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A LARGE-DEVIATION PRINCIPLE FOR BIRTH–DEATH PROCESSES WITH A LINEAR RATE OF DOWNWARD JUMPS

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Abstract

Birth–death processes form a natural class where ideas and results on large deviations can be tested. We derive a large-deviation principle under an assumption that the rate of jump down (death) grows asymptotically linearly with the population size, while the rate of jump up (birth) grows sublinearly. We establish a large-deviation principle under various forms of scaling of the underlying process and the corresponding normalization of the logarithm of the large-deviation probabilities. The results show interesting features of dependence of the rate functional upon the parameters of the process and the forms of scaling and normalization.

Keywords: Large-deviation principle; local large-deviation principle; birth–death processes; rate functional

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1. Introduction and definitions

A birth–death process is a continuous-time Markov process with states $x \in \mathbb{Z}^+ := \{0, 1, 2, \ldots\}$ (representing the population size) and with transitions occurring between neighboring states. The class of birth–death processes exhibits a remarkable balance between simplicity, allowing for analytical solutions, and complexity, showcasing a diverse range of interesting phenomena. Its versatility is accentuated by the possibility of exploring various jump rates, drawing attention from multiple research areas. Furthermore, birth–death processes find applications across diverse fields, such as information theory (involving encoding and storage of information [26, 28]), population biology, genetics, ecology (reviewed in [19, 24]), chemistry (modeling growth and extinction in systems with multiple components [11, 15, 27]),

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economics (modeling competitive production and pricing [17, 34]) and queueing system theory (explored, for example, in [22]).

In particular, birth–death processes are instrumental in exploring various aspects of large-deviation theory, which is the focus of this paper. Apart from that, we also mention connections between birth–death processes and orthogonal polynomials, as detailed in [29, 30].

In this paper we work under the assumption that the rate $\lambda(x)$ of jump $x \to x+1$ and the rate $\mu(x)$ of jump $x \to x-1$ obey the condition (1.2): $\mu(x)$ grows with x asymptotically linearly, while $\lambda(x)$ grows asymptotically sublinearly. This assumption ensures positive recurrence of the process (cf. [9]). Such processes find an application in population dynamics [10, 12]; they are also relevant in models of market interaction between ask–bid sides of a limit order book [17], energy-efficient schemes for cloud resources [23], and scenarios with an increasing number of available servers in stations [22].

Let us provide formal definitions. We consider a continuous-time Markov process $\xi(t)$, $t \geq 0$, on the state space $\mathbb{Z}^+$, starting at point 0. The process dynamics is as follows. We are given two functions, $\lambda\colon \mathbb{Z}^+ \to (0, \infty)$ giving the rate of upward jumps, and $\mu\colon \mathbb{Z}^+ \to [0, \infty)$ giving the rate of downward jumps, with $\mu(0) = 0$ and $\mu(x) > 0$ for $x \geq 1$. We set $\eta = \lambda + \mu$ for the combined jump rate. Given that $\xi(t) = x$ for some $t \geq 0$ and $x \in \mathbb{Z}^+$, the value of the process remains unchanged for an exponentially distributed random time τ_x of rate $\eta(x)$. At time $t + \tau_x$ the process jumps to either $x+1$ or $x-1$ with the probabilities

$$\mathbb{P}(\xi(t+\tau_x) = x+1) = \frac{\lambda(x)}{\eta(x)}, \qquad \mathbb{P}(\xi(t+\tau_x) = x-1) = \frac{\mu(x)}{\eta(x)}. \tag{1.1}$$

For the case where $x = 0$, the only feasible transition is to site 1. The key assumption is that there exist constants $P, Q > 0$ and $l \in [0, 1)$ such that

$$\lim_{x\to\infty} \frac{\lambda(x)}{x^l} = P, \qquad \lim_{x\to\infty} \frac{\mu(x)}{x} = Q. \tag{1.2}$$

We focus on the large-deviation principle (LDP) for the family of processes

$$\xi_T(t) = \frac{\xi(tT)}{\varphi(T)}, \qquad 0 \leq t \leq 1, \tag{1.3}$$

for subexponential (1.4), exponential (1.5), or superexponential (1.6) growth of the value $\varphi(T)$. Here, $T > 0$ is a time-scaling parameter, and $\varphi\colon (0, \infty) \to (0, \infty)$ is a Lebesgue-measurable function referred to as a scaling function. We assume that $\lim_{T\to\infty} \varphi(T) = \infty$.

The space where we will establish the large-deviation principle is $\mathbb{L} = \mathbb{L}_1[0, 1]$, with the standard metric $\rho(f, g) = \int_0^1 |f(t) - g(t)|\, \mathrm{d}t$, $f, g \in \mathbb{L}$. Let $\mathfrak{B} = \mathfrak{B}_{(\mathbb{L},\rho)}$ denote the Borel σ-algebra in $(\mathbb{L}, \rho)$; for a set $\mathbb{B} \in \mathfrak{B}$, $\mathrm{cl}(\mathbb{B})$ and $\mathrm{int}(\mathbb{B})$ stand for the closure and the interior of $\mathbb{B}$, respectively.

Recall the notions and definitions we need (see, for more details, [5–7, 25, 31, 32]). In Definitions 1.1 and 1.2 we attempt to cover a variety of situations occurring in the context of the current paper. In these definitions we use a Lebesgue-measurable function $\psi\colon (0, \infty) \to (0, \infty)$ satisfying $\lim_{T\to\infty} \psi(T) = \infty$, and a $\mathfrak{B}$-measurable functional $I\colon \mathbb{G} \to [0, \infty]$ where $\mathbb{G} \subseteq \mathbb{L}$ and $\mathbb{G} \in \mathfrak{B}$. Given $\mathbb{A} \subseteq \mathbb{G}$ such that $\mathbb{A} \in \mathfrak{B}$, we set $I(\mathbb{A}) = \inf_{y\in\mathbb{A}} I(y)$, with $I(\varnothing) = \infty$. Furthermore, ψ is referred to as a normalizing function and I as a large-deviation (LD) rate functional.

Definition 1.1. Let $\mathbb{G} \subseteq \mathbb{L}$ and $\mathbb{G} \in \mathfrak{B}$. Let a family of random processes $\xi_T(\cdot)$, $T > 0$, be defined as in (1.3) for some scaling function φ. We say that this family satisfies a

$(\mathbb{G}, \mathbb{L}, \rho)$-local large-deviation principle ($(\mathbb{G}, \mathbb{L}, \rho)$-LLDP) with an LD functional $I: \mathbb{G} \to [0, \infty]$ and the normalizing function ψ if, for all $f \in \mathbb{G}$,

$$\lim_{\varepsilon\to 0} \limsup_{T\to\infty} \frac{1}{\psi(T)} \ln \mathbb{P}(\xi_T(\cdot) \in \mathbb{U}_\varepsilon(f)) = \lim_{\varepsilon\to 0} \liminf_{T\to\infty} \frac{1}{\psi(T)} \ln \mathbb{P}(\xi_T(\cdot) \in \mathbb{U}_\varepsilon(f)) = -I(f),$$

where $\mathbb{U}_\varepsilon(f) = \{g \in \mathbb{L}: \rho(f, g) < \varepsilon\}$.

Definition 1.2. Let $\xi_T(\cdot)$, $T > 0$, be family of random processes defined as in (1.3) for some scaling function φ. We say that this family satisfies an $(\mathbb{L}, \rho)$-LDP with a normalizing function ψ and an LD functional $I: \mathbb{L} \to (0, \infty]$ if, whenever set $\mathbb{B} \subseteq \mathbb{L}$ and $\mathbb{B} \in \mathfrak{B}$,

$$\limsup_{T\to\infty} \frac{1}{\psi(T)} \ln \mathbb{P}(\xi_T(\cdot) \in \mathbb{B}) \le -I(\mathrm{cl}(\mathbb{B})),$$
$$\liminf_{T\to\infty} \frac{1}{\psi(T)} \ln \mathbb{P}(\xi_T(\cdot) \in \mathbb{B}) \ge -I(\mathrm{int}(\mathbb{B})).$$

Definition 1.3. Let $\xi_T(\cdot)$, $T > 0$, be family of random processes defined as in (1.3) for some scaling function φ. We say that this family is exponentially tight (ET) on $(\mathbb{L}, \rho)$ with a normalizing function ψ if, for any $C > 0$, there exists a compact set $\mathbb{K}_C \subseteq \mathbb{L}$ such that

$$\limsup_{T\to\infty} \frac{1}{\psi(T)} \ln \mathbb{P}(\xi_T(\cdot) \notin \mathbb{K}_C) \le -C.$$

If a family $\xi_T(\cdot)$, $T > 0$, and a functional I satisfy Definitions 1.2 and 1.3 (in particular, if family $\xi_T(\cdot)$, $T > 0$, is ET) then, for all $c \ge 0$, the set $\{f \in \mathbb{L}: I(f) \le c\}$ is compact in $(\mathbb{L}, \rho)$. In this case, we say that I is a 'good rate functional' (cf. [5, Section 1.2], [7, Section 2.2]). In this paper the ET property is established in Lemma A.6. It is known (see, for example, [20]) that if the trajectories of random processes $\xi_T(\cdot)$ belong to a Polish space then the ET property is a necessary condition for the goodness of functional I. Note that this holds true in our setting.

[33] established an LLDP for a family of processes (1.3) with the scaling function $\varphi(T) = T$, while [13] did so for the case of subexponential asymptotics of $\varphi(T)$ when

$$\lim_{T\to\infty} \frac{\ln \varphi(T)}{T} = 0. \tag{1.4}$$

In this latter case, the family (1.3) is not ET (we discuss this in Section 4). Consequently, the LDP is not available in the whole of $(\mathbb{L}, \rho)$.

In the present paper we consider two complementary conditions:

$$\text{there exists a constant } k \in (0, \infty) \text{ such that } \lim_{T\to\infty} \frac{\ln \varphi(T)}{T} = k; \tag{1.5}$$

$$\lim_{T\to\infty} \frac{\ln \varphi(T)}{T} = \infty. \tag{1.6}$$

The form of the LD functional depends on which condition is assumed, (1.5) or (1.6), cf. Section 2, Theorems 2.1 and 2.2. An emerging question is why the scalings (1.5) or (1.6) lead to the large-deviation principle, while the scaling (1.4) does not. We explain this in Section 4.

Let us discuss what is currently known outside condition (1.2); cf. [13]. Suppose that $\lambda(x) \sim Px^l$ and $\mu(x) \sim Qx^m$, where $0 \le l < m$. If $m \in (0, 1)$ then three cases emerge, depending on a

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condition upon scaling function φ, and the form of the rate functional is different in each of these cases. If we assume that $m > 1$ then only an LLDP will take place, so the three cases will be reduced to one. Also, from [13] it follows that an LLDP holds true when rates $\lambda(x)$ and $\mu(x)$ are regularly varying functions. Separately, notice the case where $\lambda(x) = P$ and $\mu(x) = Q$ where P and Q are positive constants. Here, process $\xi(t)$ is compound Poisson, for which the LD asymptotics are well known [3, 14, 16].

This paper contains four sections. In Section 2 we state our main result, Theorem 2.2, and a trio of auxiliary assertions (Lemmas 2.1–2.3). Section 3 is dedicated to the derivation of Theorem 2.2 from Lemmas 2.1–2.3 and the proofs of these lemmas. Section 4 contains a discussion of the results obtained. Finally, in the Appendix we prove some additional technical assertions (Lemmas A.1–A.6) used in the proof or interpretation of the obtained results.

