOF. Suppose, by contradiction, that there exists a strata-preserving complex structure J. Then $\dim \mathfrak{n}_j \in 2\mathbb{N}$ for all $j \geq 1$. However, $\dim \mathfrak{n}_1 = 2$ implies that $\dim \mathfrak{n}_2 = 1$, which contradicts the assumption that $\dim \mathfrak{n}_2 \in 2\mathbb{N}$. Hence, $\mathfrak{n}$ does not have a strata-preserving complex structure. □

REMARK 3.5. Let $\mathfrak{n}$ be a step 3 stratified Lie algebra with a strata-preserving complex structure. Arguing in a similar way as in Proposition 3.4, we conclude that $\dim \mathfrak{n} \neq 4$ or 6.

We show that there always exists a stratification on a step 2 nilpotent Lie algebra with a strata-preserving complex structure J.

THEOREM 3.6. *Let $\mathfrak{n}$ be a step 2 nilpotent Lie algebra with a complex structure J. Suppose that $\mathfrak{c}_1(\mathfrak{n})$ is J-invariant. Then $\mathfrak{n}$ admits a J-invariant stratification.*

PROOF. Define a J-invariant inner product ψ by

$$\psi(X, Y) = \phi(X, Y) + \phi(JX, JY) \quad \text{for all } X, Y \in \mathfrak{n},$$

where ϕ is an inner product on $\mathfrak{n}$. We show that there exists a stratification on $\mathfrak{n}$ such that $\mathfrak{n}_1$ and $\mathfrak{n}_2$ are J-invariant. Define $\mathfrak{n}_2 = [\mathfrak{n}, \mathfrak{n}]$ and $\mathfrak{n}_1 = \mathfrak{n}_2^{\perp}$, the orthogonal complement of $\mathfrak{n}_2$ with respect to ψ. Then $\mathfrak{n}_2 = \mathfrak{c}_1(\mathfrak{n})$ is J-invariant and by definition, $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$. Also note that

$$\mathfrak{n}_2 = [\mathfrak{n}_1 \oplus \mathfrak{n}_2, \mathfrak{n}_1 \oplus \mathfrak{n}_2] = [\mathfrak{n}_1, \mathfrak{n}_1].$$

This implies that $\mathfrak{n}_1$ generates $\mathfrak{n}$. Thus, J is a complex structure that preserves both $\mathfrak{n}_1$ and $\mathfrak{n}_2$. □

REMARK 3.7.

(i) Let $\mathfrak{g}$ be an arbitrary Lie algebra. A complex structure J on $\mathfrak{g}$ is called *bi-invariant* if $J[X, Y] = [JX, Y]$ for all $X, Y \in \mathfrak{g}$. That is, $J \circ \mathrm{ad} = \mathrm{ad} \circ J$. A complex structure J is called *Abelian* if $[X, Y] = [JX, JY]$ for all $X, Y \in \mathfrak{g}$. See, for example, [2, 14]. Notice that J preserves all terms of $\mathfrak{c}_j(\mathfrak{n})$ and $\mathfrak{c}^j(\mathfrak{n})$ if J is bi-invariant, while if J is Abelian, J only preserves all terms of $\mathfrak{c}^j(\mathfrak{n})$.

(ii) Suppose that $\mathfrak{n}$ is a step k stratified Lie algebra with a bi-invariant complex structure J. From (3.1), $\mathfrak{c}_j(\mathfrak{n}) = \bigoplus_{j+1 \leq l \leq k} \mathfrak{n}_l$ and it is clear that $\dim \mathfrak{n}_j \in 2\mathbb{N}$ for all $j \in \{1, \ldots, k\}$.

