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THE ELEPHANT RANDOM WALK IN THE TRIANGULAR ARRAY SETTING

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Abstract

Gut and Stadmüller (2021, 2022) initiated the study of the elephant random walk with limited memory. Aguech and El Machkouri (2024) published a paper in which they discuss an extension of the results by Gut and Stadtmüller (2022) for an 'increasing memory' version of the elephant random walk without stops. Here we present a formal definition of the process that was hinted at by Gut and Stadtmüller. This definition is based on the triangular array setting. We give a positive answer to the open problem in Gut and Stadtmüller (2022) for the elephant random walk, possibly with stops. We also obtain the central limit theorem for the supercritical case of this model.

Keywords: Random walk with memory; limit theorems; phase transition

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1. Introduction

In recent years there has been a lot of interest in the study of the elephant random walk (ERW) since it was introduced in [15]; see the excellent thesis [13] for a detailed bibliography. The standard ERW is described as follows. Let $p \in (0, 1)$ and $s \in [0, 1]$. We consider a sequence $X_1, X_2, \ldots$ of random variables taking values in $\{+1, -1\}$ given by

$$X_1 = \begin{cases} +1 & \text{with probability } s, \\ -1 & \text{with probability } 1-s; \end{cases} \tag{1.1}$$

$\{U_n : n \geq 1\}$ a sequence of independent random variables, independent of X_1, with U_n having a uniform distribution over $\{1, \ldots, n\}$; and, for $n \in \mathbb{N} := \{1, 2, \ldots\}$,

$$X_{n+1} = \begin{cases} +X_{U_n} & \text{with probability } p, \\ -X_{U_n} & \text{with probability } 1-p. \end{cases} \tag{1.2}$$

The ERW $\{W_n\}$ is defined by $W_n = \sum_{k=1}^{n} X_k$ for $n \in \mathbb{N}$.

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Gut and Stadmüller [9, 10] studied a variation of this model as described in [9, Section 3.2]; [1] also studied a similar variation of the model as described in [1, Section 2]. We present the two different formalizations of models given in [1, 10]; our work is based on the first formalization.

1.1. Triangular array setting

Consider a sequence $\{m_n : n \in \mathbb{N}\}$ of positive integers satisfying

$$1 \le m_n \le n \quad \text{for each } n \in \mathbb{N}. \tag{1.3}$$

Let $X_1, X_2, \ldots$ be the sequence defined by (1.1) and (1.2). We define a triangular array of random variables $\{\{S_k^{(n)} : 1 \le k \le n\} : n \in \mathbb{N}\}$ as follows. Let $\{Y_k^{(n)} : 1 \le k \le n\}$ be random variables with

$$Y_k^{(n)} = \begin{cases} X_k & \text{for } 1 \le k \le m_n, \\ X_k^{(n)} & \text{for } m_n < k \le n, \end{cases} \tag{1.4}$$

where, for $m_n < k \le n$,

$$X_k^{(n)} = \begin{cases} +X_{U_{k,n}} & \text{with probability } p, \\ -X_{U_{k,n}} & \text{with probability } 1-p. \end{cases} \tag{1.5}$$

Here, $\mathcal{U}_n := \{U_{k,n} : m_n < k \le n\}$ is an independent and identically distributed (i.i.d.) collection of uniform random variables over $\{1, \ldots, m_n\}$, and $\{\mathcal{U}_n : n \in \mathbb{N}\}$ is an independent collection. Finally, for $1 \le k \le n$ let $S_k^{(n)} := \sum_{i=1}^{k} Y_i^{(n)}$. We note that for fixed $n \in \mathbb{N}$, the sequence $\{S_k^{(n)} : 1 \le k \le n\}$ is a random walk with increments in $\{+1, -1\}$. However, the sequence $\{S_n^{(n)} : n \in \mathbb{N}\}$ does not have such a representation. We study properties of the sequence $\{T_n : n \in \mathbb{N}\}$ given by

$$T_n := S_n^{(n)}. \tag{1.6}$$

The process $\{T_n : n \in \mathbb{N}\}$ was called the *ERW with gradually increasing memory* in [10], where

$$\lim_{n \to \infty} m_n = +\infty. \tag{1.7}$$

1.2. Linear setting

In this setting the ERW $W'_{n+1} := W'_n + Z_{n+1}$ is given by the increments

$$W'_1 = Z_1 = \begin{cases} +1 & \text{with probability } s, \\ -1 & \text{with probability } 1-s, \end{cases}$$

$$Z_{n+1} = \begin{cases} +Z_{V_n} & \text{with probability } p, \\ -Z_{V_n} & \text{with probability } 1-p, \end{cases}$$

where V_n is a uniform random variable over $\{1, \ldots, m_n\}$, and $\{V_n : n \in \mathbb{N}\}$ is an independent collection.

Remark 1.1. We note here that the dependence structure in the definitions of T_n and W'_n are different and as such, results obtained for T_n need not carry to those obtained for W'_n. The

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error in [1, Theorem 2 (3)] is due to the use of the linear setting for their equation (3.20), while working in the triangular array setting. In particular, there is a mistake in the expression of $\overline{M}_\infty$ on [1, p. 14], which was fixed in the subsequent corrigendum. Their results in the corrected version agree with the results obtained here, although the methods used are different; this paper also provides additional results not obtained by them.

In the next section we present the statement of our results, and in Sections 3 and 4 we prove the results. In Section 5 and thereafter we study similar questions about the ERW with stops and present our results.

2. Results for the ERW in the triangular array setting

Before we state our results, we give a short synopsis of the results for the standard ERW $\{W_n\}$ [3, 4, 6–8, 11, 14]. Let $\alpha := 2p - 1$.

