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SATISFICING SOLUTIONS TO A MONETARY POLICY PROBLEM

A Viability Theory Approach

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Herbert A. Simon, 1978 Economics Nobel Prize laureate, talked about *satisficing* (his neologism) rather than *optimizing* as being what economists really need. Indeed, optimization might be an unsuitable solution procedure (in that it suggests a unique "optimal" solution) for problems where many solutions could be satisfactory. We think that looking for an applicable monetary policy is a problem of this kind because there is no unique way in which a central bank can achieve a desired inflation (unemployment, etc.) path. We think that it is viability theory, which is a relatively young area of mathematics, that rigorously captures the essence of satisficing. We aim to use viability analysis to analyze a simple macro policy model and show how some robust adjustment rules can be endogenously obtained.

Keywords: Dynamic Systems, Viability Theory, Macroeconomic Modeling

1. INTRODUCTION

The aim of this paper is to explore the usefulness of viability theory for the analysis and synthesis of an economic state-constrained decision-making problem. The problem concerns inflation targeting in a closed economy.

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Herbert A. Simon, 1978 Economics Nobel Prize laureate, talked about *satisficing* [his neologism (1955)] rather than *optimizing* solutions as being what economists really need. We share Simon's view in that we believe that *some* economic agents may not seek unique optimal solutions. For example, the central bank governor in a country where the allowable inflation band has been legislated, a national park director responsible for biodiversity of the fauna. Each of them will strive to satisfy several objectives, many of them consisting of ensuring that the key outcomes (e.g., inflation or the number of bears) remain within some normative bounds. The bounds might result from some felicity function optimization, but the governor or park director will perceive them as exogenously specified. We think that economic theory that follows the Simon prescription may bring modeling closer to how these people actually behave.

We also think that it is viability theory, which is a relatively young area of mathematical analysis [see Aubin (1997) and Aubin et al. (2000)] that rigorously captures the essence of satisficing. If so, viability theory is an appropriate tool for achieving a satisficing solution to many economic problems. We aim to demonstrate this by solving a stylized central bank macroeconomic problem.[1] The solution will enable us to analyze the system's evolution within certain normative constraints (such as a desired inflation band) rather than the system's convergence to a steady state, as is the case for many traditional (optimal) monetary policy solutions. On the basis of the system's evolution we will be able to propose some robust satisficing monetary policies.

It is common knowledge [see, e.g., Başar and Bernhard (1991), Deissenberg (1987)] that a strategy that maximizes a given objective function on paper may not deliver expected results in real life because of parameter and model uncertainty. A way out has been computation of robust solutions, usually calculated as min-max strategies, as in Başar and Bernhard (1991), Deissenberg (1987), and Rustem (1994). This approach has been successfully applied to the design of *robust* monetary policies, see, for example, Hansen et al. (2006) and Žaković et al. (2002, 2006).

Strategies delivered by viability theory are *alternative* robust strategies. They are based on an evolutionary analysis of admissible system trajectories rather than robust (i.e., min-max) optimization.

It is the precautionary character of policy advice based on viability theory that makes it robust. Our policies, which are obtained through a viability analysis and which we call *satisficing*, are precautionary (or "preventative") in that they are based on the economic system's inertia. This makes them naturally forward-looking and suitable for any future circumstances. This is so because knowledge of the system's inertia enables us to detect (and avoid) regions of economic conditions (such as large output gap[2] or accelerating inflation) where control of the system is difficult or impossible.

Moreover, computation of a satisficing policy requires fewer parameters to be calibrated or estimated than of an optimizing policy. In this, the former is more robust than any other policy, which is computed for a model that requires more parameters. This feature among others will render our policies less vulnerable to the Lucas (1976) critique than the optimizing policies.

Generically, an evolutionary analysis enabled by viability theory proposes policies that are less invasive (i.e., attempting to change the status quo) than those delivered through optimization. In particular, a viability-theory-based strategy will advocate adjustment rules that may be "passive" [like those in Benhabib et al. (2001)] for a large number of the economy states and "active" for some critical states only. Also, in contrast to optimal policies, which are usually unique, the viable control planner will have the flexibility to endeavor to attain other aims in regions where the passive satisficing policies can be used. We wish to point out that, because of the possibility of a passive policy choice, the same equations embedded in our viability approach may be less vulnerable to the Lucas critique than those embedded in the standard approach.

Finally, we believe that an evolutionary analysis gives us more insight into the system's (macro) economics than a customary equilibrium analysis. The insight is gained (mainly) through the disclosure of the economy states from which attainment of the central bank's objectives is problematic. In particular, our analysis will establish from which states of the economy the avoidance of a *liquidity trap* is impossible.

In the next section, we provide an introduction to viability theory; we define the basic notions and solve two simple viability problems. In Section 3, we apply the theory to a simple macroeconomic model and provide justification for the above observations on the differences between optimization and the viability theory approach. The paper ends with concluding remarks.

2. WHAT IS VIABILITY THEORY?

2.1. Meaning

Suppose there is given a closed set K in the state space that may represent some normative constraints. The basic problem that viability theory attempts to solve is whether, for a given initial state, a control strategy exists that prevents the system from leaving the constraint set. The *viability kernel* for the closed set K is the (largest) subset of K that contains initial conditions for which such a strategy exists. The kernel will be defined formally in Definition 2.1.

Consider a dynamic economic system with several state variables. At time $t \in \Theta \equiv [0,\ T] \subset \mathbf{R}^+$, where T can be finite or infinite, the state[3] variables are

$$x(t) \equiv [x_1(t), x_2(t), \dots x_N(t)]' \in \mathbf{R}^N, \qquad \forall t \in \Theta$$

and the controls (or actions) are

$$u(t) \equiv [u_1(t), u_2(t), \dots u_M(t)]' \in \mathbf{R}^M, \qquad \forall t \in \Theta.$$

Imposition of normative restrictions on states and strategies means that

$$\forall t \in \Theta, \quad x(t) \in K, \quad \text{and} \quad u(t) \in U,$$

where symbols K, U represent sets of constraints that the state and control variables need to satisfy. In general, the control constraints depend on x; however, for simplicity, we will avoid the notation $U(x)$.

The state evolves according to system dynamics $f(\cdot,\cdot)$ and controls $u(t)$ as follows:

$$\dot{x}(t) = f(x(t), u(t)), \qquad t \in \Theta, \quad x(t) \subset \mathbf{R}^N, \quad u(t) \in U \subset \mathbf{R}^M. \qquad \mathbf{(1)}$$

Evidently, we are dealing with controlled dynamics; that is, at every state $x(t)$, the system's *velocity* $\dot{x}(t)$ depends on action $u(t)$. We will be looking for such $u(t)$, for which $x(t) \in K$ for all $t \in \Theta$.[4]

In economic terms, the last relationship tells us that at time t, for a given composition of x (capital, labor, technology, etc.), the extent of growth (or decline), or steady state stability, is dependent on the map $f : \mathbf{R}^N \times \mathbf{R}^M \twoheadrightarrow \mathbf{R}^N$, whose values are limited by the scope of the system's dynamics f and controls contained in U.

A viability theory analysis attempts to establish nonemptiness of a *viability kernel*, which is a collection of loci for initial conditions of viable evolutions $x(t), t \in \Theta$.

DEFINITION 2.1. *The viability kernel of the constraint set K for the control set U is the set of initial conditions $x_0 \in K$ denoted*[5] *as V and defined as follows:*

$$V \equiv \{x_0 \in K : \exists x(t) \text{ solution to (1) with } x(0) = x_0 \text{ s.t. } x(t) \in K, \forall t \in \Theta\}. \qquad \mathbf{(2)}$$

In other words, we know that if a trajectory begins inside the viability kernel V then we have sufficient controls to keep this trajectory in the constraint set K for $t \in \Theta$. See Figure 1 for an illustration of the viability idea.[6]

The state constraint set K is represented by the yellow (or light shadowed) round shape contained in the state space (where X denotes the state space; here, $X \equiv \mathbf{R}^2$). The solid and dash-dotted lines symbolize system evolution.

The viability kernel for the constraint set K, given controls from set U and the system dynamics f, is the purple (darker) shadowed contour denoted V. The system evolution represented by the trajectories that start inside the kernel (dashed lines) are viable in K; that is, they remain in K. This is not the property of the other trajectories (dash-dotted lines) that start outside the kernel. They leave K in finite time $<T$.

We can now say what we understand as a viability problem, and define what we mean by its solution.

Given the system dynamics $f(\cdot,\cdot)$, the sets of constraints K and U, and the horizon T, the associated viability problem consists of establishing the existence of the viability kernel V.

DEFINITION 2.2. *When the kernel is nonempty $V \neq \emptyset$, we say that the viability problem has a solution; otherwise, the viability problem has no solution.*

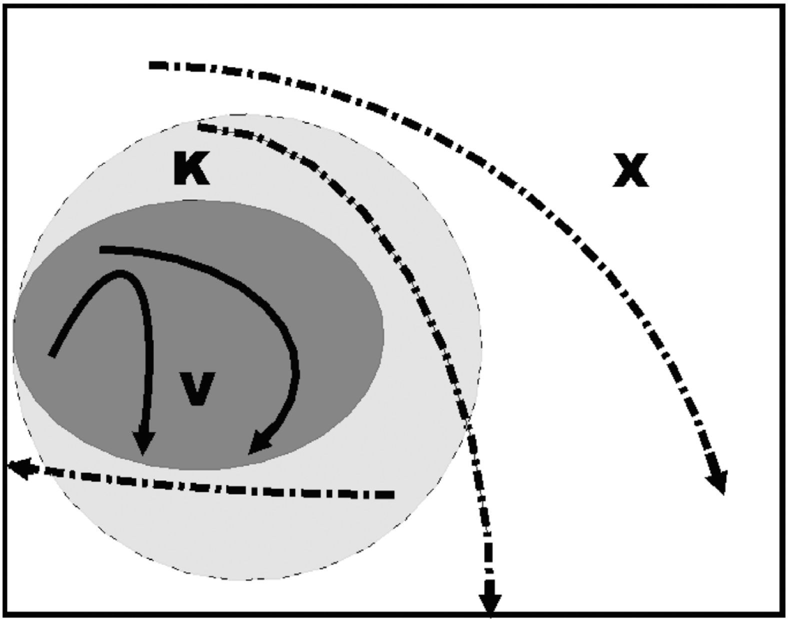

FIGURE 1. The viable and nonviable trajectories.

Let us observe that the viability kernel is a "broad" concept that allows a uniform treatment of problems defined on infinite or finite horizons. In particular, a generically nonstationary solution to a finite-horizon policy problem,[7] which might consist in reaching a target in finite time, can be analyzed as a viable control problem [e.g., see, Doyen and Saint-Pierre (1997)].

2.2. Linear dynamics example

Viability kernels can be characterized relatively easily for some typical dynamic systems [see Aubin (1997), Aubin et al. (2000), and Cardaliaguet et al. (1999)].