PROPOSITION 3.8. *Let $\mathfrak{n}$ be a step k stratified Lie algebra with a strata-preserving complex structure J. Then $J\mathfrak{c}_j(\mathfrak{n}) = \mathfrak{c}_j(\mathfrak{n})$ for all $j \geq 0$ and J is nilpotent of step k.*

PROOF. We first show that $J\mathfrak{c}_j(\mathfrak{n}) = \mathfrak{c}_j(\mathfrak{n})$ for all $j \geq 0$. Recall, from (2.1), that $\mathfrak{c}_j(\mathfrak{n}) = \bigoplus_{j+1 \leq l \leq k} \mathfrak{n}_l$ and hence $J\mathfrak{c}_j(\mathfrak{n}) = \mathfrak{c}_j(\mathfrak{n})$ for all $j \geq 0$. By Corollary 2.15, J is nilpotent of step k. □

It is known that every step 2 nilpotent Lie algebra may be stratified (see, for example, [12]). We will provide another proof in Theorem 3.9, that every complex structure on a step 2 nilpotent Lie algebra is nilpotent of step 2 or 3 (compare [7,

TABLE 1. Nilpotency of J.

J	Strata-preserving	Nonstrata-preserving
$J\mathfrak{z} = \mathfrak{z}$	J nilpotent of step 2	J nilpotent of step 2
$J\mathfrak{z} \neq \mathfrak{z}$	J nilpotent of step 2	J nilpotent of step 3

Theorem 1.3] and [15, Proposition 3.3]). In what follows, we denote by $\mathfrak{k} = \mathfrak{n}_2 \cap J\mathfrak{n}_2$ the largest J-invariant subspace contained in $\mathfrak{n}_2$ and we also remind the reader that $\mathfrak{d}^1 = \mathfrak{z} \cap J\mathfrak{z}$ is the largest J-invariant subspace contained in $\mathfrak{z}$.

THEOREM 3.9. *Let* $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$ *be a step* 2 *nilpotent Lie algebra with a complex structure* J *and a* J*-invariant inner product* ψ.

(i) *If* $\mathfrak{k} = \{0\}$, *then* $\mathfrak{d}_1$ *is Abelian and* J *is nilpotent of step* 2.
(ii) *If* $\{0\} \neq \mathfrak{k} \subset \mathfrak{n}_2$, *then* J *is nilpotent of step* 3.
(iii) *If* $\mathfrak{n}_2 = \mathfrak{k}$, *then* J *is strata-preserving and nilpotent of step* 2.

In conclusion, J *is nilpotent of either step* 2 *or* 3.

PROOF. We start with parts (i) and (ii) together. Suppose that $J\mathfrak{n}_2 \neq \mathfrak{n}_2$. Then, $\mathfrak{p}_2 = [J\mathfrak{n}_2, \mathfrak{n}] \subseteq \mathfrak{n}_2$. For all $Z_2 \in \mathfrak{n}_2$ and $X, JX \in \mathfrak{n}$, by the Newlander–Nirenberg condition,

$$[J\mathfrak{n}_2, \mathfrak{n}] \ni [JZ_2, JX] = J[JZ_2, X] \in J[J\mathfrak{n}_2, \mathfrak{n}]. \tag{3.2}$$

This implies that $\mathfrak{p}_2$ is J-invariant in $\mathfrak{n}_2$. We now consider the following two possibilities.

(i) If $\mathfrak{k} = \{0\}$, then from (3.2), $\mathfrak{p}_2 = \{0\}$. By Theorem 2.13, J is nilpotent of step 2.
(ii) If $\{0\} \neq \mathfrak{k} \subset \mathfrak{n}_2$, since $\{0\} \neq \mathfrak{p}_2 \subseteq \mathfrak{k}$ and $J\mathfrak{p}_2 \subset \mathfrak{n}_2$, then by definition, $\mathfrak{p}_3 = \{0\}$. By Theorem 2.13, J is nilpotent of step 3.

Finally, for part (iii), suppose that $\mathfrak{n}_2 = \mathfrak{k}$. We find that J preserves $\mathfrak{n}_2$. By Theorem 3.6, J is strata-preserving. From Corollary 3.8, J is nilpotent of step 2.

In conclusion, J is either nilpotent of step 2 or 3. □

REMARK 3.10.