- For $\alpha \in (-1, 1)$ i.e. $p \in (0, 1)$,

$$\lim_{n\to\infty} \frac{W_n}{n} = 0 \quad \text{almost surely (a.s.) and in } L^2. \tag{2.1}$$

- For $\alpha \in \left(-1, \frac{1}{2}\right)$, i.e. $p \in \left(0, \frac{3}{4}\right)$,

$$\frac{W_n}{\sqrt{n}} \xrightarrow{\text{d}} N\left(0, \frac{1}{1-2\alpha}\right) \quad \text{as } n \to \infty, \tag{2.2}$$

$$\limsup_{n\to\infty} \pm \frac{W_n}{\sqrt{2n \log\log n}} = \frac{1}{\sqrt{1-2\alpha}} \quad \text{a.s.} \tag{2.3}$$

- For $\alpha = \frac{1}{2}$, i.e. $p = \frac{3}{4}$,

$$\frac{W_n}{\sqrt{n \log n}} \xrightarrow{\text{d}} N(0, 1) \quad \text{as } n \to \infty, \tag{2.4}$$

$$\limsup_{n\to\infty} \pm \frac{W_n}{\sqrt{2n \log n \log\log\log n}} = 1 \quad \text{a.s.} \tag{2.5}$$

- For $\alpha \in \left(\frac{1}{2}, 1\right)$, i.e. $p \in \left(\frac{3}{4}, 1\right)$, there exists a random variable M such that

$$\lim_{n\to\infty} \frac{W_n}{n^\alpha} = M \quad \text{a.s. and in } L^2, \tag{2.6}$$

  where $\mathbb{E}[M] = \beta/\Gamma(\alpha+1)$, $\mathbb{E}[M^2] > 0$, $\mathbb{P}(M \neq 0) = 1$, and

$$\frac{W_n - Mn^\alpha}{\sqrt{n}} \xrightarrow{\text{d}} N\left(0, \frac{1}{2\alpha-1}\right) \quad \text{as } n \to \infty. \tag{2.7}$$

Our first result improves and extends [10, Theorem 3.1].

Theorem 2.1. *Let $p \in (0, 1)$ and $\alpha = 2p - 1$. Assume that $\{m_n : n \in \mathbb{N}\}$ satisfies (1.3), (1.7), and*

$$\gamma_n := \frac{m_n}{n}, \qquad \lim_{n\to\infty} \gamma_n = \gamma \in [0, 1]. \tag{2.8}$$

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(i) *If* $\alpha \in \left(-1, \frac{1}{2}\right)$, *i.e.* $p \in \left(0, \frac{3}{4}\right)$, *then*

$$\frac{\gamma_n T_n}{\sqrt{m_n}} \xrightarrow{\mathrm{d}} N\left(0, \frac{\{\gamma + \alpha(1-\gamma)\}^2}{1-2\alpha} + \gamma(1-\gamma)\right) \quad \text{as } n \to \infty. \tag{2.9}$$

(ii) *If* $\alpha = \frac{1}{2}$, *i.e.* $p = \frac{3}{4}$, *then*

$$\frac{\gamma_n T_n}{\sqrt{m_n \log m_n}} \xrightarrow{\mathrm{d}} N\left(0, \frac{(1+\gamma)^2}{4}\right) \quad \text{as } n \to \infty. \tag{2.10}$$

(iii) *If* $\alpha \in \left(\frac{1}{2}, 1\right)$, *i.e.* $p \in \left(\frac{3}{4}, 1\right)$, *then*

$$\lim_{n\to\infty} \frac{\gamma_n T_n}{(m_n)^\alpha} = \{\gamma + \alpha(1-\gamma)\}M \quad \text{a.s. and in } L^2, \tag{2.11}$$

where M *is the random variable in (2.6). Moreover,*

$$\frac{\gamma_n T_n - M\{\gamma_n + \alpha(1-\gamma_n)\}(m_n)^\alpha}{\sqrt{m_n}} \xrightarrow{\mathrm{d}} N\left(0, \frac{\{\gamma + \alpha(1-\gamma)\}^2}{2\alpha - 1} + \gamma(1-\gamma)\right) \quad \text{as } n \to \infty. \tag{2.12}$$

Remark 2.1. If $\alpha = \gamma = 0$ then the right-hand side of (2.9) is $N(0,0)$, which we interpret as the delta measure at 0. Our result (2.12) differs from [1, Theorem 2 (3)]; the reason for this is given in Remark 1.1.

The next theorem concerns the strong law of large numbers and its refinement.

Theorem 2.2 *Let* $p \in (0, 1)$ *and* $\alpha = 2p - 1$. *Assume that* $\{m_n : n \in \mathbb{N}\}$ *satisfies (1.3), (1.7), and (2.8). Then*

$$\lim_{n\to\infty} \frac{T_n}{n} = 0 \quad \text{a.s.} \tag{2.13}$$

Actually, we obtain the following sharper result: If $c \in \left(\max\left\{\alpha, \frac{1}{2}\right\}, 1\right)$ *then*

$$\lim_{n\to\infty} \frac{\gamma_n T_n}{(m_n)^c} = 0 \quad \text{a.s.} \tag{2.14}$$

3. Proof of Theorem 2.1

Throughout this section we assume that (1.3), (1.7), and (2.8) hold.