A commemorative note It is with great sadness and sorrow that the rest of the authors report of the loss of our remarkable collaborator and friend Nikita Vvedenskaya (1930–2022). Until her last days she actively worked on this project, and her contribution was essential and irreplaceable. We will miss her dearly.

2. Notation, and the main result

We denote by $\mathbb{V} = \mathbb{V}[0, 1]$ the set of non-negative measurable functions $f: [0, 1] \mapsto [0, \infty)$ of a finite variation. Given $f \in \mathbb{V}$, let $\operatorname{Var} f$ be the total variation of f.

Next, $\mathbb{C} = \mathbb{C}[0, 1]$ is the space of continuous functions on [0, 1]. From now on we let $\mathbb{G}$ be the set of functions $f \in \mathbb{C}$ such that $f(0) = 0$ and $f(t) > 0$ for $t > 0$.

The following result follows from [13].

Theorem 2.1. *Assume conditions* (1.2) *and* (1.4). *Then the family* $\xi_T(\cdot)$, $T > 0$, *defined as in* (1.3) *satisfies a* $(\mathbb{G}, \mathbb{L}, \rho)$*-LLDP with the normalizing function* $\psi(T) = T\varphi(T)$ *and the LD functional* $I(f) = Q \int_0^1 f(t)\, \mathrm{d}t$.

Given $f \in \mathbb{V}$, we use the following decomposition into monotone increasing and decreasing components:

$$f(t) = f^+(t) - f^-(t), \quad f^+(0) = f(0), \quad f^-(0) = 0; \quad \operatorname{Var} f = \operatorname{Var} f^+ + \operatorname{Var} f^-. \tag{2.1}$$

Such a decomposition is unique (cf. [21, Chapter 1, §4]).

Denote by $\mathbb{D} = \mathbb{D}[0, 1]$ the space of càdlàg functions on [0,1) with left limits at $t = 1$. Observe that for every $f \in \mathbb{V}$ there exists a function $f_{\mathbb{D}} \in \mathbb{D}$ such that $\rho(f, f_{\mathbb{D}}) = 0$.

We now introduce the main result of this paper.

Theorem 2.2. *Assume condition* (1.2).

(i) Under condition (1.5) *the family* $\xi_T(\cdot)$, $T > 0$, *defined as in* (1.3) *satisfies an* $(\mathbb{L}, \rho)$*-LDP with the normalizing function* $\psi(T) = \varphi(T) \ln \varphi(T)$ *and the good LD functional* $I: \mathbb{L} \to [0, \infty]$ *where*

$$I(f) = \begin{cases} \dfrac{Q}{k} \displaystyle\int_0^1 f(s)\, \mathrm{d}s + (1 - l) f^+_{\mathbb{D}}(1), & f \in \mathbb{V}, \\ \infty, & f \notin \mathbb{V}. \end{cases}$$

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(ii) Under condition (1.6) *the family* $\xi_T(\cdot)$, $T > 0$, *defined as in* (1.3) *satisfies an* $(\mathbb{L}, \rho)$-*LDP with the normalizing function* $\psi(T) = \varphi(T) \ln \varphi(T)$ *and the good LD functional* $I \colon \mathbb{L} \to [0, \infty]$ *where*

$$I(f) = \begin{cases} (1-l)f_{\mathbb{D}}^{+}(1), & f \in \mathbb{V}, \\ \infty, & f \notin \mathbb{V}. \end{cases}$$

Before we pass to the proof, let us make some comments. Note that the LDP in the space of right-continuous functions with the Skorokhod metric is not obtained since the set of functions with total variation bounded by a constant is non-compact in this space. On the other hand, it seems that the results of this paper will hold for the space of functions without second-kind discontinuities equipped with the Borovkov metric (cf. [1, 2, 4]). It is also worth mentioning that, in contrast with the classical results, in our case the LD functional $I(f)$ does not contain the integral of the convex function of the derivative of the absolutely continuous component of the function f.

The proof of Theorem 2.2 uses the auxiliary assertions on Lemmas 2.1–2.3. Let us introduce some additional notions. Given $T > 0$, denote by $\mathbb{X}_T$ the set of right-continuous functions $u \colon [0, T] \to \mathbb{Z}^+$, with $u(0) = 0$, having a finite number of jumps $n\,(u)$, where every jump has size ± 1. This gives the set of trajectories for the birth–death process $\xi(t)$, $t \in [0, T]$. We speak below of measures on $(\mathbb{X}_T, \mathfrak{X}_T)$, where $\mathfrak{X}_T$ is a standard Borel σ-algebra in $\mathbb{X}_T$.

Next, consider a continuous-time Markov process $\zeta(t)$, $t \in [0, T]$, on the state space $\mathbb{Z}$, with the full jump rate 1, jump size ± 1, and probabilities of jumps $1/2$. There is a positive probability that this process lives in $\mathbb{X}_T$. In Lemma 2.1 and later we refer to the two processes as ξ and ζ.

Lemma 2.1. (cf. [17, 33].) *The distribution of the random process* ξ *on* $\mathbb{X}_T$ *is absolutely continuous with respect to that of a process* ζ. *The corresponding Radon–Nikodym density* $\mathbf{p} = \mathbf{p}_T$ *on* $\mathbb{X}_T$ *has the form*

$$\mathbf{p}(u) = \begin{cases} 2^{n(u)} \left(\displaystyle\prod_{i=1}^{n(u)} \mathrm{e}^{-(\eta(u(t_{i-1}))-1)\tau_i} \nu(u(t_{i-1}), u(t_i)) \right) \mathrm{e}^{-(\eta(u(t_{n(u)})-1))(T-t_{n(u)})}, & n(u) \geq 1, \\ \mathrm{e}^{-(\eta(0)-1)T}, & n(u) = 0, \end{cases}$$

where $\eta(x) = \lambda(x) + \mu(x)$, $x \in \mathbb{Z}^+$; *cf.* (1.1). *Here we suppose that the function* $u \in \mathbb{X}_T$ *has jumps at time points* $0 < t_1 < \cdots < t_{n(u)} < T$ *and set* $\tau_i = t_i - t_{i-1}$, *with* $t_0 = 0$. *Further, the value* $\nu(u(t_{i-1}), u(t_i))$ *is given by*

$$\nu(u(t_{i-1}), u(t_i)) = \begin{cases} \lambda(u(t_{i-1})), & u(t_i) - u(t_{i-1}) = 1; \\ \mu(u(t_{i-1})), & u(t_i) - u(t_{i-1}) = -1. \end{cases}$$

Let $N_T(\zeta)$ be the number of jumps in process $\zeta(t)$ on the interval $[0,T]$. The claim of Lemma 2.1 is equivalent to the fact that, for any measurable set $\mathbb{H} \subseteq \mathbb{X}_T$,

$$\mathbb{P}(\xi \in \mathbb{H}) = \mathrm{e}^{T} \mathbb{E}\big[\mathrm{e}^{-A_T(\zeta)} \exp\{B_T(\zeta) + N_T(\zeta) \ln 2\} \mathbf{1}(\zeta \in \mathbb{H})\big]. \tag{2.2}$$

Here,

$$A_T(\zeta) := \int_0^T \eta(\zeta(t))\,\mathrm{d}t = \begin{cases} \sum_{i=1}^{N_T(\zeta)} \eta(\zeta(t_{i-1}))\tau_i + \eta\big(\zeta\big(t_{N_T(\zeta)}\big)\big)\big(T - t_{N_T(\zeta)}\big), & N_T(\zeta) \geq 1, \\ \eta(0)T, & N_T(\zeta) = 0; \end{cases} \tag{2.3}$$

$$B_T(\zeta) := \begin{cases} \sum_{i=1}^{N_T(\zeta)} \ln\left(\nu(\zeta(t_{i-1}), \zeta(t_i))\right), & N_T(\zeta) \geq 1, \\ 0, & N_T(\zeta) = 0. \end{cases}$$

The symbols $\mathbf{1}(\cdot)$ and $\mathbf{1}[\,\cdot\,]$ represent indicators of events in the σ-algebra $\mathfrak{B}$.

The representation in (2.2) is used in the analysis of the value $\ln \mathbb{P}(\xi_T(\cdot) \in \mathbb{U}_\varepsilon(f))$. We set $\zeta_T(t) := \zeta(tT)/\varphi(T)$, $t \in [0, 1]$. In what follows, we write ξ_T, ζ_T instead of $\xi_T(\cdot)$, $\zeta_T(\cdot)$, and A_T, B_T, N_T instead of $A_T(\zeta)$, $B_T(\zeta)$, $N_T(\zeta)$.

The proof of Theorem 2.2 is based on the analysis of $\mathbf{p}_T$. This is a common method in LD theory, particularly, in the specification of an LD functional. Namely, we analyze the Radon–Nikodym density $\mathbf{p}_T$ on the event $\{\zeta_T \in \mathbb{U}_\varepsilon(f)\}$ and prove a $(\mathbb{V}, \mathbb{L}, \rho)$-LLDP by using the independence of increments in process ζ_T, together with the Stirling formula and properties of the functional space $(\mathbb{L}, \rho)$; see also Lemmas 2.2 and 2.3 and their proofs in the Appendix. Next, we prove that the family ξ_T is ET (cf. Lemma A.6). Then, by using a standard implication LLDP plus ET $\Rightarrow$ LDP (cf. [5, Lemma 4.1.23], [20]), we obtain an $(\mathbb{L}, \rho)$-LDP for processes ξ_T.

Lemma 2.2. *Assume condition* (1.2) *and one of conditions* (1.5) *or* (1.6)*. Then, for all* $f \in \mathbb{V}$ *with* $\rho(f, 0) > 0$,

$$\lim_{\varepsilon\to 0} \limsup_{T\to\infty} \frac{\ln \mathbb{E}[\exp\{B_T + N_T \ln 2\}\mathbf{1}(\zeta_T \in \mathbb{U}_\varepsilon(f))]}{\varphi(T)\ln\varphi(T)} \leq (1-l)f^+_{\mathbb{D}}(1).$$

Lemma 2.3. *Assume condition* (1.2) *and one of conditions* (1.5) *or* (1.6)*. Then, for all* $f \in \mathbb{V}$ *with* $\rho(f, 0) > 0$,

$$\lim_{\varepsilon\to 0} \liminf_{T\to\infty} \frac{\ln \mathbb{E}[\exp\{B_T + N_T \ln 2\}\mathbf{1}(\zeta_T \in \mathbb{U}_\varepsilon(f))]}{\varphi(T)\ln\varphi(T)} \geq (1-l)f^+_{\mathbb{D}}(1).$$

3. Proofs of Theorem 2.2 and Lemmas 2.2 and 2.3

Proof of Theorem 2.2. First, consider the case where $\rho(f, 0) = 0$. Obviously,

$$\lim_{\varepsilon\to 0} \limsup_{T\to\infty} \frac{1}{\varphi(T)\ln\varphi(T)} \ln \mathbb{P}\big(\xi_T \in \mathbb{U}_\varepsilon(f)\big) \leq 0 = I(f).$$

It is easy to see that

$$\lim_{\varepsilon\to 0} \liminf_{T\to\infty} \frac{1}{\varphi(T)\ln\varphi(T)} \ln \mathbb{P}\big(\xi_T \in \mathbb{U}_\varepsilon(f)\big) \geq \lim_{\varepsilon\to 0} \liminf_{T\to\infty} \frac{1}{\varphi(T)\ln\varphi(T)} \ln \mathbb{P}\left(\sup_{t\in[0,1]} \xi_T(t) = 0\right)$$
$$= \lim_{\varepsilon\to 0} \liminf_{T\to\infty} \frac{1}{\varphi(T)\ln\varphi(T)} \ln \mathrm{e}^{-\lambda(0)T} = 0.$$

Now, suppose that $\rho(f, 0) > 0$. We start by evaluating A_T. As follows from (2.3),

$$A_T := \int_0^T \eta(\zeta(t))\, \mathrm{d}t = T \int_0^1 \eta(\varphi(T)\zeta_T(s))\, \mathrm{d}s. \tag{3.1}$$

Condition (1.2) implies that, for any given $\varepsilon, \gamma \in (0, 1)$, for T large enough,

$$Q(1-\gamma) \le \frac{\eta(\varphi(T)(\zeta_T(s) \vee \varepsilon))}{\varphi(T)(\zeta_T(s) \vee \varepsilon)} \le Q(1+\gamma). \tag{3.2}$$

Here and below, $a \vee b = \max(a, b)$. Furthermore, the values ε and γ will tend to zero.