Consider the linear dynamics system defined as follows:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} v_x \\ v_y \end{bmatrix}. \tag{3}$$

The state variables are x, y; the instrument set U is the unit ball

$$U \equiv \left\{ (v_x, v_y) : v_x^2 + v_y^2 \leq 1, \quad (v_x, v_y) \in \mathbf{R}^2 \right\}, \tag{4}$$

where v_x adds to speed in direction x and v_y adds to speed in direction y.

We can see that the further from the origin the system is, the faster it moves along direction y. This might remind us of a paddling boat on a river approaching

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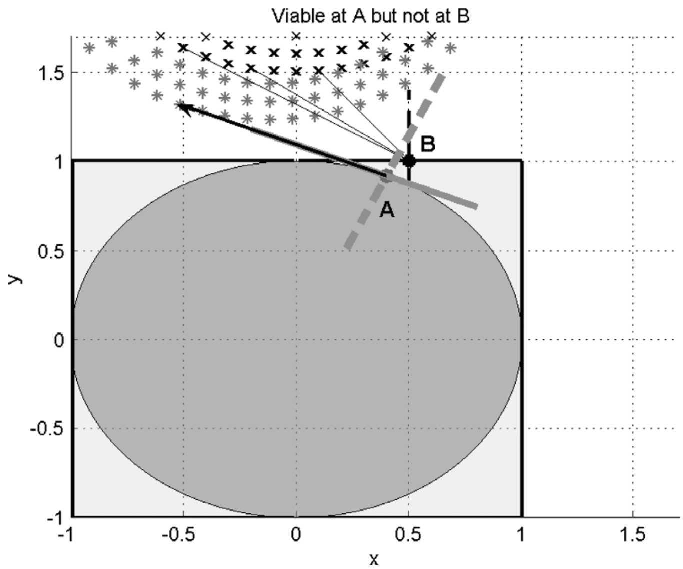

FIGURE 2. A geometric characterization of viability.

a waterfall. The further we are from calm waters ($y = 0$) the faster we move. Intuitively, given our limited strength, we sense the existence of a point of no return, after which we will not be able to paddle away from the waterfall.

Notice that this situation is reminiscent of what might occur if an economy is *hot*. The larger output gap, the higher inflation, the larger output gap, and so forth. Given limited instruments at the central bank's disposal, too hot an economy might be bound to suffer from a spiraling inflation. Clearly, determination of the point of no return on the river and of the combination of output gap and inflation for the economy is important for a safe journey. In viability theory, these points are determined once the viability kernel is established.

First, we will check if the rectangle

$$K \equiv \left\{(x, y) \, : \, \max(|x|, |y|) \leq 1, \; (x, y) \in \mathbf{R}^2\right\}$$

is a viability kernel. If it is, we will be certain that we can prevent the system to escape from K using actions from U.

Figure 2 provides an illustration of the problem and of some geometric properties required for viability.

Consider the frontier point **B** of the rectangle. The velocities from U are constrained (see (4)) and generate evolution directions denoted by "x" (crosses); the normal[8] at **B** is the thin black line. We see that all angles are acute and that there is no vector at **B** that would point inside the rectangle (or form an obtuse angle with the normal). We see that for any $(v_x, v_y) \in U$, we are unable to "return" to the rectangle from point **B**. This means that point **B** is not viable. The same

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reasoning can be repeated at many points of rectangle K. The conclusion will be that the rectangle cannot be the viability kernel.

However, we can prove that disc V,

$$V \equiv \left\{(x, y) : x^2 + y^2 \leq 1, \ (x, y) \in \mathbf{R}^2\right\} \tag{5}$$

(delimited by the circle of radius 1 centered at origin), is a viability kernel for the sets K, U and the system dynamics (3).

Indeed, we can see that there are velocities at point **A** that generate evolution directions represented by "*" (stars), which point inside the disc (form an obtuse angle with the normal going through this point). This means that there exist some $(v_x, v_y) \in U$ for which we can turn the system so that it remains in V. This reasoning can be repeated at any point of disc V. We could also prove that no point $(x, y) \in K \setminus V$ is viable and conclude that disc V is the viability kernel.

The evolution direction at **A** that keeps the system inside V is represented by the black vector at **A** pointing left. This direction is a consequence of use of some outermost velocities $(v_x, v_y) \in U$. Presumably, this control is extreme (or outermost) in that there is no other velocity vector that would generate a more obtuse angle with the normal at **A**. In fact, the pair of velocities that generate the black vector satisfy $v_x^2 + v_y^2 = 1$.

A comparison between sets K and V (the latter is a viability kernel; the former is not) tells us that at **B**, the system moves too fast to be controlled through $(v_x, v_y) \in U$. However, the same control set contains elements that are sufficient to restrain the system, should we apply them early,[9] that is, when the process is within V.

2.3. Satisficing Policies

In economic situations in which a planner may be identified (e.g., a central bank), a viability kernel can be used to select policies that keep the dynamic process x inside the closed constraint set K.

Once the kernel is established, choosing a satisficing policy is a simple procedure. This idea can be illustrated using Figure 2. We can see that there are controls $v_x^2 + v_y^2 \leq 1$ that keep trajectory x in $V \subset K$. In particular, we know that even if $x(t)$ is at the frontier of V, the outermost or extreme control is sufficient to prevent the system trajectory from leaving V.

If V denotes the viability kernel of constraints K for dynamics f, then the following generic policy rule can be formulated [see *regulation maps* in Aubin (1997)]:

$$\begin{cases} \qquad \forall x \in V \ \text{ apply instrument } u \in W \\ \text{where } W \equiv \left\{u \in U \ : \ f(x, u) \text{ is a direction tangent or inward to } V\right\}. \end{cases} \tag{6}$$

So W is a set of instruments available at x that keep the system evolution inside V.

For a given viability problem this rule will be decomposed into two normative directives: within the interior of the viability kernel V, every (admissible) control can be used;[10] on the boundary of the kernel $\mathsf{fr}\,V$, a specific instrument (path) must be followed[11]. We will identify this instrument for each viability problem solved in this paper (see Sections 2.4, and 4.4).

Let us briefly look at what kind of actions a central bank planner undertakes.

Routinely, every given time interval, the planner announces a cash interest rate. A Taylor rule or an optimizing rule[12] might be used to determine the new interest rate. The latter usually equals the old interest rate plus or minus a fraction of a percentage point. Although this might look simple, the process leading to the rate determination is typically based on optimization of a loss function that contains a significant number of parameters calibrated and/or estimated.

The effect of the central bank optimization process is similar to the application of the satisficing policy: either maintains x (e.g., inflation) in K. However, as will be explained later, fewer (subjective) parameters are needed to establish V than to compute a minimizing solution to the bank loss function. Also, the possibility of the use of any control in the interior of the viability kernel offers the planner a possibility of striveing to achieve other goals (*e.g.*, political) that were not used for the specification of K. (Perhaps they were difficult to specify mathematically, or they arose after the viability kernel had been established.) This is not the case of an optimal solution, which remains optimal for the original problem formulation only.

Should there be uncertainty regarding the model parameters, a sensitivity analysis needs to be performed to establish to what extent the system's dynamics is affected by the uncertainties. Once established, the current position of the system will be generalized from $x(t)$ to $x(t) + b_1(x(t), \kappa(t))$, where $b_1(\cdot, \cdot)$ is a ball centered at $x(t)$ with a radius $\kappa(t)$ that will result from a robustness analysis of (1).

When the model is subjected to shocks whose magnitude can be estimated (or whose distribution is known), the viability kernel will have to be such that $x(t) + b_2(x(t), \varepsilon(t)) \in K$, where radius $\varepsilon(t)$ will depend on the shock.[13] Then the above policy prescription can be followed.

2.4. An Analytical Solution to a Viability Problem

To optimize or not. . . . In a number of situations, an analytical description of a viability kernel is possible. This is the case for some simple economic models. We will describe analytically a viability kernel for a stylized scarce commodity consumption problem. We will see how viability theory might be useful in deriving rules that otherwise could be thought of as behavioral. We will also suggest a possible optimization problem as a counterpart for the viability problem to highlight some particular features of the latter.

Our commodity is of limited supply $M > 0$. The limit $M > 0$ could be the total electricity supply from a large trader, a city bus carriers total passenger capacity, a monopolistic oil provider's supply, etc. In each of the above cases, even if the commodity supplier is a private firm, it will be held responsible by the government for the commodity's delivery collapse. The fines applied by the government are usually very large and can threaten the firm's survival.[14] On the other hand, a zero demand for the commodity would also be disastrous to the supplier. Hence, ideally from the supplier's (and government's) point of view, the demand for the commodity should be contained between 0 and M.

The commodity that we are interested in is such that one can develop an "appetite" for it.[15] That is, one usually wants to consume more of the commodity in the next period than now. (However, losing appetite is also possible.) This will be the case for electricity, public transport, chocolate, drugs, heating oil, and many more. For example, if we find oil economical this winter, we might want to use more of it next year.

Let $x(t)$ denote demand for the commodity at time $t \in \Theta$. Suppose that the changes in the demand are proportional[16] to the amount consumed at present. So the demand variation can be written as $\dot{x} = ax(t)$, where $a > 0$. Clearly, if our appetite is not modified, we will soon want to consume more than M and thus exhaust the supply.

Think of compensation $p(t)x(t)$ [where $p(t) \geq 0$ is price] paid to the commodity provider. Paying the compensation should slow down the demand as follows:[17]

$$\dot{x}(t) = ax(t) - p(t)x(t). \qquad \textbf{(7)}$$

If price $p(t)$ changes with $x(t)$ as a result of an appropriate pricing strategy $p(x)$, then there is a chance that consumption will slow down and supply will not be exhausted.

This suggests that the supplier might want to manage demand for the commodity through price $p(t)$ so that $\underline{x} \leq x(t) \leq M$, where $\underline{x}$ is some minimum demand that guarantees the supplier's survival. [For simplicity, we will normalize demand and write the above constraint as $0 \leq x(t) \leq M$.]

A plausible optimization problem of the commodity supplier could thus be

- given the state equation (7), compute $p(t) \in [0, \overline{p}]$ such that

$$\int_0^T e^{-\rho t} \left\{ p(t)x(t) - W_1[\max(0, x(t) - M)]^2 - W_2[\max(0, -x(t))]^2 \right\} dt \qquad \textbf{(8)}$$

is maximized, where ρ is the discount rate and $W_1, W_2 > 0$ are penalty coefficients. Price $\overline{p}$ is some maximum price above which consumers switch to a different product (or technology). Notice that we have assumed away the supply cost (or absorbed it in $p(t)$).

Maximization of (8) in $p(t)$ is a difficult problem[18] to solve. It is an optimal control problem with a nonlinear nondifferentiable objective function. Moreover, it contains three arbitrary parameters.