Proof. Let $\mathcal{F}_n$ be the σ-algebra generated by $X_1, \ldots, X_n$. For $n \in \mathbb{N}$, the conditional distribution of X_{n+1} given the history up to time n is

$$\begin{aligned}\mathbb{P}(X_{n+1} = \pm 1 \mid \mathcal{F}_n) &= \frac{\#\{k = 1, \ldots, n : X_k = \pm 1\}}{n} \cdot p + \frac{\#\{k = 1, \ldots, n : X_k = \mp 1\}}{n} \cdot (1-p) \\ &= \frac{1}{2}\left(1 \pm \alpha \cdot \frac{W_n}{n}\right).\end{aligned}$$

For each $n \in \mathbb{N}$, let $\mathcal{G}^{(n)}_{m_n} = \mathcal{F}_\infty := \sigma(\{X_i : i \in \mathbb{N}\}) = \sigma(\{X_1\} \cup \{U_i : i \in \mathbb{N}\})$ and

$$\mathcal{G}^{(n)}_k := \sigma(\{X_i : i \in \mathbb{N}\} \cup \{X^{(n)}_i : m_n < i \le k\}) = \sigma(\{X_1\} \cup \{U_i : i \in \mathbb{N}\} \cup \{U_{i,n} : m_n < i \le k\})$$

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for $k \in (m_n, n] \cap \mathbb{N}$. From (1.5), we can see that the conditional distribution of $X_k^{(n)}$ for $k \in (m_n, n] \cap \mathbb{N}$ is given by

$$\mathbb{P}(X_k^{(n)} = \pm 1 \mid \mathcal{G}_{k-1}^{(n)}) = \frac{1}{2}\left(1 \pm \alpha \cdot \frac{W_{m_n}}{m_n}\right).$$

(This corresponds to [10, (2.2)].) From this we have that

$$\mathbb{E}[X_k^{(n)} \mid \mathcal{F}_\infty] = \alpha \cdot \frac{W_{m_n}}{m_n} \tag{3.1}$$

for each $k \in (m_n, n] \cap \mathbb{N}$, and

$$\mathbb{E}[T_n - W_{m_n} \mid \mathcal{F}_\infty] = \sum_{k=m_n+1}^{n} \mathbb{E}[X_k^{(n)} \mid \mathcal{F}_\infty] = \alpha(n - m_n) \cdot \frac{W_{m_n}}{m_n}. \tag{3.2}$$

We introduce

$$A_n := \mathbb{E}[T_n \mid \mathcal{F}_\infty], \qquad B_n := T_n - A_n. \tag{3.3}$$

Noting that

$$A_n = W_{m_n} + \mathbb{E}[T_n - W_{m_n} \mid \mathcal{F}_\infty] = \frac{W_{m_n}}{\gamma_n} \cdot \{\gamma_n + \alpha(1 - \gamma_n)\}, \tag{3.4}$$

we have

$$\gamma_n T_n = \gamma_n(A_n + B_n) = c_n W_{m_n} + \gamma_n B_n, \tag{3.5}$$

where $c_n = c_n(\alpha) := \gamma_n + \alpha(1 - \gamma_n)$.

First, we prove Theorem 2.1(i). Assume that $\alpha \in \left(-1, \frac{1}{2}\right)$. By (3.4) and (2.2),

$$\frac{\gamma_n A_n}{\sqrt{m_n}} = \frac{c_n W_{m_n}}{\sqrt{m_n}} \xrightarrow{\text{d}} \{\gamma + \alpha(1 - \gamma)\} \cdot N\left(0, \frac{1}{1 - 2\alpha}\right) \quad \text{as } n \to \infty.$$

In terms of characteristic functions, this is equivalent to, for $\xi \in \mathbb{R}$,

$$\mathbb{E}\left[\exp\left(\frac{\mathrm{i}\xi \gamma_n A_n}{\sqrt{m_n}}\right)\right] \to \exp\left(-\frac{\xi^2}{2} \cdot \frac{\{\gamma + \alpha(1 - \gamma)\}^2}{1 - 2\alpha}\right) \quad \text{as } n \to \infty. \tag{3.6}$$

Now we turn to $\{B_n\}$. Unless specified otherwise, all the results on $\{B_n\}$ hold for all $\alpha \in (-1, 1)$. Since, for each $n \in \mathbb{N}$, $X_k^{(n)}$ for $k \in (m_n, n] \cap \mathbb{N}$ are independent and identically distributed under $\mathbb{P}(\,\cdot \mid \mathcal{F}_\infty)$, so

$$B_n = \sum_{k=m_n+1}^{n} \{X_k^{(n)} - \mathbb{E}[X_k^{(n)} \mid \mathcal{F}_\infty]\} \tag{3.7}$$

is a sum of centered i.i.d. random variables. The conditional variance of $X_k^{(n)}$ for

$$\begin{aligned} \mathbb{V}[X_k^{(n)} \mid \mathcal{F}_\infty] &= \mathbb{E}[(X_k^{(n)})^2 \mid \mathcal{F}_\infty] - (\mathbb{E}[X_k^{(n)} \mid \mathcal{F}_\infty])^2 \\ &= \begin{cases} 0 & \text{for } k \in [1, m_n] \cap \mathbb{N}, \\ 1 - \alpha^2 \cdot \left(\dfrac{W_{m_n}}{m_n}\right)^2 & \text{for } k \in (m_n, n] \cap \mathbb{N}. \end{cases} \end{aligned} \tag{3.8}$$

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We have

$$\mathbb{V}\left[\frac{\gamma_n B_n}{\sqrt{m_n}} \mid \mathcal{F}_\infty\right] = \frac{(\gamma_n)^2}{m_n} \cdot (n - m_n) \cdot \left\{1 - \alpha^2 \cdot \left(\frac{W_{m_n}}{m_n}\right)^2\right\} = \gamma_n(1-\gamma_n) \cdot \left\{1 - \alpha^2 \cdot \left(\frac{W_{m_n}}{m_n}\right)^2\right\}, \tag{3.9}$$

which converges to $\gamma(1-\gamma)$ as $n \to \infty$ a.s. by (2.1). Based on this observation, we prove the following result.