Let us upper-bound the integral in (3.1). Suppose $\zeta_T \in \mathbb{U}_\varepsilon(f)$. Then the right bound in (3.2) implies that, for any $\varepsilon, \gamma \in (0, 1)$, if T is large enough, we have the inequalities

$$\begin{aligned}
\int_0^1 \eta(\varphi(T)\zeta_T(s))\, \mathrm{d}s &\le \int_0^1 \eta(\varphi(T)(\zeta_T(s) \vee \varepsilon))\, \mathrm{d}s \\
&\le \varphi(T)(1+\gamma)Q \int_0^1 (\zeta_T(s) \vee \varepsilon)\, \mathrm{d}s \\
&\le \varphi(T)(1+\gamma)Q \int_0^1 (|\zeta_T(s) - f(s)| + f(s) + \varepsilon)\, \mathrm{d}s \\
&\le \varphi(T)(1+\gamma)Q \int_0^1 f(s)\, \mathrm{d}s + 2\varphi(T)(1+\gamma)Q\varepsilon.
\end{aligned} \tag{3.3}$$

Next, consider a lower bound for the integral in (3.1). Due to the asymptotic character of condition (1.2), we need some caution when dealing with the regions where the scaled process approaches level zero. We set

$$H := \{t \in [0, 1] : f(t) > 0\}, \qquad H_\varepsilon := \left\{t \in [0, 1] : f(t) \ge \varepsilon + \sqrt{\varepsilon}\right\}, \tag{3.4}$$

$$G_\varepsilon := \left\{t \in [0, 1] :\ \zeta_T(t) < \varepsilon, f(t) \ge \varepsilon + \sqrt{\varepsilon}\right\}. \tag{3.5}$$

If $\zeta_T \in \mathbb{U}_\varepsilon(f)$, the left-hand bound in (3.2) implies that, once more, for any given small ε and γ within the interval (0,1), and with a sufficiently large value of T, we have

$$\begin{aligned}
\int_0^1 \eta(\varphi(T)\zeta_T(s))\, \mathrm{d}s &\ge \varphi(T)(1-\gamma)Q \int_{H_\varepsilon \setminus G_\varepsilon} \zeta_T(s)\, \mathrm{d}s \\
&\ge \varphi(T)(1-\gamma)Q \int_{H_\varepsilon \setminus G_\varepsilon} f(s)\, \mathrm{d}s - \varphi(T)(1-\gamma)Q \int_{H_\varepsilon \setminus G_\varepsilon} |\zeta_T(s) - f(s)|\, \mathrm{d}s \\
&\ge \varphi(T)(1-\gamma)Q \int_{H_\varepsilon \setminus G_\varepsilon} f(s)\, \mathrm{d}s - \varphi(T)(1-\gamma)Q\varepsilon.
\end{aligned} \tag{3.6}$$

If $\zeta_T \in \mathbb{U}_\varepsilon(f)$, the Lebesgue measure of the set G_ε defined by (3.5) has the following upper bound. Since $f(s) - \zeta_T(s) \ge \sqrt{\varepsilon}$ for all $s \in G_\varepsilon$ we have

$$L(G_\varepsilon) = \int_{G_\varepsilon} \mathrm{d}s \le \int_0^1 \frac{|\zeta_T(s) - f(s)|}{\sqrt{\varepsilon}}\, \mathrm{d}s = \frac{\rho(\zeta_T, f)}{\sqrt{\varepsilon}} \le \sqrt{\varepsilon}. \tag{3.7}$$

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By virtue of (2.2), (3.1), (3.3), and (3.6), we obtain that, for T large enough,

$$\exp\left\{T - T\varphi(T)(1-\gamma)Q\int_{H_\varepsilon\setminus G_\varepsilon} f(s)\,\mathrm{d}s + T\varphi(T)(1-\gamma)Q\varepsilon\right\}\mathbb{E}[\exp\{B_T + N_T\ln 2\}\mathbf{1}(\zeta_T\in\mathbb{U}_\varepsilon(f))]$$
$$\geq \mathbb{P}(\xi_T(\cdot)\in\mathbb{U}_\varepsilon(f))$$
$$\geq \exp\left\{T - T\varphi(T)(1+\gamma)Q\int_0^1 f(s)\,\mathrm{d}s - 2T\varphi(T)(1+\gamma)Q\varepsilon\right\}\mathbb{E}[\exp\{B_T + N_T\ln 2\}\mathbf{1}(\zeta_T\in\mathbb{U}_\varepsilon(f))]. \tag{3.8}$$

The bounds in (3.8) conclude an initial part of the proof of Theorem 2.2. Subsequent parts establish assertions (i) and (ii) based on (3.8), while assuming conditions (1.5) and (1.6), respectively.

First, assume condition (1.5). According to the upper bound in (3.8), for any $\varepsilon, \gamma\in(0, 1)$,

$$\begin{aligned}\limsup_{T\to\infty}\frac{1}{\varphi(T)\ln\varphi(T)}\ln\mathbb{P}(\xi_T(\cdot)\in\mathbb{U}_\varepsilon(f)) &\leq -\frac{Q(1-\gamma)}{k}\int_{H_\varepsilon\setminus G_\varepsilon} f(s)\,\mathrm{d}s + \frac{Q(1-\gamma)}{k}\varepsilon\\ &\quad + \limsup_{T\to\infty}\frac{1}{\varphi(T)\ln\varphi(T)}\ln\mathbb{E}[\exp\{B_T+N_T\ln 2\}\mathbf{1}(\zeta_T\in\mathbb{U}_\varepsilon(f))].\end{aligned} \tag{3.9}$$

Because of (3.7), $\lim_{\varepsilon\to 0} L(G_\varepsilon) = 0$, and, by the definition of H and H_ε (see (3.4)), $H_\varepsilon\subseteq H$ and $\lim_{\varepsilon\to 0} L(H\setminus H_\varepsilon) = 0$,

$$\int_0^1 f(s)\,\mathrm{d}s = \int_H f(s)\,\mathrm{d}s + \int_{[0,1]\setminus H} f(s)\,\mathrm{d}s = \int_H f(s)\,\mathrm{d}s.$$

From this and (3.9), for any $\gamma\in(0, 1)$,

$$\begin{aligned}\lim_{\varepsilon\to 0}\limsup_{T\to\infty}\frac{\ln\mathbb{P}(\xi_T(\cdot)\in\mathbb{U}_\varepsilon(f))}{\varphi(T)\ln\varphi(T)} &\leq -\frac{Q(1-\gamma)}{k}\int_H f(s)\,\mathrm{d}s + \lim_{\varepsilon\to 0}\limsup_{T\to\infty}\frac{\ln\mathbb{E}[\exp\{B_T+N_T\ln 2\}\mathbf{1}(\zeta_T\in\mathbb{U}_\varepsilon(f))]}{\varphi(T)\ln\varphi(T)}\\ &= -\frac{Q(1-\gamma)}{k}\int_0^1 f(s)\,\mathrm{d}s + \lim_{\varepsilon\to 0}\limsup_{T\to\infty}\frac{\ln\mathbb{E}[\exp\{B_T+N_T\ln 2\}\mathbf{1}(\zeta_T\in\mathbb{U}_\varepsilon(f))]}{\varphi(T)\ln\varphi(T)}.\end{aligned}$$

Passing to the limit $\gamma\to 0$ and using Lemma 2.2, we get, for $f\in\mathbb{V}$,

$$\lim_{\varepsilon\to 0}\limsup_{T\to\infty}\frac{\ln\mathbb{P}(\xi_T(\cdot)\in\mathbb{U}_\varepsilon(f))}{\varphi(T)\ln\varphi(T)} \leq -\frac{Q}{k}\int_0^1 f(s)\,\mathrm{d}s - (1-l)f_{\mathbb{D}}^{+}(1).$$

Because of the lower bound in (3.8), and using an argument similar to the one above, together with Lemma 2.3, we obtain, for $f\in\mathbb{V}$,

$$\lim_{\varepsilon\to 0}\liminf_{T\to\infty}\frac{\ln\mathbb{P}(\xi_T(\cdot)\in\mathbb{U}_\varepsilon(f))}{\varphi(T)\ln\varphi(T)} \geq -\frac{Q}{k}\int_0^1 f(s)\,\mathrm{d}s - (1-l)f_{\mathbb{D}}^{+}(1).$$

This completes the proof of $(\mathbb{V}, \mathbb{L}, \rho)$-LLDP in assertion (i).

Now, assume condition (1.6). Then the bound in (3.8), along with Lemmas 2.2 and 2.3, implies that

$$\lim_{\varepsilon\to 0}\limsup_{T\to\infty}\frac{\ln \mathbb{P}(\xi_T(\cdot)\in \mathbb{U}_\varepsilon(f))}{\varphi(T)\ln\varphi(T)}=\lim_{\varepsilon\to 0}\liminf_{T\to\infty}\frac{\ln \mathbb{P}(\xi_T(\cdot)\in \mathbb{U}_\varepsilon(f))}{\varphi(T)\ln\varphi(T)}=-(1-l)f^+_{\mathbb{D}}(1).$$

This completes the proof of $(\mathbb{V}, \mathbb{L}, \rho)$-LLDP in assertion (ii).

Furthermore, Lemma A.6 implies the ET property for the family $\xi_T(\cdot)$, $T>0$, and the fact that $I(f)=\infty$ for $f\in\mathbb{L}\setminus\mathbb{V}$ under any of the conditions in (1.5) or (1.6). As a result, we get an LDP under each of conditions (1.5) and (1.6). □

The proofs of Lemmas 2.2 and 2.3 are based on upper and lower bounds for the expected value $E := \mathbb{E}(\exp\{B_T+N_T(\zeta)\ln 2\}\mathbf{1}[\zeta_T\in\mathbb{U}_\varepsilon(f)])$.

Proof of Lemma 2.2. Given $a\in(0,\infty)$, let $\mathbb{V}_a$ be the set of functions $f\in\mathbb{V}$ with $0\le f(0)\le a$ and $\mathrm{Var} f\le a$. Next, given $C\in(0,\infty)$, define the set $\mathbb{K}_C := \mathbb{V}_{a(C)}$ with $a(C) := 3C/(1-l)$. According to Lemma A.1, $\mathbb{K}_C$ is compact in $(\mathbb{L}, \rho)$. We write $E\le E_1+E_2$, where

$$E_1 := \mathbb{E}\big(\exp\{B_T+N_T(\zeta)\ln 2\}\mathbf{1}\big[\zeta_T\in\mathbb{U}_\varepsilon(f)\cap\mathbb{K}_C\big]\big),$$
$$E_2 := \mathbb{E}\big(\exp\{B_T+N_T(\zeta)\ln 2\}\mathbf{1}\big[\zeta_T\in\mathbb{K}^{\mathrm{c}}_C\big]\big),$$

and $\mathbb{K}^{\mathrm{c}}_C$ represents the complement of set $\mathbb{K}_C$.

