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THE ESSENTIAL NORMS OF COMPOSITION OPERATORS ON WEIGHTED DIRICHLET SPACES

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Abstract

Let φ be an analytic self-map of the unit disc. If φ is analytic in a neighbourhood of the closed unit disc, we give a precise formula for the essential norm of the composition operator C_φ on the weighted Dirichlet spaces $\mathcal{D}_\alpha$ for $\alpha > 0$. We also show that, for a univalent analytic self-map φ of $\mathbb{D}$, if φ has an angular derivative at some point of $\partial\mathbb{D}$, then the essential norm of C_φ on the Dirichlet space is equal to one.

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1. Introduction

Let $\mathbb{D}$ be the open unit disc in the complex plane $\mathbb{C}$ and let $H(\mathbb{D})$ denote the collection of analytic functions on $\mathbb{D}$. Throughout this paper, φ denotes a nonconstant analytic function on $\mathbb{D}$, with $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. Thus φ induces a composition operator C_φ on $H(\mathbb{D})$ defined by the equation $C_\varphi(f) = f \circ \varphi$ for $f \in H(\mathbb{D})$.

For $\alpha > -1$, the weighted Dirichlet space $\mathcal{D}_\alpha$ is defined by

$$\mathcal{D}_\alpha = \Big\{ f \in H(\mathbb{D}) : \|f\|^2_{\mathcal{D}_\alpha} = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \, dA_\alpha(z) < \infty \Big\},$$

where $dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha \, dA(z)$ and dA denotes the normalised area measure on $\mathbb{D}$. When $\alpha = 0$, we replace the notation $\mathcal{D}_0$ by $\mathcal{D}$, which is called the Dirichlet space.

The weighted Bergman spaces A^2_α $(\alpha > -1)$ are defined by

$$A^2_\alpha = \Big\{ f \in H(\mathbb{D}) : \|f\|^2_{A^2_\alpha} = \int_{\mathbb{D}} |f(z)|^2 \, dA_\alpha(z) < \infty \Big\}.$$

The Hardy space H^2 is defined by

$$H^2 = \Big\{ f \in H(\mathbb{D}) : \|f\|^2_{H^2} = \sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta/2\pi < \infty \Big\}.$$

It is well known that $\mathcal{D}_1 = H^2$ and $\mathcal{D}_\alpha = A^2_{\alpha-2}$ for $\alpha > 1$.

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Let X be a Banach space. The essential norm of C_φ on X, denoted by $\|C_\varphi\|_{e,X}$, is the distance from C_φ to the subspace consisting of all compact operators, namely,

$$\|C_\varphi\|_{e,X} = \inf\{\|C_\varphi - K\| : K \text{ is compact on } X\}.$$

The essential norms of composition operators on $\mathcal{D}_\alpha$ were characterised by J. Shapiro [9] in terms of generalised Nevanlinna counting functions. In this paper, Shapiro also gave an exact formula for the essential norm of C_φ on H^2. Cima and Matheson [2] gave another exact formula for the essential norm of C_φ on H^2 based on the Aleksandrov measure of φ. Poggi-Corradini [8] considered A^2_α ($\alpha = 0, 1$) and obtained a similar result to the one in [9] for H^2, using generalised Nevanlinna counting functions and the theory of zero-divisors. In fact, as pointed out by the authors in [1], Shimorin's results on zero-divisors [11–13] mean that Corradini's technique also applies for $-1 < \alpha \le 1$. An exact formula for the essential norm of C_φ on $\mathcal{D}_\alpha$ is still unknown, except in the cases stated above.

Now let φ be analytic in a neighbourhood of the closed unit disc. Cowen [3, Theorem 2.4] showed that

$$M \le \|C_\varphi\|^2_{e,H^2} \le 4M,$$

where

$$M = \max\Big\{\sum_{\varphi(e^{i\theta})=\varsigma} |\varphi'(e^{i\theta})|^{-1} : |\varsigma| = 1\Big\}.$$

Furthermore, employing the Aleksandrov measure and the angular derivative of φ (to be defined in the next section), Cima and Matheson [2, page 63] proved that

$$\|C_\varphi\|^2_{e,H^2} = M.$$

The main result of this paper extends the result in [3] to characterise the essential norm of C_φ on $\mathcal{D}_\alpha$ ($\alpha > 0$), where φ is holomorphic in a neighbourhood of the closed unit disc. We use the angular derivative and generalised Nevanlinna counting functions. Our result is explicit and should be readily applicable.