Lemma 3.1. *For* $\gamma \in [0, 1]$*,*

$$\mathbb{E}\left[\exp\left(\frac{\mathrm{i}\xi\gamma_n B_n}{\sqrt{m_n}}\right) \mid \mathcal{F}_\infty\right] \to \exp\left(-\frac{\xi^2}{2} \cdot \gamma(1-\gamma)\right) \quad \text{as } n \to \infty \text{ a.s.} \tag{3.10}$$

Proof. Because B_n is the sum (3.7) of centered i.i.d. random variables under $\mathbb{P}(\cdot \mid \mathcal{F}_\infty)$, by (3.8) we have

$$\mathbb{E}\left[\exp\left(\frac{i\xi\gamma_n B_n}{\sqrt{m_n}}\right) \mid \mathcal{F}_\infty\right] = \left[1 - \frac{\xi^2\gamma_n}{2n} \cdot \left\{1 - \alpha^2 \cdot \left(\frac{W_{m_n}}{m_n}\right)^2\right\} + o\left(\frac{\gamma_n}{n}\right)\right]^{n-m_n} \quad \text{as } n \to \infty \text{ a.s.}$$

Note that $\gamma_n/n \to 0$ and $(\gamma_n/n) \cdot (n - m_n) = \gamma_n(1-\gamma_n) \to \gamma(1-\gamma)$ as $n \to \infty$. Now (3.10) follows from this together with (2.1). □

From (3.5), (3.6), and (3.10), together with the bounded convergence theorem, we can see that

$$\mathbb{E}\left[\exp\left(\frac{i\xi\gamma_n T_n}{\sqrt{m_n}}\right)\right] = \mathbb{E}\left[\exp\left(\frac{i\xi\gamma_n A_n}{\sqrt{m_n}}\right) \cdot \mathbb{E}\left[\exp\left(\frac{i\xi\gamma_n B_n}{\sqrt{m_n}}\right) \mid \mathcal{F}_\infty\right]\right]$$

converges to

$$\exp\left(-\frac{\xi^2}{2} \cdot \frac{\{\gamma + \alpha(1-\gamma)\}^2}{1-2\alpha}\right) \cdot \exp\left(-\frac{\xi^2}{2} \cdot \gamma(1-\gamma)\right)$$

as $n \to \infty$. This gives (2.9).

The proof of Theorem 2.1(ii) is along the same lines as that of (i), and is actually simpler. Assume that $\alpha = \frac{1}{2}$. As $c_n\left(\frac{1}{2}\right) = (1+\gamma_n)/2$, from (3.4) and (2.4) we have

$$\frac{\gamma_n A_n}{\sqrt{m_n \log m_n}} = \frac{c_n W_{m_n}}{\sqrt{m_n \log m_n}} \xrightarrow{\mathrm{d}} \frac{1+\gamma}{2} \cdot N(0, 1) \quad \text{as } n \to \infty.$$

Also, from (3.9) and (2.1), we have

$$\mathbb{E}\left[\left(\frac{\gamma_n B_n}{\sqrt{m_n}}\right)^2\right] = \gamma_n(1-\gamma_n)\left\{1 - \alpha^2 \cdot \mathbb{E}\left[\left(\frac{W_{m_n}}{m_n}\right)^2\right]\right\} \to \gamma(1-\gamma) \quad \text{as } n \to \infty. \tag{3.11}$$

This implies that $\gamma_n B_n/\sqrt{m_n \log m_n} \to 0$ as $n \to \infty$ in L^2. By Slutsky's lemma, we obtain (2.10).

Finally, we prove Theorem 2.1(iii). Assume that $\alpha \in \left(\frac{1}{2}, 1\right)$. By (3.4) and (2.6),

$$\frac{\gamma_n A_n}{(m_n)^\alpha} = \frac{c_n W_{m_n}}{(m_n)^\alpha} \to \{\gamma + \alpha(1-\gamma)\}M \tag{3.12}$$

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as $n \to \infty$ a.s. and in L^2. Noting that $\gamma_n B_n/(m_n)^\alpha \to 0$ as $n \to \infty$ in L^2, by (3.11) we obtain the L^2-convergence in (2.11). From (4.2), which will be proved in the next section, the almost-sure convergence in (2.11) follows. To prove (2.12), by (3.5) we have

$$\gamma_n T_n - c_n \cdot M \cdot (m_n)^\alpha = c_n\{W_{m_n} - M \cdot (m_n)^\alpha\} + \gamma_n B_n. \tag{3.13}$$

Note that M is $\mathcal{F}_\infty$-measurable. Using (2.7), (3.10), and (3.13), we obtain (2.12) similarly to the proof of (2.9). □

4. Proof of Theorem 2.2

In this section we assume that (1.3), (1.7), and (2.8) hold.

Proof. First we give almost-sure bounds for $\{B_n\}$.