However, if the fines to pay for disrupting supply are *very* large, then W_1, W_2 are *very* big and hence the problem boils down to maintaining the demand and price in the rectangle $[0,\ M] \times [0,\ \overline{p}]$. Notice that

- given the state equation (7), keeping

$$(x(t),\ p(t)) \in [0,\ M] \times [0,\ \overline{p}], \quad \forall t \in \Theta \tag{9}$$

may thus be a satisfactory solution to the supplier's problem. We observe that the arbitrary constants ρ, W_1, W_2 do not enter this problem specification.

We will solve problem (9) through a viability analysis and call its (Markovian) solution $p(x(t)) \in [0, \overline{p}]$ *satisficing*.

Viability kernel. If the price of the commodity is *sticky*,[19] then the price path that will guarantee the next period's supply might be nontrivial. In other words, if demand is accelerating but there is a limitation on the price changes, then not all price paths will ensure that $x(t) \leq M\ \forall t$, that is, demand satisfaction. In viability parlance, one would say that not all price paths are viable.

From here on, we assume that the commodity price can be controlled by the supplier and that the price process is sticky. The demand changes according to (7). Due to perishability or nonstockability of the commodity, we cannot buy more of it than we are able to consume at once. The maximum commodity supply is $M > 0$. The problem of how $p(t)$ should behave so that $0 \leq x(t) \leq M$ for $t \in \Theta$ is typical of viability analysis. With bounds imposed on $p(t)$ and its speed, a solution to that problem is rather simple.

With all that we have said about the rectangle $[0,\ M] \times [0,\ \overline{p}]$ and the price process stickiness, we have the following description[20] of the system at hand:

$$\begin{cases} \dot{x} = ax(t) - p(t)x(t) \\ \dot{p} = u\ \in\ [-c,\ c]. \end{cases} \tag{10}$$

We will look at the problem variables (x, p) in the phase space; see Figure 3.

In the figure, the upper bound on p is assumed as 0.2; $M = 1$. The light shaded rectangle whose vertices are (0,0),(1,0),(1,0.2), (0,0.2) is the constraint set K. We will suppose that $a = 0.05$ and $c = 0.001$.

If price $p(t) = a$, quantities $x(t)$ are steady; see (10). Hence, line $p(t) = a$ defines a collection of steady states. However, each of them is unstable. Indeed, below this line, $p < a \Rightarrow \dot{x} > 0$ and with a constant price we drift to the right; see the dash-dotted line starting at point C. Above the line, $p > a \Rightarrow \dot{x} < 0$, and hence we drift to the left; see the dash-dotted line starting at C'. It is clear that if we are off steady state we will hit one of the physical constraints on x in finite time. To prevent exhaustion ($x = M$) or extinction ($x = 0$), $\dot{p}$ cannot be zero that is, the price needs to vary.

Intuitively, to steer away from M, for $p < a$, we should increase p. Similarly, for $p > a$, we should decrease p to avoid $x = 0$. The dotted lines show the system

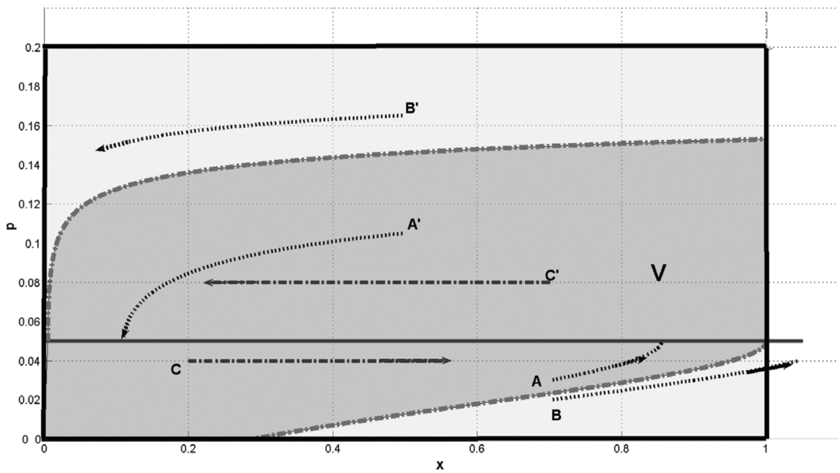

FIGURE 3. An analytical derivation of a viability kernel.

trajectories when prices change at full velocity $c = 0.001$ (for those that start at A and B) and $c = -0.001$ (for those that start at A' and B'). For points such as A and A' the strategy of changing p at full speed is sufficient to prevent x from crossing the boundary of the constraint set K. However, if we start from other points such as B or B' we are bound to violate one of the constraints even if the price change is fastest.

To solve the viability problem we need to compute a collection of (x, p) from which a feasible price policy can maintain the system within the constraints. In other words, we need to establish the viability kernel[21] V, where f is defined through (10).

The viability kernel will be established here through an explicit calculation of the critical trajectories that bring the system, at a full speed c (or $-c$), to a steady state that intersects with the constraints. The critical trajectories are the thick lines that originate at $(1, 0.05)$ and $(0, 0.05)$, which are the last acceptable steady states. The resulting viability kernel V is the dark shadowed area between those lines.[22] They have been easily computed by running the system backward in time at full speed from the points where a steady state intersects with the constrains.

Now, we can see how a viability analysis could help the supplier to establish a viable price strategy. Should there be no other considerations regarding price, the policy[23] should be (where $\operatorname{fr} V$ is the kernel's boundary and hence $V \backslash \operatorname{fr} V$ is its interior):

$$\begin{cases} \text{if } (x, p) \in V \backslash \text{fr}\, V, \text{ apply any feasible change of price} & u \in [-c, c]; \\ \text{if } (x, p) \in \left\{ (\text{fr}\, V) \bigcap \{(x, p) : p < a\} \right\}, \text{ increase price} & u = c; \\ \text{if } (x, p) \in \left\{ (\text{fr}\, V) \bigcap \{(x, p) : p > a\} \right\}, \text{ decrease price} & u = -c, \end{cases} \tag{11}$$

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where $\bigcap$ denotes intersection. Should a steady state ($p = a$) once be achieved, this might be maintained by the following policy rule:

$$\text{if } (x, p) \in \Big\{V \bigcap \{(x, p) : p = a\}\Big\}, \quad \text{keep price steady } u = 0. \tag{12}$$

As observed earlier [after rule (6)], the policy advice is passive, $u \in [-c, c]$, if the current point is within the viability kernel (see the first "if" in (11)); the policy is active, that is, price p changes with the highest velocity $|c|$, if the current point is at the viability kernel's boundary (see the second and third "if" in (11)). We notice that if the policy is passive, some other goals[24] could be realized.

We notice that the kernel boundaries have an attractive economic interpretation. For example, consider the lower part of V. For a given price p and consumption x such that point p, x is below the steady state line, consumption of a rational consumer can grow until the limit of the viability kernel is reached. Then the boundary tells the agent in which way the prices will evolve so that $x \leq M$. This is important for the consumer, who might be solving their own optimization (or viability) problem, where a demand law will be part of the problem specification. Notice that this dynamic inverse demand law has been endogenously obtained.

3. A MACROECONOMIC MODEL

3.1. A Viability Theory Problem

Realistically, what a typical central bank wants to achieve is the maintenance of a few key macroeconomic variables within some bounds. Usually, the bank realizes its multiple targets using optimizing solutions that result from minimization of the bank's loss function. Typically, the loss function includes penalties for violating an allowable inflation band and also for a nonsmooth interest adjustment. The solution, which minimizes the loss function, is unique for a given selection of the loss function parameters. In that, it does not allow for alternative strategies.

Our intention is to apply viability theory to the bank's problem. Keeping variables of interest in a constrained set sounds very much like the viability theory problem illustrated in Figure 1. We will establish the economy's kernel, the subset of the constraint set K, inside which the economy evolution can be contained given the economic dynamics and instruments available to the central bank.

In the next section we will describe a stylized monetary rules model [inspired by Svensson (2002) and Walsh (2003)]. We will then solve a viability theory problem formulated for that model. We will show that the solutions obtained through viability theory do not suffer from the drawbacks typical of their optimizing counterparts.

3.2. A Central Bank Problem

Suppose a central bank is using short-term nominal interest rate $i(t)$ as an instrument to control inflation $\pi(t)$ and, to a lesser extent, output gap $y(t)$. A model that relates these variables may look like this [see Walsh (2003), p. 508) where the time step $h = 1$):

$$y(t) = a_1\, y(t-h) + a_2\, y(t-2h) - a_3\big(i(t-h) - E_{t-h}\pi(t)\big) + u(t) \tag{13}$$

$$\pi(t) = \pi(t-h) + \gamma y(t) + \eta(t), \tag{14}$$

where $y(t)$ is output gap, $u(t)$ and $\eta(t)$ are serially uncorrelated disturbances (or *shocks*: in aggregate demand and inflation, respectively) with means equal to zero, a_1, a_2, a_3, and γ are calibrated parameters, and E_{t-h} is the expectation operator.

Equation (13) represents the aggregate spending relationship. It corresponds to a traditional IS function where aggregate demand is inversely related to the real interest rate $r(t-h) = i(t-h) - E_{t-h}\pi(t)$. Note that, in our aggregate spending specification, time-t spending depends on the lagged value of the real interest rate. Because monetary policy affects aggregate demand via real interest rate, the assumption that time-t spending depends on lagged real interest rate will imply a lagged response of output to monetary policy changes. This reflects a long-standing view that many macroeconomic variables do not respond instantaneously to monetary policy shocks [see Friedman (1968)]. The interest rate relevant to aggregate spending decisions would be the long-term rate, which is ralated to the short-term rate via the term structure relationship. To minimize the number of variables in our exposition, we do not distinguish between the long-term and short-term interest rates.

Equation (14) captures the inflation-adjustment process driven by the size of the output gap. In the canonical New Keynesian specification, current inflation $\pi(t)$ depends on the expected future inflation $E_t\pi(t+h)$. Furthermore, in the Fuhrer-Moore (1995) model of multiperiod, overlapping nominal contracts, current inflation depends on both past inflation and expected future inflation. However, several empirical works [e.g., Fuhrer (1997)] suggest that the expected inflation term is empirically unimportant once lagged inflation is included in the inflation adjustment equation. Considering this, and also to simplify our exposition, we ignore the expected inflation term.

Hence, our model is a simplified version of the Rudebusch-Svensson (1999) model, with output gap and inflation that are state variables. They are driven by shocks in aggregate demand, inflation, and nominal interest rate.