Theorem 1.1. *Suppose that φ is analytic in a neighbourhood of the closed unit disc and $\alpha > 0$. Then*

$$\|C_\varphi\|^2_{e,\mathcal{D}_\alpha} = \max\Big\{\sum_{\varphi(e^{i\theta})=\varsigma} |\varphi'(e^{i\theta})|^{-\alpha} : |\varsigma| = 1\Big\}.$$

From the Julia–Carathéodory theorem, we obtain the following result on $\mathcal{D}$.

Theorem 1.2. *Suppose that φ is univalent and has an angular derivative at some point $\eta \in \partial\mathbb{D}$. Then*

$$\|C_\varphi\|_{e,\mathcal{D}} = 1.$$

2. Prerequisites

In this section, we collect results that are needed for the proofs of the theorems.

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2.1. Generalised Nevanlinna counting functions. The generalised Nevanlinna counting function for φ is defined by

$$N_{\varphi,\gamma}(\omega) = \sum_{z\in\varphi^{-1}(\omega)} (1-|z|^2)^\gamma \quad \text{for all } \omega\in\mathbb{D},\ \gamma\geq 0,$$

where $N_{\varphi,\gamma}(\omega)=0$ if $\omega\notin\varphi(\mathbb{D})$. In particular,

$$N_{\varphi,0}(\omega)=n_\varphi(\omega)$$

is called the multiplicity of φ at ω.

2.2. The change of variable formula (see [4, Theorem 2.32]). For any analytic self-map φ of $\mathbb{D}$ and any $f\in\mathcal{D}_\alpha$,

$$\begin{aligned}\|f\circ\varphi\|^2_{\mathcal{D}_\alpha} &= |f(\varphi(0))|^2 + \int_{\mathbb{D}} |(f\circ\varphi)'(z)|^2\,dA_\alpha(z)\\ &= |f(\varphi(0))|^2 + (\alpha+1)\int_{\mathbb{D}} |f'(\omega)|^2 N_{\varphi,\alpha}(\omega)\,dA(\omega).\end{aligned}$$

2.3. The pseudo-hyperbolic disc (see [14, page 61]). For $a\in\mathbb{D}$, define φ_a by

$$\varphi_a(z)=\frac{a-z}{1-\bar{a}z} \quad \text{for all } z\in\mathbb{D}.$$

For $0<r<1$, the pseudo-hyperbolic disc

$$D(a,r)\stackrel{\text{def}}{=}\{z\in\mathbb{D}:|\varphi_a(z)|<r\}=\varphi_a(r\mathbb{D})$$

is a Euclidean disc with centre and radius given by

$$C=\frac{1-r^2}{1-r^2|a|^2}a, \qquad R=\frac{1-|a|^2}{1-r^2|a|^2}r. \tag{2.1}$$

It is easy to check that

$$\varphi_a'(z)=-\frac{1-|a|^2}{(1-\bar{a}z)^2}$$

and

$$1-|\varphi_a(z)|^2=\frac{(1-|a|^2)(1-|z|^2)}{|1-\bar{a}z|^2}.$$

2.4. Angular derivative (see [4, pages 50–51]). Firstly, recall the notation for nontangential approach regions (see [9, page 383]). For $\eta\in\partial\mathbb{D}$ and $0<\rho<1$, let $S_\rho(\eta)$ be the convex hull of the disc $\rho\mathbb{D}$ and the point η. For $0<r<1$, let $S_{\rho,r}(\eta)=S_\rho(\eta)\setminus r\mathbb{D}$.

Secondly, if f is a function defined on $\mathbb{D}$ and $\eta\in\partial\mathbb{D}$, then

$$\angle\lim_{z\to\eta} f(z)=L$$

means that $f(z)\to L$ as $z\to\eta$ through any nontangential approach region $S_\rho(\eta)$. In this case, we say that L is the nontangential limit of f at η.

Lastly, φ is said to have an angular derivative at $\eta\in\partial\mathbb{D}$ if there is $\xi\in\partial\mathbb{D}$ so that

$$\angle\lim_{z\to\eta}\frac{\varphi(z)-\xi}{z-\eta}$$

exists. We call the limit the angular derivative of φ at η, and denote it by $\varphi'(\eta)$.