(i) If J is nilpotent of step 3, then there does not necessarily exist a J-invariant stratification.
(ii) We recall, from [7, Theorem 1.3], if $\mathfrak{z}$ is not J-invariant, then J is nilpotent of step 3. From Theorem 3.9, we derive Table 1.

From Table 1, if J is nilpotent of step 2, then J is either strata-preserving or centre-preserving. More precisely, we conclude that either $\mathfrak{k} = \mathfrak{n}_2 \cap J\mathfrak{n}_2 = \{0\}$ or $J\mathfrak{n}_2 = \mathfrak{n}_2$. Indeed, if $\mathfrak{n}$ is step 2 nilpotent Lie algebra with a nilpotent complex structure J of step 2, J may not be strata-preserving.

An even dimensional nilpotent Lie algebra with $\dim \mathfrak{c}_1(\mathfrak{n}) = 1$ has step 2. There does not exist a J-invariant stratification for dimensional reasons. We give the following result for $\dim \mathfrak{c}_1(\mathfrak{n}) \geq 2$.

THEOREM 3.11. *Let* $\mathfrak{n}$ *be a step* 2 *stratified Lie algebra with a complex structure* J.

(i) *Suppose that* $\dim \mathfrak{n}_2 = 2$. *Then*

  (a) J *is nilpotent of step* 2;
  (b) *if* $\dim \mathfrak{d}^1 = 2$, *then* $J\mathfrak{n}_2 = \mathfrak{n}_2$.

(ii) *Suppose that* $\dim \mathfrak{n}_2 = 2l$ *for some* $l \geq 2 \in \mathbb{N}$. *Furthermore, assume that* $\dim \mathfrak{d}^1 \leq 4l - 2$ *and* $J\mathfrak{n}_2 \neq \mathfrak{n}_2$. *Then* J *is nilpotent of step* 3.

PROOF. By Theorem 3.9, J is nilpotent of either step 2 or 3.

Start with part (i). Assume that $\dim \mathfrak{n}_2 = 2$. For part (a), notice that J could be either strata-preserving or not. If J is strata-preserving, by Theorem 3.9(iii), J is nilpotent of step 2. Otherwise, J is not strata-preserving. Since $\dim \mathfrak{n}_2 = 2$, it follows that $\mathfrak{k} = \{0\}$. Then by Theorem 3.9(i), J is Abelian and hence nilpotent of step 2.

Next, for part (b), recall that $\mathfrak{d}^1 = \mathfrak{z} \cap J\mathfrak{z}$ is the largest J-invariant subspace of $\mathfrak{z}$. Suppose that $\mathfrak{n}_2$ is not J-invariant. Then $\mathfrak{k} = \{0\}$. From part (i), J is nilpotent of step 2. It follows, from Theorem 2.13, that $\mathfrak{d}_2 = \{0\}$ and $\mathfrak{d}_1 \subseteq \mathfrak{d}^1$. But $\dim \mathfrak{d}_1 = \dim \mathfrak{n}_2 \oplus J\mathfrak{n}_2 = 4 > \dim \mathfrak{d}^1$. This is a contradiction. Hence, $J\mathfrak{n}_2 = \mathfrak{n}_2$.

We now show part (ii). Notice that $l \neq 1$. Otherwise $\dim \mathfrak{n}_2 = \dim \mathfrak{d}^1 = 2$. This implies that $J\mathfrak{n}_2 = \mathfrak{n}_2$. Suppose, by contradiction, that J is not nilpotent of step 3. Hence, J is nilpotent of step 2. Then from Remark 3.10(ii), $\mathfrak{k} = \{0\}$ and by definition, $\mathfrak{d}_1 = \mathfrak{n}_2 \oplus J\mathfrak{n}_2 \subseteq \mathfrak{d}^1$. However, $\dim \mathfrak{d}_1 = 4l > \dim \mathfrak{d}^1$. This is a contradiction. Hence, $\mathfrak{k} \neq \{0\}$. By Theorem 3.9(ii), J is nilpotent of step 3. □

REMARK 3.12. We can extend the statement of part (i) into a higher step stratification as follows. Let $\mathfrak{n}$ be a step k stratified Lie algebra with a nilpotent complex structure J of step k. Suppose that $\dim \mathfrak{n}_k = 2$ and $\dim \mathfrak{d}^1 = 2$. Then $J\mathfrak{n}_k = \mathfrak{n}_k$.