Let us upper-bound the term E_1. Obviously, process $\zeta(t)$ can be represented as $\zeta(t)=\zeta^+(t)-\zeta^-(t)$, where ζ^+ and ζ^- are independent Poisson processes of rate $\frac12$, with $\mathbb{E}(\zeta^+(t))=\mathbb{E}(\zeta^-(t))=t/2$. Note that if $\zeta_T\in\mathbb{K}_C$ then, by virtue of (1.2), for any $\gamma\in(0,1)$ and T large enough, we can upper-bound B_T as follows:

$$\begin{aligned}B_T&=\sum_{i=1}^{N_T(\zeta)}\ln\big(\nu(\zeta(t_{i-1}),\zeta(t_i))\big)\\&\le \zeta^-(T)\ln\big(Q\varphi(T)a(C)(1+\gamma)\big)+\zeta^+(T)\ln\big(P\varphi^l(T)a(C)(1+\gamma)\big)=: B^-_T+B^+_T.\end{aligned}\tag{3.10}$$

Recall that the processes ζ^- and ζ^+ are independent and non-decreasing. Also note that $N_T(\zeta)\le a(C)\varphi(T)$. Because of this, and due to representation in (3.10), Lemmas A.4 and A.5 imply that

$$\begin{aligned}E_1&\le \mathrm{e}^{a(C)\varphi(T)\ln 2}\mathbb{E}\Big(\mathrm{e}^{B_T}\mathbf{1}\Big[\zeta^-_T(1)\ge f^-_{\mathbb{D}}(1)-\delta(\varepsilon)\Big]\mathbf{1}\Big[\zeta^+_T(1)\ge f^+_{\mathbb{D}}(1)-\delta(\varepsilon)\Big]\Big)\\&\le \mathrm{e}^{a(C)\varphi(T)\ln 2}\mathbb{E}\Big(\mathrm{e}^{B^-_T}\mathbf{1}\Big[a(C)\ge\zeta^-_T(1)\ge f^-_{\mathbb{D}}(1)-\delta(\varepsilon)\Big]\Big)\mathbb{E}\Big(\mathrm{e}^{B^+_T}\mathbf{1}\Big[\zeta^+_T(1)\ge f^+_{\mathbb{D}}(1)-\delta(\varepsilon)\Big]\Big).\end{aligned}\tag{3.11}$$

Here, $\lim_{\varepsilon\to 0}\delta(\varepsilon)=0$, and $\zeta^+_T(t) := \zeta^+(tT)/\varphi(T)$, $\zeta^-_T(t) := \zeta^-(tT)/\varphi(T)$.

Observe that, as $\rho(f,0)>0$, for $\varepsilon>0$ small enough, $f^+_{\mathbb{D}}(1)-\delta(\varepsilon)>0$. By utilizing the definition of B^-_T in (3.10) and once more taking advantage of the boundedness of the total number of jumps $N_T(\zeta)\le a(C)\varphi(T)$, we can deduce, for sufficiently large values of T, that

$$\begin{aligned}&\mathbb{E}\Big(\mathrm{e}^{B^-_T}\mathbf{1}\Big[a(C)\ge\zeta^-_T(1)\ge f^-_{\mathbb{D}}(1)-\delta(\varepsilon)\Big]\Big)\\&\quad\le\sum_{k=0}^{\lfloor\varphi(T)a(C)\rfloor}\exp\{k\ln(Q\varphi(T)a(C)(1+\gamma))\}\frac{\mathrm{e}^{-T/2}(T/2)^k}{k!}\\&\quad\le\exp\{\lfloor\varphi(T)a(C)\rfloor\ln(Qa(C)(1+\gamma))\}\sum_{k=0}^{\lfloor\varphi(T)a(C)\rfloor}\mathrm{e}^{k\ln\varphi(T)}\frac{\mathrm{e}^{-T/2}(T/2)^k}{k!},\end{aligned}\tag{3.12}$$

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where $\lfloor b \rfloor$ denotes the integer part of b. To streamline the upcoming calculations, we write $g_1(T) := \lfloor \varphi(T)a(C) \rfloor \ln (Qa(C)(1+\gamma))$.

According to Lemma A.7, the terms in the last sum in (3.12) constitute an increasing sequence for T large enough (specifically, when $T > 2a(C)$), and their maximum value is attained in the final term:

$$\max_{0\le k\le \lfloor \varphi(T)a(C)\rfloor} \mathrm{e}^{k\ln \varphi(T)} \frac{\mathrm{e}^{-T/2}(T/2)^k}{k!} = \exp\{\lfloor \varphi(T)a(C)\rfloor \ln \varphi(T)\} \frac{\mathrm{e}^{-T/2}(T/2)^{\lfloor \varphi(T)a(C)\rfloor}}{\lfloor \varphi(T)a(C)\rfloor !}.$$

Therefore, continuing from (3.12), for a sufficiently large T we obtain

$$\mathbb{E}\Big(\mathrm{e}^{B_T^-}\mathbf{1}\Big[a(C) \ge \zeta_T^-(1) \ge f_{\mathbb{D}}^-(1) - \delta(\varepsilon)\Big]\Big) \le (\lfloor \varphi(T)a(C)\rfloor + 1) \frac{\exp\{\lfloor \varphi(T)a(C)\rfloor \ln \varphi(T) + g_1(T) + g_2(T)\}}{\lfloor \varphi(T)a(C)\rfloor !}, \tag{3.13}$$

where $g_2(T) := \lfloor \varphi(T)a(C)\rfloor \ln (T/2) - T/2$. Finally, the fact that $g_1(T) + g_2(T) = o(\varphi(T)\ln \varphi(T))$, together with the Stirling approximation, guarantee that the right-hand side of (3.13) is

$$(\lfloor \varphi(T)a(C)\rfloor + 1) \frac{\exp\{\lfloor \varphi(T)a(C)\rfloor \ln \varphi(T) + g_1(T) + g_2(T)\}}{\lfloor \varphi(T)a(C)\rfloor !} = \mathrm{e}^{o(\varphi(T)\ln \varphi(T))} \tag{3.14}$$

as $T \to \infty$.

By the definition of B_T^+ in (3.10), the expected value $\mathbb{E}\Big(\mathrm{e}^{B_T^+}\mathbf{1}\Big[\zeta_T^+(1) \ge f_{\mathbb{D}}^+(1) - \delta(\varepsilon)\Big]\Big)$ in (3.11) is bounded in the following manner. For T large enough,

$$\begin{aligned}
&\mathbb{E}\Big(\mathrm{e}^{B_T^+}\mathbf{1}\Big[\zeta_T^+(1) \ge f_{\mathbb{D}}^+(1) - \delta(\varepsilon)\Big]\Big) \\
&\le \sum_{k=\lfloor \varphi(T)(f_{\mathbb{D}}^+(1)-\delta(\varepsilon))\rfloor}^{\infty} \exp\left\{k\ln\left(P\varphi^l(T)a(C)(1+\gamma)\right)\right\} \frac{\mathrm{e}^{-T/2}(T/2)^k}{k!} \\
&\le \sum_{k=\lfloor \varphi(T)(f_{\mathbb{D}}^+(1)-\delta(\varepsilon))\rfloor}^{\infty} \exp\left\{k\ln\left(P\varphi^l(T)a(C)(1+\gamma)\right) - k\ln k + k\ln(\mathrm{e}T/2)\right\} \\
&\le \sum_{k=\lfloor \varphi(T)(f_{\mathbb{D}}^+(1)-\delta(\varepsilon))\rfloor}^{\infty} \exp\{lk\ln \varphi(T) - k\ln k + 2k\ln(\mathrm{e}T/2)\} \\
&\le \sum_{k=\lfloor \varphi(T)(f_{\mathbb{D}}^+(1)-\delta(\varepsilon))\rfloor}^{\infty} \exp\left\{lk\ln \varphi(T) - k\ln(\lfloor \varphi(T)(f_{\mathbb{D}}^+(1)-\delta(\varepsilon))\rfloor) + 2k\ln(\mathrm{e}T/2)\right\} \\
&\le \sum_{k=\lfloor \varphi(T)(f_{\mathbb{D}}^+(1)-\delta(\varepsilon))\rfloor}^{\infty} \exp\{-(1-l)k\ln \varphi(T) + 3k\ln(\mathrm{e}T/2)\} \\
&= \sum_{k=\lfloor \varphi(T)(f_{\mathbb{D}}^+(1)-\delta(\varepsilon))\rfloor}^{\infty} \exp\{-k\big((1-l)\ln \varphi(T) - 3\ln(\mathrm{e}T/2)\big)\} \\
&= \frac{\exp\left\{-\lfloor \varphi(T)(f_{\mathbb{D}}^+(1)-\delta(\varepsilon))\rfloor((1-l)\ln \varphi(T) - 3\ln(\mathrm{e}T/2))\right\}}{1-\exp\{-(1-l)\ln \varphi(T) + 3\ln(\mathrm{e}T/2)\}}.
\end{aligned} \tag{3.15}$$

From the bounds in (3.11), (3.14), and (3.15) we get that, for T large enough,

$$E_1 \leq \exp\left\{-(1-l)\big(f^+_{\mathbb{D}}(1) - \delta(\varepsilon)\big)\varphi(T)\ln\varphi(T) + o(\varphi(T)\ln\varphi(T))\right\}. \tag{3.16}$$

Further, Lemma A.6 implies that, for T large enough,

$$E_2 \leq \exp\{-C\varphi(T)\ln\varphi(T) + o(\varphi(T)\ln\varphi(T))\}. \tag{3.17}$$

Choosing $C > (1-l)\big(f^+_{\mathbb{D}}(1) - \delta(\varepsilon)\big)$ and using the inequalities $E \leq E_1 + E_2$, (3.16), and (3.17), we obtain

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\limsup_{T\to\infty}\frac{\ln E}{\varphi(T)\ln\varphi(T)} \\
&\qquad \leq \lim_{\varepsilon\to 0}\limsup_{T\to\infty}\frac{\ln\left(2\exp\{-(1-l)\big(f^+_{\mathbb{D}}(1) - \delta(\varepsilon)\big)\varphi(T)\ln\varphi(T) + o(\varphi(T)\ln\varphi(T))\}\right)}{\varphi(T)\ln\varphi(T)} \\
&\qquad \leq -\lim_{\varepsilon\to 0}(1-l)\big(f^+_{\mathbb{D}}(1) - \delta(\varepsilon)\big) = -(1-l)f^+_{\mathbb{D}}(1).
\end{aligned}$$

□

Proof of Lemma 2.3. Here the goal is to establish a lower bound for E. As usual, obtaining lower bounds is a more difficult task. Let us outline the idea of the proof. The main step is to extract from the event $\{\zeta_T \in \mathbb{U}_\varepsilon(f)\}$ a smaller event:

$$\{\zeta_T \in \mathbb{U}_\varepsilon(f)\} \supset \left\{\int_0^\Delta |\zeta_T(t) - \widetilde{g}_\varepsilon(t)|\,\mathrm{d}t < \varepsilon/4,\ \sup_{t\in[\Delta,1]}|\zeta_T(t) - \widetilde{g}_\varepsilon(t)| < \varepsilon/8\right\}.$$

Here, Δ is a constant that depends on ε, and $\widetilde{g}_\varepsilon$ is a continuous function such that:

- $\widetilde{g}_\varepsilon$ is close to f in the ρ metric (in the proof, $\rho(f, \widetilde{g}_\varepsilon) < 3\varepsilon/4$);
- the variation $\operatorname{Var}\widetilde{g}_\varepsilon$ is close to the variation of f;
- $\widetilde{g}_\varepsilon$ is equal to a small constant δ on the interval $[0, \Delta]$ (in the proof, $\delta = \varepsilon/4$) and $\widetilde{g}_\varepsilon$ is greater than δ on $[\Delta, 1]$.