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2.5. Julia–Carathéodory Theorem (see [10, page 57] or [4, Theorem 2.44]). Suppose that φ is an analytic self-map of $\mathbb{D}$ and that $\eta \in \partial\mathbb{D}$. Then the following three statements are equivalent:

(1) $d(\eta) = \liminf_{z\to\eta}(1 - |\varphi(z)|)/(1 - |z|) < \infty$;
(2) φ has finite angular derivative $\varphi'(\eta)$ at η; and
(3) both φ and φ' have (finite) nontangential limits at η and $|\xi| = 1$, where $\xi = \lim_{r\to 1}\varphi(r\eta)$.

Moreover, when these conditions hold:

(4) $d(\eta) > 0$ in (1); and
(5) $\varphi'(\eta) = d(\eta)\overline{\eta}\xi$ and $d(\eta) = \angle\lim_{z\to\eta}(1 - |\varphi(z)|)(1 - |z|)$.

The next lemma is a geometric consequence of the Julia–Carathéodory theorem.

Lemma 2.1 [9, Corollary 3.2]. *Suppose that φ has an angular derivative at some point $\eta \in \partial\mathbb{D}$ and that ξ is the nontangential limit of φ at η. Then, for each pair σ, ρ with $0 < \sigma < \rho < 1$, there exists t with $0 < t < 1$ such that*

$$S_{\sigma,t}(\xi) \subseteq \varphi(S_\rho(\eta)).$$

Moreover, we deduce the following corollary.

Corollary 2.2. *Suppose that φ has an angular derivative at some point $\eta \in \partial\mathbb{D}$ and that ξ is the nontangential limit of φ at η. Fix r with $0 < r < 1$ and let $\sigma = 2r/(1 + r^2)$. Then, for each ρ with $\sigma < \rho < 1$, there exist t, h with $0 < t, h < 1$ such that*

$$D(a, r) \subseteq S_{\sigma,t}(\xi) \subseteq \varphi(S_\rho(\eta)) \quad \text{for all } a \in E_h,$$

where $E_h = \{a : \arg a = \arg\xi, h < |a| < 1\}$.

Proof. Suppose that $\sigma < \rho < 1$. By Lemma 2.1, there exists t with $0 < t < 1$ such that

$$S_{\sigma,t}(\xi) \subseteq \varphi(S_\rho(\eta)).$$

To finish the proof, it suffices to show that $D(a, r) \subseteq S_\sigma(\xi)$ (combining this with the fact that for any $z \in D(a, r)$, z tends to ξ if a is close to ξ). Fix $z \in D(a, r)$. If the straight line in $\mathbb{D}$ through z ends at ξ, making an angle $\theta_z < \pi/2$ with the radius to that point, then

$$\sup_{z\in D(a,r)} \sin(\theta_z) \le \frac{R}{1 - |C|} = \frac{r(1 - |a|^2)}{1 - |a|^2 r^2 - |a| + |a| r^2} = \frac{r(1 + |a|)}{1 + |a| r^2} \le \frac{2r}{1 + r^2} = \sigma,$$

after substituting the values for C and R given in (2.1). This implies $D(a, r) \subseteq S_\sigma(\xi)$ and completes the proof. □

Lemma 2.3. *Suppose that $\alpha > 0$ and that φ is analytic in a neighbourhood of the closed unit disc, with $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. For $\lambda \in \partial\mathbb{D}$, if $\{\zeta_j\}_{j=1}^n$ is the set of all preimages of φ at λ in the unit circle $\partial\mathbb{D}$, then*

$$\angle\lim_{a\to\lambda} \frac{N_{\varphi,\alpha}(a)}{(1 - |a|^2)^\alpha} = \sum_{j=1}^n |\varphi'(\zeta_j)|^{-\alpha}.$$

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Proof. Since φ is analytic in a neighbourhood of $\overline{\mathbb{D}}$, $\varphi'(\zeta_j)$ exists, and by the Julia–Carathéodory theorem, $\varphi'(\zeta_j) = d(\zeta_j)\overline{\zeta_j}\lambda \neq 0$ for $j = 1, 2, \ldots, n$. Moreover, there exists $\gamma > 1$ such that $\varphi - \lambda$ has no zeros on $\partial(\gamma\mathbb{D})$ and $\{\zeta_j\}_{j=1}^n$ are all the zeros of $\varphi - \lambda$ in $\gamma\mathbb{D}$. Thus we may define

$$\delta = \min_{\omega\in\partial(\gamma\mathbb{D})} |\varphi(\omega) - \lambda| > 0.$$

If $|a - \lambda| < \delta/2$ and $\omega \in \partial(\gamma\mathbb{D})$, then

$$|a - \lambda| < |\varphi(\omega) - \lambda|.$$

By Rouché's Theorem, $\varphi - a$ must have n zeros in $\gamma\mathbb{D}$.