COROLLARY 3.13. *Let* $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$ *be a step* 2 *stratified Lie algebra with a complex structure* J *such that* $\dim \mathfrak{n}_2 = 2$. *Then* J *is centre-preserving or strata-preserving or both. Furthermore, suppose that* $2 \leq \dim \mathfrak{z} \leq 3$ *or* $\dim \mathfrak{z} = 4$ *and* $J\mathfrak{z} \neq \mathfrak{z}$. *Then there exists a* J*-invariant stratification.*

PROOF. By Theorem 3.11, J is nilpotent of step 2. Then by Table 1, $J\mathfrak{n}_2 = \mathfrak{n}_2$ or $J\mathfrak{z} = \mathfrak{z}$ or both if $\mathfrak{n}_2 = \mathfrak{z}$.

Furthermore, $\dim \mathfrak{d}^1 = 2$ since $2 \leq \dim \mathfrak{z} \leq 3$ or $\dim \mathfrak{z} = 4$ and $J\mathfrak{z} \neq \mathfrak{z}$. By Theorem 3.11(ii), $J\mathfrak{n}_2 = \mathfrak{n}_2$. Furthermore, by Theorem 3.6, there exists a J-invariant stratification. □

Suppose that $\mathfrak{n}$ is a 6-dimensional step 2 nilpotent Lie algebra with a complex structure. In [4, Table 1], there is a complete classification of complex structures on these algebras. However, no information is provided on whether or not J preserves the strata.

COROLLARY 3.14 [1, 4]. *Let* $\mathfrak{n}$ *be a* 6*-dimensional step* 2 *nilpotent Lie algebra with a complex structure* J *such that* $\dim \mathfrak{c}_1(\mathfrak{n}) = 2$. *Then* $\mathfrak{n}$ *admits a* J*-invariant stratification.*

PROOF. By Theorem 3.9 and Proposition 2.18, J is nilpotent and $2 \leq \dim \mathfrak{z} \leq 4$. If $\dim \mathfrak{z} = 4, \dim \mathfrak{c}_1(\mathfrak{n}) = 1$ and J is not strata-preserving due to dimensional reasons. We omit this case. Next, assume that $\dim \mathfrak{z} \leq 3$. The result is a direct consequence of Corollary 3.13. □

In what follows, we focus on higher step stratified Lie algebras with complex structures.

PROPOSITION 3.15. *Let* $\mathfrak{n}$ *be a step* 3 *stratified Lie algebra with a complex structure* J*. Suppose that* $J\mathfrak{n}_3 = \mathfrak{n}_3$*. Then* J *is nilpotent of step* 3.

PROOF. By the definition of the descending central series $\mathfrak{p}_j$ in (2.3), it follows that $\{0\} \neq \mathfrak{p}_2 = \mathfrak{n}_3 + [J\mathfrak{c}_1(\mathfrak{n}), \mathfrak{n}]$. On the one hand, suppose that $[J\mathfrak{c}_1(\mathfrak{n}), \mathfrak{n}] = \{0\}$. We deduce that $\mathfrak{p}_2 = \mathfrak{n}_3$ and hence $\mathfrak{p}_3 = \{0\}$ by definition. Using Theorem 2.13, J is nilpotent of step 3. On the other hand, suppose that $[J\mathfrak{c}_1(\mathfrak{n}), \mathfrak{n}] \neq \{0\}$. Then by the Newlander–Nirenberg condition, for all $U \in \mathfrak{c}_1(\mathfrak{n})$ and $X, JX \in \mathfrak{n}$,

$$0 \neq \underbrace{[JU, JX] - J[JU, X]}_{\in [J\mathfrak{c}_1(\mathfrak{n}),\mathfrak{n}] + J[J\mathfrak{c}_1(\mathfrak{n}),\mathfrak{n}]} = \underbrace{[U, X] + J[U, JX]}_{\in \mathfrak{n}_3}.$$