Then the expected value E will be lower-bounded by a product E_+E_- (see (3.24)), where E_+ (respectively, E_-) controls the variations of ζ^+ and $\widetilde{g}^+_\varepsilon$ (respectively, ζ^- and $\widetilde{g}^-_\varepsilon$). Finally, the quantity E_+ will give us the bound (3.27) claimed in the lemma.

Let us proceed with the formal proof. First, consider the case where $\operatorname{Var} f_{\mathbb{D}} > 0$. We start by proving the existence of the function $\widetilde{g}_\varepsilon$. We construct $\widetilde{g}_\varepsilon$ by using an auxiliary function g (see below). From the point of view of future arguments, it is convenient to set $2\Delta := \sup\{t\colon \operatorname{Var} f_{\mathbb{D}} \leq \varepsilon/2\}$. Observe that since $\operatorname{Var} f_{\mathbb{D}} > 0$, we have $2\Delta < 1$ for $\varepsilon > 0$ small enough and, as $f_{\mathbb{D}}$ is right-continuous at 0, we also have that $2\Delta > 0$.

Define

$$g(t) := \begin{cases} \dfrac{\varepsilon}{4} \vee f(0), & t \in [0, 2\Delta), \\ f(t) + \dfrac{\varepsilon}{2}, & t \in [2\Delta, 1], \end{cases}$$

Recall that $a \vee b = \max(a, b)$. It is easy to see that $\rho(f, g) \leq \varepsilon/2$. Note that function g is convenient because it does not vanish on $[0, 1]$.

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Let us decompose $g_{\mathbb{D}}$ into an increasing and a decreasing component:

$$\begin{aligned} g_{\mathbb{D}}(t) &= g_{\mathbb{D}}^+(t) - g_{\mathbb{D}}^-(t), && \operatorname{Var} g_{\mathbb{D}} = \operatorname{Var} g_{\mathbb{D}}^+ + \operatorname{Var} g_{\mathbb{D}}^-, \\ g_{\mathbb{D}}^+(t) &= \frac{\varepsilon}{4} \vee f(0) && \text{for } t \in [0, 2\Delta), \\ g_{\mathbb{D}}^-(t) &= 0, && \text{for } t \in [0, 2\Delta). \end{aligned}$$

From the definition of the constant 2Δ it follows that

$$0 \le \operatorname{Var} f_{\mathbb{D}}^+ - \operatorname{Var} g_{\mathbb{D}}^+ \le \frac{\varepsilon}{2}. \tag{3.18}$$

The functions $g_{\mathbb{D}}^+$ and $g_{\mathbb{D}}^-$ are monotone and continuous on $[0, 2\Delta)$ and left-continuous at the end point 1. Also,

$$\inf_{t\in[0,2\Delta)} \big(g_{\mathbb{D}}^+(t) - g_{\mathbb{D}}^-(t)\big) = \frac{\varepsilon}{4} \vee f(0), \qquad \inf_{t\in[2\Delta,1]} \big(g_{\mathbb{D}}^+(t) - g_{\mathbb{D}}^-(t)\big) \ge \frac{\varepsilon}{2}.$$

Hence, there exist monotone continuous functions $\widetilde{g}_\varepsilon^+$ and $\widetilde{g}_\varepsilon^-$ such that

$$\begin{aligned} &\widetilde{g}_\varepsilon^+(t) = g_{\mathbb{D}}^+(t) = \frac{\varepsilon}{4} \vee f(0), \quad \widetilde{g}_\varepsilon^-(t) = g_{\mathbb{D}}^-(t) = 0 && \text{if } t \in [0, \Delta], \\ &\widetilde{g}_\varepsilon^+(t) \ge g_{\mathbb{D}}^+(t), \quad \widetilde{g}_\varepsilon^-(t) \le g_{\mathbb{D}}^-(t) && \text{if } t \in (\Delta, 1), \end{aligned}$$

$\widetilde{g}_\varepsilon^+(1) = g_{\mathbb{D}}^+(1)$, $\widetilde{g}_\varepsilon^-(1) = g_{\mathbb{D}}^-(1)$, and $\rho\big(\widetilde{g}_\varepsilon^+, g_{\mathbb{D}}^+\big) + \rho\big(\widetilde{g}_\varepsilon^-, g_{\mathbb{D}}^-\big) < \varepsilon/4$. Let $\widetilde{g}_\varepsilon := \widetilde{g}_\varepsilon^+ - \widetilde{g}_\varepsilon^-$. Then $\rho(\widetilde{g}_\varepsilon, g) < \varepsilon/4$ and

$$\inf_{t\in[0,1]} \widetilde{g}_\varepsilon(t) = \inf_{t\in[0,1]} \big(\widetilde{g}_\varepsilon^+(t) - \widetilde{g}_\varepsilon^-(t)\big) \ge \inf_{t\in[0,1]} \big(g_{\mathbb{D}}^+(t) - g_{\mathbb{D}}^-(t)\big) \ge \frac{\varepsilon}{4}.$$

Using the decomposition $\zeta(t) = \zeta^+(t) - \zeta^-(t)$, we have, for T large enough,

$$\begin{aligned} E &:= \mathbb{E}\big(\exp\{B_T + N_T(\zeta)\ln 2\}\mathbf{1}[\zeta_T \in \mathbb{U}_\varepsilon(f)]\big) \\ &\ge \mathbb{E}\bigg(e^{B_T}\mathbf{1}\bigg[\sup_{t\in[0,\Delta)} \zeta_T(t) < \widetilde{g}_\varepsilon(\Delta), \sup_{t\in[\Delta,1]} |\zeta_T(t) - \widetilde{g}_\varepsilon(t)| < \frac{\varepsilon}{8}\bigg]\bigg) \\ &\ge \mathbb{E}\bigg(e^{B_T}\mathbf{1}\bigg[\zeta^-(\Delta T) = 0, \zeta^+(\Delta T) = \lfloor\varphi(T)\widetilde{g}_\varepsilon(\Delta)\rfloor, \sup_{t\in[\Delta,1]} |\zeta_T(t) - \widetilde{g}_\varepsilon(t)| < \frac{\varepsilon}{8}\bigg]\bigg). \end{aligned} \tag{3.19}$$

Write

$$\begin{aligned} \mathbb{W}_1 &:= \bigg\{\sup_{t\in[\Delta,1]} \big|\zeta_T^-(t) - \widetilde{g}_\varepsilon^-(t)\big| < \frac{\varepsilon}{16}\bigg\}, \\ \mathbb{S}_1 &:= \{\zeta^-(\Delta T) = 0\}, \\ \mathbb{W}_2 &:= \bigg\{\sup_{t\in[\Delta,1]} \big|\zeta_T^+(t) - \zeta_T^+(\Delta) - \big(\widetilde{g}_\varepsilon^+(t) - \widetilde{g}_\varepsilon^+(\Delta)\big)\big| < \frac{\varepsilon}{16}\bigg\}, \\ \mathbb{S}_2 &:= \big\{\zeta^+(\Delta T) = \big\lfloor\varphi(T)\widetilde{g}_\varepsilon(\Delta)\big\rfloor\big\}. \end{aligned}$$

From the bound in (3.19) it follows that

$$E \ge \mathbb{E}\big[e^{B_T}\mathbf{1}(\mathbb{W}_1 \cap \mathbb{W}_2 \cap \mathbb{S}_1 \cap \mathbb{S}_2)\big]. \tag{3.20}$$

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Let us first estimate B_T from below. According to the definition of ν in Lemma 2.1, the sum B_T can always be separated into two sums: the one over negative jumps, $\sum_{(-)}$, and the one over positive jumps, $\sum_{(+)}$:

$$
\begin{aligned}
B_T &= \sum_{i=1}^{N_T(\zeta)} \ln\left(\nu(\zeta(t_{i-1}), \zeta(t_i)\right) \\
&= \sum_{i=1}^{N_T(\zeta)} \ln\left(\nu(\zeta(t_{i-1}), \zeta(t_i)\right)\mathbf{1}[\zeta(t_{i-1}) > \zeta(t_i)] + \sum_{i=1}^{N_T(\zeta)} \ln\left(\nu(\zeta(t_{i-1}), \zeta(t_i)\right)\mathbf{1}[\zeta(t_{i-1}) < \zeta(t_i)] \\
&=: \sum_{(-)} + \sum_{(+)}.
\end{aligned}
$$

The lower bound for $\sum_{(-)}$ is constructed in the following way. Note that, according to $\mathbb{S}_1$, there are no negative jumps of the process ζ during the time interval $[0, T\Delta]$ and, according to $\mathbb{W}_1 \cap \mathbb{W}_2$, the process belongs to an $\varepsilon/8$ neighborhood (in the uniform metric) of the function $\widetilde{g}_\varepsilon$. Then, due to (1.2), for any $\gamma \in (0, 1)$ and T sufficiently large we obtain

$$
\sum_{(-)} \geq \zeta^-(T) \ln\left(\frac{\varepsilon Q\varphi(T)(1-\gamma)}{8}\right). \tag{3.21}
$$

Let us bound sum $\sum_{(+)}$ from below. Note that, for each $r > 1$, any trajectory from the event $\mathbb{S}_1 \cap \mathbb{S}_2 \cap \mathbb{W}_2$ has $\zeta^+(T) - \lfloor \varphi(T)\widetilde{g}_\varepsilon(\Delta)/r \rfloor$ positive jumps, when the trajectory is not lower than $\varepsilon/8r$. Thus, on these jumps, for any $\gamma \in (0, 1)$ and T sufficiently large, $\nu(\zeta(t_{i-1}), \zeta(t_i)) \geq \varepsilon P\varphi^l(T)(1-\gamma)/8r$.

On the remaining $\lfloor \varphi(T)\widetilde{g}_\varepsilon(\Delta)/r \rfloor$ positive jumps, $\nu(\zeta(t_{i-1}), \zeta(t_i)) \geq \lambda_{\min}$; here, $\lambda_{\min} := \min_{x\in\mathbb{Z}^+} \lambda(x)$. Finally,

$$
\sum_{(+)} \geq \left(\zeta^+(T) - \left\lfloor \frac{\varphi(T)\widetilde{g}_\varepsilon(\Delta)}{r} \right\rfloor\right) \ln\left(\frac{\varepsilon P\varphi^l(T)(1-\gamma)}{8r}\right) + \left\lfloor \frac{\varphi(T)\widetilde{g}_\varepsilon(\Delta)}{r} \right\rfloor \ln \lambda_{\min}. \tag{3.22}
$$

The new parameter r introduced will further tend to infinity, $r \to \infty$.