Now fix σ, ρ with $0 < \sigma < \rho < 1$ and choose $0 < t < 1$ such that the $S_j = S_{\rho,t}(\zeta_j)$ are disjoint for $1 \le j \le n$. By Lemma 2.1, $\bigcap_{j=1}^n \varphi(S_j)$ contains $S_{\sigma,s}(\lambda)$ for some s with $0 < s < 1$. If we pick s sufficiently large so that $|a - \lambda| < \delta/2$ for every $a \in S_{\sigma,s}(\lambda)$, then $\varphi - a$ has exactly n zeros in $\gamma\mathbb{D}$. Since $a \in \bigcap_{j=1}^n \varphi(S_j)$, it follows that $\varphi - a$ has exactly n zeros in $\mathbb{D}$.

For $a \in S_{\sigma,s}(\lambda)$ and $1 \le j \le n$, choose the preimage $z^{(j)}(a)$ of a that lies in S_j. Then

$$N_{\varphi,\alpha}(a) = \sum_{j=1}^{n} (1 - |z^{(j)}(a)|^2)^\alpha. \tag{2.2}$$

By the Schwarz lemma [5, Lemma 1.2], for any analytic mapping $\phi : \mathbb{D} \to \mathbb{D}$,

$$\frac{|\phi'(z)|}{1 - |\phi(z)|^2} \le \frac{1}{1 - |z|^2} \quad \text{for all } z \in \mathbb{D}. \tag{2.3}$$

This ensures that $z^{(j)}(a) \to \zeta_j$ through S_j for each j, as $a \to \lambda$ through $S_{\sigma,s}(\lambda)$. Thus, again by the Julia–Carathéodory theorem,

$$\lim_{a\to\lambda,\, a\in S_{\sigma,s}(\lambda)} \frac{(1 - |z^{(j)}(a)|^2)^\alpha}{(1 - |a|^2)^\alpha} = |\varphi'(\zeta_j)|^{-\alpha}.$$

Combining this with (2.2),

$$\angle \lim_{a\to\lambda} \frac{N_{\varphi,\alpha}(a)}{(1 - |a|^2)^\alpha} = \sum_{j=1}^{n} |\varphi'(\zeta_j)|^{-\alpha},$$

which completes the proof. □

Note that there exists a sequence $\{a_m\}$ in $\mathbb{D}$ such that $|a_m| \to 1$ and

$$\limsup_{|a|\to 1} \frac{N_{\varphi,\alpha}(a)}{(1 - |a|^2)^\alpha} = \lim_{|a_m|\to 1} \frac{N_{\varphi,\alpha}(a_m)}{(1 - |a_m|^2)^\alpha}.$$

By selecting an appropriate subsequence, if necessary, we may assume that a_m converges to some point $\xi \in \partial\mathbb{D}$. Thus

$$\limsup_{|a|\to 1} \frac{N_{\varphi,\alpha}(a)}{(1 - |a|^2)^\alpha} = \lim_{a_m\to\xi} \frac{N_{\varphi,\alpha}(a_m)}{(1 - |a_m|^2)^\alpha}. \tag{2.4}$$

This remark leads to the next proposition.

Proposition 2.4. *Suppose that $\alpha > 0$ and that φ is analytic in a neighbourhood of the closed unit disc, with $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. Then*

$$\lim_{a_m \to \xi} \frac{N_{\varphi,\alpha}(a_m)}{(1-|a_m|^2)^\alpha} = \max\Big\{ \sum_{\varphi(e^{i\theta})=\varsigma} |\varphi'(e^{i\theta})|^{-\alpha} : |\varsigma| = 1 \Big\}. \tag{2.5}$$

Remark 2.5. If, for each $\varsigma \in \partial\mathbb{D}$, the preimage of φ at ς does not exist, then we define

$$\max\Big\{ \sum_{\varphi(e^{i\theta})=\varsigma} |\varphi'(e^{i\theta})|^{-\alpha} : |\varsigma| = 1 \Big\} = 0.$$

Proof of Proposition 2.4.