Assume[25] that $a_2 = 0$ in (13), call a the "new" coefficient with the one-lag term, and apply the expectation operator E_{t-h} to both (13) and (14). We obtain

$$E_{t-h}y(t) = a\, E_{t-h}y(t-h) - a_3(E_{t-h}i(t-h) - E_{t-h}\pi(t)) \tag{15}$$

$$E_{t-h}\pi(t) = E_{t-h}\pi(t-h) + \gamma E_{t-h}y(t). \tag{16}$$

At time $t-h$, the expectations are identical with the observations, so

$$E_{t-h}y(t) = ay(t-h) - a_3(i(t-h) - E_{t-h}\pi(t)) \tag{17}$$

$$E_{t-h}\pi(t) = \pi(t-h) + \gamma E_{t-h}y(t). \tag{18}$$

Assume differentiability of the inflation and output gap processes. If so, for small h,

$$E_{t-h}y(t) = y(t-h) + \dot{y}h \quad \textbf{(19)}$$

$$E_{t-h}\pi(t) = \pi(t-h) + \dot{\pi}h. \quad \textbf{(20)}$$

These relationships tell us that agents forecast the expected values using extrapolations. This corresponds to the basic learning process [compare Honkapohja and Mitra (2006)].

Substituting in (17) and (18) (and omitting the time index $t-h$) yields

$$y + \dot{y}h = ay - a_3(i - (\pi + \dot{\pi}h)) \quad \textbf{(21)}$$

$$\pi + \dot{\pi}h = \pi + \gamma(y + \dot{y}h). \quad \textbf{(22)}$$

From (22), $\dot{\pi}h = \gamma y + \gamma \dot{y}h$. Allowing for that and for $\alpha h = a - 1$, $\xi h = a_3$, $\zeta h = \gamma$, and then dividing by h, we get the following inflation and output gap dynamics for $h \to 0$:

$$\frac{dy}{dt} = \alpha y(t) - \xi(i(t) - \pi(t)) \quad \textbf{(23)}$$

$$\frac{d\pi}{dt} = \zeta y(t). \quad \textbf{(24)}$$

The equations, which constitute the above model, are continuous time equivalents of the aggregate demand equation (13) and the Phillips curve (14). They say that output gap constitutes a sticky process (23) driven by the real interest rate, i.e., the difference between the interest and inflation rates, and that the inflation rate (24) changes proportionally to the output gap. We will calibrate the model in Section 3.3.

Notice that the model (23)–(24) is suitable for managing output gap and inflation.[26] For example, if output gap were positive and growing, increasing $i(t)$ (which is the central bank instrument) in (23) would slow down the output gap. However, the inflation will not start diminishing as long as output gap is positive.[27] We observe that the central bank could exploit inertia of the controlled processes to restrain their evolutions.

3.3. Parameter Values

We use the following parameter values [see Walsh (2003)[28]]:

$$\xi = \frac{a_3}{h}\bigg|_{h=1} = 0.35, \qquad \zeta = \frac{\gamma}{h}\bigg|_{h=1} = 0.002.$$

Regarding α, we know that $\alpha = \frac{a-1}{h}\big|_{h=1}$, but, first we need to say what the value of a is. In general, it is impossible to approximate a second-order process (13) by a first-order process $y(t) = ay(t-h) - a_3(i(t-h) - E_{t-h}\pi(t)) + u(t)$. However,

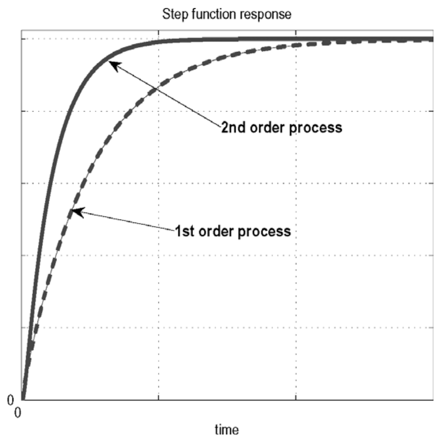

FIGURE 4. The second-order and first-order process responses.

(13) stalibilizes long-term if perturbed (e.g., by a step function); see Figure 4, solid line. The other line represents the first-order process response, for which we have chosen a such that, after some time, the responses are approximately at the same level.

The solid line shows a step-function response of (13) with Fuhrer's $a_1 = 1.53, a_2 = -0.55$ [see Fuhrer (1994) and Walsh (2003)]. The value of a for the dashed line is $a = 0.98$,[29] so $\alpha = -0.02$. Hence the macroeconomic model that we will analyze is

$$\frac{dy}{dt} = -0.02\, y(t) - 0.35(i(t) - \pi(t)) \tag{25}$$

$$\frac{d\pi}{dt} = 0.002\, y(t). \tag{26}$$

The model enables us to analyze the behavior of the economy for which it was calibrated.

3.4. The Constraints

Usually there is little doubt as to what the *politically* desired inflation bounds are. For example, in New Zealand, the inflation band has been legislated to be confined to $[0.01,\ 0.03]$. There is less agreement about what the desired output gap should be. We will assume a rather wide interval for output gap to reflect a lesser concern of the central bank for $y(t)$ (e.g., $y(t) \in [-0.04,\ 0.04]$).

Similarly to the desired size of the output gap, the instrument set composition also depends on political decisions. We will assume[30] that $i(t) \in [0,\ 0.07]$. An $\mathbf{R}^3$ region, within which the *meta*-system[31] trajectory $[y(t), \pi(t), i(t)]$ will have

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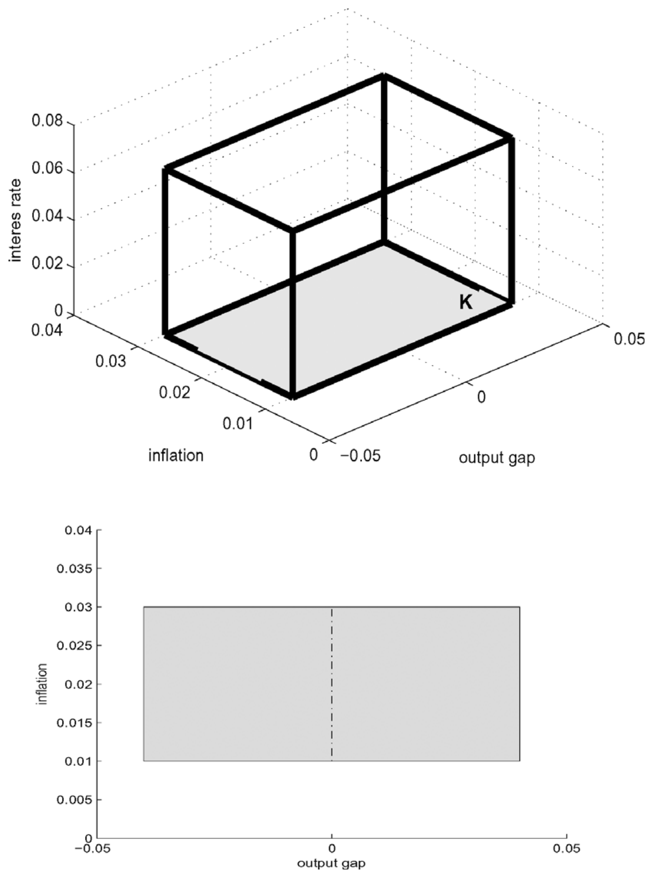

FIGURE 5. Constraint set and projection.

to be contained, is shown in Figure 5. So the constraint set K is

$$K \equiv \{(y(t), \pi(t), i(t)) : -0.04 \leq y(t) \leq 0.04,\ 0.01 \leq \pi(t) \leq 0.03,\ 0.0 \leq i(t) \leq 0.07\}. \quad \textbf{(27)}$$

See $K \subset \mathbf{R}^3$ in Figure 5 upper panel and its two-dimensional projection in the lower panel.

Independent of keeping the interest range constrained, many central banks are worried about the interest rate *smoothness* [see, e.g., Amato and Laubach (1999)]. That concern is usually modeled by adding $w(i(t) - i(t-h))^2,\ w > 0$ to the loss function. In continuous time, limiting the interest rate "velocity"

$$u \equiv \frac{di}{dt} \quad \textbf{(28)}$$

will produce a smooth time profile of $i(t)$. Bearing in mind that the central bank's announcements concern *changes* in interest rate, we will treat u as the bank's

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control. Observation that the changes are usually made every quarter and that the typical change is $1/4\%$, the bank's control set U will be defined as

$$U \equiv \{u \,:\, u(t) \in [-0.005,\ 0.005]\}; \tag{29}$$

that is, the interest rate can drop, or increase, between 0 and 0.5% per quarter.

Hence, the dynamic system to analyze the relationship between the interest rate, inflation, and output gap needs to be augmented by the interest rate velocity constraint and will now look as follows:

$$\frac{dy}{dt} = -0.02\, y(t) - 0.35\,(i(t) - \pi(t)), \tag{30}$$

$$\frac{d\pi}{dt} = 0.002\, y(t), \tag{31}$$

$$\frac{di}{dt} = u \in [-0.005,\ 0.005]. \tag{32}$$

3.5. Robustness of Model

As we have mentioned in the Introduction, a viability model of the central bank problem needs less subjectively assessed parameters than the corresponding optimization model. In particular, a viability model (30)–(32)(plus (27)) does not require any weight that the bank loss function necessitates. Neither is the discount rate needed. The bounds of the constraint set are either legislated or identifiable in a rather nonobjectionable manner. If there is not much concern for limits of a variable, as for output gap, then they can be set "large."

Consequently, the boundaries of the constraint set K, within which the economy can move, convey information about the desired evolution of the economy in a more objective fashion than the loss function weights and discount factor tell us about agents preferences.

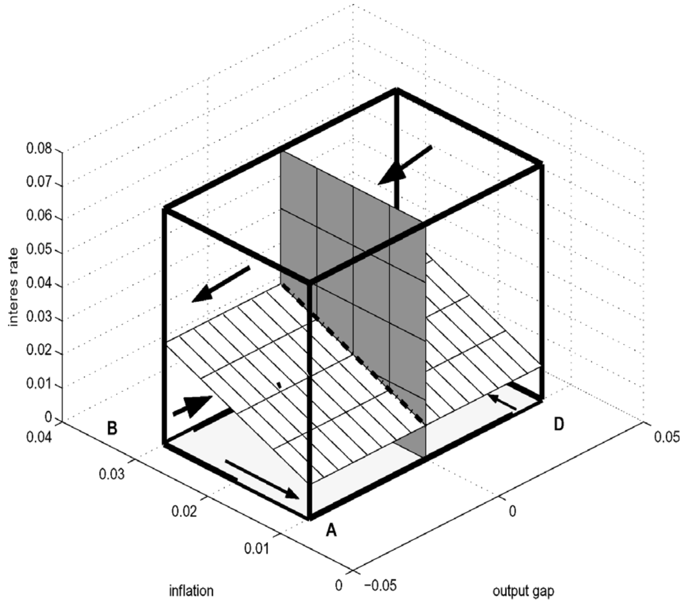
4. VIABLE SOLUTIONS

We will perform a viability theory analysis using the model (30)–(32). This is *computational economics* and the results will be parameter-specific; however, the procedure can easily be repeated for any plausible parameter selection compatible with the institutional framework.