Thus, on the event $\mathbb{W}_1 \cap \mathbb{W}_2 \cap \mathbb{S}_1 \cap \mathbb{S}_2$, due to (3.21) and (3.22), we obtain that, for any $\gamma \in (0, 1)$, $r > 1$, and T large enough,

$$
\begin{aligned}
B_T &= \sum_{i=1}^{N_T(\zeta)} \ln\left(\nu(\zeta(t_{i-1}), \zeta(t_i))\right) \\
&\geq \zeta^-(T) \ln\left(\frac{\varepsilon Q\varphi(T)(1-\gamma)}{8}\right) + \left\lfloor \frac{\varphi(T)\widetilde{g}_\varepsilon(\Delta)}{r} \right\rfloor \ln \lambda_{\min} \\
&\quad + \left(\zeta^+(T) - \left\lfloor \frac{\varphi(T)\widetilde{g}_\varepsilon(\Delta)}{r} \right\rfloor\right) \ln\left(\frac{\varepsilon P\varphi^l(T)(1-\gamma)}{8r}\right) \\
&\geq \zeta^-(T) \ln\left(J\varphi(T)\right) + \left\lfloor \frac{\varphi(T)\widetilde{g}_\varepsilon(\Delta)}{r} \right\rfloor \ln \lambda_{\min} + \left(\zeta^+(T) - \left\lfloor \frac{\varphi(T)\widetilde{g}_\varepsilon(\Delta)}{r} \right\rfloor\right) \ln\left(J\varphi^l(T)\right),
\end{aligned} \tag{3.23}
$$

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where $J := \min(\varepsilon P(1-\gamma)/8r, \varepsilon Q(1-\gamma)/8)$. From (3.20), (3.23), and the independence of processes ζ^+ and ζ^-, we get

$$E \geq \mathbb{E}\bigg(\exp\bigg\{\bigg(\zeta^+(T) - \bigg\lfloor \frac{\varphi(T)\widetilde{g}_\varepsilon(\Delta)}{r} \bigg\rfloor\bigg) \ln(J\varphi^l(T)) + \bigg\lfloor \frac{\varphi(T)\widetilde{g}_\varepsilon(\Delta)}{r} \bigg\rfloor \ln \lambda_{\min}\bigg\}\mathbf{1}(\mathbb{S}_2 \cap \mathbb{W}_2)\bigg) \\ \times \mathbb{E}(\exp\{\zeta^-(T)\ln(J\varphi(T))\}\mathbf{1}(\mathbb{S}_1 \cap \mathbb{W}_1)) =: E_+E_-. \tag{3.24}$$

Let us lower-bound the value E_+. Consider a partition $\Delta = t_0 < t_1 < \cdots < t_m = 1$ such that $\max_{i=1,\ldots,m}\big(\widetilde{g}^+_\varepsilon(t_i) - \widetilde{g}^+_\varepsilon(t_{i-1})\big) \leq \varepsilon/32$ and $\min_{i=1,\ldots,m}\big(\widetilde{g}^+_\varepsilon(t_i) - \widetilde{g}^+_\varepsilon(t_{i-1})\big) > 0$. By virtue of the independence of increments in process ζ^+, for T large enough,

$$\begin{aligned}
E_+ &\geq \mathbb{E}\bigg(\exp\bigg\{\bigg(\zeta^+(T) - \bigg\lfloor \frac{\varphi(T)\widetilde{g}_\varepsilon(\Delta)}{r} \bigg\rfloor\bigg) \ln(J\varphi^l(T)) + \bigg\lfloor \frac{\varphi(T)\widetilde{g}_\varepsilon(\Delta)}{r} \bigg\rfloor \ln(\lambda_{\min})\bigg\} \\
&\qquad \times \mathbf{1}[\zeta^+(\Delta T) = \lfloor \varphi(T)\widetilde{g}_\varepsilon(\Delta)\rfloor] \\
&\qquad \times \prod_{i=1}^m \mathbf{1}[\zeta^+(Tt_i) - \zeta^+(Tt_{i-1}) = \big\lfloor\big(\widetilde{g}^+_\varepsilon(t_i) - \widetilde{g}^+_\varepsilon(t_{i-1})\big)\varphi(T)\big\rfloor]\bigg) \\
&\geq \mathbb{E}\bigg(\exp\bigg\{\lfloor \varphi(T)\widetilde{g}^+_\varepsilon(1)\rfloor \ln(J\varphi^l(T)) - 2\bigg\lfloor \frac{\varphi(T)\widetilde{g}_\varepsilon(\Delta)}{r} \bigg\rfloor \ln(J\varphi^l(T))\bigg\} \\
&\qquad \times \mathbf{1}[\zeta^+(\Delta T) = \lfloor \varphi(T)\widetilde{g}_\varepsilon(\Delta)\rfloor] \\
&\qquad \times \prod_{i=1}^m \mathbf{1}\big[\zeta^+(Tt_i) - \zeta^+(Tt_{i-1}) = \big\lfloor\big(\widetilde{g}^+_\varepsilon(t_i) - \widetilde{g}^+_\varepsilon(t_{i-1})\big)\varphi(T)\big\rfloor\big]\bigg) \\
&= \exp\bigg\{\lfloor \varphi(T)\widetilde{g}^+_\varepsilon(1)\rfloor \ln(J\varphi^l(T)) - 2\bigg\lfloor \frac{\varphi(T)\widetilde{g}_\varepsilon(\Delta)}{r} \bigg\rfloor \ln(J\varphi^l(T))\bigg\} \\
&\quad \times \mathbb{P}(\zeta^+(\Delta T) = \lfloor \varphi(T)\widetilde{g}_\varepsilon(\Delta)\rfloor) \\
&\quad \times \prod_{i=1}^m \mathbb{P}\big(\zeta^+(Tt_i) - \zeta^+(Tt_{i-1}) = \big\lfloor\big(\widetilde{g}^+_\varepsilon(t_i) - \widetilde{g}^+_\varepsilon(t_{i-1})\big)\varphi(T)\big\rfloor\big).
\end{aligned} \tag{3.25}$$

With the help of the Stirling formula, we get, for T large enough,

$$\begin{aligned}
&\mathbb{P}(\zeta^+(\Delta T) = \lfloor \varphi(T)\widetilde{g}_\varepsilon(\Delta)\rfloor)\prod_{i=1}^m \mathbb{P}\big(\zeta^+(Tt_i) - \zeta^+(Tt_{i-1}) = \big\lfloor\big(\widetilde{g}^+_\varepsilon(t_i) - \widetilde{g}^+_\varepsilon(t_{i-1})\big)\varphi(T)\big\rfloor\big) \\
&= \frac{\mathrm{e}^{-T\Delta/2}(T\Delta/2)^{\lfloor \widetilde{g}_\varepsilon(\Delta)\varphi(T)\rfloor}}{\lfloor \widetilde{g}_\varepsilon(\Delta)\varphi(T)\rfloor!} \prod_{i=1}^m \frac{\mathrm{e}^{-T(t_i - t_{i-1})/2}(T(t_i - t_{i-1})/2)^{\lfloor(\widetilde{g}^+_\varepsilon(t_i) - \widetilde{g}^+_\varepsilon(t_{i-1}))\varphi(T)\rfloor}}{\big\lfloor\big(\widetilde{g}^+_\varepsilon(t_i) - \widetilde{g}^+_\varepsilon(t_{i-1})\big)\varphi(T)\big\rfloor!} \\
&\geq \prod_{i=1}^m \exp\bigg\{-\frac{T(t_i - t_{i-1})}{2} - \big(\widetilde{g}^+_\varepsilon(t_i) - \widetilde{g}^+_\varepsilon(t_{i-1})\big)\varphi(T)\ln\big(\big(\widetilde{g}^+_\varepsilon(t_i) - \widetilde{g}^+_\varepsilon(t_{i-1})\big)\varphi(T)\big)\bigg\} \\
&\quad \times \exp\bigg\{-\frac{T\Delta}{2} - \widetilde{g}_\varepsilon(\Delta)\varphi(T)\ln(\widetilde{g}_\varepsilon(\Delta)\varphi(T))\bigg\}
\end{aligned}$$

$$\geq \prod_{i=1}^{m} \exp\left\{-\frac{T(t_i - t_{i-1})}{2} - \left(\widetilde{g}_\varepsilon^+(t_i) - \widetilde{g}_\varepsilon^+(t_{i-1})\right)\varphi(T) \ln\left(\widetilde{g}_\varepsilon^+(1)\varphi(T)\right)\right\}$$
$$\times \exp\left\{-\frac{T\Delta}{2} - \widetilde{g}_\varepsilon^+(\Delta)\varphi(T) \ln\left(\widetilde{g}_\varepsilon^+(1)\varphi(T)\right)\right\}$$
$$\geq \exp\left\{-T - \widetilde{g}_\varepsilon^+(1)\varphi(T) \ln\left(\widetilde{g}_\varepsilon^+(1)\varphi(T)\right)\right\}. \tag{3.26}$$

From the bounds in (3.25) and (3.26), it follows that, for T large enough,

$$E_+ \geq \exp\bigg\{-\widetilde{g}_\varepsilon^+(1)\varphi(T)(1-l)\ln\varphi(T) - T - \varphi(T)\widetilde{g}_\varepsilon^+(1)|\ln(J)| - 3\left\lfloor\frac{\varphi(T)\widetilde{g}_\varepsilon(\Delta)}{r}\right\rfloor \ln\left(J\varphi^l(T)\right)\bigg\}. \tag{3.27}$$

Next, let us lower-bound the quantity E_-. If $\widetilde{g}_\varepsilon^-(1) = 0$ then

$$E_- \geq \mathbb{P}(\zeta^-(T) = 0) = \mathrm{e}^{-T/2}. \tag{3.28}$$

If $\widetilde{g}_\varepsilon^-(1) > 0$ then we consider the partition $\Delta = t_0 < t_1 < \cdots < t_m = 1$ such that $\max_{i=1,\ldots,m}(\widetilde{g}_\varepsilon^-(t_i) - \widetilde{g}_\varepsilon^-(t_{i-1})) \leq \varepsilon/32$ and $\min_{i=1,\ldots,m}(\widetilde{g}_\varepsilon^-(t_i) - \widetilde{g}_\varepsilon^-(t_{i-1})) > 0$. By using the independence of increments in process ζ^- and the Stirling formula, for T large enough,

$$E_- \geq \mathbb{E}\bigg(\mathrm{e}^{\zeta^-(T)\ln(J\varphi(T))}\mathbf{1}(\zeta^-(\Delta T) = 0)$$
$$\times \prod_{i=1}^{m} \mathbf{1}\left[\zeta^-(Tt_i) - \zeta^-(Tt_{i-1}) = \left\lfloor(\widetilde{g}_\varepsilon^-(t_i) - \widetilde{g}_\varepsilon^-(t_{i-1}))\varphi(T)\right\rfloor\right]\bigg)$$
$$\geq \exp\{\varphi(T)\widetilde{g}_\varepsilon^-(1)\ln(J\varphi(T)) - m\ln(J\varphi(T))\}\mathbb{P}(\zeta^-(\Delta T) = 0)$$
$$\times \prod_{i=1}^{m} \mathbb{P}\left(\zeta^-(Tt_i) - \zeta^-(Tt_{i-1}) = \lfloor(\widetilde{g}_\varepsilon^-(t_i) - \widetilde{g}_\varepsilon^-(t_{i-1}))\varphi(T)\rfloor\right)$$
$$\geq \exp\left\{\varphi(T)\widetilde{g}_\varepsilon^-(1)\ln(J\varphi(T)) - m\ln(J\varphi(T))\right\}$$
$$\times \exp\left\{-T - \widetilde{g}_\varepsilon^-(1)\varphi(T)\ln\left(\widetilde{g}_\varepsilon^-(1)\varphi(T)\right)\right\}$$
$$= \exp\left\{-T - \widetilde{g}_\varepsilon^-(1)\varphi(T)\ln\widetilde{g}_\varepsilon^-(1) + \widetilde{g}_\varepsilon^-(1)\varphi(T)\ln J - m\ln(J\varphi(T))\right\}. \tag{3.29}$$

From (3.20) and (3.27)–(3.29) we obtain that, for any $r > 1$ and $\varepsilon > 0$ small enough,

$$\liminf_{T\to\infty} \frac{\ln E}{\varphi(T)\ln\varphi(T)} \geq -\widetilde{g}_\varepsilon^+(1)(1-l) - \frac{3l\widetilde{g}_\varepsilon(\Delta)}{r}.$$

Passing to the limit $r \to \infty$ yields

$$\liminf_{T\to\infty} \frac{\ln E}{\varphi(T)\ln\varphi(T)} \geq -\widetilde{g}_\varepsilon^+(1)(1-l).$$

By definition, $\widetilde{g}_\varepsilon^+(1) = g_{\mathbb{D}}^+(1)$. Also, by virtue of (3.18), $|g_{\mathbb{D}}^+(1) - f_{\mathbb{D}}^+(1)| \leq \varepsilon/2$. This gives

$$\lim_{\varepsilon\to 0}\liminf_{T\to\infty} \frac{\ln E}{\varphi(T)\ln\varphi(T)} \geq -(1-l)f_{\mathbb{D}}^+(1).$$

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Therefore, Lemma 2.3 has been proven when $\operatorname{Var} f_{\mathbb{D}} > 0$.