Case I: φ has no angular derivative at every point η in $\partial\mathbb{D}$. Since φ is analytic in a neighbourhood of the closed unit disc, if there exists $\lambda \in \partial\mathbb{D}$ such that $\varphi(\eta) = \lambda$ for some $\eta \in \partial\mathbb{D}$, then $\varphi'(\eta)$ exists, which is a contradiction. Therefore,

$$\max_{z \in \mathbb{D}} |\varphi(z)| < 1.$$

This implies that (2.5) is valid.

Case II: φ has an angular derivative at some point on the unit circle. By Lemma 2.3,

$$\limsup_{|a|\to 1} \frac{N_{\varphi,\alpha}(a)}{(1-|a|^2)^\alpha} \geq \max\Big\{ \sum_{\varphi(e^{i\theta})=\varsigma} |\varphi'(e^{i\theta})|^{-\alpha} : |\varsigma| = 1 \Big\} > 0. \tag{2.6}$$

Conversely, from (2.4) and (2.6),

$$\lim_{a_m \to \xi} \frac{N_{\varphi,\alpha}(a_m)}{(1-|a_m|^2)^\alpha} > 0.$$

Combining this with the fact that φ is analytic in the neighbourhood of $\overline{\mathbb{D}}$, we can find a sufficiently large $M > 0$ so that, for $m > M$, the preimage of φ at a_m exists and ξ is a value of φ at some point in $\partial\mathbb{D}$.

Suppose that $\{\zeta_j\}_{j=1}^n$ is the set of all preimages of φ at ξ in the unit circle. As shown above (see the proof of Lemma 2.3), $\varphi'(\zeta_j) \neq 0$ for $j = 1, 2, \dots, n$ and there is a Euclidean disc $B(\xi, \delta)$ such that $\varphi - a$ has at most n zeros in $\mathbb{D}$ for every $a \in B(\xi, \delta) \cap \mathbb{D}$. Recall that φ is analytic in the neighbourhood of $\overline{\mathbb{D}}$, so φ preserves angles at ζ_j for $1 \leq j \leq n$. Hence we can choose $\epsilon > 0$ and define Ω_j ($j = 1, 2, \dots, n$) by

$$\Omega_j = \{z \in \mathbb{D} : |z - \zeta_j| < \epsilon\}$$

so that:

(1) $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$;

(2) $\varphi(\Omega_j)$ is a simply connected region internally tangential to the circle at ξ for $1 \leq j \leq n$, and $\Omega = \bigcap_{j=1}^n \varphi(\Omega_j) \neq \emptyset$; and

(3) $\Omega \subseteq B(\xi, \delta)$.

Fix j with $1 \le j \le n$. If there is a subsequence $\{b_s\}$ of $\{a_m\}$ such that, for every s, the preimage $z^{(j)}(b_s)$ of b_s lies in Ω_j, then (2.3) ensures that $z^{(j)}(b_s) \to \zeta_j$ through Ω_j as $b_s \to \xi$ through $\varphi(\Omega_j)$. Thus, by the Julia–Carathéodory theorem once more,

$$\lim_{b_s\to\xi} \frac{(1-|z^{(j)}(b_s)|^2)^\alpha}{(1-|b_s|^2)^\alpha} \le \limsup_{z\to\zeta_j}\Big(\frac{1-|z|^2}{1-|\varphi(z)|^2}\Big)^\alpha = |\varphi'(\zeta_j)|^{-\alpha}. \tag{2.7}$$

In what follows, it suffices to show that if there is a subsequence $\{c_s\}$ of $\{a_m\}$ such that $c_s \in \Omega$ for every s, then

$$\lim_{c_s\to\xi} \frac{N_{\varphi,\alpha}(c_s)}{(1-|c_s|^2)^\alpha} \le \max\Big\{\sum_{\varphi(e^{i\theta})=\lambda} |\varphi'(e^{i\theta})|^{-\alpha} : |\lambda| = 1\Big\}.$$

In this case, where $c_s \in \Omega$ for every s, choose the preimage $z^{(j)}(c_s)$ of c_s that lies in Ω_j for $1 \le j \le n$. Then

$$N_{\varphi,\alpha}(c_s) = \sum_{j=1}^{n}(1-|z^{(j)}(c_s)|^2)^\alpha.$$

From (2.7),

$$\lim_{c_s\to\xi} \frac{N_{\varphi,\alpha}(c_s)}{(1-|c_s|^2)^\alpha} \le \max\Big\{\sum_{\varphi(e^{i\theta})=\lambda} |\varphi'(e^{i\theta})|^{-\alpha} : |\lambda| = 1\Big\}. \tag{2.8}$$

Thus, combining (2.4) and (2.6) with (2.8) completes the proof of the proposition.