4.1. A Steady State and Transition Analysis

First, let us examine the existence of steady states of (30)–(32) and assess their stability.

Add the plane $y = 0$ and another one $i = \pi$ to Figure 5; see Figure 6 top panel. The steady states are at the intersection of those planes; see the dash-dotted line in Figure 6 top panel.



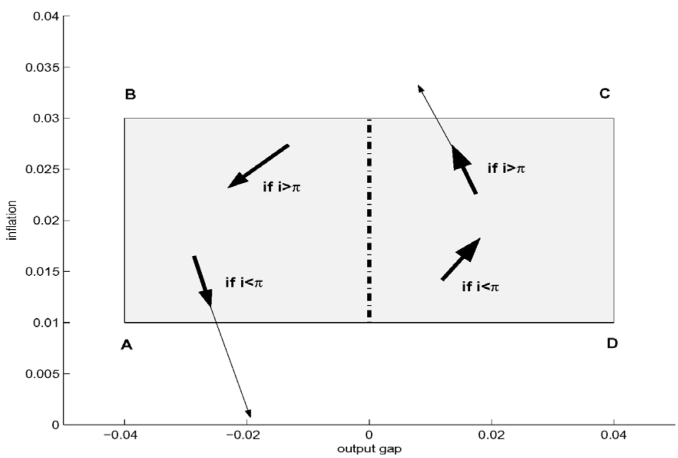


FIGURE 6. Constraint set K and typical evolutions.

In the top panel we see that when the output gap is negative ($y < 0$), inflation decreases (arrow points toward **A**); reciprocally, inflation increases if the output gap is positive (arrow points away from **A**). Above the plane $i = \pi$, interest rate dominates inflation, which decreases the output gap (above $i = \pi$, arrows point left). Below $i = \pi$, where inflation is higher than interest rate, the arrows point right, which means that the output gap increases. (Basically, for moderate values of y, output gap diminishes where real interest rate is positive.)

The bottom panel shows a few typical evolutions' directions in the two-dimensional (y, π) plane.

4.2. Two Precarious Situations

The introductory models discussed in Sections 2.2 and 2.4 are two-dimensional. Because of this "low" dimensionality, the viability kernels were easily obtained for those problems. In particular, we used a geometric characterization of the kernel in Section 2.2, whereas in Section 2.4 we derived the kernel analytically. The central-bank viable-control problem for system dynamics (30)–(32) and constraint set (27) is essentially three-dimensional and thus is a "complex" problem, for which it is very difficult to specify the analytical form of the kernel [compare Martinet and Doyen (2007)].[32] The geometric representation in three dimensions is also difficult.

In this paper, rather than computing the viability kernel for the entire constraint set (27) and system dynamics (30)–(32), thus characterizing the central bank's viable policies for *any* point in K, we will establish satisficing controls for two important monetary control problems critical for the functioning of the bank. The first concerns a *liquidity trap*. The other relates to an economy that is *hot*.

These problems occur in the three-dimensional space (see upper panel Figure 6) and it is where the viability kernel lives. We will analyze the problems in this space; however, we will also try to examine the evolutions' projections onto two-dimensional space (bottom panel of Figure 6). We hope that this will help preserve the transparency of our results.

We use Figure 6, bottom panel, to sketch what a two-dimensional analysis can tell us about the system evolutions. Around corner **A**, inflation is low and output gap negative. It is evident from the figure that if the bank lowers the interest rate "too late," which is represented by the lower arrow pointing downward rather then rightward, the economy may drift (with negative output gap) toward zero inflation, where no instrument exists to lift the output.[33]

The locus of points relevant to a "hot" economy is corner **C** (large and positive output gap and high inflation). Intuitively, when the economy is close to **C**, there is little the bank can do to prevent the economy from exceeding the 3% boundary. To keep the economy in K, the bank must turn the economy to the left by increasing i early.

Situations such as these ask for the determination of a collection of points from where the control from U (defined in (29)) is sufficient for $y(t), \pi(t)$ to avoid leaving K in finite time. In the next section we will determine the collections of points, presumably close to **A** and **C**, respectively, from which the dynamic system (30)–(32) can be controlled so that the liquidity trap and exceeding the upper inflation boundary are avoided.

4.3. Geometric Characterization

We need to say how we will characterize the collections of points in the state space from where viable controls exist. We will use geometric characterization, helped by the analytical calculations of the *limiting* trajectories, that is, those on which some critical feasible controls are applied.

In general [see, e.g., Aubin (1997), Cardaliaguet et al. (1999), Martinet and Doyen (2007), and Smirnov (2002)], the characterization of viable controls relies on an analysis of the compatibility of the system's dynamics with the geometry of the constraint set. A point in K is accepted as an element of the kernel's frontier if an evolution direction at this point is tangent to, or pointing inward into, the viability kernel. This condition is equivalent to the request that an evolution direction form an obtuse angle with the normal to the kernel's frontier. To confirm that a point is a member of the kernel V, a nonempty intersection between two cones is sufficient: the first is the cone containing the system's feasible velocities at x, and the other is the normal (or "contingent") cone to $x \in V$.

All these conditions can be easily illustrated by the analysis of the evolution directions in Figure 2. For example, point **A** was confirmed as a member of V because there existed an evolution from this point that did not leave V. We noted that this evolution formed a 90° angle (limit obtuse). Equivalently, one evolution's direction intersected with one ray of the normal cone to V at **A**. A similar analysis could also be performed in Figure 3. Here, the boundaries $\mathrm{fr}\, V$ were obtained analytically by "running" system (10) backward in time from two steady states lying on the boundary of the constraint set K with the highest velocity. Hence, the nonempty intersection between the cone containing the system's feasible velocities and the normal cone to the boundary was trivially fulfilled. Any evolution that starts below the boundary (such as that from **B** in Figure 3) must leave K in finite time because the boundary was obtained as the limit trajectory, on which the prices grow with the highest allowable speed. Hence, there is no velocity to bend the trajectory upwards before the K limit ($x = 1$) is reached. Reciprocally, any evolution that originates above the boundary (such as that from **A** in Figure 3) can be maintained in V simply because if it dropped to $\mathrm{fr}\, V$ the fastest price change would carry it to a steady state $x = 1$, $p = a$.

We will use the above observations to comment on viable controls for the three-dimensional problem of the central bank, which we will solve in the subsequent sections. We will explicitly compute the limiting trajectories by running the system backward in time from a point within the constraint set, which the central bank may consider as a target, perhaps intermediate, using the maximum feasible speed. Then we will analyze the trajectories' neighborhood to decide where the satisficing controls exist.

4.4. Satisficing Controls

We will discuss the existence of satisficing controls for the two important economic situations that were sketched in Section 4.2.

Liquidity trap. We will now determine the kernel's boundaries for the "south-west" corner (**A**). As in the case discussed in Section 2.4 (see Figure 3), we need

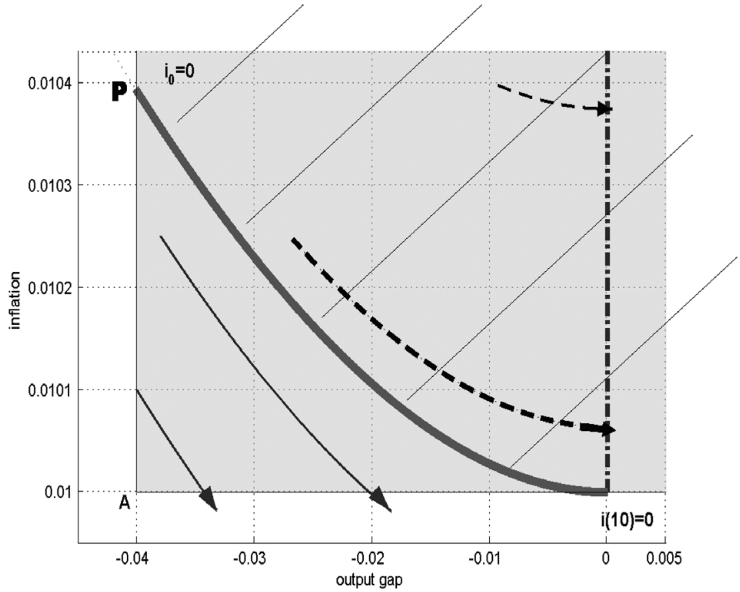


FIGURE 7. Corner **A**: viability kernel for *zero*-interest-rate policy.

to say what states (perhaps intermediate) the central bank might want to achieve when the economy is in recession ($y(t) < 0$) and the inflation and interest rates are low and such that the realization of a sizeable negative real interest rate $i(t) - \pi(t)$ is impossible.

If $i(t) - \pi(t) \approx 0$ and $y(t) < 0$, a long recession is looming; see equations (30), (31). However, even if the interest rate is zero, a small positive inflation creates a small negative real interest rate and the economy will go out of a recession. Thus it will be of interest to the bank to identify the economy states from which the zero-interest-rate policy guarantees a recovery without sliding to deflation.

In Figure 7 a limiting system evolution is shown as the solid line on which the zero-interest-rate policy is applied, that is, $i_0 = i(10) = 0$, where 10 [quarters] is the time needed for the economy to move from point **P** to $y = 0, \pi = 1\%$. This is the point from which the economy will not slide to recession. Indeed, the economy will recover: with the negative real interest rate and output gap zero, the output gap growth is positive; hence inflation will start rising, output will become positive, etc., see equations (30), (31). Eventually, the bank will be able to apply a nonzero interest policy. We therefore believe that this point ($y = 0, \pi = 1\%, i = 0$) can represent a bank's intermediate target.

The same limiting evolution is shown as one of the solid lines in the three-dimensional Figure 8. It starts from around the corner **A** (marked $i_0 = 0$) and lies flat on the K set floor $i = 0$. (Notice that, in this scale, point **A** appears overlapping with point **P**.)

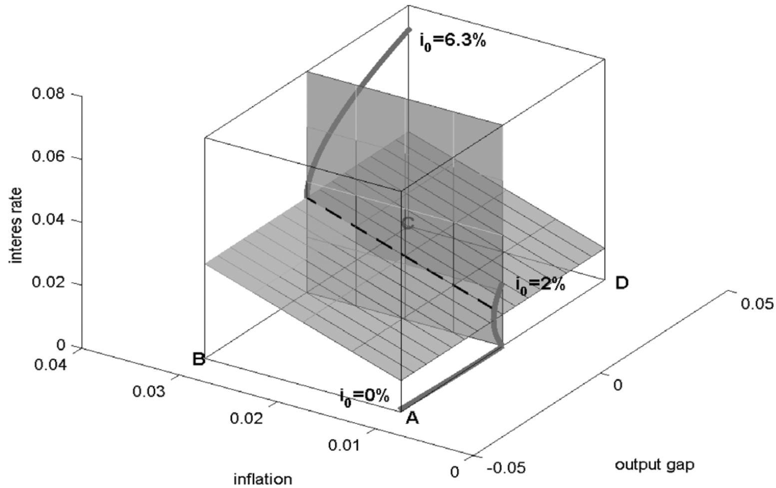


FIGURE 8. Viability problem presentation in $\mathbf{R}^3$.