Hence, $[J\mathfrak{c}_1(\mathfrak{n}), \mathfrak{n}] \subseteq \mathfrak{n}_3$. This implies that $\mathfrak{p}_2 \subseteq \mathfrak{n}_3$ and therefore $J\mathfrak{p}_2 \subseteq \mathfrak{n}_3$. Then $\mathfrak{p}_3 = [\mathfrak{p}_2, \mathfrak{n}] + [J\mathfrak{p}_2, \mathfrak{n}] = \{0\}$. Again by Theorem 2.13, J is nilpotent of step 3. □

PROPOSITION 3.16. *Let* $\mathfrak{n}$ *be an* 8*-dimensional step* 3 *stratified Lie algebra with a complex structure* J *such that* $2\dim \mathfrak{n}_3 = \dim \mathfrak{c}_1(\mathfrak{n}) = 4$*. Suppose that* $J\mathfrak{n}_3 \neq \mathfrak{n}_3$ *and* $\dim \mathfrak{z} \leq 3$*. Then* J *is nilpotent of step* 4*. Furthermore,* $\mathfrak{d}_2 = \mathfrak{n}_3 \oplus J\mathfrak{n}_3$.

PROOF. Since $\mathfrak{n}_3 \subseteq \mathfrak{z}, \dim \mathfrak{z} \geq 2$. By [10, Corollary 3.12], J is nilpotent. Then using Remark 2.4(i), $3 \leq j_0 \leq 4$, where j_0 is the nilpotent step of J. Suppose, by contradiction, that J is nilpotent of step 3. It follows, from equation (2.4), that $\mathfrak{n}_3 + J\mathfrak{n}_3 \subseteq \mathfrak{d}_2 \subseteq \mathfrak{d}^1 \subseteq \mathfrak{z}$. On the one hand, since $\dim \mathfrak{z} \leq 3, \dim \mathfrak{d}^1 = 2$. On the other hand, since $J\mathfrak{n}_3 \neq \mathfrak{n}_3$ and $\dim \mathfrak{n}_3 = 2, \mathfrak{n}_3 \cap J\mathfrak{n}_3 = \{0\}$ and therefore $\dim \mathfrak{n}_3 \oplus J\mathfrak{n}_3 = 4 > \dim \mathfrak{d}^1$. This is a contradiction. So J is nilpotent of step 4.

We now show that $\mathfrak{d}_2 = \mathfrak{n}_3 \oplus J\mathfrak{n}_3$. It is sufficient to show that $\mathfrak{d}_2 \subseteq \mathfrak{n}_3 \oplus J\mathfrak{n}_3$. By definition,

$$\mathfrak{d}_2 = [\mathfrak{d}_1, \mathfrak{n}] + J[\mathfrak{d}_1, \mathfrak{n}] = \operatorname{span}\{[T, X] + J[T', X'] : \text{for all } T, T' \in \mathfrak{d}_1, \ \text{for all } X, X' \in \mathfrak{n}\}.$$

For all $T, T' \in \mathfrak{d}_1$, we may write $T = U + JV$ and $T' = U' + JV'$ where $U, V, U', V' \in \mathfrak{c}_1(\mathfrak{n})$. Then

$$0 \neq [T, X] + J[T', X'] = \underbrace{[U, X] + J[U', X']}_{\in \mathfrak{n}_3 \oplus J\mathfrak{n}_3} + [JV, X] + J[JV', X']. \tag{3.3}$$

By the Newlander–Nirenberg condition,

$$0 \neq \underbrace{[JV, X] + J[JV, JX]}_{\in [J\mathfrak{c}_1(\mathfrak{n}),\mathfrak{n}] + J[J\mathfrak{c}_1(\mathfrak{n}),\mathfrak{n}]} = J[V, X] - [V, X] \in \mathfrak{n}_3 \oplus J\mathfrak{n}_3.$$