Lemma 4.1 *For any $\alpha \in (-1, 1)$ and $\gamma \in [0, 1]$,*

$$\limsup_{n\to\infty} \frac{\gamma_n B_n}{\sqrt{2\gamma_n(1-\gamma_n)m_n \log n}} \le 1 \quad \text{a.s.} \tag{4.1}$$

In particular, for any $c \in \left(\frac{1}{2}, 1\right)$,

$$\lim_{n\to\infty} \frac{\gamma_n B_n}{(m_n)^c} = 0 \quad \text{a.s.} \tag{4.2}$$

Proof. Note that $|X_k^{(n)} - \mathbb{E}[X_k^{(n)} \mid \mathcal{F}_\infty]| \le 1$ for each $1 \le k \le n$. For $\lambda \in \mathbb{R}$, it follows from Azuma's inequality [2] that

$$\begin{aligned}\mathbb{E}[\exp(\lambda\gamma_n B_n) \mid \mathcal{F}_\infty] &= \mathbb{E}\left[\exp\left(\lambda\gamma_n \sum_{k=m_n+1}^{n} \left\{X_k^{(n)} - \mathbb{E}[X_k^{(n)} \mid \mathcal{F}_\infty]\right\}\right) \mid \mathcal{F}_\infty\right] \\ &\le \exp\left((\lambda\gamma_n)^2(n-m_n)/2\right),\end{aligned}$$

and

$$\mathbb{P}(|\gamma_n B_n| \ge x) \le 2\exp\left(-\frac{x^2}{2\gamma_n(1-\gamma_n)m_n}\right) \quad \text{for } x > 0.$$

Taking $x = \sqrt{2(1+\varepsilon)\gamma_n(1-\gamma_n)m_n \log n}$ for some $\varepsilon > 0$, we have

$$\sum_{n=1}^{\infty} \mathbb{P}\left(|\gamma_n B_n| \ge \sqrt{2(1+\varepsilon)\gamma_n(1-\gamma_n)m_n \log n}\right) \le \sum_{n=1}^{\infty} \frac{2}{n^{1+\varepsilon}}.$$

This, together with the Borel–Cantelli lemma, implies (4.1). To obtain (4.2) note that, for $c \in \left(\frac{1}{2}, 1\right)$,

$$\frac{2\gamma_n(1-\gamma_n)m_n \log n}{(m_n)^{2c}} = \frac{2(1-\gamma_n)\log n}{n(m_n)^{2c-2}} \le \frac{2(1-\gamma_n)\log n}{n^{2c-1}} \to 0 \quad \text{as } n \to \infty,$$

where we used $2c-2 < 0 < 2c-1$ and $m_n \le n$. □

We now prove (2.14) in Theorem 2.2. Equation (2.13) is readily derived from (2.14). For the case $\alpha \in \left(-1, \frac{1}{2}\right)$, from (2.3) and (3.4) we have

$$\limsup_{n\to\infty} \frac{\gamma_n A_n}{\sqrt{2m_n \log\log m_n}} \le \frac{\gamma + \alpha(1-\gamma)}{\sqrt{1-2\alpha}} \quad \text{a.s.}$$

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For the case $\alpha = \frac{1}{2}$, from (2.5) and (3.4) we have

$$\limsup_{n\to\infty} \frac{\gamma_n A_n}{\sqrt{2m_n \log m_n \log\log\log m_n}} \leq \frac{1+\gamma}{2} \quad \text{a.s.}$$

By (4.2), if $\alpha \in \left(-1, \frac{1}{2}\right]$ then (2.14) holds for any $c \in \left(\frac{1}{2}, 1\right)$. As for the case $\alpha \in \left(\frac{1}{2}, 1\right)$, almost-sure convergence in (2.11) follows from (3.12) and (4.2). Thus, (2.14) holds for any $c \in (\alpha, 1)$. □

5. The ERW with stops in the triangular array setting

Let $s \in [0, 1]$, and assume that $p, q, r \in [0, 1)$ satisfy $p + q + r = 1$. We consider a sequence $X_1, X_2, \ldots$ of random variables taking values in $\{+1, 0, -1\}$ given by

$$X_1 = \begin{cases} +1 & \text{with probability } s, \\ -1 & \text{with probability } 1 - s; \end{cases} \tag{5.1}$$

$\{U_n : n \geq 1\}$ a sequence of independent random variables, independent of X_1, with U_n having a uniform distribution over $\{1, \ldots, n\}$; and, for $n \in \mathbb{N}$,

$$X_{n+1} = \begin{cases} X_{U_n} & \text{with probability } p, \\ -X_{U_n} & \text{with probability } q, \\ 0 & \text{with probability } r. \end{cases} \tag{5.2}$$

The *ERW with stops* $\{W_n\}$ is defined by $W_n = \sum_{k=1}^{n} X_k$ for $n \in \mathbb{N}$. Note that if $r = 0$ then it is the standard ERW defined in Section 1. Hereafter we assume that $r \in (0, 1)$.

The ERW with stops was introduced in [12]. To describe the limit theorems obtained in [5], it is convenient to introduce the new parameters $\alpha := p - q$ and $\beta := 1 - r$, where $\beta \in (0, 1)$ and $\alpha \in [-\beta, \beta]$. Let Σ_n be the number of moves up to time n, i.e.

$$\Sigma_n := \sum_{k=1}^{n} \mathbf{1}_{\{X_k \neq 0\}} = \sum_{k=1}^{n} X_k^2 \quad \text{for } n \in \mathbb{N}.$$

It is shown in [5] that

$$\lim_{n\to\infty} \frac{\Sigma_n}{n^\beta} = \Sigma > 0 \quad \text{a.s. and in } L^2, \tag{5.3}$$

where Σ has a Mittag–Leffler distribution with parameter β. We turn to the central limit theorem for $\{W_n\}$ in [5].