On this line, the economy evolves from $y(0) < 0$ to $y(T) \geq 0$ (where $T = 10$ quarters) and the inflation is decreasing to 1%. To calculate this trajectory, we have run (30)–(32) backward from $y(T) = 0, \pi(T) = 0.01, i(T) = 0$ with $u = 0$ until the output gap lower boundary was reached (point **P**). This trajectory (solid line) delimits a viable control area (marked by the thin lines), from where the zero-interest-rate policy guarantees achievement of a non-negative output gap (in 10 quarters or less). Any state to the left of the solid line does not have that property.

We can see that keeping the economy in the viable control area kernel can prevent a liquidity trap. An economy, once it drops outside this region can remain for long at a negative output gap level and become deflationary (in some monetary policy jargon, people say that the economy falls into a liquidity trap).

Figure 7, jointly with equations (30)–(32), can help us understand why the minimum allowable inflation level should be kept positive: if inflation is positive, then the interest rate can be made smaller (positive or zero) than the inflation rate, and hence a negative real interest rate can be achieved. This helps output gap to grow. Should inflation be nonpositive, output gap will grow more slowly or not at all.

However, the current interest rate (i.e., one from which the bank starts combating recession) can be high, and dropping it to zero may create a shock if (32) is not satisfied. To avoid this, the bank needs to change the interest smoothly. If this happens, the negative real interest rate cannot be realized instantaneously and some further output-gap shrinking phase will take place. This may last long for high interest rates (obviously, it takes longer to lower a high interest rate to zero than a low interest rate).[34] It is therefore of importance for the bank to identify the economy states from which the policy of the fastest drop of interest rate moves the

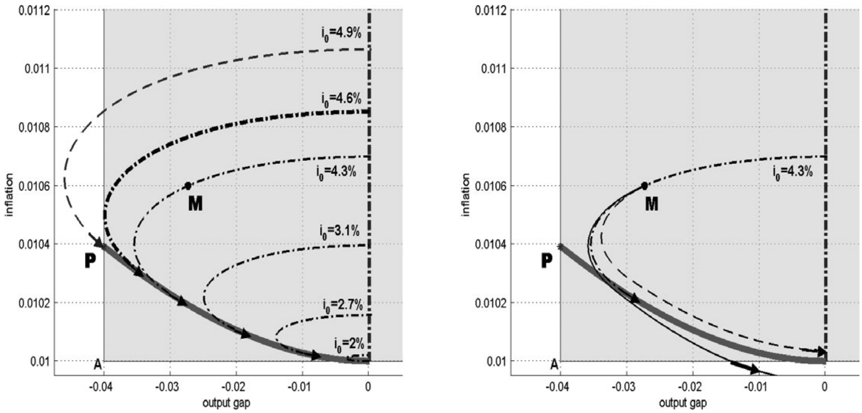


FIGURE 9. Corner **A**: evolutions for maximum-speed recovery.

economy to point **P** (or the trajectory **P**→ (0, 0.01)); see Figure 7), from which the zero-interest-rate policy will lead the economy to recovery.

Notice the dashed trajectory in Figure 9, which starts at neutral output gap with $i_0 = 4.9\%$ and joins smoothly the zero-interest trajectory at point **P**. On this trajectory (and also on the other trajectories obtained for negative output gap and positive real interest rate), output gap shrinks before it starts growing. We can see that, for a recovery that would avoid this rather dramatic output gap decline (and the y-lower bound violation), the interest rate $i(t)$ for $y(t) < 0$ should not be too high.

In this figure, we can also see that an economic evolution that starts with $i_0 = 4.6\%$ (the figure top dash-dotted trajectory) satisfies the output gap lower limit. The other dash-dotted lines show the fastest interest rate drop evolutions from neutral output gap for several other inflation rates. (The lowest such trajectory, marked $i_0 = 2\%$, was also shown in Figure 8.) We see that maintenance by the central bank of a relationship between the inflation and interest rates is crucial for the fastest recovery.

To highlight this relationship, we consider an economy characterized by output gap -2.73% and inflation 1.06%, which is the point **M** on the evolution that crosses the neutral output gap with $i_0 = 4.3\%$. If the corresponding interest rate is 2.77% at this point, then the evolution follows the trajectory marked 4.3% in the right panel and reaches a recovery state within the constraint set K. If, however, the interest rate is 2.85%, then the economy evolves on the solid line and the inflation lower boundary is violated. But, if the interest rate is 2.62%, then the economy recovers inside K. We infer that for evolutions that do not escape from K, the interest rate has to be lower than some critical level, which depends on output gap and inflation.

In Figure 9, left panel, we show more evolution trajectories that bring the economy to zero interest rate with the maximal interest rate drop. Each of these

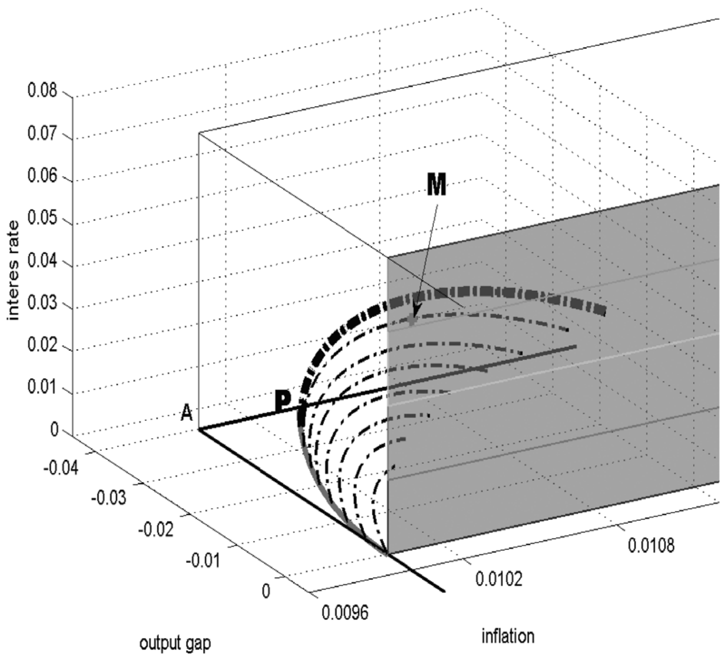


FIGURE 10. Corner **A** blowup.

trajectories was computed analytically; each reaches zero interest rate on the separating evolution **P**→ (0, 0.01) smoothly with $u = -0.005$. On these trajectories, the overlap between the velocities' cone and the normal ("contingent") cone on the viability kernel's boundary is trivially fulfilled. Hence, there are satisficing controls to bring the economy to $y = 0, \pi = 1\%, i = 0$ from the points that are inside of each trajectory and no such controls exist for points that are outside the trajectories.

To better explain what the viability kernel about corner **A** is, we need to examine the system evolutions in $\mathbf{R}^3$, which we started in Figure 8. Figures 10–12 are relevant to understanding the existence of a critical interest rate level for a given combination of inflation and output gap. We also use these figures to identify an area for which there exist viable controls that take the economy away from a liquidity trap.

Figure 10 shows the maximum-speed recovery evolutions in $\mathbf{R}^3$, commented on earlier in Figure 9. The front semitransparent wall is the neutral output gap plane. Behind the wall, a three-dimensional surface is emerging, delimited by the recovery evolution trajectories: the outermost trajectory is the evolution corresponding to $i_0 = 4.6\%$, which is the thickest dash-dotted line in Figure 9 passing near **P**. The innermost trajectory corresponds to $i_0 = 0\%$. If the economy is to recover speedily and remain in the rectangular box of constraints K, then the economy's three coordinates, output gap, inflation, and interest rate, must remain behind this

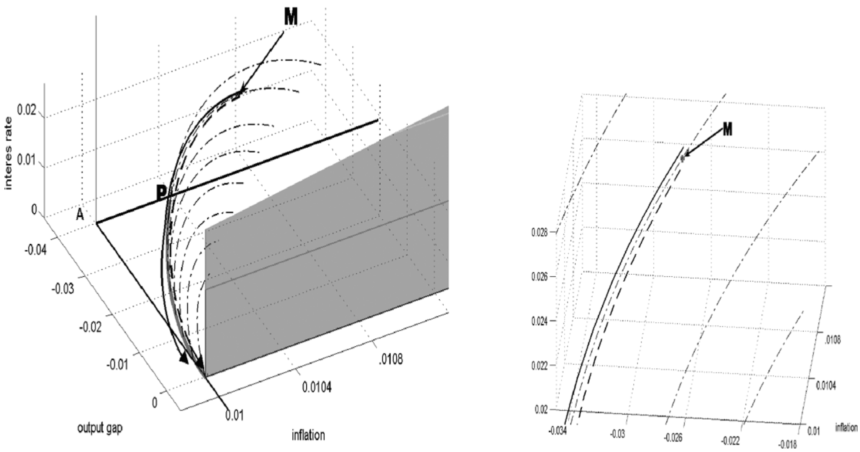


FIGURE 11. Corner **A** blowup (continuation).

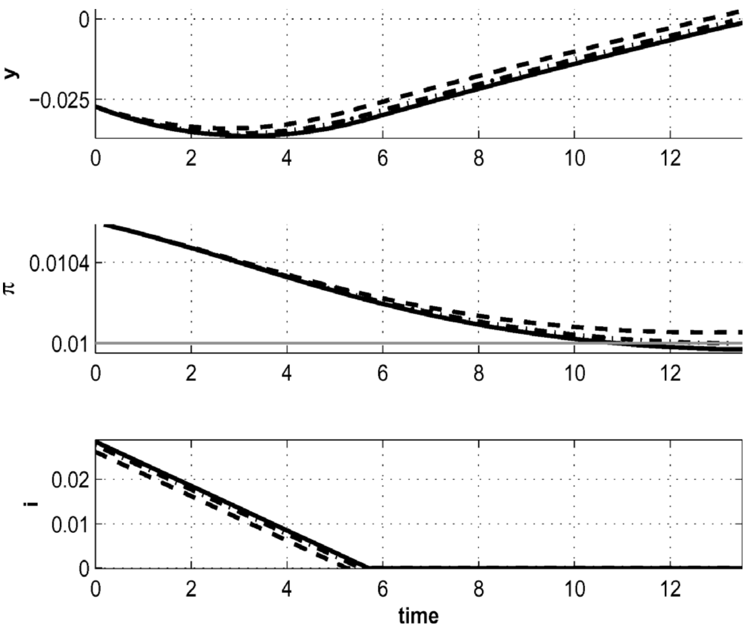


FIGURE 12. Time profiles of output gap, inflation, interest rate, and interest rate changes, originating from point **M**.

surface. In other words, the area behind this surface contains the economy states from which the viable evolutions control the economy away from negative output gap and low inflation.