In the case where $\operatorname{Var} f_{\mathbb{D}} = 0$, we have $f_{\mathbb{D}}(t) = f(0) > 0$, $t \in [0, 1]$. It is easy to see that $\{\zeta_T \in \mathbb{U}_\varepsilon(f)\} \supseteq \mathbb{O}$, where

$$\mathbb{O} := \left\{\zeta^+\left(\frac{\varepsilon}{2f(0)}T\right) = \lfloor f(0)\varphi(T)\rfloor,\ \zeta^+(T) - \zeta^+\left(\frac{\varepsilon}{2f(0)}T\right) = 0,\ \zeta^-(T) = 0\right\}.$$

The rest of the proof is reduced to a lower bound for B_T on event $\mathbb{O}$, which essentially repeats the above argument. For brevity, we omit it here. □

4. Discussion

It is instructive to discuss Theorems 2.1 and 2.2 in connection with the question mentioned in Section 1: why under condition (1.4) do we get only an LLDP whereas (1.5) or (1.6) lead to an LDP? Consider an example where $\lambda(x) = P$, $\mu(x) = Qx$. In this case we can write down a probability distribution for process ξ at the time point T explicitly [8]:

$$\mathbb{P}(\xi(T) = x) = \frac{(a(T))^x}{x!} \mathrm{e}^{-a(T)}, \qquad x \in \mathbb{Z}^+,$$

where $a(T) = (P/Q)\big(1 - \mathrm{e}^{-QT}\big)$. Following on from this, if $f(1) > 0$ then

$$\lim_{\varepsilon\to 0}\lim_{T\to\infty} \frac{1}{\varphi(T)\ln\varphi(T)} \ln \mathbb{P}(\xi_T(1) \in [f(1) - \varepsilon, f(1) + \varepsilon]) = -f(1)$$

under any of conditions (1.4)–(1.6).

Consequently, under condition (1.4) the normalizing function $\psi(T) = \varphi(T)\ln\varphi(T)$ in the LDP on the state space $\mathbb{Z}^+$ is different from the function $\psi(T) = T\varphi(T)$ figuring in the LLDP on the functional space $\mathbb{L}$. In other words, for any càdlàg function $f \not\equiv 0$ we have

$$\lim_{\varepsilon\to 0}\lim_{T\to\infty} \frac{1}{\varphi(T)\ln\varphi(T)} \ln \frac{\mathbb{P}(\xi_T \in \mathbb{U}_\varepsilon(f))}{\mathbb{P}(\xi_T(1) \in [f(1) - \varepsilon, f(1) + \varepsilon])} = -\infty.$$

This is why under condition (1.4) the family of processes $\xi_T(\cdot)$, $T > 0$, lacks the ET property in any reasonable functional space.

However, under condition (1.5) or (1.6) the normalizing functions coincide, and we manage to get an LDP in the functional space $(\mathbb{L}, \rho)$ as stated in Theorem 2.2.

Also, note that Theorem 2.2 allows us to get a rough asymptotic for the probability that a trajectory of ξ_T crosses a level $a > 0$. Indeed, with the help of (2.2) and an argument similar to the one used in the proof of (3.15) we have that, under any of conditions (1.5) and (1.6),

$$\begin{aligned}
&\limsup_{T\to\infty} \frac{1}{\varphi(T)\ln\varphi(T)} \ln \mathbb{P}\left(\sup_{t\in[0,1]} \xi_T(t) \ge a\right) \\
&\le \limsup_{T\to\infty} \frac{1}{\varphi(T)\ln\varphi(T)} \ln\left(\mathrm{e}^T \mathbb{E}\left(\mathrm{e}^{-A_T(\zeta)} \exp\{B_T(\zeta) + N_T(\zeta)\ln 2\}\mathbf{1}\left(\sup_{t\in[0,1]} \zeta_T(t) \ge a\right)\right)\right) \\
&\le \limsup_{T\to\infty} \frac{1}{\varphi(T)\ln\varphi(T)} \ln \mathbb{E}(\exp\{B_T(\zeta) + N_T(\zeta)\ln 2\}\mathbf{1}(\zeta_T^+(1) \ge a)) \le -(1 - l)a.
\end{aligned}$$

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Since process ξ_T is càdlàg, and the set $\{f \in \mathbb{L}\colon \text{ess sup}_{t\in[0,1]} f(t) > a\}$ is open, Theorem 2.2 implies that, under (1.5) or (1.6),

$$\begin{aligned}\liminf_{T\to\infty} \frac{1}{\varphi(T)\ln\varphi(T)} \ln \mathbb{P}\Big(\sup_{t\in[0,1]} \xi_T(t) \ge a\Big) &\ge \liminf_{T\to\infty} \frac{1}{\varphi(T)\ln\varphi(T)} \ln \mathbb{P}\Big(\sup_{t\in[0,1]} \xi_T(t) > a\Big)\\ &= \liminf_{T\to\infty} \frac{1}{\varphi(T)\ln\varphi(T)} \ln \mathbb{P}\Big(\text{ess}\sup_{t\in[0,1]} \xi_T(t) > a\Big)\\ &\ge -\inf_{f\colon \text{ess sup}_{t\in[0,1]} f(t)>a} I(f)\\ &= -\inf_{f\colon \text{ess sup}_{t\in[0,1]} f(t)>a} (1-l) f^+_{\mathbb{D}}(1) = -(1-l)a.\end{aligned}$$

Thus,

$$\lim_{T\to\infty} \frac{1}{\varphi(T)\ln\varphi(T)} \ln \mathbb{P}\Big(\sup_{t\in[0,1]} \xi_T(t) \ge a\Big) = -(1-l)a.$$

Appendix A. Auxiliary results

In this section we establish some auxiliary assertions.

Lemma A.1. *For any fixed $C > 0$ the set $\mathbb{V}_C$ is compact in $(\mathbb{L}, \rho)$.*

Proof. The Helly theorem [18] implies that from every sequence $f_n \in \mathbb{V}_C$ we can extract a subsequence f_{n_k} convergent as $k \to \infty$ almost surely to some $f \in \mathbb{V}_C$. Applying the Lebesgue dominated convergence theorem yields

$$\lim_{k\to\infty} \int_0^1 |f_{n_k}(t) - f(t)|\,\mathrm{d}t = \int_0^1 \lim_{k\to\infty} |f_{n_k}(t) - f(t)|\,\mathrm{d}t = 0. \qquad \square$$

Let $\mathbb{M} = \mathbb{M}[0, 1]$ denote the set of non-decreasing functions on [0, 1].

Lemma A.2. *Suppose that the function $f \in \mathbb{V}$ is represented as $f(t) = g_1(t) - g_2(t)$, where $g_1, g_2 \in \mathbb{M}$. Then, for any $0 \le t_1 < t_2 \le 1$, $g_1(t_2) - g_1(t_1) \ge f^+(t_2) - f^+(t_1)$.*

Proof. Assume the opposite; then there exist $0 \le t_1 < t_2 \le 1$ such that $g_1(t_2) - g_1(t_1) < f^+(t_2) - f^+(t_1)$. Observe that we will then also have $g_2(t_2) - g_2(t_1) < f^-(t_2) - f^-(t_1)$.

Let $\text{Var}_{[t_1,t_2]}$ stand for the variation on $[t_1, t_2]$. Since the variation of a sum does not exceed the sum of the variations, we obtain

$$\text{Var}_{[t_1,t_2]} g_1 + \text{Var}_{[t_1,t_2]} g_2 = g_1(t_2) - g_1(t_1) + g_2(t_2) - g_2(t_1) \ge \text{Var}_{[t_1,t_2]} f.$$

On the other hand,

$$g_1(t_2) - g_1(t_1) + g_2(t_2) - g_2(t_1) < f^+(t_2) - f^+(t_1) + f^-(t_2) - f^-(t_1) = \text{Var}_{[t_1,t_2]} f.$$

The contradiction completes the proof. $\square$

Let $\mathbb{K}$ be a compact set in $(\mathbb{L}, \rho)$. Consider a family of functions $u_T(t)$, $t \in [0, 1]$, $T > 0$, such that $u_T(t) := u_T^+(t) - u_T^-(t)$, where $u_T^+, u_T^- \in \mathbb{M} \cap \mathbb{K}$ for all T. Given $f \in \mathbb{M}$, set

$$\mathbb{B}_f := \{g \in \mathbb{L}\colon g(t_2) - g(t_1) \ge f(t_2) - f(t_1) \text{ for all } 0 \le t_1 < t_2 \le 1\}.$$

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The following lemma then holds.

Lemma A.3. *Suppose that, for $f \in \mathbb{V}$,*

$$\lim_{T\to\infty} \int_0^1 |u_T(t) - f(t)|\,\mathrm{d}t = 0. \tag{A.1}$$

Then, for the functions $f^{\pm} \in \mathbb{M}$ figuring in decomposition (2.1),

$$\lim_{T\to\infty} \inf_{g\in\mathbb{B}_{f^+}} \int_0^1 |u_T^+(t) - g(t)|\,\mathrm{d}t = 0, \qquad \lim_{T\to\infty} \inf_{g\in\mathbb{B}_{f^-}} \int_0^1 |u_T^-(t) - g(t)|\,\mathrm{d}t = 0.$$

Proof. Let us prove that $\lim_{T\to\infty} \inf_{g\in\mathbb{B}_{f^+}} \int_0^1 |u_T^+(t) - g(t)|\,\mathrm{d}t = 0$. Suppose the opposite; then there exists $\gamma > 0$ such that, for any $M > 0$, there exists $T > M$ such that $\inf_{g\in\mathbb{B}_{f^+}} \int_0^1 |u_T^+(t) - g(t)|\,\mathrm{d}t \geq \gamma$. Since the functions u_T^+ lie in a compact interval, this inequality implies that there exists a subsequence T_M and a function $\tilde{g}$ such that

$$\lim_{M\to\infty} \int_0^1 |u_{T_M}^+(t) - \tilde{g}(t)|\,\mathrm{d}t = 0, \qquad \inf_{g\in\mathbb{B}_{f^+}} \int_0^1 |\tilde{g}(t) - g(t)|\,\mathrm{d}t \geq \gamma,$$

and $\tilde{g}_{\mathbb{D}} \in \mathbb{M}$ because the functions $u_{T_M}^+$, $M = 1, 2, \ldots$, are monotone in t.