Hence, $[JV, X] + J[JV', X'] \in \mathfrak{n}_3 \oplus J\mathfrak{n}_3$. From (3.3), $[T, X] + J[T', X'] \in \mathfrak{n}_3 \oplus J\mathfrak{n}_3$. Hence $\mathfrak{d}_2 \subseteq \mathfrak{n}_3 \oplus J\mathfrak{n}_3$. In conclusion, $\mathfrak{d}_2 = \mathfrak{n}_3 \oplus J\mathfrak{n}_3$. □

Acknowledgements

The results of this paper are contained in the author's Master of Science thesis at the University of New South Wales, prepared under the supervision of Michael Cowling and Alessandro Ottazzi. I would like to give deep thanks to both of them, for guiding and sharing their views on mathematics. They also provided very useful comments and suggestions on the project.

References

[1] A. Andrada, M. L. Barberis and I. Dotti, 'Classification of abelian complex structures on 6-dimensional Lie algebras', *J. Lond. Math. Soc. (2)* **83**(1) (2011), 232–255.

[2] M. L. Barberis and I. Dotti, 'Abelian complex structures on solvable Lie algebras', *J. Lie Theory* **14**(1) (2004), 25–34.

[3] L. A. Cordero, M. Fernández, A. Gray and L. Ugarte, 'Compact nilmanifolds with nilpotent complex structures: Dolbeault cohomology', *Trans. Amer. Math. Soc.* **352**(12) (2000), 5405–5433.

[4] L. A. Cordero, M. Fernández, A. Gray and L. Ugarte, 'Nilpotent complex structures', *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM* **95**(1) (2001), 45–55.

[5] M. G. Cowling, J. Li, A. Ottazzi and Q. Wu, 'Conformal and CR mappings on Carnot groups', *Proc. Amer. Math. Soc. Ser. B* **7** (2000), 67–81.

[6] M. G. Cowling and A. Ottazzi, 'Structure of stratified groups I. Product decompositions', *J. Lie Theory* **27**(1) (2017), 177–183.

[7] Q. Gao, Q. Zhao and F. Zheng, 'Maximal nilpotent complex structures', *Transformation Groups*, to appear. https://doi.org/10.1007/s00031-021-09688-3.

[8] J. Hilgert and K.-H. Neeb, *Structure and Geometry of Lie Groups*, Springer Monographs in Mathematics (Springer, New York, 2012).

[9] A. W. Knapp, *Lie Groups Beyond an Introduction*, 2nd edn, Progress in Mathematics, 140 (Birkhäuser, Boston, MA, 2002).

[10] A. Latorre, L. Ugarte and R. Villacampa, 'The ascending central series of nilpotent Lie algebras with complex structure', *Trans. Amer. Math. Soc.* **372**(6) (2019), 3867–3903.

[11] A. Latorre, L. Ugarte and R. Villacampa, 'Complex structures on nilpotent Lie algebras with one-dimensional center', Preprint, 2020, arXiv:2011.09916.

[12] E. Le Donne, 'A primer on Carnot groups: homogenous groups, Carnot–Carathéodory spaces, and regularity of their isometries', *Anal. Geom. Metr. Spaces* **5**(1) (2017), 116–137.

[13] A. Newlander and L. Nirenberg, 'Complex analytic coordinates in almost complex manifolds', *Ann. of Math. (2)* **65** (1957), 391–404.

[14] E. Remm, 'Non-existence of complex structures on filiform Lie algebras', *An. Univ. Vest Timiş. Ser. Mat.-Inform.* **39** (Special Issue: Mathematics) (2001), 391–399.

[15] S. Rollenske, 'Geometry of nilmanifolds with left-invariant complex structure and deformations in the large', *Proc. Lond. Math. Soc. (3)* **99**(2) (2009), 425–460.

JUNZE ZHANG, School of Mathematics and Statistics,
University of New South Wales, Sydney, New South Wales 2052, Australia
e-mail: z5055984@ad.unsw.edu.au