Figure 11 shows the details concerning the difference between the evolutions starting inside and outside the viability kernel. The two trajectories shown in

Figure 9 as a solid line and a dashed line are now presented in $\mathbf{R}^3$. It is apparent that the solid-line evolution that begins at $i = 2.85\%$ starts from outside the critical trajectory (see the right panel) and leaves the constraint set K in finite time (see the left panel). However, the dashed-line evolution from $i = 2.62\%$ begins inside and leads to the fastest interest rate drop (speedy) recovery. We can see the time profile of each of the evolution's coordinates in Figure 12.

The three time profiles in Figure 12 represent the three cases shown in two dimensions in Figure 9 and in three dimensions in Figure 11 (focus on the right panels). They differ by the original interest rate, which, at time zero, is 2.77% on the dash-dotted line (the one that passes through **M**), 2.85% on the solid line, and 2.62% on the dashed line.

For the dash-dotted line, the fastest reduction of the interest rate leads the output gap to neutral without violating the lower inflation bound. If the evolution starts from $i_0 = 2.62\%$, which is inside the 3D area shown in Figure 10, then the neutral output gap is also achieved without violating the lower inflation bound. However, if the evolution starts from $i_0 = 2.85\%$, which is outside the 3D area shown in Figure 10, then even with the fastest reduction of the interest rate, the lower inflation bound is crossed at time $t \approx 10.8$. This illustrates that there is no viable control from a point that is outside the critical evolution. (Here, i_0 on the solid line is 2.85%, which is more than the limit $i_0 = 2.77\%$; see the dash-dotted line.)

We infer from the above[35] that the interior of the area displayed in Figure 10 is the viability kernel for the area of K around corner **A**[36].

A warning can be taken from the above figures that even mildly negative output gaps can lead to a liquidity trap if inflation is very low and if the bank starts late to control the economy, that is, the interest rates are relatively high.

This is mainly due to the inertia of the economic processes considered here. Keeping the process evolution inside a viability kernel guarantees that the instrument (here, nominal short-term interest rate) will be applied sufficiently early so that uncontrollable fallouts can be avoided.

We also notice that very low inflation could harm viability of the system. In other words, it will be a slow process for the economy to recover from a liquidity trap when inflation is low. This observation reinforces a rather commonly accepted central bank canon: not to consider zero or very low inflation as a goal of its monetary policy [compare Nishiyama (2003)].

Overheating. We will now determine a set of points around the north-east corner (C), for which the satisficing controls exist. Here, we will restrict our analysis to situations in which interest rates dominate inflation, that is, the economy is above the 45° line in Figure 6 (top panel) and output gap decreases. This monotonic movement, from right to left, helps us analyze the economy in $\mathbf{R}^2$ (output gap—inflation), rather than in $\mathbf{R}^3$ (output gap—inflation—interest rate).

Let us consider which point of the economy the central bank may want to target, at least temporarily, if inflation runs high and output gap is positive. Obviously, any point from which the upper boundary on inflation will be exceeded will not be aimed for. This excludes any point on this boundary as a target, even transitory, for as long as $y(t) > 0$ [see equation (31)]. Hence, the bank should aim at $y(t) \leq 0$. However, strictly negative output gap $y(t) < 0$ means a recession (or a beginning of it) thus the bank should use a policy conducive to $y(t) = 0$, rather than $y(t) \leq 0$. As the highest allowable inflation in our example is 3%, the bank ought to aim at $y(T) = 0, \pi(T) = 0.03$ for some $T > 0$.

Presumably, with inflation close to the upper limit, the nominal interest rate is also high and close to its maximum. Keeping it high can control the inflation, but if the real interest rate $i(t) - \pi(t)$ is positive and rather large, a recession is probable [see equation (30) and Figure 9; also bear in mind footnote 34]. Hence the bank should try to achieve $y(T) = 0, \pi(T) = 0.03$ with the interest rate that minimizes the recessionary and inflationary pressures. This means that the bank will intend $i(T) = 0.03$, which characterizes a steady state, at which all pressures vanish.

It is therefore of great importance for the bank to recognize which is the limiting system evolution that reaches $y(T) = 0, \pi(T) = 0.03, i(T) = 0.03$ in finite time T and with the fastest allowable interest rate reduction $u = di/dt = -0.005$. The limiting property of this system's trajectory will be such that, for a given level of the interest rate that is higher than inflation (as was assumed at the beginning of this section), from the points that are above this trajectory, violation of the inflation upper limit, or output gap lower limit, will happen in finite time. Conversely, for evolutions from states that are on or below this trajectory, the violation will not happen and the steady state $y(T) = 0, \pi(T) = 0.03, i(T) = 0.03$ will be achieved (again, for a given level of interest rate that is higher than inflation). We will demonstrate these features of the economy's evolutions by analyzing the system's dynamics (30)–(32) and Figures 13, 14.

We have determined this critical trajectory by running (30)–(32) backward from $y(T) = 0, \pi(T) = 0.03, i(T) = 0.03$ with $u = -.005$. The trajectory is shown in Figure 13 (and also in the three-dimensional Figure 8; see the system trajectory that starts at $i_0 = 6.3\%$).

Part of a booming economy's characterization is that the interest rate is close to the limit, which was set at 7% for this economy; see (27). In Figure 13, on the limiting trajectory, we control the economy from when the interest rate was 6.3%, the output gap was maximal (4%), and inflation run high but still below 3%. The interest rate was gradually eased and, after less than 7 quarters, reached 3%, the same value as inflation. On this trajectory, we run the economy forward from the point that was reached when we ran it backwards; hence, not surprisingly, these two evolutions coincide with the same thick trajectory.

We will further analyze the economic dynamics depicted in Figure 13 and conclude that the evolution represented by the thick solid line belongs to the viability kernel's boundary (where the kernel and boundary are three-dimensional objects). We will see that a 2D projection of the kernel lies left from this line (marked by the thin lines).

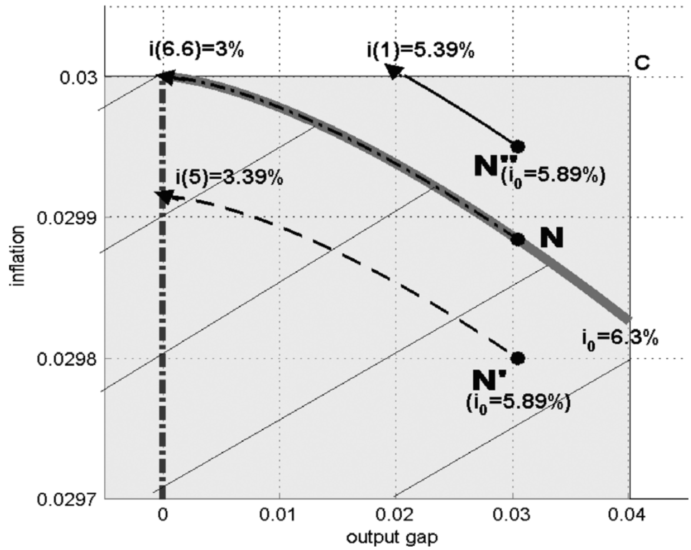


FIGURE 13. Viable and nonviable trajectories around corner **C**.

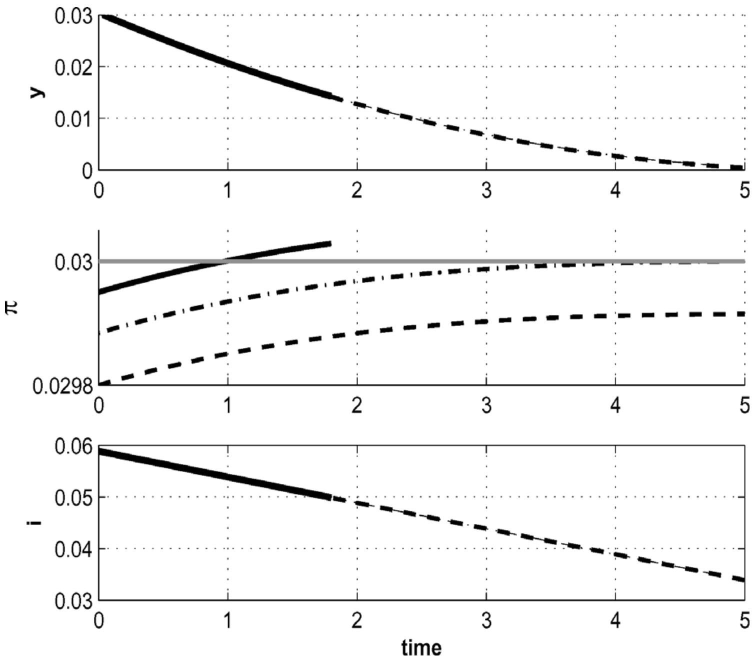


FIGURE 14. Time profiles of output gap, inflation, interest rate, and interest rate changes, originating from points **N**, **N**′, and **N**″.

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Figure 13 also shows what would happen if the bank did not combat inflation early and let it leave the kernel. Consider point **N** lying on the limiting evolution. Suppose the interest rate is 5.89%, which will be the interest rate of an economy controlled by the maximum interest rate drop from the point marked $i_0 = 6.3\%$. If the economy is controlled from point **N** in this way, it will continue evolving on the thick line until the steady state marked by $i(6.6) = 3\%$ is reached. This evolution is also shown in the time domain in Figure 14; see the dash-dotted line.

Next, suppose that inflation has crossed the limiting evolution and that the economy is at point **N**″ and the interest rate is 5.89% (as at point **N**). If the bank started raising the interest rate (or keeping it constant), the economy would cross the neutral output gap with high interest rate. Inflation would start diminishing but the lower bound on output gap would be broken; see Figure 9. Alternatively, the bank could drop the interest rate with the maximum speed. This was a *viable* policy from point **N** but is not from **N**″. We can see this by observing the maximum interest drop evolution that is represented by the solid line originating from **N**″, which crosses the inflation upper limit. Also, see the solid line[37] in Figure 14, which represents this evolution's time profile. We can see that the violation of the 3% inflation limit will occur in about one quarter.

However, if the economy is below the limiting evolution, application of the allowable instruments can prevent violation of the inflation upper limit. For example, an evolution from **N**′ managed by the maximum interest rate drop reaches the neutral output gap in 5 quarters and the corresponding interest level is such that the violation of the lower bounds on output gap and inflation is *not* forthcoming (compare Figure 9). This evolution, originating from point **N**′, is indicated by the dashed line in both Figures 13 and 14.