- For $\alpha \in [-\beta, \beta/2)$,

$$\frac{W_n}{\sqrt{\Sigma_n}} \xrightarrow{\text{d}} N\left(0, \frac{\beta}{\beta - 2\alpha}\right) \quad \text{as } n \to \infty. \tag{5.4}$$

- For $\alpha = \beta/2$,

$$\frac{W_n}{\sqrt{\Sigma_n \log \Sigma_n}} \xrightarrow{\text{d}} N(0, 1) \quad \text{as } n \to \infty. \tag{5.5}$$

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- For $\alpha \in (\beta/2, \beta]$, there exists a random variable M such that

$$\lim_{n\to\infty} \frac{W_n}{n^\alpha} = M \quad \text{a.s. and in } L^2 \tag{5.6}$$

and

$$\frac{W_n - Mn^\alpha}{\sqrt{\Sigma_n}} \xrightarrow{\text{d}} N\left(0, \frac{\beta}{2\alpha - \beta}\right) \quad \text{as } n \to \infty, \tag{5.7}$$

where $\mathbb{P}(M > 0) > 0$.

Next, we define the sequence $\{T_n\}$ as in (1.6); however, $Y_k^{(n)}$ and $X_k^{(n)}$ of (1.4) and (1.5) are defined with $\{X_i\}$ as in (5.1) and (5.2) instead of (1.1) and (1.2). We call this the *ERW with stops in the triangular array setting*.

Our first result of this section is an extension of [10, Theorem 4.1]. We note here that [10] allows X_1 to take value 0 with probability r, unlike this paper. As such they have an extra δ_0 in their results for the case $\gamma = 0$.

Theorem 5.1. *Let $\beta \in (0, 1)$ and $\alpha \in [-\beta, \beta]$. Assume that $\{m_n : n \in \mathbb{N}\}$ satisfies (1.3), (1.7), and (2.8).*

(i) *If $\alpha \in [-\beta, \beta/2)$ then*

$$\frac{\gamma_n T_n}{\sqrt{\Sigma_{m_n}}} \xrightarrow{\text{d}} N\left(0, \frac{\beta\{\gamma + \alpha(1-\gamma)\}^2}{\beta - 2\alpha} + \beta\gamma(1-\gamma)\right) \quad \text{as } n \to \infty. \tag{5.8}$$

(ii) *If $\alpha = \beta/2$ then*

$$\frac{\gamma_n T_n}{\sqrt{\Sigma_{m_n} \log \Sigma_{m_n}}} \xrightarrow{\text{d}} N(0, \{\gamma + \beta(1-\gamma)/2\}^2) \quad \text{as } n \to \infty. \tag{5.9}$$

(iii) *If $\alpha \in (\beta/2, \beta]$ then*

$$\lim_{n\to\infty} \frac{\gamma_n T_n}{(m_n)^\alpha} = \{\gamma + \alpha(1-\gamma)\}M \quad \text{in } L^2, \tag{5.10}$$

where M is the random variable in (5.6). Moreover,

$$\frac{\gamma_n T_n - M \cdot \{\gamma_n + \alpha(1-\gamma_n)\} \cdot (m_n)^\alpha}{\sqrt{\Sigma_{m_n}}} \xrightarrow{\text{d}} N\left(0, \frac{\beta\{\gamma + \alpha(1-\gamma)\}^2}{2\alpha - \beta} + \beta\gamma(1-\gamma)\right) \quad \text{as } n \to \infty. \tag{5.11}$$

Remark 5.1. Unlike the results in [10], we have a random normalization in the results above. This is because we consider the general case $\gamma \in [0, 1]$. We can obtain the L^4-convergence in (5.10) using Burkholder's inequality as in [1, (3.15)].

We also consider the process $\{\Xi_n : n \in \mathbb{N}\}$ defined by $\Xi_n := \sum_{k=1}^n \{X_k^{(n)}\}^2$ for $n \in \mathbb{N}$. The next theorem is an improvement of [10, Theorem 4.2].

Theorem 5.2. *Under the same conditions as in Theorem 5.1, we have*

$$\lim_{n\to\infty} \frac{\gamma_n \Xi_n}{(m_n)^\beta} = \{\gamma + \beta(1-\gamma)\}\Sigma \quad \text{in } L^2, \tag{5.12}$$

where Σ is defined in (5.3).

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The strong law of large numbers and its refinement can also be obtained for the ERW with stops.

Theorem 5.3 *Under the same conditions as in Theorem 5.1, we have (2.13). In addition, (2.14) holds for* $c \in \big(\max\{\alpha, \tfrac{1}{2}\}, 1\big)$.

Remark 5.2. Assume that $\beta \in \big(\tfrac{1}{2}, 1\big)$. As a by-product of the proof of Theorem 5.3, we can prove the a.s. convergence in (5.12). The a.s. convergence in (5.10) is valid for $\alpha \in \big(\tfrac{1}{2}, \beta\big]$.

6. Proof of Theorem 5.1

Proof. Noting that $p = (\beta + \alpha)/2$ and $q = (\beta - \alpha)/2$, for $n \in \mathbb{N}$ we have

$$\mathbb{P}(X_{n+1} = \pm 1 \mid \mathcal{F}_n) = \frac{\#\{k = 1, \ldots, n\colon X_k = \pm 1\}}{n} \cdot p + \frac{\#\{k = 1, \ldots, n\colon X_k = \mp 1\}}{n} \cdot q$$
$$= \frac{1}{2}\left(\beta \cdot \frac{\Sigma_n}{n} \pm \alpha \cdot \frac{W_n}{n}\right).$$

For $k \in (m_n, n] \cap \mathbb{N}$, we have

$$\mathbb{P}\big(X_k^{(n)} = \pm 1 \mid \mathcal{G}_{k-1}^{(n)}\big) = \frac{1}{2}\left(\beta \cdot \frac{\Sigma_{m_n}}{m_n} \pm \alpha \cdot \frac{W_{m_n}}{m_n}\right), \tag{6.1}$$
$$\mathbb{P}\big(\{X_k^{(n)}\}^2 = 1 \mid \mathcal{G}_{k-1}^{(n)}\big) = \beta \cdot \frac{\Sigma_{m_n}}{m_n}. \tag{6.2}$$

From (6.1), we see that (3.1) and (3.2) continue to hold in this setting. Defining $\{A_n\}$ and $\{B_n\}$ by (3.3), we note that they satisfy (3.4) and (3.5).