Therefore, it follows from (A.1) that $\lim_{M\to\infty} \int_0^1 |u_{T_M}^-(t) - (\tilde{g}(t) - f(t))|\,\mathrm{d}t = 0$. Then, since $u_T^- \in \mathbb{M}$, from this it follows that $\hat{g}_{\mathbb{D}}(t) := \tilde{g}_{\mathbb{D}}(t) - f_{\mathbb{D}}(t)$ also belongs to $\mathbb{M}$. Hence, $f_{\mathbb{D}}(t) = \tilde{g}_{\mathbb{D}}(t) - \hat{g}_{\mathbb{D}}(t)$, where $\tilde{g}_{\mathbb{D}} \notin B_{f^+}$ and $\hat{g}_{\mathbb{D}} \in \mathbb{M}$, which contradicts Lemma A.2.

In a similar fashion we can prove that $\lim_{T\to\infty} \inf_{g\in\mathbb{B}_{f^-}} \int_0^1 |u_T^-(t) - g(t)|\,\mathrm{d}t = 0$. □

The following result is a direct corollary of Lemma A.3.

Lemma A.4. *Let $\mathbb{K}$ be a compact set in $(\mathbb{L}, \rho)$. There exists $\delta(\varepsilon) > 0$ such that $\lim\limits_{\varepsilon\to 0} \delta(\varepsilon) = 0$ and for every $u \in \mathbb{K} \cap \mathbb{U}_\varepsilon(f)$ and u^+, u^- from the decomposition $u = u^+ - u^-$ (cf. (2.1)) the distances between $u^{\pm}$ and $\mathbb{B}_{f^{\pm}}$ satisfy*

$$\rho\big(u^+, \mathbb{B}_{f^+}\big) < \delta(\varepsilon), \quad \rho(u^-, \mathbb{B}_{f^-}) < \delta(\varepsilon).$$

Lemma A.5. *Suppose the function $u \in \mathbb{V}$ is increasing on $[0, 1]$. Let $\mathbb{B}_\varepsilon := \{g \in \mathbb{M} \colon \rho(g, u) < \varepsilon\}$. Then there exists $\delta(\varepsilon) > 0$ such that $\inf_{g\in\mathbb{B}_\varepsilon} g(1) \geq u_{\mathbb{D}}(1) - \delta(\varepsilon)$ and $\lim_{\varepsilon\to 0} \delta(\varepsilon) = 0$.*

Proof. Since $u_{\mathbb{D}}$ is increasing and left-continuous at $t = 1$, there exists a function $\gamma(\Delta) > 0$ such that $\lim_{\Delta\to 0} \gamma(\Delta) = 0$ and $\sup_{t\in[1-\gamma(\Delta),1]} (u_{\mathbb{D}}(1) - u_{\mathbb{D}}(t)) < \Delta$. Let us choose $\Delta(\varepsilon)$ so that $\Delta(\varepsilon)\gamma(\Delta(\varepsilon)) \geq \varepsilon$ and $\lim_{\varepsilon\to 0} \Delta(\varepsilon) = 0$. Put $\delta(\varepsilon) := 3\Delta(\varepsilon)$. Suppose that $\inf_{g\in\mathbb{B}_\varepsilon} g(1) < u_{\mathbb{D}}(1) - \delta(\varepsilon)$. Then the condition $\inf_{g\in\mathbb{B}_\varepsilon} \rho(g, u) < \varepsilon$ implies that there exists a function $g \in \mathbb{B}_\varepsilon$ such that

$$\begin{aligned}
\varepsilon > \int_{1-\gamma(\Delta(\varepsilon))}^1 |g(t) - u_{\mathbb{D}}(t)|\,\mathrm{d}t &\geq \int_{1-\gamma(\Delta(\varepsilon))}^1 (|g(t) - u_{\mathbb{D}}(1)| - |u_{\mathbb{D}}(1) - u_{\mathbb{D}}(t)|)\,\mathrm{d}t \\
&> \int_{1-\gamma(\Delta(\varepsilon))}^1 |g(t) - u_{\mathbb{D}}(1)|\,\mathrm{d}t - \Delta(\varepsilon)\gamma(\Delta(\varepsilon)) \\
&\geq \int_{1-\gamma(\varepsilon)}^1 |g(1) - u_{\mathbb{D}}(1)|\,\mathrm{d}t - \Delta(\varepsilon)\gamma(\Delta(\varepsilon)) \\
&> 2\gamma(\Delta(\varepsilon))\Delta(\varepsilon) - \Delta(\varepsilon)\gamma(\Delta(\varepsilon)) > \varepsilon.
\end{aligned}$$

This contradiction completes the proof of the lemma. □

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Lemma A.6. (The ET property) *Let condition* (1.5) *or* (1.6) *be satisfied. Then, for any* $C > 0$, *there exists a set* $\mathbb{K}_C \subset \mathbb{L}$, *compact in* $(\mathbb{L}, \rho)$, *such that*

$$\limsup_{T\to\infty} \frac{1}{\psi(T)} \ln \mathbb{P}\big(\xi_T(\cdot) \in \mathbb{K}_C^{\mathrm{c}}\big) \le -C,$$

where $\mathbb{K}_C^{\mathrm{c}} = \mathbb{L} \setminus \mathbb{K}_C$ *and* $\psi(T) = \varphi(T) \ln \varphi(T)$.

Proof. Take $\mathbb{K}_C := \mathbb{V}_{a(C)}$, where $a(C) := 3C/(1-l)$. Then

$$\begin{aligned}
\mathbb{P}(\xi_T \in \mathbb{K}_C^{\mathrm{c}}) &\le \mathrm{e}^T \mathbb{E}\bigg(\exp\{B_T + N_T \ln 2\}\mathbf{1}\bigg[\zeta_T \in \mathbb{K}_C^{\mathrm{c}}, \inf_{t\in[0,1]} \zeta_T(t) \ge 0\bigg]\bigg) \\
&\le \mathrm{e}^T \mathbb{E}\bigg(\exp\{B_T + N_T \ln 2\}\mathbf{1}\bigg[N_T \ge a(C)\varphi(T), \inf_{t\in[0,1]} \zeta_T(t) \ge 0\bigg]\bigg) \\
&= \mathrm{e}^T \sum_{r=\lfloor a(C)\varphi(T)\rfloor}^{\infty} \mathbb{E}\bigg(\exp\{B_T + N_T \ln 2\}\mathbf{1}\bigg[N_T = r, \inf_{t\in[0,1]} \zeta_T(t) \ge 0\bigg]\bigg) \\
&\le \mathrm{e}^T \sum_{r=\lfloor a(C)\varphi(T)\rfloor}^{\infty} \mathbb{E}\bigg(\exp\{B_T + N_T \ln 2\}\mathbf{1}\bigg[N_T = r, \zeta^+(T) \ge \frac{r}{2}\bigg]\bigg),
\end{aligned} \tag{A.2}$$

where the first inequality comes from (2.2), removing the A_T; the second inequality comes from the observation that the process should have at least $a(C)\varphi(T)$ jumps during time interval [0, T] to belong to the set $\mathbb{K}_C^{\mathrm{c}}$. The last inequality means that if the number of jumps in the time interval [0, 1] is r, then to guarantee the inequality $\inf_{t\in[0,1]} \zeta_T(t) \ge 0$ the number of positive jumps should be at least $r/2$.

Let us upper-bound B_T on the event $\{\omega\colon N_T = r, \zeta^+(T) \ge r/2\}$ with $r \ge \lfloor a(C)\varphi(T)\rfloor$. From condition (1.2) it follows that, for any $\gamma > 0$ and T large enough,

$$\begin{aligned}
B_T = \sum_{i=1}^{r} \ln\left(\nu(\zeta(t_{i-1}), \zeta(t_i))\right) &\le \zeta^-(T) \max_{1\le i\le r} \ln\left(\mu(i)\right) + \zeta^+(T) \max_{1\le i\le r} \ln\left(\lambda(i)\right) \\
&\le \zeta^-(T) \ln\left((1+\gamma)Qr\right) + \zeta^+(T) \ln\left((1+\gamma)Pr^l\right) \\
&= \zeta^-(T) \ln r + l\zeta^+(T) \ln r + r \ln M \\
&= (r - (1-l)\zeta^+(T)) \ln r + r \ln M \\
&\le \frac{r}{2}(1+l) \ln r + r \ln M.
\end{aligned} \tag{A.3}$$

Here, $M := (1+\gamma)^2 (Q \vee 1)(P \vee 1)$.

By using (A.2), (A.3), and the Stirling formula, we obtain that, for $r \ge \lfloor a(C)\varphi(T)\rfloor$ and T large enough,

$$\begin{aligned}
\mathbb{E}\bigg(\exp\{B_T + N_T \ln 2\}\mathbf{1}\bigg[N_T = r, \zeta^+(T) \ge \frac{r}{2}\bigg]\bigg) &\le \exp\bigg\{\frac{r}{2}(1+l) \ln r + r \ln(2M)\bigg\}\mathbb{P}(N_T = r) \\
&\le \mathrm{e}^{-T} \exp\bigg\{\frac{r}{2}(1+l) \ln r - r \ln r + r \ln(2TM\mathrm{e})\bigg\} \\
&= \mathrm{e}^{-T} \exp\bigg\{-\frac{r}{2}(1-l) \ln r + r \ln(2TM\mathrm{e})\bigg\} \\
&\le \mathrm{e}^{-T} \exp\bigg\{-\frac{r}{3}(1-l) \ln r\bigg\},
\end{aligned} \tag{A.4}$$

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where the last inequality is a consequence of the fact that under any of the conditions (1.5) or (1.6) the term $r \ln(2TMe)$ is $o((r/2)(1-l)\ln(r))$ as T tends to infinity. The inequalities (A.2) and (A.4) imply that

$$\limsup_{T\to\infty} \frac{1}{\psi(T)} \ln \mathbb{P}\big(\xi_T \in \mathbb{K}_C^{\mathrm{c}}\big) \le \lim_{T\to\infty} \frac{1}{\varphi(T)\ln\varphi(T)} \ln \exp\bigg\{-\frac{\lfloor a(C)\varphi(T)\rfloor}{3}(1-l)\ln\varphi(T)\bigg\} = -C. \qquad \square$$

Set $g_k := \mathrm{e}^{k\ln\varphi(T)}\mathrm{e}^{-T/2}(T/2)^k/k!$.

Lemma A.7. *For any $C > 0$ and $T > 2C$,* $\max_{0\le k\le C\varphi(T)} g_k = g_{\lfloor C\varphi(T)\rfloor}$.

Proof. Given $1 \le k \le \lfloor C\varphi(T)\rfloor$ where $T > 2C$, we have

$$\frac{g_k}{g_{k-1}} = \mathrm{e}^{\ln\varphi(T)}\frac{T/2}{k} = \frac{\varphi(T)T}{2k} \ge \frac{\varphi(T)T}{2\lfloor C\varphi(T)\rfloor} > 1.$$

Thus, the sequence g_k increases for $0 \le k \le \lfloor C\varphi(T)\rfloor$. $\square$

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Competing interests

There were no competing interests to declare which arose during the preparation or publication process of this article.

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