From the above analysis (and from many more economic dynamics simulations not reported here), we conclude that the thick solid line in Figure 13 belongs to the viability kernel's boundary. Furthermore, we deduce that there exist satisficing controls for the economy's states that are below this trajectory, whereas they do not exist for the states above it.

5. CONCLUDING REMARKS

We have considered a simple macroeconomic model. An analysis based on viability theory enabled us to discuss how a central bank monetary policy can be established. We have endogenously derived the following satisficing policy recommendations:

(I) if $y(t), \pi(t)$ are well inside V then apply $i(t) + hu,\ u \in [-0.005,\ 0.005]$ for every time interval h;

(II) otherwise apply (assuming feasible) $i(t) - 0.005h$ if $y < 0$ or $i(t) + 0.005h$ if $y > 0$, where V is the viability kernel (e.g., see the 3D area in Figure 10 behind the surface spanned by the limit evolutions).

These recommendations are in line with policy (6); in particular, (II) is extreme in that it calls for the full speed interest rate changes. The recommendations should be understood as guidelines for the central bank's governor.

We explain "well inside" as follows. We believe that there are two states an economy can be in: well inside the viability kernel and close to its boundaries.[38] An assessment of in which of them the economy is in will obviously depend on the bank governor's judgment. Our graphs are helpful in the assessment. They can tell the governor where the economy is expected to move, given current conditions and the applied instruments. If, at $t + h$, the economy is expected to remain in the kernel, then the economy state at t is well inside the kernel.

The distinction between these two states of the economy is needed for the governor to decide which size of instrument u to apply. With our model, the governor can assess where the economy is expected to be at time $t + h$ and what options he or she will then have. We believe that the choices made in this manner will be less arbitrary than the optimal ones that rely on the loss function weights and discount rate, which are subjective parameters.

The satisficing policy choices can be modified to allow for measurement errors, parameter uncertainty (In α, ζ, etc.), and shocks, even if the system's dynamics is deterministic[39]. In broad terms, a ball around each point of the trajectory in the y, π plane (see, e.g., Figure 6) might be constructed where the ball size was proportional to the degree of uncertainty (e.g., measured by standard deviations). The conditions to apply rules (I), (II) can be modified: if the ball does not intersect the viability kernel's boundary, apply (I); else apply (II).

Usually, the bank's policy established through a viability analysis should appear more credible to economic agents than its optimized counterpart. This will be so because the former depends on fewer arbitrary parameters than does the latter. This and the observation about passivity of a *viable* (or "satisficing") policy, coming below, renders the latter less vulnerable to the Lucas critique (1976) than its optimizing counterpart.

As was observed before, policy advice based on viability theory may be passive, that is, recommend $u = 0$ [see (I)] for a wide range of states of the economy that are well inside the kernel V. This will mean that the instrument realizations may not change even if y, π have changed. However, this would not be the case if a loss-function optimizing policy were implemented where each combination of y, π implied a different i. If the bank chooses $u = 0$, the private sector will have no need to change its behavior; hence the bank's model parameters, which depend generically on this behavior, will not change. In consequence, the bank policy will remain time-consistent.

In general, a policy based on a viability analysis is precautionary in that it controls the system away from regions of adverse economic conditions (such as large negative output gap or accelerating inflation) where control of the system is difficult or impossible. Hence, a viable policy is naturally forward-looking and thus attractive for uncertain conditions, where uncertainty might be due to the parameter values or stochastic shocks. We can say that policy advice based on viability theory takes a compromise into account between the instrument timing and its strength. This might be paraphrased by saying that an early application of a light instrument can replace (and, probably, be more efficient than) a late use of a heavy instrument.

Future work will concentrate on open economy models, which will take exchange rate uncertainty into account. A starting point for an analysis might be Clément-Pitiot and Doyen (1999), which computes viable policies that can keep the exchange rate in a target zone; also see Krawczyk and Kim (2004a), where exchange rate enters the model as a nuisance agent's control. Extensions of the model, in which the parameters would become new metastate variables controlled by more nuisance agents, will produce viable policies robust to private sector agents' reactions, and hence even more resistant to the Lucas critique than those computed in this paper.

We can also envisage an interest in viability kernels from international economic agencies (such as the International Monetary Fund). Given economic data from a country, its kernel can be established. An assessment of the current state of the economy where it is relative to the kernel could help the agency in its advisory (or lending) decisions regarding this country.

NOTES

1. This paper and its earlier version [Krawczyk and Kim (2004b)], along with the reports [Saint-Pierre (2001) and Clément-Pitiot and Saint-Pierre (2006)] where an endogenous business cycle was studied, and [Clément-Pitiot and Doyen (1999)], where an exchange rate dynamics analysis was carried out, pioneer viability theory application to macroeconomics. For viability theory applications to environmental economics see Bene et al. (2001), De Lara et al. (2006), Martinet and Doyen (2007), and Martinet et al., in press; see Pujal and Saint-Pierre (2006) and references provided there for applications to financial analysis.

2. We adopt the common meaning of output gap, that is, the log deviation of actual output from the normal or potential level.

3. The state may be generalized to metastate, which will comprise instruments (flows) along the usual stock variables. If so, the system's controls will be velocities of the instruments; see later in the paper the system of equations (10) and also the system of equations (30)–(32).

4. We do not rely on the use of set-valued maps and differential inclusions in this paper, which is standard in mathematical papers about viable controls. However, we signal that one can view the system velocity as the point-to-set map at $x(t) \in K$

$$F(x) \equiv \{f(x,u), u \in U\}$$

and that the dynamics (1) can be rewritten as the differential inclusion $\dot{x}(t) \in F(x(t))$. See Aubin (1997), Aubin et al. (2000), and, in particular, Smirnov (2002) for theorems on the existence of solutions to differential inclusions.

5. We stress that one determines a viability kernel given the constraint set (K), system dynamics (f), and horizon (T). Some authors use the rather complex expression $V_f^K(T)$ or $\mathrm{Viab}_f^K(T)$ to denote a kernel. We will use simple V and highlight for which constraints, dynamics, and horizon the kernel is determined.

6. For autonomous systems.

7. Notice that traditional monetary-policy optimization models [e.g., Walsh (2003)] are modeled and solved as infinite-horizon and stationary.

8. Actually, proximal normal. Given a convex set and a point outside the set, in a normed space, a proximal normal is the direction of a vector that connects the point to a nearest point of the set. A normal is just a direction in which a support functional is positive. We assume our problems are formulated in normed spaces and will use the term *normal*.

9. Suppose the evolution started at the origin.

10. See Aubin (1997, p. 99) and Martinet and Doyen (2007, Section 3.4.1.3).

11. Unless a steady state has been reached. Also, see *ibid.*

12. See, for example, Walsh (2003).

13. This radius might equal a standard deviation shock magnitude. It may also equal the size of the shock that occurs "once in 100 years," etc.

14. Recently, the main gas supplier in Wellington, New Zealand was obliged to pay substantial compensation to customers for a breakdown in gas delivery; also, a transmission line burnout in Auckland resulted in heavy fines the electricity grid operator had to pay its patronage.

15. Aubin discusses (1997) several simple viability problems. In particular, an affine and bilinear system dynamics is considered; see pp. 46–51. Our model is inspired by the former but has a structure similar to that of the latter, with the economic interpretation expanded.

16. We believe that you may not like oil or chocolate at the beginning even if it is costless and you will only gradually develop a taste for it. On the other hand, linearity of the changes is not crucial for the subsequent analysis.

17. We do not pretend to derive equation (7). We conjecture that it might capture the consumption habit of x.

18. Maximization of (8) in a Markovian (i.e., feedback) strategy $p(x(t)) \in [0, \overline{p}]$ would be even more difficult.

19. For example, a lot of demand could make the clearing price rise faster than inflation. However, the supplier may not want to feel some kind of social condemnation caused by the prices behaving too differently from inflation. Also, the supplier may not want to change the *menu* too frequently. All that would mean that the price would not change instantaneously.

20. This is a differential inclusion where the right-hand sides define the correspondence F referred to in footnote 4.

21. The time of keeping the evolution viable is not important in this and subsequent examples, hence we will drop index T from notation.

22. More precisely, the boundary of V is the x-axis from $0 + \varepsilon$ (where $\varepsilon > 0$ and small) to where it intersects with the positive-velocity critical trajectory, then this trajectory, then $x = 1$, and then the negative-velocity critical trajectory until $(0, 0 + \varepsilon)$.

23. Notice that the policy advice for (x, p) that belong to the kernel's frontier is to apply an *extreme* price strategy, for which the price changes are maximal.

24. Not included in K; for example, integral (8) could be minimized.

25. In Section 3.3, we explain how we have assigned a value to new coefficient a in (17). We could have used the full equation (13) in our study. However, this would have increased the state space dimensionality and made the viability analysis less transparent. As this paper's main purpose is to show how viability theory can be applied in macroeconomics, we prefer to use a lower order system.

26. For the time being, we assume the Lucas critique away. We will come back to it in the Concluding Remarks.

27. In this simple model, inflation grows for any positive output gap. However, adding exchange rate to the model, as in Krawczyk and Kim (2004a), helps to understand why this might not always be the case.

28. The parameter values come from a table published in Walsh (2003), which quotes maximum likelihood estimates for a model originally studied by Fuhrer (1994).

29. Notice that $1.53 - 0.55 = 0.98$.

30. The interest rate upper bound of 7% appears to be attained in New Zealand in 2004-4. We could extend it to better reflect this country's conditions. However, in other countries (e.g., the United States), 7% seems a plausible upper bound for the interest rate and we will keep it at that level in this introductory study.

31. See note 3.

32. For high-dimension problems, algorithms can lead to numerical determination of kernels; see, for example, Saint-Pierre (1994).

33. This is what is meant by a liquidity trap: the economy remains in an area where output gap is negative and inflation is close to zero (positive or negative). We again cite McCallum (2004) for an analysis of a liquidity trap problem performed through an established method. Also, notice that Nishiyama (2003) is a recent publication where a liquidity trap problem is analyzed in state space.

34. Violation of the lower bound on output gap is imminent if the initial interest rate is high relative to inflation measured at the same time.

35. And from many more simulated evolutions not shown here for lack of space.

36. We can now better appreciate that the areas in Figure 7 and in Figure 13, marked by the thin lines, are $\mathbf{R}^2$ projections of the respective viability kernels, which generically live in three dimensions; that is, they depend generically on interest rate.

37. The output gap evolutions from $\mathbf{N}$, $\mathbf{N}'$, and $\mathbf{N}''$ are different from each other. However, in this figure's scale, they appear indistinguishable.

38. If the economy was outside V, the special crisis control would have to be applied to bring the economy to V. Crisis control can also be computed using viability analysis; see, for example, Bene et al. (2001) and Martinet and Doyen (2007). However, this problem surpasses the scope of this paper and will be dealt with in subsequent papers.

39. Viability theory can also deal with explicit stochastic models.

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