We prepare a lemma about $\{B_n\}$.

Lemma 6.1. *Under the assumption of Theorem 5.1:*

(i) For $\alpha \in [-\beta, \beta]$ and $\xi \in \mathbb{R}$,

$$\mathbb{E}\left[\exp\left(\frac{\mathrm{i}\xi\gamma_n B_n}{\sqrt{\Sigma_{m_n}}}\right) \mid \mathcal{F}_\infty\right] \to \exp\left(-\frac{\xi^2}{2} \cdot \beta\gamma(1-\gamma)\right) \quad \text{as } n \to \infty \text{ a.s.}, \tag{6.3}$$
$$\mathbb{E}\left[\exp\left(\frac{\mathrm{i}\xi\gamma_n B_n}{\sqrt{\Sigma_{m_n}\log\Sigma_{m_n}}}\right) \mid \mathcal{F}_\infty\right] \to 1 \quad \text{as } n \to \infty \text{ a.s.} \tag{6.4}$$

(ii) If $\alpha \in (\beta/2, \beta]$ then $\gamma_n B_n/(m_n)^\alpha \to 0$ as $n \to \infty$ in L^2.

Proof. Note that $\mathbb{E}[B_n \mid \mathcal{F}_\infty] = 0$. By (6.1) and (6.2),

$$\mathbb{V}[X_k^{(n)} \mid \mathcal{F}_\infty] = \beta \cdot \frac{\Sigma_{m_n}}{m_n} - \alpha^2 \cdot \left(\frac{W_{m_n}}{m_n}\right)^2$$

for $k \in (m_n, n] \cap \mathbb{N}$. As in (3.9), we have

$$\mathbb{V}[B_n \mid \mathcal{F}_\infty] = (n - m_n) \cdot \left\{\beta \cdot \frac{\Sigma_{m_n}}{m_n} - \alpha^2 \cdot \left(\frac{W_{m_n}}{m_n}\right)^2\right\} = \frac{1 - \gamma_n}{\gamma_n} \cdot \left\{\beta\Sigma_{m_n} - \alpha^2 \cdot \frac{W_{m_n}^2}{m_n}\right\}. \tag{6.5}$$

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From this,

$$\mathbb{V}\left[\frac{\gamma_n B_n}{\sqrt{\Sigma_{m_n}}} \mid \mathcal{F}_\infty\right] = \gamma_n(1-\gamma_n)\cdot\left\{\beta - \alpha^2\cdot\frac{(m_n)^\beta}{\Sigma_{m_n}}\cdot\left(\frac{W_{m_n}}{(m_n)^{(1+\beta)/2}}\right)^2\right\}. \tag{6.6}$$

For any $\beta \in (0, 1)$ and $\alpha \in [-\beta, \beta]$, we show that

$$\lim_{n\to\infty} \frac{W_{m_n}}{(m_n)^{(1+\beta)/2}} = 0 \quad \text{a.s.} \tag{6.7}$$

Indeed, if $\alpha \in [-\beta, \beta/2)$ then

$$\limsup_{n\to\infty} \frac{W_n}{\sqrt{2n^\beta \log\log n}} = \sqrt{\frac{\beta\Sigma}{\beta-2\alpha}} \quad \text{a.s.} \tag{6.8}$$

by [5, (3.5)]. If $\alpha = \beta/2$ then

$$\limsup_{n\to\infty} \frac{W_n}{\sqrt{2n^\beta \log n \log\log\log n}} = \sqrt{\beta\Sigma} \quad \text{a.s.} \tag{6.9}$$

by [5, (3.13)]. If $\alpha \in (\beta/2, \beta]$ then $W_{m_n}/(m_n)^\alpha \to M$ as $n \to \infty$ a.s. by (5.6), and $(1+\beta)/2 > \alpha$ since $2\alpha - \beta \le \beta < 1$. In any case we have (6.7). Since $(m_n)^\beta/\Sigma_{m_n} \to 1/\Sigma$ as $n \to \infty$ a.s. by (5.3), we see that (6.6) converges to $\beta\gamma(1-\gamma)$ as $n \to \infty$ a.s., and

$$\mathbb{V}\left[\frac{\gamma_n B_n}{\sqrt{\Sigma_{m_n}\log\Sigma_{m_n}}} \mid \mathcal{F}_\infty\right] \to 0 \quad \text{as } n \to \infty \text{ a.s.}$$

By a similar computation to Lemma 4.1, we obtain (6.3) and (6.4) in (i).