The following corollary is a direct consequence of (2.4) and Proposition 2.4.

Corollary 2.6. *Suppose that $\alpha > 0$ and that φ satisfies the hypotheses of Proposition 2.4. Then*

$$\limsup_{|a|\to 1} \frac{N_{\varphi,\alpha}(a)}{(1-|a|^2)^\alpha} = \max\Big\{\sum_{\varphi(e^{i\theta})=\varsigma} |\varphi'(e^{i\theta})|^{-\alpha} : |\varsigma| = 1\Big\}.$$

3. Proof of Theorems 1.1 and 1.2

In what follows, we assume that $\alpha \ge 0$.

3.1. The upper estimate. Suppose K_n takes f to the nth partial sum of its Taylor series: that is,

$$(K_n f)(z) = \sum_{j=0}^{n} a_j z^j \quad \text{where } f(z) = \sum_{j=0}^{\infty} a_j z^j \in \mathcal{D}_\alpha.$$

Let $R_n = I - K_n$, where I is identity operator on $\mathcal{D}_\alpha$. It is clear that K_n is compact on $\mathcal{D}_\alpha$. Hence

$$\|C_\varphi\|_{e,\mathcal{D}_\alpha} = \|C_\varphi(K_n + R_n)\|_{e,\mathcal{D}_\alpha} \le \|C_\varphi R_n\|. \tag{3.1}$$

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For any $f \in \mathcal{D}_\alpha$, it follows from the change of variable formula that

$$\|C_\varphi R_n f\|^2_{\mathcal{D}_\alpha} = |R_n f(\varphi(0))|^2 + \int_{\mathbb{D}} |(R_n f)'(\varphi(z))|^2 |\varphi'(z)|^2 \, dA_\alpha(z)$$
$$= |R_n f(\varphi(0))|^2 + (\alpha+1) \int_{\mathbb{D}} |(R_n f)'(\omega)|^2 N_{\varphi,\alpha}(\omega) \, dA(\omega).$$

Fix $0 < r_0 < 1$. Then

$$\|C_\varphi R_n f\|^2_{\mathcal{D}_\alpha} = |R_n f(\varphi(0))|^2 + (\alpha+1) \int_{\mathbb{D}\setminus r_0\mathbb{D}} |(R_n f)'(\omega)|^2 N_{\varphi,\alpha}(\omega) \, dA(\omega)$$
$$+ (\alpha+1) \int_{r_0\mathbb{D}} |(R_n f)'(\omega)|^2 N_{\varphi,\alpha}(\omega) \, dA(\omega).$$

From [4, pages 133–135],

$$\limsup_{n\to\infty} \sup_{\|f\|_{\mathcal{D}_\alpha}\le 1} \Big(\int_{r_0\mathbb{D}} |(R_n f)'(\omega)|^2 N_{\varphi,\alpha}(\omega) \, dA(\omega) + |R_n f(\varphi(0))|^2 \Big) = 0.$$

This implies that

$$\limsup_{n\to\infty} \|C_\varphi R_n\|^2 \le (\alpha+1) \sup_{\|f\|_{\mathcal{D}_\alpha}\le 1} \int_{\mathbb{D}\setminus r_0\mathbb{D}} |f'(\omega)|^2 N_{\varphi,\alpha}(\omega) \, dA(\omega)$$
$$\le \sup_{\omega\in\mathbb{D}\setminus r_0\mathbb{D}} \frac{N_{\varphi,\alpha}(\omega)}{(1-|\omega|^2)^\alpha} \sup_{\|f\|_{\mathcal{D}_\alpha}\le 1} \int_{\mathbb{D}\setminus r_0\mathbb{D}} |f'(\omega)|^2 \, dA_\alpha(\omega)$$
$$\le \sup_{\omega\in\mathbb{D}\setminus r_0\mathbb{D}} \frac{N_{\varphi,\alpha}(\omega)}{(1-|\omega|^2)^\alpha}.$$