Next, we consider (ii). By (6.5),

$$\mathbb{E}\left[\left(\frac{\gamma_n B_n}{(m_n)^\alpha}\right)^2\right] = \gamma_n(1-\gamma_n)\cdot\left\{\beta\cdot\frac{\mathbb{E}[\Sigma_{m_n}]}{(m_n)^{2\alpha}} - \alpha^2\cdot\frac{\mathbb{E}[(W_{m_n})^2]}{(m_n)^{1+2\alpha}}\right\}.$$

From [5, (A.6)], $\mathbb{E}[(W_n)^2] \sim n^{2\alpha}/\{(2\alpha-\beta)\Gamma(2\alpha)\}$ as $n \to \infty$. On the other hand, from [5, (4.4)] we can see that $\mathbb{E}[\Sigma_n] \sim n^\beta/\Gamma(1+\beta)$ as $n \to \infty$. Noting that $\beta < 2\alpha$, we have (ii). □

Assume that $\alpha \in [-\beta, \beta/2)$. By (3.4) and (5.4), we have

$$\frac{\gamma_n A_n}{\sqrt{\Sigma_{m_n}}} = \frac{c_n W_{m_n}}{\sqrt{\Sigma_{m_n}}} \xrightarrow{\text{d}} \{\gamma + \alpha(1-\gamma)\}\cdot N\left(0, \frac{\beta}{\beta-2\alpha}\right) \quad \text{as } n \to \infty.$$

Combining this and (6.3), we can prove (5.8) by the same method as for (2.9). Next, we consider the case $\alpha = \beta/2$. By (3.4) and (5.5), we have

$$\frac{\gamma_n A_n}{\sqrt{\Sigma_{m_n}\log\Sigma_{m_n}}} = \frac{c_n W_{m_n}}{\sqrt{\Sigma_{m_n}\log\Sigma_{m_n}}} \xrightarrow{\text{d}} \{\gamma + \alpha(1-\gamma)\}\cdot N(0, 1) \quad \text{as } n \to \infty.$$

This together with (6.4) gives (5.9). As for the case $\alpha \in (\beta/2, \beta]$, by (3.4) and (5.6),

$$\frac{\gamma_n A_n}{(m_n)^\alpha} = \frac{c_n W_{m_n}}{(m_n)^\alpha} \to \{\gamma + \alpha(1-\gamma)\}M \quad \text{as } n \to \infty \text{ a.s. and in } L^2.$$

Now (5.10) follows from Lemma 6.1(ii). The proof of (5.11) is almost identical to that of (2.12): use (3.5), (5.7), and (6.3). □

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7. Proof of Theorem 5.2

Proof. Put $A'_n := \mathbb{E}[\Xi_n \mid \mathcal{F}_\infty]$ and $B'_n := \Xi_n - A'_n$. Using (6.2), we can see that $\gamma_n A'_n = c_n(\beta)\Sigma_{m_n}$, which together with (5.3) imply

$$\frac{\gamma_n A'_n}{(m_n)^\beta} = \frac{c_n(\beta)\Sigma_{m_n}}{(m_n)^\beta} \to \{\gamma + \beta(1-\gamma)\} \cdot \Sigma \quad \text{as } n \to \infty \text{ a.s. and in } L^2.$$

As for B'_n, again by (6.2) we can see that

$$\begin{aligned}\mathbb{V}\left[\left(\frac{\gamma_n B'_n}{(m_n)^\beta}\right)^2 \mid \mathcal{F}_\infty\right] &= \frac{(\gamma_n)^2}{(m_n)^{2\beta}} \cdot \sum_{k=m_n+1}^{n} \mathbb{V}[\{X_k^{(n)}\}^2 \mid \mathcal{F}_\infty] \\ &= \frac{(\gamma_n)^2}{(m_n)^{2\beta}} \cdot (n-m_n) \cdot \beta \cdot \frac{\Sigma_{m_n}}{m_n} \cdot \left(1 - \beta \cdot \frac{\Sigma_{m_n}}{m_n}\right), \\ \mathbb{E}\left[\left(\frac{\gamma_n B'_n}{(m_n)^\beta}\right)^2\right] &= \frac{\beta\gamma_n(1-\gamma_n)}{(m_n)^\beta} \cdot \mathbb{E}\left[\frac{\Sigma_{m_n}}{(m_n)^\beta} \cdot \left(1 - \beta \cdot \frac{\Sigma_{m_n}}{m_n}\right)\right].\end{aligned}$$

Since $\beta < 1$ and $\Sigma_{m_n}/(m_n)^\beta$ converges to Σ in L^2 by (5.3), we have

$$\mathbb{E}\left[\frac{\Sigma_{m_n}}{(m_n)^\beta} \cdot \left(1 - \beta \cdot \frac{\Sigma_{m_n}}{m_n}\right)\right] = \mathbb{E}\left[\frac{\Sigma_{m_n}}{(m_n)^\beta}\right] - \frac{\beta}{(m_n)^{1-\beta}} \cdot \mathbb{E}\left[\left(\frac{\Sigma_{m_n}}{(m_n)^\beta}\right)^2\right] \to \mathbb{E}[\Sigma] \quad \text{as } n \to \infty.$$

Noting that $\beta > 0$, this shows that $\gamma_n B'_n/(m_n)^\beta \to 0$ as $n \to \infty$ in L^2, which completes the proof. □

8. Proof of Theorem 5.3

Proof. The proof of Lemma 4.1 is based on the fact that $|X_k^{(n)} - \mathbb{E}[X_k^{(n)} \mid \mathcal{F}_\infty]| \le 1$. Thus, $\{B_n\}$ for the ERW with stops in the triangular array setting also satisfies (4.2) for any $c \in \left(\frac{1}{2}, 1\right)$. If $\alpha \in [-\beta, \beta/2]$ then, from (3.4), (6.8), and (6.9), we can see that $\gamma_n A_n = o(n^c)$ for any $c \in (\beta/2, 1)$. If $\alpha \in (\beta/2, \beta]$ then (3.4) and (5.6) imply that $\gamma_n A_n = o(n^c)$ for any $c \in (\alpha, 1)$. In any case, (2.14) holds for $c \in \left(\max\left\{\alpha, \frac{1}{2}\right\}, 1\right)$. □

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There were no competing interests to declare which arose during the preparation or publication process of this article.

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