Letting $r_0 \to 1$ and combining (3.1) and the preceding formula,

$$\|C_\varphi\|^2_{e,\mathcal{D}_\alpha} \le \limsup_{|a|\to 1} \frac{N_{\varphi,\alpha}(a)}{(1-|a|^2)^\alpha}. \tag{3.2}$$

3.2. The lower estimate. Suppose that $a \in \mathbb{D}$. Let

$$f_a^\alpha(z) = (1-|a|^2)^{1/2\alpha+1} \int_0^z \frac{d\omega}{(1-\bar{a}\omega)^{\alpha+2}} \quad \text{for all } z \in \mathbb{D}.$$

Clearly, $\|f_a^\alpha\|_{\mathcal{D}_\alpha} = 1$ and f_a^α converges pointwise to zero on $\mathbb{D}$ as $|a| \to 1$. By [4, Corollary 1.3], f_a^α converges to zero weakly on $\mathcal{D}_\alpha$, and hence

$$\lim_{|a|\to 1} \|K f_a^\alpha\|_{\mathcal{D}_\alpha} = 0$$

for any compact operator K on $\mathcal{D}_\alpha$. This yields

$$\|C_\varphi - K\| \ge \limsup_{|a|\to 1} \|(C_\varphi - K) f_a^\alpha\|_{\mathcal{D}_\alpha} \ge \limsup_{|a|\to 1} \|C_\varphi f_a^\alpha\|_{\mathcal{D}_\alpha},$$

which implies that

$$\|C_\varphi\|_{e,\mathcal{D}_\alpha} \ge \limsup_{|a|\to 1} \|C_\varphi f_a^\alpha\|_{\mathcal{D}_\alpha}. \tag{3.3}$$

By the change of variable formula,

$$\begin{aligned}
\|C_\varphi f_a^\alpha\|_{\mathcal{D}_\alpha}^2 &= |f_a^\alpha(\varphi(0))|^2 + \int_{\mathbb{D}} |(f_a \circ \varphi)'(z)|^2 \, dA_\alpha(z) \\
&= |f_a^\alpha(\varphi(0))|^2 + (\alpha+1) \int_{\mathbb{D}} |f_a'(\omega)|^2 N_{\varphi,\alpha}(\omega) \, dA(\omega) \\
&\ge (\alpha+1) \int_{\mathbb{D}} \frac{(1-|a|^2)^{\alpha+2}}{|1-\bar{a}\omega|^{2\alpha+4}} N_{\varphi,\alpha}(\omega) \, dA(\omega) \\
&= (\alpha+1) \int_{\mathbb{D}} \frac{(1-|a|^2)^{\alpha}}{|1-\bar{a}\omega|^{2\alpha}} |\varphi_a'(\omega)|^2 N_{\varphi,\alpha}(\omega) \, dA(\omega) \\
&= \int_{\mathbb{D}} \frac{N_{\varphi,\alpha}(\varphi_a(z))}{(1-|\varphi_a(z)|^2)^\alpha} \, dA_\alpha(z). \qquad (3.4)
\end{aligned}$$

3.3. Proof of Theorem 1.1. Suppose that $\alpha > 0$ and that φ is analytic in a neighbourhood of the closed unit disc. If

$$\max\left\{ \sum_{\varphi(e^{i\theta})=\varsigma} |\varphi'(e^{i\theta})|^{-\alpha} : |\varsigma| = 1 \right\} = 0,$$

then Theorem 1.1 follows from Corollary 2.6 and (3.2). Thus, in what follows, we assume that

$$\max\left\{ \sum_{\varphi(e^{i\theta})=\varsigma} |\varphi'(e^{i\theta})|^{-\alpha} : |\varsigma| = 1 \right\} > 0$$

and choose $\xi_0 \in \partial\mathbb{D}$ such that

$$\sum_{\varphi(e^{i\theta})=\xi_0} |\varphi'(e^{i\theta})|^{-\alpha} = \max\left\{ \sum_{\varphi(e^{i\theta})=\varsigma} |\varphi'(e^{i\theta})|^{-\alpha} : |\varsigma| = 1 \right\}. \tag{3.5}$$

On the one hand, Corollary 2.6 and (3.2) give

$$\|C_\varphi\|_{e,\mathcal{D}_\alpha}^2 \le \max\left\{ \sum_{\varphi(e^{i\theta})=\varsigma} |\varphi'(e^{i\theta})|^{-\alpha} : |\varsigma| = 1 \right\}.$$

On the other hand, if we fix r with $0 < r < 1$, then by (3.4),

$$\|C_\varphi f_a^\alpha\|_{\mathcal{D}_\alpha}^2 \ge \int_{r\mathbb{D}} \frac{N_{\varphi,\alpha}(\varphi_a(z))}{(1-|\varphi_a(z)|^2)^\alpha} \, dA_\alpha(z). \tag{3.6}$$

Now choose a sequence $\{a_k\} \subseteq \mathbb{D}$ so that $\arg a_k = \arg \xi_0$ and $a_k \to \xi_0$ as $k \to \infty$. By Corollary 2.2 and Lemma 2.3,

$$\lim_{k\to\infty} \frac{N_{\varphi,\alpha}(\varphi_{a_k}(z))}{(1-|\varphi_{a_k}(z)|^2)^\alpha} = \sum_{\varphi(e^{i\theta})=\xi_0} |\varphi'(e^{i\theta})|^{-\alpha} \quad \text{for all } z \in r\mathbb{D}.$$

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Thus, by the Lebesgue dominated convergence theorem,

$$\lim_{k\to\infty}\int_{r\mathbb{D}}\frac{N_{\varphi,\alpha}(\varphi_{a_k}(z))}{(1-|\varphi_{a_k}(z)|^2)^\alpha}\,dA_\alpha(z)=(1-(1-r^2)^{\alpha+1})\sum_{\varphi(e^{i\theta})=\xi_0}|\varphi'(e^{i\theta})|^{-\alpha}.$$

Combining this with (3.6),

$$\limsup_{|a|\to1}\|C_\varphi f_a^\alpha\|^2_{\mathcal{D}_\alpha}\geq(1-(1-r^2)^{\alpha+1})\sum_{\varphi(e^{i\theta})=\xi_0}|\varphi'(e^{i\theta})|^{-\alpha}.$$

Let $r\to1$. By (3.3) and (3.5),

$$\|C_\varphi\|^2_{e,\mathcal{D}_\alpha}\geq\max\Big\{\sum_{\varphi(e^{i\theta})=\varsigma}|\varphi'(e^{i\theta})|^{-\alpha}:|\varsigma|=1\Big\}.$$

This completes the proof of Theorem 1.1.

3.4. Proof of Theorem 1.2. Since φ is univalent, (3.2) yields

$$\|C_\varphi\|^2_{e,\mathcal{D}}\leq1. \tag{3.7}$$

This result, with a different proof, can also be found in [6, Proposition 2.4].

On the other hand, suppose that φ has an angular derivative at $\eta\in\partial\mathbb{D}$ and that ξ is the nontangential limit of φ at η. Fix r with $0<r<1$. By (3.3) and (3.4),

$$\|C_\varphi\|^2_{e,\mathcal{D}}\geq\limsup_{|a|\to1}\int_{r\mathbb{D}}n_\varphi(\varphi_a(z))\,dA(z)\geq\limsup_{\arg a=\arg\xi,\,a\to\xi}\int_{r\mathbb{D}}n_\varphi(\varphi_a(z))\,dA(z). \tag{3.8}$$

By Corollary 2.2, there exists h with $0<h<1$ such that

$$D(a,r)\subseteq\varphi(\mathbb{D})\quad\text{for all } a\in E_h,$$

which implies that

$$n_\varphi(\varphi_a(z))=1\quad\text{for all } z\in r\mathbb{D}\text{ and } b\in E_h.$$

It follows from (3.7) and (3.8) that

$$r^2\leq\|C_\varphi\|^2_{e,\mathcal{D}}\leq1.$$

Letting $r\to1$ gives

$$\|C_\varphi\|_{e,\mathcal{D}}=1,$$

which completes the proof.

In [7, Theorem 5.3], MacCluer and Shapiro showed that if C_φ is bounded on $\mathcal{D}_\gamma$ for some γ with $-1<\gamma<0$, then C_φ is compact on $\mathcal{D}$ if and only if φ does not have an angular derivative at any point of $\partial\mathbb{D}$. Hence, by Theorem 1.2, we have this corollary.

Corollary 3.1. *Suppose that φ is univalent and that C_φ is bounded on $\mathcal{D}_\gamma$ for some γ with $-1<\gamma<0$. Then C_φ is not compact on $\mathcal{D}$ if and only if*

$$\|C_\varphi\|_{e,\mathcal{D}}=1.$$

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